

SUBJECT : ENGINEERING PHYSICS.

YEAR : 1st Year

SEMESTER - 1

COURSE CONTENTS.

UNIT-I.

Introduction.

⇒ coordinate system:

→ Cartesian

→ Cylindrical and spherical coordinate system.

→ Transformations.

⇒ Differential calculus:

→ Gradient

→ Divergence.

→ curl and their physical significance.

⇒ Integral calculus:-

→ line

→ surface &

→ Volume Integrals.

⇒ Integral theorem:-

→ Gauss and Stokes theorems.

→ curvilinear coordinates.

⇒ Second derivatives:-

→ Laplacian

Coordinate system:

Physics is one of the fundamental natural sciences studying the laws of inanimate nature. Obviously, studying physics is an exciting and challenging adventure. The word physics comes from a Greek term meaning nature, and therefore physics should be a science dedicated to be study of all natural phenomenon.

The nature are, say, universe consists of material bodies which are in permanent interaction and motion. All the observable natural phenomenon are governed by certain laws. The study and the explanation of the laws governing the connections between the various processes and phenomenon are the fundamental aim of every branch of physics. And in general of science. The analysis of the interaction of material bodies and of the laws of electromagnetic phenomenon is the aim of physics.

The motion of the substance has various forms:

Mechanical, electromagnetic, thermal etc. The mechanical form of motion of the matter is the simplest one. It consists of the movement of the bodies or their parts relative to one another. We can see movements of bodies everywhere in our daily life. The laws of mechanical motion are studied in mechanics. The other branches of physics cannot be studied without mechanics ~~as~~ since various displacements are observed almost in all physical phenomenon.

Motion of a body or particle occurs both in space and in time. According to Newton, time is absolute and flows on without relation to the presence of any physical object or event. In this absolute space and time concept of Newton, motion of an object is characterised by change of its position in space as time evolves. Today, we know that in fact there is no object at rest in this absolute space with respect to which one can define the position of another object. Obviously at any instant of time, the position of an object is always defined by the observer with respect to himself. An observer, always first creates a local space, in which the positions of other objects are then described. This local space, to which is attached the observer, is called "Reference frame / -frame of reference".

The concept of rest, motion or acceleration is meaningful only when a frame of reference is specified. Newton's first law really ^{a statement} about reference frame. Actually the motion of a body can be described only relative to some other body.

Frame of reference and coordinate systems:

A frame of reference is a real or conditionally rigid body with respect to which the motion of the body being studied is to be considered. Rigidly fixed in the frame of reference is some kind of coordinate system so that the position of each point of a moving body can be uniquely determined by the three coordinates of the point. The following systems of coordinates are most frequently employed in mechanics.

(1) Rectangular (or) Cartesian Coordinate System (x, y, z):

In this coordinate system, the three dimensions are represented by the three axes x, y and z perpendicular to each other.

The coordinates of any point in space are taken as distances from the origin (O) along the three axes x, y and z and written as (x, y, z). This system is generally used where independent rectangular coordinate system is possible in space.

a) The left handed b) The right handed.

which cannot be made to coincide with each other by any means of translational or rotation.

Cartesian coordinate system by convention is chosen to be right handed as shown in the figure.

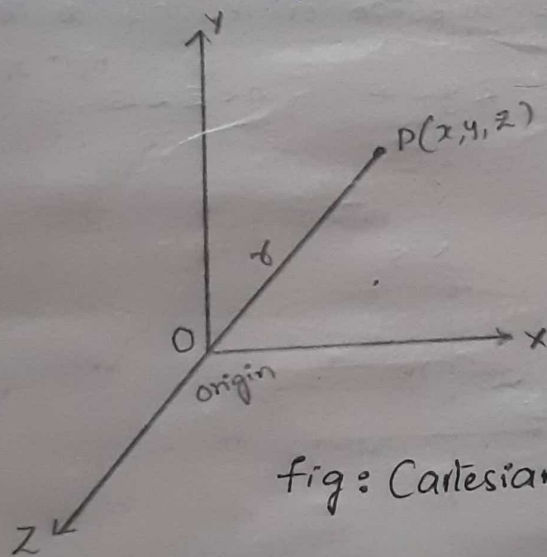


Fig: Cartesian coordinate system.

The position of the particle 'P' represented by the position vector $\vec{r}$. $\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

The vectorial increment 'dr' in ' $\vec{r}$ ' can be written as $d\vec{r}$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

The instantaneous velocity of 'P' in cartesian coordinate system is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

The magnitude velocity of particle P is given by

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Similarly, the instantaneous acceleration of P in rectangular coordinate system is given by

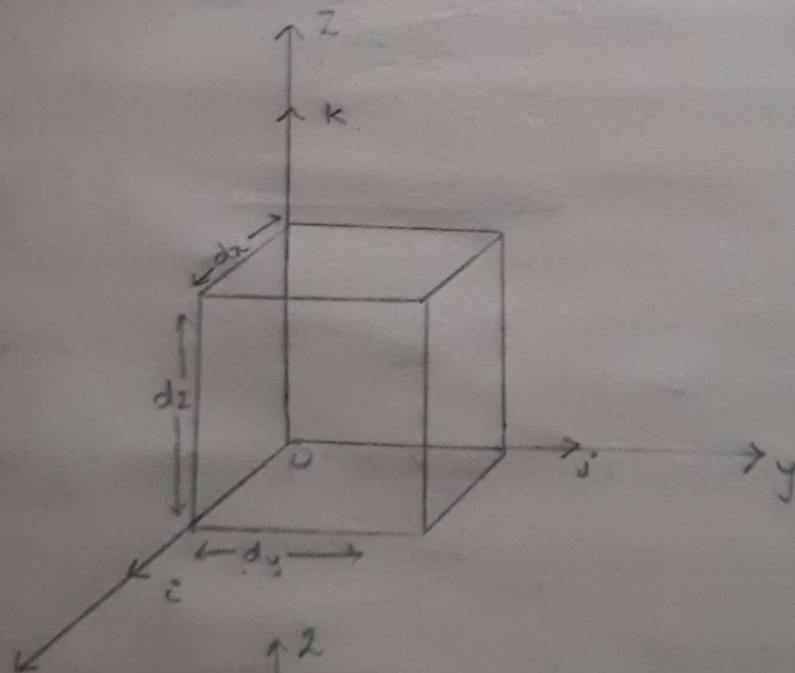
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k}$$

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

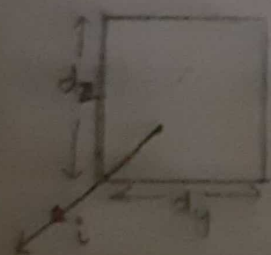
The magnitude acceleration of P is given by

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

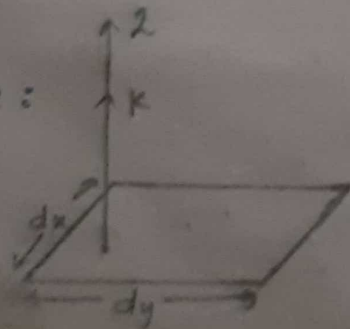
The ranges of the cartesian (or) rectangular coordinate variables (x, y, z) are $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$.



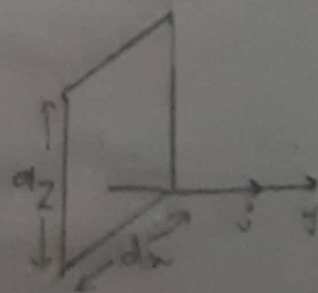
Differential surface:



$$ds_x = dydz \hat{i}$$



$$ds_z = dx dy \hat{k}$$



$$ds_y = dx dz \hat{j}$$

Differential volume :

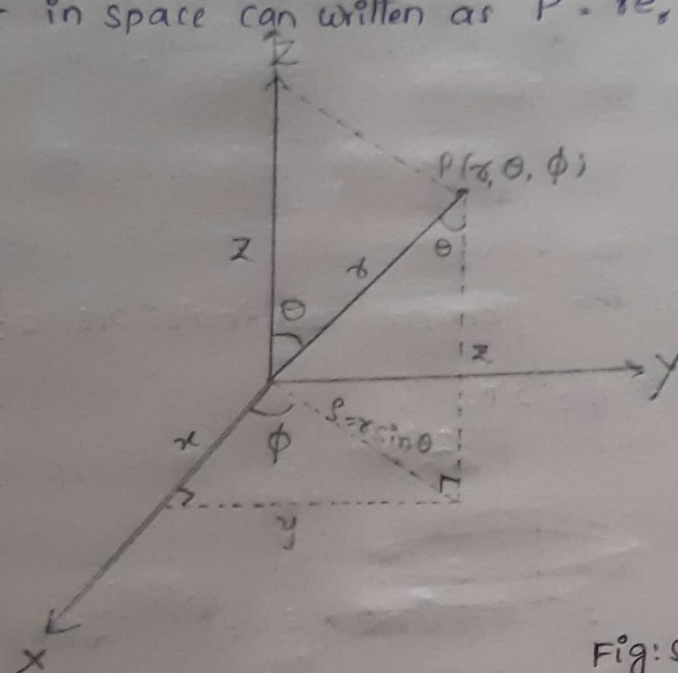
$$[dv = dr dy dz]$$

Spherical Coordinate Systems -

The coordinates of a point, say 'P', in this coordinate system are represented by radial vector r , the Zenithal altitude (or) polar angle (θ) & Azimuthal (or) longitudinal angle ' ϕ ' as shown in figure.

In spherical coordinate system the unit vectors are $\hat{e}_r, \hat{e}_\theta$ & $\hat{e}_\phi$

Any point in space can be written as $\vec{P} = r\hat{e}_r + \theta\hat{e}_\theta + \phi\hat{e}_\phi$



$$\cos \theta = \frac{z}{r}$$

$$z = r \cos \theta$$

$$\sin \theta = \frac{\rho}{r}$$

$$\rho = r \sin \theta$$

$$\tan \theta = \frac{r \sin \theta}{z}$$

Fig: Spherical polar coordinate system

The spherical polar coordinates (r, θ, ϕ) are related to the rectangular (or) Cartesian polar coordinates (x, y, z)

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\therefore r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan \phi = \frac{y}{x}$$

$$\cos \phi = \frac{x}{r \sin \theta}$$

$$x = r \sin \theta \cos \phi$$

$$\sin \phi = \frac{y}{r \sin \theta}$$

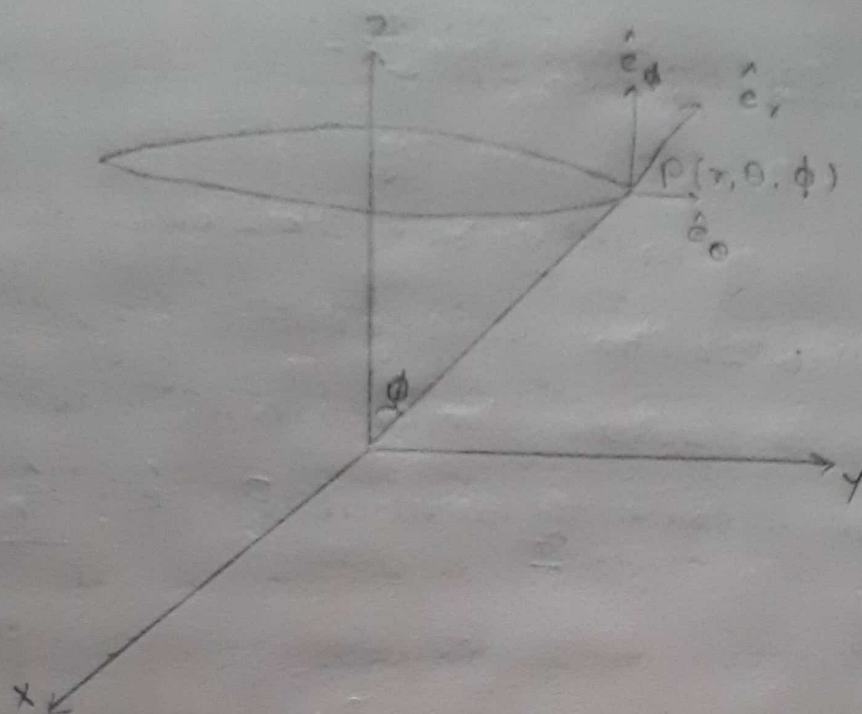
$$y = r \sin \theta \sin \phi$$

$$\tan \phi = \frac{y}{x}$$

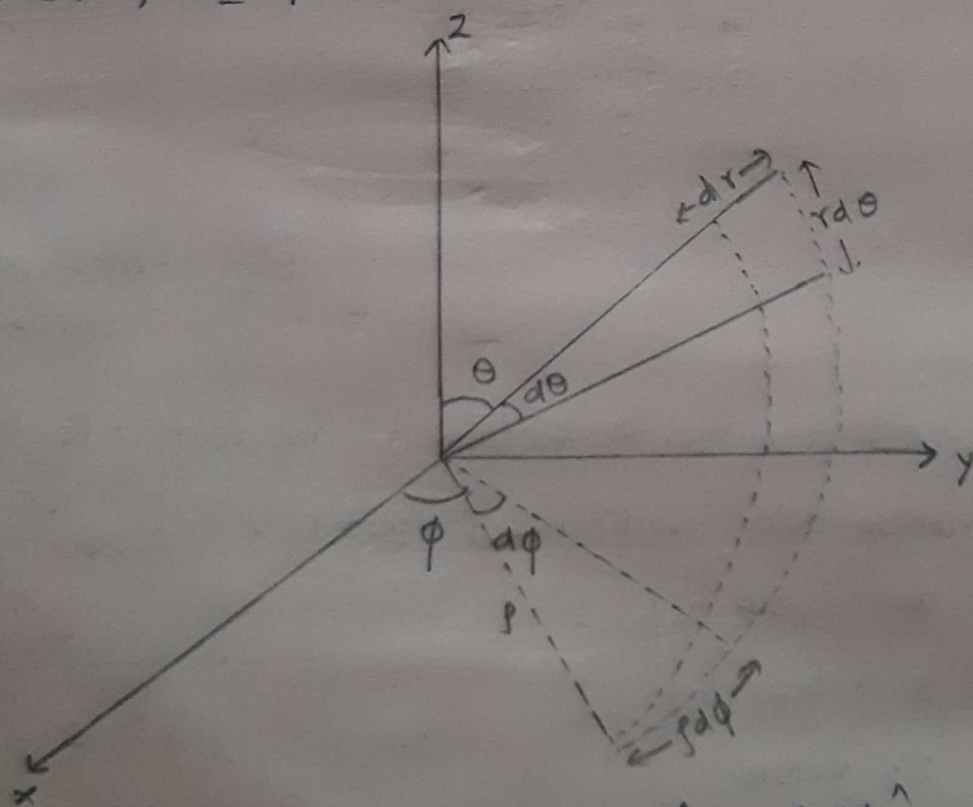
$$\sqrt{x^2 + y^2} = r \sin \theta$$

$$\tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

The unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ extended respectively in the direction of r increasing, θ increasing and ϕ increasing as shown in the figure.



The ranges of the spherical coordinates are $0 \leq r \leq \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$.



Differential length $d\vec{e} = dr\hat{e}_r + r d\theta\hat{e}_\theta + r \sin\theta d\phi\hat{e}_\phi$

$$d\vec{l} = dr\hat{e}_r + r d\theta\hat{e}_\theta + r \sin\theta d\phi\hat{e}_\phi \quad (\because r \sin\theta)$$

Differential surface along r -direction,

$$\begin{aligned} ds_r &= r d\theta \cdot r \sin\theta d\phi \hat{e}_r \\ &= r^2 \sin\theta d\theta d\phi \hat{e}_r \end{aligned}$$

Differential surface along θ -direction,

$$\begin{aligned} ds_\theta &= dr \cdot r \sin\theta d\phi \hat{e}_\theta \\ &= r \sin\theta dr d\phi \hat{e}_\theta \end{aligned}$$

Differential surface along ϕ -direction,

$$\begin{aligned} ds_\phi &= ~~r dr~~ dr \cdot r d\theta \hat{e}_\phi \\ &= r dr d\theta \hat{e}_\phi \end{aligned}$$

Differential volume :-

$$\begin{aligned} dv &= dr \cdot r d\theta \cdot r \sin\theta d\phi \\ &= r^2 \sin\theta dr d\theta d\phi \end{aligned}$$

Properties :-

The unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ can be expressed in terms of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ as follows:

$$\hat{e}_r = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}$$

$$\hat{e}_\theta = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}$$

$$\hat{e}_\phi = ~~\cos\theta~~ (-\sin\phi) \hat{i} + ~~\sin\theta~~ \cos\phi \hat{j}$$

$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_r \cdot \hat{e}_\phi = \hat{e}_\theta \cdot \hat{e}_\phi = 0$$

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$$

$$\hat{e}_\theta \times \hat{e}_\phi = \hat{e}_r$$

$$\hat{e}_\phi \times \hat{e}_r = \hat{e}_\theta$$

The spherical polar coordinates are very convenient in those problems of physics where there is no preferred direction and

the force in physical problem is spherically symmetrical.

Examples:

1. Coulomb force due to a point charge.
2. Gravitational force due to a point mass.

Cylindrical coordinate system :-

It is also called cylindrical circular coordinate system.

In cylindrical coordinate system the three unit vectors are $\hat{e}_\rho$, $\hat{e}_\phi$ and $\hat{e}_z$ along ρ , ϕ , and z directions respectively.

In this system, the position of particle $P(\rho, \phi, z)$ relative to origin 'O' is described by the two circular polar coordinates (ρ, ϕ) define in the plane $z=0$ and the third coordinate being the z coordinate itself.

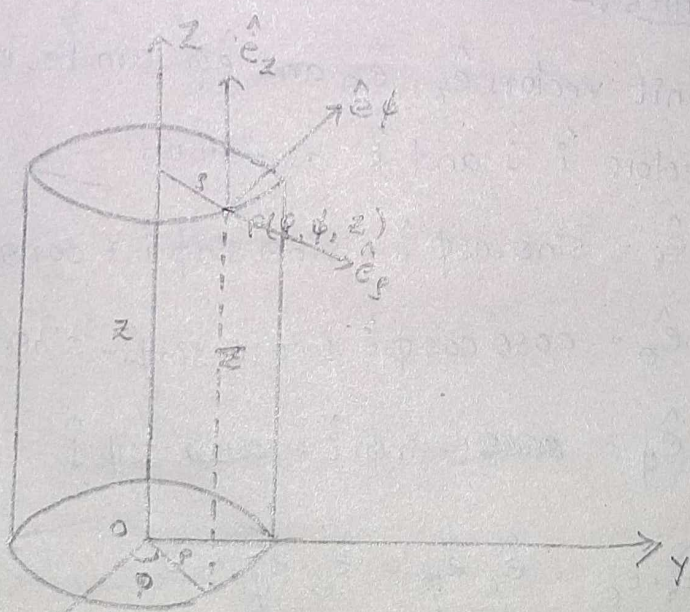


Fig: cylindrical coordinate system.

Here ' ρ ' is the radius of the cylinder passing through point 'P'. ' ϕ ' is the azimuthal angle which is measured from the x-axis in the xy-plane. And z is the same as in the Cartesian coordinate system.

The position of the particle 'P'

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z\hat{k}$$

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \tan \phi &= y/x \end{aligned}}$$

from fig.

$$\therefore \cos \phi = \frac{x}{\rho}$$

$$\Rightarrow x = \rho \cos \phi$$

$$\sin \phi = \frac{y}{\rho}$$

$$y = \rho \sin \phi$$

A vector $\vec{P}$ in cylindrical coordinate system can be written as (ρ, ϕ, z) .

$$\vec{P} = \rho \hat{e}_\rho + \phi \hat{e}_\phi + z \hat{e}_z$$

When $\hat{e}_\rho$, $\hat{e}_\phi$ and $\hat{e}_z$ are the unit vectors in the ρ, ϕ, z directions as shown in the figure.

The differential length in the cylindrical coordinate system is given by:

$$d\vec{l} = d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z$$

The differential surface in cylindrical coordinate system is given by

$$d\vec{S}_\rho = \rho d\phi dz \hat{e}_\rho$$

$$d\vec{S}_\phi = d\rho dz \hat{e}_\phi$$

$$d\vec{S}_z = d\rho \rho d\phi \hat{e}_z$$

The differential volume in cylindrical coordinate system is given by

$$dV = d\rho \rho d\phi dz = \rho d\rho d\phi dz$$

The coordinate surfaces are:

1. cylinders co-axial with the z -axis ($\rho = \text{constant}$)
2. Planes through the z -axis ($\phi = \text{constant}$)
3. planes perpendicular to the z -axis ($z = \text{constant}$)

The ranges of cylindrical coordinates are:

$$0 \leq \rho < \infty, \quad 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty$$

Vector properties:

$$\hat{e}_r \times \hat{e}_\phi = \hat{e}_z$$

$$\hat{e}_\phi \times \hat{e}_z = \hat{e}_r$$

$$\hat{e}_z \times \hat{e}_r = \hat{e}_\phi$$

$$\hat{e}_r \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_r = 0$$

cylindrical coordinates are preferred to describe the system where there is an axis of symmetry. The rotation of a particle about an axis is good example of this system of coordinates.

Problem:

1. If $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\hat{a}$ is the unit vector of $\vec{A}$, then find the ~~value~~ $\hat{a}$?

Sol: Given, $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$.

We know that,

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|}$$

$$= \frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{2^2 + 3^2 + 1}} = \frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{4+9+1}}$$

$$\hat{a} = \frac{2\hat{i} + 3\hat{j} + \hat{k}}{\sqrt{14}}$$

$$\hat{a} = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$$

* Convert $(2, 3, 5)$ into spherical coordinates.

Sol:- Given cartesian coordinates are $(2, 3, 5)$.

Spherical coordinates $(\rho, \theta, \phi) = ?$

$$\text{We know, } \rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$\therefore \rho = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$$

$$\theta = \tan^{-1} \frac{\sqrt{4+9}}{5} = \tan^{-1} 0.72 = 35.79$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \left(\frac{3}{2} \right) = 56.30993$$

* Find the cylindrical coordinates of the point $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

Sol:- Given cartesian coordinates are (x, y, z) is $(3, 4, 1)$

cylindrical coordinates are $(\rho, \phi, z) = ?$

$$\text{We know, } \rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{4}{3} = 53.13$$

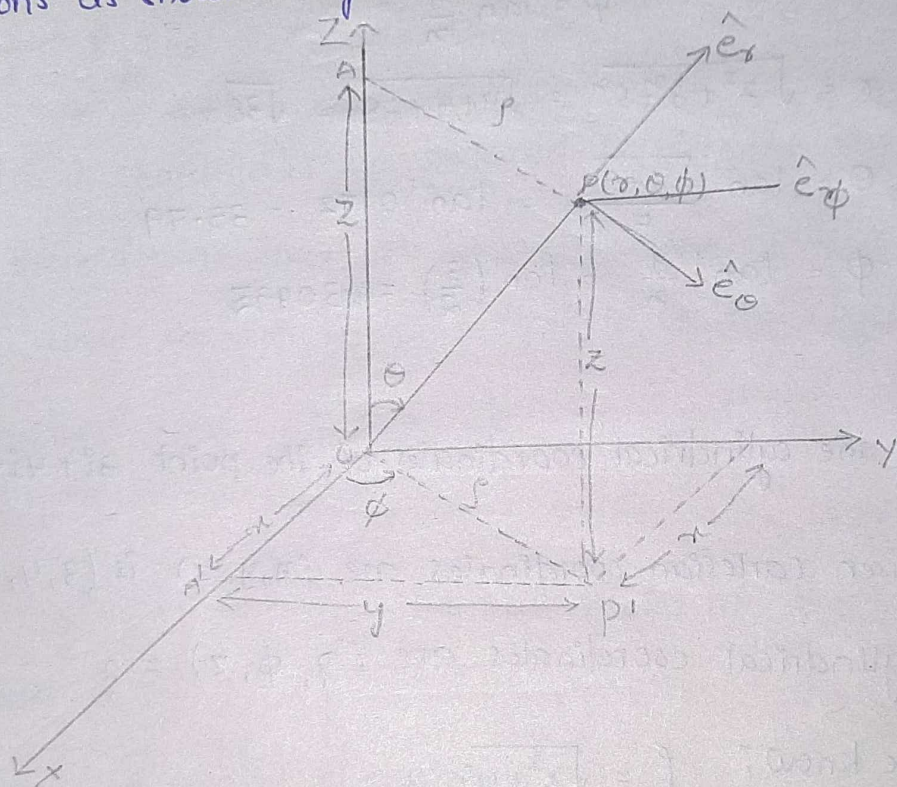
$$z = 1$$

$$\therefore (\rho, \phi, z) = (5, 53.1, 1)$$

Transformations:

1. From Cartesian coordinate system to spherical coordinate system:

Consider a point $P(r, \theta, \phi)$ in the spherical coordinate system. The unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ are the unit vectors along r, θ, ϕ directions as shown in figure.



Let us project the point $P(r, \theta, \phi)$ on to the x - y plane.

From figure $\triangle OPA$,

$$\sin \theta = \frac{\rho}{r} \Rightarrow \rho = r \sin \theta \rightarrow (1)$$

$$\cos \theta = \frac{z}{r} \Rightarrow z = r \cos \theta \rightarrow (2)$$

$$\tan \theta = \frac{\sqrt{x^2 + y^2}}{z} = \frac{\rho}{z}$$

$$\therefore \theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right) \rightarrow (3)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$r = \sqrt{x^2 + y^2 + z^2} \rightarrow (4)$$

$$(\because \rho = \sqrt{x^2 + y^2})$$

similarly from $\triangle OPA'$,

$$\cos \phi = \frac{x}{\rho} \Rightarrow x = \rho \cos \phi \longrightarrow (5)$$

$$\sin \phi = \frac{y}{\rho} \Rightarrow y = \rho \sin \phi \longrightarrow (6)$$

$$\tan \phi = \frac{y}{x} \Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow (7)$$

$$\rho = \sqrt{x^2 + y^2} \longrightarrow (8)$$

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right), \phi = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow (9)$$

$$\text{and } x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \longrightarrow (10)$$

Eqns (9) and (10) represents the transformation of coordinates from Cartesian to spherical and spherical to Cartesian respectively.

The unit vectors $\hat{i}, \hat{j}, \hat{k}$ and $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ are related as follows:

$$\hat{i} = \sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi$$

$$\hat{j} = \sin \theta (-\sin \phi) \hat{e}_r + \cos \theta (-\sin \phi) \hat{e}_\theta - \cos \phi \hat{e}_\phi$$

$$\hat{k} = \sin \theta \sin \phi \hat{e}_r + \cos \theta \sin \phi \hat{e}_\theta + \cos \theta \hat{e}_\phi$$

$$\hat{k} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta$$

$$\hat{e}_r = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

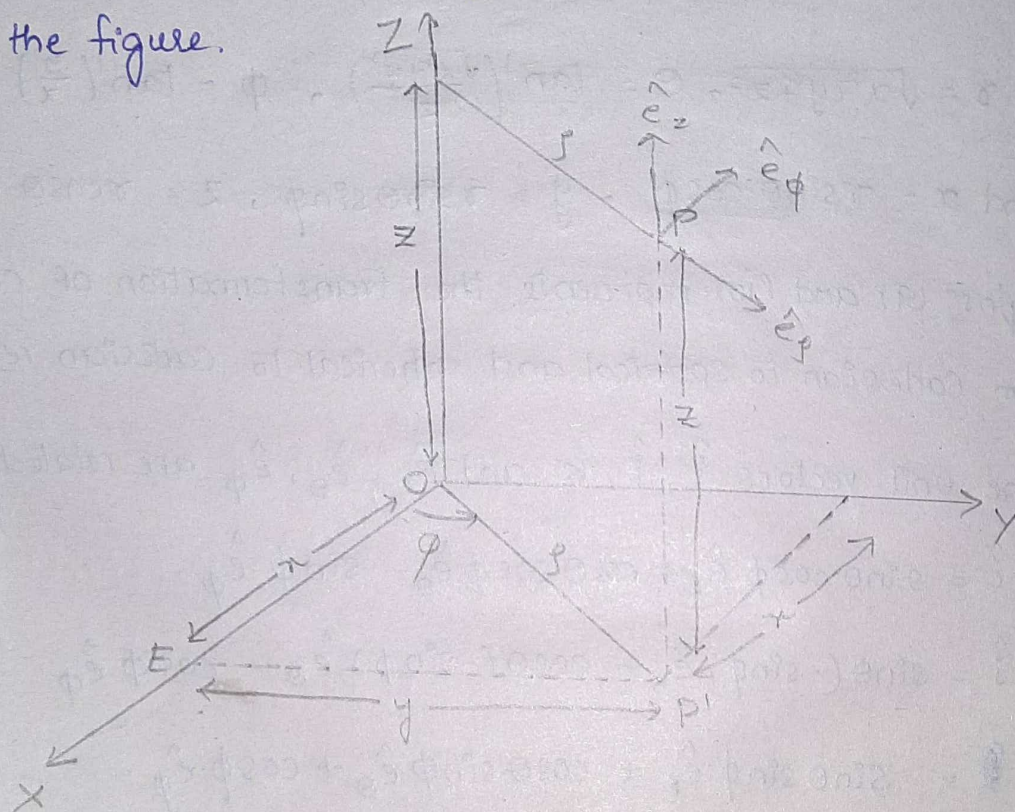
$$\hat{e}_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

$$\hat{e}_\phi = \cos \theta (-\sin \phi) \hat{i} + \cos \theta \cos \phi \hat{j}$$

$$\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

2. From Cartesian to Cylindrical Coordinate system:-

Consider a point $P(\rho, \phi, z)$ in a cylindrical coordinate system as shown in figure. $\hat{e}_\rho$, $\hat{e}_\phi$ and $\hat{e}_z$ are the unit vectors along ρ , ϕ , z directions respectively. The relationship between the values of Cartesian coordinates to cylindrical coordinates are obtained from the figure.



From $\Delta OEP'$,

$$\cos \phi = \frac{x}{\rho} \Rightarrow x = \rho \cos \phi \longrightarrow (1)$$

$$\sin \phi = \frac{y}{\rho} \Rightarrow y = \rho \sin \phi \longrightarrow (2)$$

$$\tan \phi = \frac{y}{x} \Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow (3)$$

$$\rho = \sqrt{x^2 + y^2} \longrightarrow (4)$$

$$z = z \longrightarrow (5)$$

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\left(\frac{y}{x}\right), z = z \longrightarrow (6)$$

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z \longrightarrow (7)$$

Eq's (6) & (7) represents the transformations of coordinates from Cartesian to cylindrical & cylindrical to Cartesian respectively.

The unit vectors $\hat{i}, \hat{j}, \hat{k}$ and $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z$ are related as follows:

$$\hat{i} = \cos \phi \hat{e}_\rho - \sin \phi \hat{e}_\phi$$

$$\hat{j} = \sin \phi \hat{e}_\rho + \cos \phi \hat{e}_\phi$$

$$\hat{k} = \hat{k} \text{ (or) } \hat{e}_z$$

$$\hat{e}_\rho = \cos \phi \hat{i} + \sin \phi \hat{j}$$

$$\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

$$\hat{e}_z = \hat{k} \text{ (or) } \hat{e}_z$$

Table:

From To	Cartesian	Spherical	Cylindrical
Cartesian	$x = x$ $y = y$ $z = z$	$x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$	$x = \rho \cos \phi$ $y = \rho \sin \phi$ $z = z$
Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right)$ $\phi = \tan^{-1} \left(\frac{y}{x} \right)$	$r = r$ $\theta = \theta$ $\phi = \phi$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1} \left(\frac{\rho}{z} \right)$ $\phi = \phi$
Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1} \left(\frac{y}{x} \right)$ $z = z$	$\rho = r \sin \theta$ $\phi = \phi$ $z = r \cos \theta$	$\rho = \rho$ $\phi = \phi$ $z = z$

Problems:-

1. Convert a point $(5, 60^\circ, 7)$ to cartesian co-ordinate?

Sol:- The given coordinate system is in cylindrical system.

$$(\rho, \phi, z) = (5, 60^\circ, 7) \therefore \rho = 5, \phi = 60^\circ, \text{ and } z = 7.$$

conversion of a point from cylindrical to cartesian is given by

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z.$$

$$\therefore x = 5 \cos 60^\circ = 5 \times \frac{1}{2} = \frac{5}{2} = 2.5.$$

$$y = 5 \sin 60^\circ = 5 \times \frac{\sqrt{3}}{2} = 2.5\sqrt{3} = 4.33.$$

$$z = z = 7.$$

$$\therefore (x, y, z) = (2.5, 4.33, 7)$$

2. Convert a point $(6, 45^\circ, 60^\circ)$ to cartesian and then cartesian to cylindrical?

sol:- The given coordinate system is in spherical system.

$$(\rho, \theta, \phi) = (6, 45^\circ, 60^\circ) \therefore \rho = 6, \theta = 45^\circ, \phi = 60^\circ.$$

spherical to cartesian, (x, y, z)

$$\begin{aligned} x &= \rho \sin \theta \cos \phi = 6 \sin 45^\circ \cos 60^\circ \\ &= 6 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{3}{\sqrt{2}} = 2.12 \end{aligned}$$

$$\begin{aligned} y &= \rho \sin \theta \sin \phi = 6 \sin 45^\circ \sin 60^\circ \\ &= 6 \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{\sqrt{2}} = 3.67 \end{aligned}$$

$$\begin{aligned} z &= \rho \cos \theta = 6 \cos 45^\circ \\ &= 6 \cdot \frac{1}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 4.24 \end{aligned}$$

~~spherical~~ cartesian coordinates are $(x, y, z) = (2.12, 3.67, 4.24)$

Cartesian to cylindrical, (ρ, ϕ, z)

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} = \sqrt{(2.12)^2 + (3.67)^2} = \sqrt{4.66 + 13.46} = \sqrt{18.12} \\ &= 4.25 \end{aligned}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3.67}{2.12}\right) = \tan^{-1}(1.73) = 59.97^\circ$$

$$z = z = 4.24$$

$\therefore$ cylindrical coordinates (ρ, ϕ, z) are $(4.25, 59.97^\circ, 4.24)$

Differential Calculus :-

In electromagnetics we come across two types of function of fields. They are 1. scalar field and 2. vector field.

1. scalar field :

A scalar quantity which can be specified at every point in a region of space is a scalar field.

Examples :-

Temperature field in a given volume,

pressure-field of air in atmosphere,

An example of scalar field in electromagnetism is the electric potential.

2. vector field :

A vector quantity which can be specified at every point in a region (or) space is called a vector field.

Examples :-

Gravitational force field of the earth,

Electric field due to a point charge,

Flow of water in a pipe.

Gradient :-

The del operator plays a very important role in vector calculus. It is not a vector.

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

"Del (∇)" is used to define gradient of a scalar field, divergence and curl of a vector field.

Let $\phi(x, y, z)$ a scalar function of position of a scalar point of coordinates (x, y, z) . The gradient of a scalar function ϕ is defined as

$$\text{grad } \phi \text{ (or) } \nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \phi$$

$$= \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

The del operator is a vector operator, when operated with a scalar (ϕ) converts the scalar into a vector.

The vector $\nabla \phi$ is called the gradient of scalar.

Physical significance for the gradient of a scalar field:
(formula for the gradient of a scalar field).

The scalar field can be mapped out by a series of level surfaces (surface where scalar field has constant values)

Let us consider two such surfaces, very close to each other as shown in figure. These surfaces are specified by constant scalar functions s and $s + ds$ respectively where ds indicates small change in s .

Let A and B are two points on these surfaces S and $S+ds$ respectively. O is a point outside the surfaces.

Let $\vec{r}$ and $\vec{r}+d\vec{r}$ be the radius vectors of points A and B with respect to origin 'O' respectively. The vector drawn from A to B will be $d\vec{r}$. [$\because AB = OB - OA = \vec{r} + d\vec{r} - \vec{r} = d\vec{r}$]

Let the normal drawn from the point A of surface S is AC. and $\hat{n}$ is the unit vector along the normal.

Now certainly AC will be minimum distance between the two surfaces. Suppose the angle between AB and AC is θ .

Now the magnitude of the rate of increase of S at point A in the direction AB is $\frac{\partial S}{\partial r}$.

Similarly, the rate of increase of S in the direction AC is $\frac{\partial S}{\partial n}$.

$$\therefore \Delta ABC, \cos \theta = \frac{AC}{AB} \text{ (or) } \partial n = \partial r \cos \theta$$

$$\cos \theta = \frac{\partial n}{\partial r}$$

$$\Rightarrow \partial n = \partial r \cos \theta$$

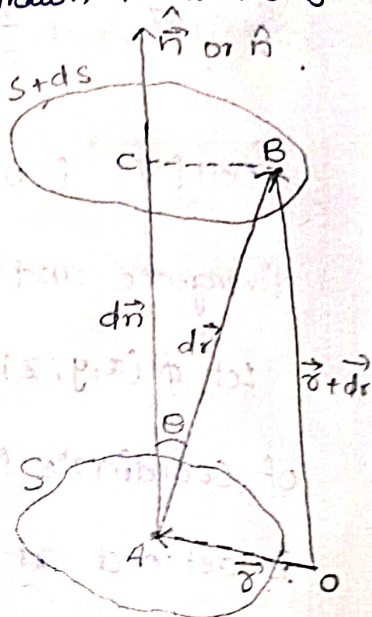
$$\therefore \frac{\partial S}{\partial n} = \frac{\partial S}{\partial r \cos \theta}$$

$$\Rightarrow \frac{\partial S}{\partial n} \cos \theta = \frac{\partial S}{\partial r} \longrightarrow (1)$$

From eq'n (1) $\frac{\partial S}{\partial r}$ is maximum when $\theta = 0^\circ$.

because $\cos \theta = 1$. so the maximum value of $\frac{\partial S}{\partial r}$ is $\frac{\partial S}{\partial n}$.

In this way the maximum rate of increase of a scalar function S at any point in a scalar field is given in magnitude and direction by the vector $\frac{\partial S}{\partial n} \hat{n}$. Where $\hat{n}$ is unit normal vector at that point.



The vector is defined as the gradient of the scalar field 'S' at that point and is written as

$$\text{grad } S = \frac{\partial S}{\partial n} \hat{n}$$

Thus the "gradient of a scalar field S is a vector whose magnitude at any point is equal to the maximum rate of increase of S at that point and whose direction is along the normal to the level surface at that point." This gives the physical significance of gradient of scalar field.

Gradient of ϕ in Rectangular coordinates:

Let $\phi(x, y, z)$ be the scalar function in cartesian coordinates.

By the direction of partial differentiation is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

By the definition of $\text{grad } \phi$, we have

$$\text{grad } \phi = \frac{\partial \phi}{\partial n} \hat{n}$$

Consider the scalar product of $\text{grad } \phi$ with an element of radius vector dx .

$$\text{We have } \text{grad } \phi \cdot dx = \frac{\partial \phi}{\partial n} \hat{n} \cdot dr$$

$$(\because \hat{n} \cdot dr = n dr \cos \theta = dr \cos \theta (\because n=1))$$

$$\text{grad } \phi \cdot dr = \frac{\partial \phi}{\partial n} dn = d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\Rightarrow \text{grad } \phi \cdot dr = \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$\therefore \text{grad } \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} = \nabla \cdot \phi$$

$$\therefore \boxed{\text{grad } \phi = \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}}$$

problems:-

1. If $S = S(x, y, z) = x^2 - x^2y + xy^2z^2$, then find the value of $\text{grad } S$ at that point $(2, -1, -3)$?

Sol:- We know that $\text{grad } S = \nabla S$

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

$$\text{grad } S = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 - x^2y + xy^2z^2)$$

$$= \hat{i} \frac{\partial}{\partial x} (x^2 - x^2y + xy^2z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 - x^2y + xy^2z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 - x^2y + xy^2z^2)$$

$$\text{grad } S = \hat{i} (2x - 2xy + y^2z^2) + \hat{j} (-x^2 + 2xyz^2) + \hat{k} (2xy^2z)$$

$\text{grad } S$ at point $(2, -1, -3)$.

$$\therefore \text{grad } S = \hat{i} [2(2) - 2(2)(-1) + (-1)^2(-3)^2] + \hat{j} [-(2)^2 + 2(2)(-1)(-3)] + \hat{k} [2(2)(-1)^2(-3)]$$

$$= \hat{i} [4 + 4 + 9] + \hat{j} [-4 - 36] + \hat{k} [-12]$$

$$\therefore \boxed{\text{grad } S = 17\hat{i} - 40\hat{j} - 12\hat{k}}$$

2. The electrical potential in a region of space is given by $V = 5x - 7x^2y + 8y^2 + 16yz - 5z$ volt where distances are measured in metre. Deduce an expression for the electric field intensity. Calculate the y-component of the field at point $(2, 4, -3)$

Sol:- We know that $E = -\text{grad } V$

$$\text{grad } V = \nabla V = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) V$$

$$\text{Given } V = 5x - 7x^2y + 8y^2 + 16yz - 5z$$

Then,

$$\text{grad } V = i \frac{\partial V}{\partial x} + j \frac{\partial V}{\partial y} + k \frac{\partial V}{\partial z}$$

$$= i \frac{\partial}{\partial x} (5x - 7x^2y + 8y^2 + 16yz - 5z)$$

$$+ j \frac{\partial}{\partial y} (5x - 7x^2y + 8y^2 + 16yz - 5z)$$

$$+ k \frac{\partial}{\partial z} (5x - 7x^2y + 8y^2 + 16yz - 5z)$$

$$= i(5 - 14xy) + j(-7x^2 + 16y + 16z) + k(16y - 5)$$

$$\therefore E = -\text{grad } V = i(14xy - 5) + j(7x^2 - 16y - 16z) + k(5 - 16y)$$

The y-component of field is

$$E_y = 7x^2 - 16y - 16z$$

At point (2, 4, -3)

$$E_y = 7(2)^2 - 16(4) - 16(-3) = 7(4) - 64 + 48$$

$$= 28 - 64 + 48 = 12 \text{ V/m}$$

$$\boxed{\therefore E_y = 12 \text{ V/m}}$$

3. In what direction from the point (1, 1, -1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is a maximum? Also find the value of this maximum directional derivative.

$$\text{grad } f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$f = x^2 - 2y^2 + 4z^2$$

$$\text{grad } f = i \frac{\partial}{\partial x} (x^2 - 2y^2 + 4z^2) + j \frac{\partial}{\partial y} (x^2 - 2y^2 + 4z^2) + k \frac{\partial}{\partial z} (x^2 - 2y^2 + 4z^2)$$

$$\text{grad } f = (2x)\hat{i} + (-4y)\hat{j} + (8z)\hat{k}$$

$$\nabla f = 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

grad f at point (1, 1, -1)

$$(\text{grad } f)_{(1,1,-1)} = i[2(1)] + j[-4(1)] + k[8(-1)]$$

$$= 2\hat{i} - 4\hat{j} - 8\hat{k}$$

So the directional derivative of f is maximum in the direction $\text{grad } f$.

The maximum value of this directional derivative is given by

$$|\text{grad } f| = \sqrt{(-2)^2 + (-4)^2 + (-8)^2} = \sqrt{4+16+64}$$

$$|\text{grad } f| = \sqrt{84} = 9.164$$

4. A field is expressed by the potential function $V = 3x^2z - xyz$. Calculate the potential at the point $(1, -2, +2)$

Sol:- $V_{\text{at}(1, -2, +2)} = 3(1)^2(-2) - (1)(-2)(2) = -6 + 4 = -2$ Volts.

5. A vector field $\vec{A}$ is represented by the function $\vec{A}$,
 $\vec{A} = i(2xy - x^4) + j(yz^2) - k(xy^2)$, then find the values of

$$\frac{\partial A}{\partial x}, \frac{\partial^2 A}{\partial x^2} \text{ and } \frac{\partial^2 A}{\partial y \partial z} ?$$

Sol:- Given that,

$$\vec{A} = \hat{i}(2xy - x^4) + \hat{j}(yz^2) - \hat{k}(xy^2)$$

$$(i) \frac{\partial \vec{A}}{\partial x} = \hat{i} \frac{\partial}{\partial x}(2xy - x^4) + \hat{j} \frac{\partial}{\partial x}(yz^2) + \hat{k} \frac{\partial}{\partial x}(-xy^2)$$

$$\frac{\partial \vec{A}}{\partial x} = \hat{i}(4xy - 4x^3) + \hat{j}(0) - \hat{k}(y^2)$$

$$\frac{\partial A}{\partial x} = \hat{i}(4xy - 4x^3) - \hat{k}y^2$$

$$(ii) \frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial A}{\partial x} \right) = \frac{\partial}{\partial x} (\hat{i}(4xy - 4x^3) - \hat{k}y^2)$$

$$= \hat{i} \frac{\partial}{\partial x}(4xy - 4x^3) - \hat{k} \frac{\partial}{\partial x}(y^2)$$

$$= \hat{i}(4y - 12x^2) - \hat{k}(0)$$

$$\frac{\partial^2 A}{\partial x^2} = (4y - 12x^2)\hat{i}$$

$$(iii) \frac{\partial^2 A}{\partial y \partial z} = \frac{\partial}{\partial z} \left(\frac{\partial A}{\partial y} \right) = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial y} (\hat{i}(2xy - x^4) + \hat{j}(yz^2) - \hat{k}(xy^2)) \right)$$

$$= \frac{\partial}{\partial z} (\hat{i}2x^2 + \hat{j}z^2 - \hat{k}xy) = \hat{j}2z$$

$$\therefore \frac{\partial^2 A}{\partial y \partial z} = 2z\hat{j}$$

6. Find the gradient of $\phi = x^2z - x^2y + xy^2z^2$ at $(-2, -1, -3)$?

Sol: Given that $\phi = x^2z - x^2y + xy^2z^2$

$$\text{grad } \phi = \nabla \phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2z - x^2y + xy^2z^2)$$

$$= i \frac{\partial}{\partial x} (x^2z - x^2y + xy^2z^2) + j \frac{\partial}{\partial y} (x^2z - x^2y + xy^2z^2) + k \frac{\partial}{\partial z} (x^2z - x^2y + xy^2z^2)$$

$$= i(2xz - 2xy + y^2z^2) + j(-x^2 + 2xy^2z) + k(x^2 + 2xy^2z)$$

$$\begin{aligned} \text{grad } \phi &= i(2(-2)(-3) - 2(-2)(-1) + (-1)^2(-3)^2) \\ &\quad + j(-(-2)^2 + 2(-2)(-1)(-3)^2) + k((-2)^2 + 2(-2)(-1)(-3)) \end{aligned}$$

$$= \hat{i}(12 - 4 + 9) + \hat{j}(-4 + 36) + \hat{k}(4 + 12)$$

$$= \hat{i}(17) + \hat{j}(32) + 16\hat{k}$$

$$= 17\hat{i} + 32\hat{j} + 16\hat{k}$$

7. If $\vec{r}$ is the position vector of a point, calculate the gradient of $\frac{1}{r}$.

Sol: We know that, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\text{grad} \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \nabla (x^2 + y^2 + z^2)^{-1/2}$$

$$= i \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-1/2} + j \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-1/2} + k \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-1/2}$$

$$= i \left(-\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2x \right) + j \left(-\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2y \right) + k \left(-\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2z \right)$$

$$= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \hat{i} - \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \hat{j} - \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \hat{k} = -\frac{\vec{r}}{r^3}$$

$$= -\frac{x}{r^3} \hat{i} - \frac{y}{r^3} \hat{j} - \frac{z}{r^3} \hat{k} = -\frac{1}{r^3} (x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}$$

$$\text{grad} \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3} = -\frac{\vec{r}}{|\vec{r}|^3} = -\frac{\hat{r}}{r^2}$$

$$\therefore \text{grad} \left(\frac{1}{r} \right) = -\frac{\hat{r}}{r^2}$$

8. Convert gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (where r is the magnitude of position vector)?

Sol: $\text{grad } r = \nabla r = \left(\hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} \right)$

$$\text{grad } r = \hat{i} \frac{\partial}{\partial x} (\sqrt{x^2 + y^2 + z^2}) + \hat{j} \frac{\partial}{\partial y} (\sqrt{x^2 + y^2 + z^2}) + \hat{k} \frac{\partial}{\partial z} (\sqrt{x^2 + y^2 + z^2})$$

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (2x) ; \quad \frac{\partial r}{\partial y} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (2y) ; \quad \frac{\partial r}{\partial z} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (2z)$$

$$\text{grad } r = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \hat{k}$$

$$= \frac{1}{\sqrt{x^2 + y^2 + z^2}} [x\hat{i} + y\hat{j} + z\hat{k}]$$

$$\text{grad } r = \frac{\vec{r}}{|\vec{r}|}$$

$\therefore \boxed{\text{grad } r = \hat{r}}$ Where $\hat{r}$ is the unit vector.

9. If $\vec{r}$ is the position vector such that $\phi = \log |\vec{r}|$. Find $\text{grad } \phi$

Sol: Given, $\phi = \log |\vec{r}|$

We know, $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\text{grad } \phi = \nabla \phi = \hat{i} \frac{\partial}{\partial x} \log |\vec{r}| + \hat{j} \frac{\partial}{\partial y} \log |\vec{r}| + \hat{k} \frac{\partial}{\partial z} \log |\vec{r}|$$

$$= \hat{i} \frac{\partial}{\partial x} [\log (\sqrt{x^2 + y^2 + z^2})] + \hat{j} \frac{\partial}{\partial y} [\log (\sqrt{x^2 + y^2 + z^2})] + \hat{k} \frac{\partial}{\partial z} [\log (\sqrt{x^2 + y^2 + z^2})]$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[\frac{2x}{2} \right] ; \quad \frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[\frac{2y}{2} \right] ; \quad \frac{\partial \phi}{\partial z} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[\frac{2z}{2} \right]$$

$$= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[\frac{2x}{2} \hat{i} + \frac{2y}{2} \hat{j} + \frac{2z}{2} \hat{k} \right]$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \hat{k}$$

$$= \frac{1}{|\vec{r}|^2} [x\hat{i} + y\hat{j} + z\hat{k}] = \frac{\vec{r}}{|\vec{r}|^2} = \frac{\hat{r}}{|\vec{r}|} \quad \left[\because \hat{r} = \frac{\vec{r}}{|\vec{r}|} \right]$$

10. If ϕ and ψ are differential scalar function of x, y, z then prove that,

$$a) \text{grad}(\phi + \psi) = \text{grad} \phi + \text{grad} \psi$$

$$b) \text{grad}(\phi \cdot \psi) = \phi \text{grad} \psi + \psi \text{grad} \phi$$

$$\text{Sol: } a) \text{grad}(\phi + \psi) = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot [\phi + \psi]$$

$$= i \frac{\partial}{\partial x} (\phi + \psi) + j \frac{\partial}{\partial y} (\phi + \psi) + k \frac{\partial}{\partial z} (\phi + \psi)$$

$$= i \frac{\partial}{\partial x} \phi + i \frac{\partial}{\partial x} \psi + j \frac{\partial}{\partial y} \phi + j \frac{\partial}{\partial y} \psi + k \frac{\partial}{\partial z} \phi + k \frac{\partial}{\partial z} \psi$$

$$= \left[i \frac{\partial}{\partial x} \phi + j \frac{\partial}{\partial y} \phi + k \frac{\partial}{\partial z} \phi \right] + \left[i \frac{\partial}{\partial x} \psi + j \frac{\partial}{\partial y} \psi + k \frac{\partial}{\partial z} \psi \right]$$

$$\therefore \text{grad}(\phi + \psi) = \text{grad} \phi + \text{grad} \psi$$

Hence proved.

$$b) \text{grad}(\phi \cdot \psi) = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] (\phi \cdot \psi)$$

$$= i \frac{\partial}{\partial x} (\phi \cdot \psi) + j \frac{\partial}{\partial y} (\phi \cdot \psi) + k \frac{\partial}{\partial z} (\phi \cdot \psi)$$

$$= i \left[\phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x} \right] + j \left[\phi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \phi}{\partial y} \right] + k \left[\phi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \phi}{\partial z} \right]$$

$$= \left[i \phi \frac{\partial \psi}{\partial x} + i \psi \frac{\partial \phi}{\partial x} + j \phi \frac{\partial \psi}{\partial y} + j \psi \frac{\partial \phi}{\partial y} + k \phi \frac{\partial \psi}{\partial z} + k \psi \frac{\partial \phi}{\partial z} \right]$$

$$= i \phi \frac{\partial \psi}{\partial x} + j \phi \frac{\partial \psi}{\partial y} + k \phi \frac{\partial \psi}{\partial z} + i \psi \frac{\partial \phi}{\partial x} + j \psi \frac{\partial \phi}{\partial y} + k \psi \frac{\partial \phi}{\partial z}$$

$$= \phi \left[i \frac{\partial \psi}{\partial x} + j \frac{\partial \psi}{\partial y} + k \frac{\partial \psi}{\partial z} \right] + \psi \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \phi \text{grad} \psi + \psi \text{grad} \phi$$

$$\therefore \text{grad}(\phi \cdot \psi) = \phi \text{grad} \psi + \psi \text{grad} \phi$$

Hence proved.

Divergence of a vector field:

The operator ∇ can be involved in the multiplication with a vector. The scalar or dot product of operator with a vector $\vec{A}$ is called as divergence. It is denoted by

$$\boxed{\text{div } A = \nabla \cdot A}$$

def: The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point.

The divergence is a scalar.

Let A be a vector function differentiable at each point (x, y, z) in a region of space.

the divergence of A is given by

$$\text{div } A = \nabla \cdot \vec{A} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (A_x \hat{i} + A_y \hat{j} + A_z \hat{k})$$

$$\Rightarrow \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\therefore \boxed{\text{div } A = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

This is the expression of divergence in cartesian coordinates

physical significance of divergence:

If $\vec{A}$ represents the velocity of a fluid (liquid/gas) then $\text{div } \vec{A}$ gives the rate of flow of the fluid at that point per unit volume. It also ~~volume~~ measure of how much that vector spreads out (diverges) from the point.

(a) If the divergence of a vector ($\text{div } \vec{A}$) is positive at that point in a fluid, then the fluid is expanding and its density at that point is falling with time or the point as a source of fluid as shown in figure.

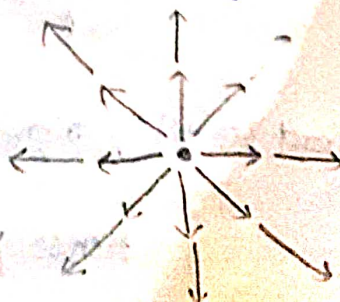


fig: divergence is positive

(b) If divergence of a vector ($\text{div } \vec{A}$) is negative, then the fluid is contracting and its density is rising or the point serves as a sink as shown in figure.

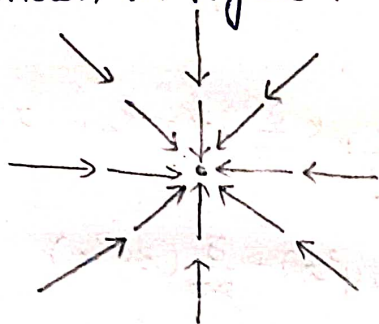


fig: Divergence is negative.

(c) If divergence of vector ($\text{div } \vec{A}$) is zero, then the fluid entering and leaving the element is the same. i.e., there is no change in the density of the fluid or the fluid is incompressible. The vector satisfying this condition is called solenoidal.

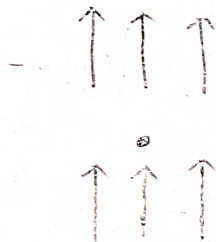


fig. Divergence is zero

problems :-

11. If $\vec{A} = \hat{i}y + \hat{j}(x^2+y^2) + \hat{k}(yz+zx)$ then find $\text{div } \vec{A}$ at point $(1, -2, 3)$.

Sol: Given that, $\vec{A} = \hat{i}y + \hat{j}(x^2+y^2) + \hat{k}(yz+zx)$

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \vec{A}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (\hat{i}y + \hat{j}(x^2+y^2) + \hat{k}(yz+zx))$$

$$= \frac{\partial y}{\partial x} + \frac{\partial}{\partial y}(x^2+y^2) + \frac{\partial}{\partial z}(yz+zx)$$

$$= 0 + 2y + y + x = 3y + x$$

$$\text{div } \vec{A} \text{ at } (1, -2, 3) = \nabla \cdot \vec{A} = 3(-2) + 1 = -6 + 1 = -5$$

$$\therefore \boxed{\text{div } \vec{A} \text{ at } (1, -2, 3) = -5}$$

12. Evaluate $\text{div } \vec{F}$, where $\vec{F} = 2x^3z\hat{i} - xy^2z\hat{j} + 3y^2x\hat{k}$?

Sol: Given that, $\vec{F} = 2x^3z\hat{i} - xy^2z\hat{j} + 3y^2x\hat{k}$.

$$\text{div } \vec{F} = ?$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (2x^3z\hat{i} - xy^2z\hat{j} + 3y^2x\hat{k})$$

$$= \frac{\partial}{\partial x} (2x^3z) - \frac{\partial}{\partial y} (xy^2z) + \frac{\partial}{\partial z} (3y^2x)$$

$$= 6x^2z - 2xyz + 0$$

$$\therefore \boxed{\text{div } \vec{F} = \nabla \cdot \vec{F} = 6x^2z - 2xyz}$$

13. If $\vec{r}$ is the position vector of a point, then find

a) $\text{div } \vec{r}$

b) $\nabla(\vec{r} \cdot \vec{A})$

Sol: We know, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

$$\text{a) } \text{div } \vec{r} = \nabla \cdot \vec{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z)$$

$$= 1 + 1 + 1 = 3$$

$$\therefore \boxed{\text{div } \vec{r} = 3}$$

$$\text{b) } \nabla(\vec{r} \cdot \vec{A}) = \nabla[(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})]$$

$$= \nabla(xA_x + yA_y + zA_z)$$

$$\Rightarrow \nabla(\vec{r} \cdot \vec{A}) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (xA_x + yA_y + zA_z)$$

$$= \hat{i} \frac{\partial}{\partial x} (xA_x + yA_y + zA_z) + \hat{j} \frac{\partial}{\partial y} (xA_x + yA_y + zA_z) + \hat{k} \frac{\partial}{\partial z} (xA_x + yA_y + zA_z)$$

$$\nabla(\vec{r} \cdot \vec{A}) = \hat{i} \frac{\partial}{\partial x} (xA_x) + \hat{j} \frac{\partial}{\partial y} (yA_y) + \hat{k} \frac{\partial}{\partial z} (zA_z)$$

$$\Rightarrow \nabla(\vec{r} \cdot \vec{A}) = i A_x \frac{\partial}{\partial x} + j A_y \frac{\partial}{\partial y} + k A_z \frac{\partial}{\partial z}$$

$$\Rightarrow \nabla(\vec{r} \cdot \vec{A}) = i A_x(1) + j A_y(1) + k A_z(1) \\ = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = \vec{A}$$

$$\therefore \boxed{\nabla(\vec{r} \cdot \vec{A}) = \vec{A}}$$

14. Find the value of $\nabla \cdot (\hat{i}(x+y) + \hat{j}(x-y) + \hat{k}(4z)) = ?$

Sol. $\nabla \cdot [\hat{i}(x+y) + \hat{j}(x-y) + \hat{k}(4z)]$
 $= (i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}) \cdot (\hat{i}(x+y) + \hat{j}(x-y) + \hat{k}(4z))$
 $= \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(4z)$
 $= 1 + (-1) + 4 = 1 - 1 + 4 = 4$

$$\therefore \nabla \cdot (\hat{i}(x+y) + \hat{j}(x-y) + \hat{k}(4z)) = 4$$

15. Find the value of $\text{div}(\vec{r}^n \hat{r})$, where $\hat{r}$ is the unit vector.

Sol. We know that, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \quad \& \quad (\vec{r}^n \hat{r}) = (x^2 + y^2 + z^2)^{\frac{n}{2}} \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} = (x^2 + y^2 + z^2)^{\frac{n-1}{2}} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\text{div}(\vec{r}^n \hat{r}) = \nabla \cdot (\hat{r}^n \hat{r}) = \nabla \cdot (\vec{r}^n \cdot \vec{r}^{-1} \vec{r}) = \nabla \cdot (\vec{r}^{n-1} \vec{r})$$

$$= [i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}] \cdot (\vec{r}^{n-1} \vec{r})$$

$$= (i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}) \cdot \vec{r}^{n-1} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x} (x \vec{r}^{n-1}) + \frac{\partial}{\partial y} (y \vec{r}^{n-1}) + \frac{\partial}{\partial z} (z \vec{r}^{n-1})$$

$$= [\vec{r}^{n-1} + x \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{\frac{n-1}{2}}] + [\vec{r}^{n-1} + y \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{\frac{n-1}{2}}] \\ + [\vec{r}^{n-1} + z \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{\frac{n-1}{2}}]$$

$$= 3r^{n-1} + x\left(\frac{n-1}{r}\right)(x^2+y^2+z^2)^{\frac{n-1}{2}-1}x + y\left(\frac{n-1}{r}\right)(x^2+y^2+z^2)^{\frac{n-1}{2}-1}y$$

$$+ z\left(\frac{n-1}{r}\right)(x^2+y^2+z^2)^{\frac{n-1}{2}-1}z$$

$$= 3r^{n-1} + x^2(n-1)(x^2+y^2+z^2)^{\frac{n-3}{2}} + y^2(n-1)(x^2+y^2+z^2)^{\frac{n-3}{2}} + z^2(n-1)(x^2+y^2+z^2)^{\frac{n-3}{2}}$$

$$= 3r^{n-1} + (n-1)\left[(x^2+y^2+z^2)^{\frac{n-3}{2}}x^2 + (x^2+y^2+z^2)^{\frac{n-3}{2}}y^2 + (x^2+y^2+z^2)^{\frac{n-3}{2}}z^2\right]$$

$$= 3r^{n-1} + (n-1)(x^2+y^2+z^2)^{\frac{n-3}{2}}[x^2+y^2+z^2]$$

$$= 3r^{n-1} + (n-1)(x^2+y^2+z^2)^{\frac{n-3}{2}+1} = 3r^{n-1} + (n-1)(x^2+y^2+z^2)^{\frac{n-1}{2}}$$

$$= 3r^{n-1} + (n-1)r^{n-1}$$

$$= r^{n-1}(3+n-1) = r^{n-1}(n+2)$$

$$= (n+2)r^{n-1}$$

$$\therefore \boxed{\operatorname{div} (r^n \hat{r}) = \nabla \cdot (r^n \hat{r}) = (n+2)r^{n-1}}$$

16. Find the value of constant C for which the vector
 $A = i(x+3y) + j(y-2z) + k(x+Cz)$ is solenoidal.

Sol:- Solenoidal means the divergence becomes zero.

$$\therefore \operatorname{div} A = \nabla \cdot A = 0$$

$$\nabla \cdot A = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot [i(x+3y) + j(y-2z) + k(x+Cz)]$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+Cz) = 0$$

$$\Rightarrow 1 + 1 + C = 0$$

$$\Rightarrow 2 + C = 0 \Rightarrow C = -2$$

$$\therefore \boxed{C = -2}$$

17. The electric field due to a point charge is expressed as $E = \frac{Q}{r^2} \hat{r}$. Show that the divergence of electric field due to that point charge is zero.

Sol: Given $E = \frac{Q}{r^2} \hat{r} = \frac{Q}{r^2} \cdot \frac{\vec{r}}{|\vec{r}|} = \frac{Q \cdot \vec{r}}{r^3}$

$$\text{div } E = \nabla \cdot E = \nabla \cdot \left(\frac{Q}{r^3} \vec{r} \right)$$

$$\Rightarrow \nabla \cdot E = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \frac{Q}{(\sqrt{x^2+y^2+z^2})^3} (xi+yj+zk)$$

$$= \frac{\partial}{\partial x} \left[\frac{Q}{(x^2+y^2+z^2)^{3/2}} x \right] + \frac{\partial}{\partial y} \left[\frac{Q}{(x^2+y^2+z^2)^{3/2}} y \right] + \frac{\partial}{\partial z} \left[\frac{Q}{(x^2+y^2+z^2)^{3/2}} z \right]$$

$$= Q \left[\frac{\partial}{\partial x} \left[\frac{x}{(\sqrt{x^2+y^2+z^2})^3} \right] + \frac{\partial}{\partial y} \left[\frac{y}{(\sqrt{x^2+y^2+z^2})^3} \right] + \frac{\partial}{\partial z} \left[\frac{z}{(\sqrt{x^2+y^2+z^2})^3} \right] \right]$$

$$= Q \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{\frac{3}{2}x}{(x^2+y^2+z^2)^{5/2}} \cdot 2x \right]$$

$$+ Q \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{\frac{3}{2}y}{(x^2+y^2+z^2)^{5/2}} \cdot 2y \right]$$

$$+ Q \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{\frac{3}{2}z}{(x^2+y^2+z^2)^{5/2}} \cdot 2z \right]$$

$$= Q \left[\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} \right] + \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} \right] + \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} \right] \right]$$

$$= Q \left\{ \frac{3}{|r|^3} - \frac{3x^2}{|r|^5} - \frac{3y^2}{|r|^5} - \frac{3z^2}{|r|^5} \right\}$$

$$= Q \left\{ \frac{3}{|r|^3} - \frac{3}{|r|^5} (x^2+y^2+z^2) \right\}$$

$$= Q \left\{ \frac{3}{|r|^3} - \frac{3}{|r|^3} (r^2) \right\}$$

$$= Q \left\{ \frac{3}{|r|^3} - \frac{3}{|r|^3} \right\}$$

$$\Rightarrow \nabla \cdot E = Q \{0\}$$

$\Rightarrow \nabla \cdot E = 0$ Hence proved.

18. Show that $\text{div grad}(R^n) = n(n+1)R^{n-2}$, where R is position vector.

Sol:- $\text{div grad}(R^n) = \nabla \cdot \nabla(R^n)$

$$\text{grad}(R^n) = \nabla(R^n) = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] R^n$$

$$= i \frac{\partial}{\partial x} R^n + j \frac{\partial}{\partial y} R^n + k \frac{\partial}{\partial z} R^n$$

$$= \hat{i} n R^{n-1} \frac{\partial R}{\partial x} + \hat{j} n R^{n-1} \frac{\partial R}{\partial y} + \hat{k} n R^{n-1} \frac{\partial R}{\partial z}$$

$$= n R^{n-1} \left[i \frac{\partial R}{\partial x} + j \frac{\partial R}{\partial y} + k \frac{\partial R}{\partial z} \right]$$

Since $R = xi + yj + zk$

$$|R| = \sqrt{x^2 + y^2 + z^2}$$

$$R^2 = x^2 + y^2 + z^2$$

$$\therefore \frac{\partial}{\partial x} (R^2) = 2R \frac{\partial R}{\partial x}$$

$$\frac{\partial}{\partial x} (x^2 + y^2 + z^2) = 2x$$

$$\Rightarrow 2R \frac{\partial R}{\partial x} = 2x \Rightarrow \frac{\partial R}{\partial x} = \frac{x}{R}$$

Similarly, $\frac{\partial R}{\partial y} = \frac{y}{R}$, $\frac{\partial R}{\partial z} = \frac{z}{R}$

$$\Rightarrow \nabla(R^n) = n R^{n-1} \left[\hat{i} \frac{x}{R} + \hat{j} \frac{y}{R} + \hat{k} \frac{z}{R} \right]$$

$$\therefore \nabla \cdot \nabla(R^n) = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot n R^{n-1} \left[i \frac{x}{R} + j \frac{y}{R} + k \frac{z}{R} \right]$$

$$= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot n R^{n-2} [xi + yj + zk]$$

$$= \frac{\partial}{\partial x} (n R^{n-2} x) + \frac{\partial}{\partial y} (n R^{n-2} y) + \frac{\partial}{\partial z} (n R^{n-2} z)$$

$$= n \left\{ R^{n-2} + x(n-2) R^{n-3} \frac{\partial R}{\partial x} \right\} + n \left\{ R^{n-2} + y(n-2) R^{n-3} \frac{\partial R}{\partial y} \right\} \\ + n \left\{ R^{n-2} + z(n-2) R^{n-3} \frac{\partial R}{\partial z} \right\}$$

$$= n \left\{ 3R^{n-2} + x(n-2) R^{n-3} \frac{\partial R}{\partial x} + y(n-2) R^{n-3} \frac{\partial R}{\partial y} + z(n-2) R^{n-3} \frac{\partial R}{\partial z} \right\}$$

$$= n \left\{ 3R^{n-2} + x(n-2) R^{n-3} \frac{x}{R} + y(n-2) R^{n-3} \frac{y}{R} + z(n-2) R^{n-3} \frac{z}{R} \right\}$$

$$= n \left\{ 3R^{n-2} + (n-2) R^{n-3} \frac{x^2}{R} + (n-2) R^{n-3} \frac{y^2}{R} + (n-2) R^{n-3} \frac{z^2}{R} \right\}$$

$$= 3nR^{n-2} + n(n-2) R^{n-3} \left[\frac{x^2 + y^2 + z^2}{R} \right]$$

$$= 3nR^{n-2} + n(n-2) R^{n-3} \left[\frac{R^2}{R} \right]$$

$$= 3nR^{n-2} + n(n-2) R^{n-3} R$$

$$= 3nR^{n-2} + n(n-2) R^{n-3} \cdot R^1 = 3nR^{n-2} + n(n-2) R^{n-3+1}$$

$$= 3nR^{n-2} + n(n-2) R^{n-2}$$

$$= nR^{n-2} (3 + (n-2)) = nR^{n-2} (3+n-2)$$

$$= nR^{n-2} (n+1)$$

$$= n(n+1) R^{n-2}$$

$$\therefore \boxed{\operatorname{div} \operatorname{grad} (R^n) = n(n+1) R^{n-2}} \quad \text{Hence proved.}$$

19. If $A = xz^3 \hat{i} - 2x^2yz \hat{j} - 2yz^4 \hat{k}$. then find $\operatorname{div} A$ at point $(1, -1, 1)$?

$$\text{Sol: } \operatorname{div} A = \nabla \cdot A = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot [xz^3 \hat{i} - 2x^2yz \hat{j} - 2yz^4 \hat{k}]$$

$$= \frac{\partial}{\partial x} [xz^3] - \frac{\partial}{\partial y} [2x^2yz] - \frac{\partial}{\partial z} [2yz^4]$$

$$= z^3 - 2x^2z - 8yz^3$$

$$\operatorname{div} A_{(1, -1, 1)} = (1)^3 - 2(1)^2(-1) - 8(-1)(1)^3 = 1 - 2 + 8 = 9 - 2 = 7$$

20. calculate the divergence for the following fields.

a) $E = a \{ 3x^2yz \hat{i} + [x^3z + yz^2(yz)^{1/2}] \hat{j} + [x^3y + y^2z(yz)^{1/2}] \hat{k} \}$

b) $B = yi - xj$.

c) $B = a \{ y\hat{i}/(x^2+y^2)^{3/2} - x\hat{k}/(x^2+y^2)^{3/2} \}$

Sol: a) Given,

$$E = a \{ 3x^2yz \hat{i} + [x^3z + yz^2(yz)^{1/2}] \hat{j} + [x^3y + y^2z(yz)^{1/2}] \hat{k} \}$$

$$\text{div } E = \nabla \cdot E = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \cdot E$$

$$= \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \cdot a [3x^2yz \hat{i} + [x^3z + yz^2(yz)^{1/2}] \hat{j} + [x^3y + y^2z(yz)^{1/2}] \hat{k}]$$

$$= \frac{\partial}{\partial x} (3ax^2yz) + \frac{\partial}{\partial y} (ax^3z + ayz^2(yz)^{1/2}) + \frac{\partial}{\partial z} (ax^3y + ay^2z(yz)^{1/2})$$

$$= 6axyz + a \frac{\partial}{\partial y} (x^3z + ay^{3/2}z^{5/2}) + \frac{\partial}{\partial z} (ax^3y + ay^{5/2}z^{3/2})$$

$$= 6axyz + \frac{3}{2} ay^{1/2}z^{5/2} + \frac{3}{2} ay^{5/2}z^{1/2}$$

$$= 3a \left(2axyz + \frac{1}{2} y^{1/2}z^{5/2} + \frac{1}{2} y^{5/2}z^{1/2} \right)$$

$$\therefore \boxed{\nabla \cdot E = 3a \left(2axyz + \frac{1}{2} y^{1/2}z^{5/2} + \frac{1}{2} y^{5/2}z^{1/2} \right)}$$

b) $\text{div } B = \nabla \cdot B = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \cdot [y\hat{i} - x\hat{j}]$

$$= \frac{\partial}{\partial x} [y] + \frac{\partial}{\partial y} (-x)$$

$$= 0 - 0$$

$$\therefore \boxed{\text{div } B = 0}$$

$$c) \quad B = a \left\{ \frac{y \hat{i}}{(x^2+y^2)^{3/2}} - \frac{x \hat{k}}{(x^2+y^2)^{3/2}} \right\}$$

$$\nabla \cdot B = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \cdot \left[\frac{ay}{(x^2+y^2)^{3/2}} \hat{i} + \frac{-ax}{(x^2+y^2)^{3/2}} \hat{k} \right]$$

$$= \frac{\partial}{\partial x} \left[\frac{ay}{(x^2+y^2)^{3/2}} \right] - \frac{\partial}{\partial y} (0) + \frac{\partial}{\partial z} \left[\frac{-ax}{(x^2+y^2)^{3/2}} \right]$$

$$= \frac{\partial}{\partial x} \left[ay \cdot (x^2+y^2)^{-3/2} \right] - \frac{\partial}{\partial z} \left[ax \cdot (x^2+y^2)^{-3/2} \right]$$

$$= \left(ay \cdot -\frac{3}{2} [x^2+y^2]^{-5/2} \cdot 2x \right) - 0$$

$$= axy (-3)(x^2+y^2)^{-5/2}$$

$$\nabla \cdot B = \frac{-3axy}{(x^2+y^2)^{5/2}} = a \left\{ \frac{-3xy}{(x^2+y^2)^{5/2}} \right\}$$

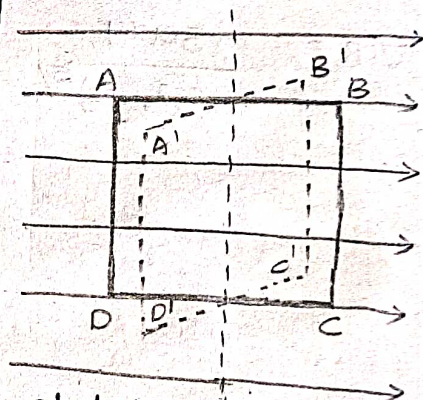
Curl of a vector field:

Consider that a vector field derived as a gradient of the scalar field. In this case, the line integral of the vector around any closed path in the field is zero such a field is called as lamellar or non curl field.

However, there are vector fields for which line integral round any closed path is non-zero. These fields can not be derived as the gradient of a scalar field, moreover they show the property of curl (or rotation)

Explanation :-

~~The curl of a vector field is defined as the maximum line integral of the vector per unit area. It is essentially a vector quantity. The direction is normal to the area.~~



Consider two areas $ABCD$ and $A'B'C'D'$ in a uniform electric field which is represented by straight parallel lines as shown in the adjacent figure. The area $A'B'C'D'$ is perpendicular to lines of force. So the contribution of line integrals is zero. For area $ABCD$, the line integrals along AD and BC are zero while the line integrals along AB and CD are not zero. This shows that there is a certain orientation of the area for which the line integral is maximum.

The curl of a vector field is defined as the maximum line integral of the vector per unit area. It is essentially a vector quantity. The direction is normal to the area.

If A is a vector function differentiable at each point (x, y, z) in a region of space, then curl (or rotation) of A expressed by the cross product of ∇ and A , i.e.,

$$\text{curl } A = \nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

This is the expression of curl in cartesian coordinates.

(or)

Curl: The maximum line integral of vector per unit area.

The cross product of an operator ∇ with ~~a~~ vector $\vec{A}$ is called as 'curl'. curl is a vector.

Let $\vec{A}$ be a vector function differentiable at any point (x, y, z) in a region of space.

The curl of $\vec{A}$ is given by

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial}{\partial y} (A_z) - \frac{\partial}{\partial z} (A_y) \right) - \hat{j} \left(\frac{\partial}{\partial x} (A_z) - \frac{\partial}{\partial z} (A_x) \right) + \hat{k} \left(\frac{\partial}{\partial x} (A_y) - \frac{\partial}{\partial y} (A_x) \right)$$

Physical significance of curl:

The curl of a vector function $\vec{V}$ ($\nabla \times \vec{V}$) is a measure of how much the vector $\vec{V}$ rotates around that point.

When curl V is zero, it means that no rotation is attached with vector $\vec{V}$, where curl $\vec{V}$ is non-zero, it means that rotation is attached with vector $\vec{V}$ as shown in below figure

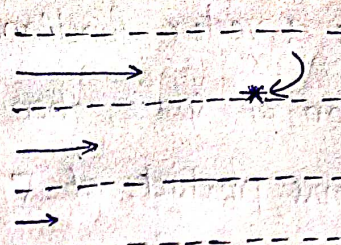
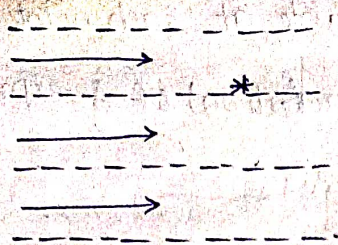


fig: a) curl is zero

fig: b) curl $\vec{v}$ is non-zero

The curl of a vector $\vec{A}$ is zero, then $\vec{A}$ is called irrotational.

Problems:-

Q1. If $\vec{r}$ is the position vector find the curl $\vec{r}$?

Sol:- We know, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\begin{aligned}\text{curl } \vec{r} &= \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y) \right) - \hat{j} \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x) \right) + \hat{k} \left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right) \\ &= \hat{i}(0) - \hat{j}(0) + \hat{k}(0) \\ &= 0\end{aligned}$$

$$\therefore \boxed{\text{curl } \vec{r} = 0}$$

Q2. Show that $\text{curl grad } \phi = 0$?

Sol:- $\text{curl grad } \phi = \nabla \times \text{grad } \phi = \nabla \times \nabla \phi$

$$\nabla \phi = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (\phi)$$

$$= \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \times \nabla \phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$\begin{aligned}
 &= i \left(\frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial y} \right) \right) - j \left(\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial x} \right) \right) + k \left(\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) \right) \\
 &= i \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right) - j \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right) + k \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right) \\
 &= i(0) - j(0) + k(0) = 0.
 \end{aligned}$$

Hence proved.

23. Show that $A = e^x(-yz \mathbf{i} + z \mathbf{j} + y \mathbf{k})$ is irrotational?

Sol. When $\text{curl } A$ is zero, then the function is called irrotational.

Given that, $A = e^x(-yz \mathbf{i} + z \mathbf{j} + y \mathbf{k})$

$$\therefore \text{curl } A = \nabla \times A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz e^x & z e^x & y e^x \end{vmatrix}$$

$$\begin{aligned}
 &= i \left(\frac{\partial}{\partial y} (y e^x) - \frac{\partial}{\partial z} (z e^x) \right) - j \left(\frac{\partial}{\partial x} (y e^x) - \frac{\partial}{\partial z} (-y z e^x) \right) \\
 &\quad + k \left(\frac{\partial}{\partial x} (z e^x) - \frac{\partial}{\partial y} (-y z e^x) \right)
 \end{aligned}$$

$$= i [e^x - e^x] - j [(-1)y e^x - (-y e^x)] + k [-z e^x - (-z e^x)]$$

$$= i [e^x - e^x] - j [-y e^x + y e^x] + k [-z e^x + z e^x]$$

$$\nabla \times A = i(0) - j(0) + k(0) = 0.$$

$\therefore$ Given A is irrotational.

24. Find constants a, b, c so that the vector A is irrotational.

$$A = i(x + 2y + az) + j(bx - 3y - z) + k(4x + cy + 2z)$$

Sol. The vector A is irrotational when $\text{curl } A = 0$.

Therefore, $\text{curl } A = \nabla \times A = 0$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0 \\
 &\quad (x+2y+az) \quad (bx-3y-z) \quad (4x+cy+2z)
 \end{aligned}$$

$$= i \left\{ \frac{\partial}{\partial y} (4x + cy + 2z) - \frac{\partial}{\partial z} (bx - 3y - z) \right\} + j \left\{ \frac{\partial}{\partial x} (4x + cy + 2z) - \frac{\partial}{\partial z} (x + 2y + az) \right\}$$

$$- j \left\{ \frac{\partial}{\partial x} (4x + cy + 2z) - \frac{\partial}{\partial z} (x + 2y + az) \right\}$$

$$+ k \left\{ \frac{\partial}{\partial x} (bx - 3y - z) - \frac{\partial}{\partial y} (x + 2y + az) \right\}$$

$$= i \{c - (-1)\} - j \{4 - a\} + k \{b - 2\} = i \{c + 1\} + j \{a - 4\} + k \{b - 2\}$$

$$\Rightarrow i \{c + 1\} + j \{a - 4\} + k \{b - 2\} = 0 \quad [\because \text{Given } \nabla \times A = 0]$$

$$\Rightarrow c + 1 = 0 ; a - 4 = 0 ; b - 2 = 0$$

$$c = -1 , a = 4 , b = 2$$

$$\therefore \boxed{a = 4, b = 2, c = -1}$$

25. Find the value of $\text{curl} (a \times r)$, where a is a constant vector.

Sol: We know, $a = a_x i + a_y j + a_z k$ & $r = x i + y j + z k$

$$a \times r = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$$= i(a_y z - a_z y) - j(a_x z - a_z x) + k(a_x y - a_y x)$$

$$\text{Now } \text{curl} (a \times r) = \nabla \times (a \times r)$$

$$\therefore \nabla \times (a \times r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_y z - a_z y) & -(a_x z - a_z x) & (a_x y - a_y x) \end{vmatrix}$$

$$= i \left(\frac{\partial}{\partial y} (a_x y - a_y x) - \frac{\partial}{\partial z} (a_y z - a_z y) \right) - j \left(\frac{\partial}{\partial x} (a_x y - a_y x) - \frac{\partial}{\partial z} (a_y z - a_z y) \right)$$

$$+ k \left(\frac{\partial}{\partial x} (a_x z - a_z x) - \frac{\partial}{\partial y} (a_y z - a_z y) \right)$$

$$= i[ax+ax] - j[-ay-ay] + k[a_z+a_z]$$

$$= i[2ax] + j[2ay] + k[2a_z]$$

$$= 2[a_x i + a_y j + a_z k] = 2a.$$

$$\therefore \boxed{\text{curl}(ax) = 2a}$$

26 If $A = xz^3 \hat{i} - 2x^2yz \hat{j} - 2yz^4 \hat{k}$, show that $\text{curl } A$ at the point $(1, -1, 1)$ is $(-4\hat{i} + 3\hat{j} + 4\hat{k})$?

Sol:

$$\text{Given, } A = xz^3 \hat{i} - 2x^2yz \hat{j} - 2yz^4 \hat{k}$$

$$\text{curl } A = \nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & -2yz^4 \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y}(-2yz^4) - \frac{\partial}{\partial z}(-2x^2yz) \right] - j \left[\frac{\partial}{\partial x}(-2yz^4) - \frac{\partial}{\partial z}(xz^3) \right] + k \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3) \right]$$

$$= i[-2z^4 - (-2x^2)] - j[0 - 3xz^2] + k[-4xyz - 0]$$

$$\nabla \times A_{(1, -1, 1)} = i[-2(1)^4 - (-2(1)^2(-1))] - j[0 - 3(1)(1)^2] + k[-4(1)(-1)(1) - 0]$$

$$= i[-2 - 2] + j[3] + k[4]$$

$$= -4\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\therefore \boxed{\nabla \times A_{(1, -1, 1)} = -4\hat{i} + 3\hat{j} + 4\hat{k}}$$

27. Calculate the curl of the fields:

$$a) E = a \left\{ 3x^2yz \hat{i} + [x^3z + yz^2(yz)^{1/2}] \hat{j} + [x^3y + y^2z(yz)^{1/2}] \hat{k} \right\}$$

Sol. $\text{curl } E = \nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a3x^2yz & ax^3z + ay^{3/2}z^{5/2} & ax^3y + ay^{5/2}z^{3/2} \end{vmatrix}$

$$= \hat{i} \left[\frac{\partial}{\partial y} [ax^3y + ay^{5/2}z^{3/2}] - \frac{\partial}{\partial z} [ax^3z + ay^{3/2}z^{5/2}] \right]$$

$$- \hat{j} \left[\frac{\partial}{\partial x} (ax^3y + ay^{5/2}z^{3/2}) - \frac{\partial}{\partial z} (3ax^2yz) \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial x} (ax^3z + ay^{3/2}z^{5/2}) - \frac{\partial}{\partial y} (3ax^2yz) \right]$$

$$= \hat{i} \left[ax^3 + \frac{5}{2} ay^{3/2}z^{3/2} \right] - \left[ax^3 + \frac{5}{2} ay^{3/2}z^{3/2} \right]$$

$$+ \hat{j} [3ax^2y - 3ax^2y] + \hat{k} [3ax^2z - 3ax^2z]$$

$$= \hat{i} \left[\cancel{ax^3} - \cancel{ax^3} + \frac{5}{2} \cancel{ay^{3/2}z^{3/2}} - \frac{5}{2} \cancel{ay^{3/2}z^{3/2}} \right] + \hat{j} [3\cancel{ax^2y} - 3\cancel{ax^2y}] + \hat{k} [3\cancel{ax^2z} - 3\cancel{ax^2z}]$$

$$= \hat{i} [0 - 0] + \hat{j} [0] + \hat{k} [0]$$

$$= 0.$$

$$\therefore \boxed{\text{curl } E = 0}$$

$$b) B = y\hat{i} - x\hat{j}$$

Sol. $\text{curl } B = \nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$

$$= \hat{i} \left[\frac{\partial}{\partial y} (0) - \frac{\partial}{\partial z} (-x) \right] - \hat{j} \left[\frac{\partial}{\partial x} (0) - \frac{\partial}{\partial z} (y) \right] + \hat{k} \left[\frac{\partial}{\partial x} (-x) - \frac{\partial}{\partial y} (y) \right]$$

$$= \hat{i} [0] - \hat{j} [0] + \hat{k} [-1 - 1] = -2\hat{k}.$$

$$c) \quad B = a \left\{ \frac{y \hat{i}}{(x^2+y^2)^{3/2}} - \frac{x \hat{k}}{(x^2+y^2)^{3/2}} \right\}$$

Sol:-

$$\text{Curl } B = \nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{ay}{(x^2+y^2)^{3/2}} & 0 & \frac{-ax}{(x^2+y^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i} \left[\frac{\partial}{\partial y} \left(\frac{-ax}{(x^2+y^2)^{3/2}} \right) - \frac{\partial}{\partial z} (0) \right] - \hat{j} \left[\frac{\partial}{\partial x} \left(\frac{-ax}{(x^2+y^2)^{3/2}} \right) - \frac{\partial}{\partial z} \left(\frac{ay}{(x^2+y^2)^{3/2}} \right) \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial x} (0) - \frac{\partial}{\partial y} \left(\frac{ay}{(x^2+y^2)^{3/2}} \right) \right]$$

$$= \hat{i} \left[\frac{\partial}{\partial y} (-ax(x^2+y^2)^{-3/2}) \right] + \hat{j} \left[\frac{\partial}{\partial x} (ax(x^2+y^2)^{-3/2}) \right] + \hat{k} \left[-\frac{\partial}{\partial y} (ay(x^2+y^2)^{-3/2}) \right]$$

$$= \hat{i} \left[ax \cdot \frac{3}{2} \cdot (x^2+y^2)^{-5/2} \cdot y \right] + \hat{j} \left[ax \cdot (-\frac{3}{2}) \cdot (x^2+y^2)^{-5/2} \cdot x \right] + \hat{k} \left[a(x^2+y^2)^{-3/2} \right]$$

$$+ \hat{k} \left[- \left(ay \cdot (-\frac{3}{2}) \cdot (x^2+y^2)^{-5/2} \cdot y \right) + \left[a(x^2+y^2)^{-3/2} \right] \right]$$

$$= \hat{i} \left[3axy \cdot \frac{1}{(x^2+y^2)^{5/2}} \right] + \hat{j} \left[-3ax^2 \frac{1}{(x^2+y^2)^{5/2}} + a \frac{1}{(x^2+y^2)^{3/2}} \right] + \hat{k} \left[a \frac{1}{(x^2+y^2)^{3/2}} - \frac{3ay^2}{(x^2+y^2)^{5/2}} \right]$$

$$= \hat{i} \left[\frac{3axy}{(x^2+y^2)^{5/2}} \right] + \hat{j} \left[\frac{a}{(x^2+y^2)^{3/2}} - \frac{3ax^2}{(x^2+y^2)^{5/2}} \right] + \hat{k} \left[\frac{3ay^2}{(x^2+y^2)^{5/2}} - \frac{a}{(x^2+y^2)^{3/2}} \right]$$

$$= \hat{i} \left[\frac{3axy}{(x^2+y^2)^{5/2}} \right] + \hat{j} \left[\frac{a(x^2+y^2) - 3ax^2}{(x^2+y^2)^{5/2}} \right] + \hat{k} \left[\frac{3ay^2 - a(x^2+y^2)}{(x^2+y^2)^{5/2}} \right]$$

$$= \hat{i} \left[\frac{3axy}{(x^2+y^2)^{5/2}} \right] + \hat{j} \left[\frac{ax^2 + ay^2 - 3ax^2}{(x^2+y^2)^{5/2}} \right] + \hat{k} \left[\frac{3ay^2 - ax^2 - ay^2}{(x^2+y^2)^{5/2}} \right]$$

$$= \hat{i} \left[\frac{3axy}{(x^2+y^2)^{5/2}} \right] + \hat{j} \left[\frac{-2ax^2 + ay^2}{(x^2+y^2)^{5/2}} \right] + \hat{k} \left[\frac{2ay^2 - ax^2}{(x^2+y^2)^{5/2}} \right]$$

$$= \frac{1}{(x^2+y^2)^{5/2}} \left[3axy \hat{i} + (ay^2 - 2ax^2) \hat{j} + (2ay^2 - ax^2) \hat{k} \right]$$

30/12

28. calculate curl A for vector $\vec{A} = \hat{i} + \hat{j} + \hat{k}$.

Sol:-

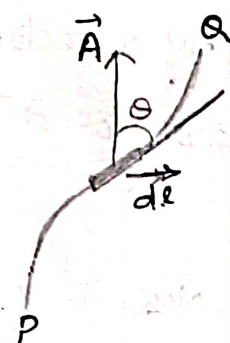
$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{vmatrix}$$
$$= \hat{i} \left[\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial z}(1) \right] - \hat{j} \left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial z}(1) \right] + \hat{k} \left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(1) \right]$$
$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$
$$= 0$$

Line Integral:

Definition:- The integration of a vector along a curve is called its line integral.

Explanation:

Let PQ be a curve drawn between two points P and Q in vector field. Let 'dl' be an element of length along the curve PQ at R. Suppose A is the vector at R making an angle θ with the direction of dl then



$$A \cdot dl = A dl \cos \theta = dl (A \cos \theta) \longrightarrow (1)$$

Equation (1) shows that the value of $A \cdot dl$ at any point of the curve is equal to the product of small element dl and the component $(A \cos \theta)$ of A along the direction of dl . The value of $A \cdot dl$ for the complete curve PQ can be obtained by integrating eqn (1) hence

$$\int_P^Q A \cdot dl = \int_P^Q dl (A \cos \theta) \longrightarrow (2)$$

Integral $\int_P^Q A \cdot dl$ is defined as line integral of A along the curve PQ.

Examples:-

1. If A represents the force acting on a particle moving along the curve from P to Q, then the line integral $\int_P^Q A \cdot dl$ represents the total work done by the force during the motion of the particle over its entire path from P to Q.
2. If A denotes the electric field intensity at any point, then the line integral represents the potential difference between P and Q.

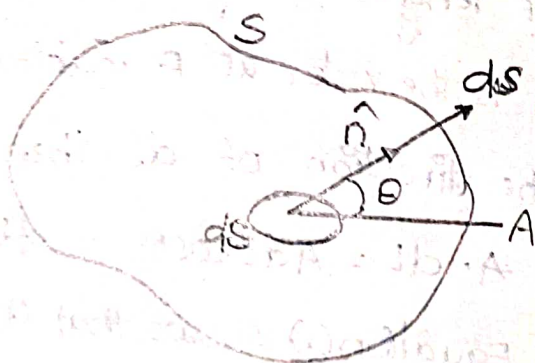
Surface integral:

Let S be a surface drawn in a vector field, and ds an infinite small element of the surface. Let $\hat{n}$ be unit positive vector normal to ds , the surface element of area ds is represented by a vector $d\mathbf{s}$ whose magnitude is $|ds|$ and direction is that of $\hat{n}$. Thus,

$$d\mathbf{s} = \hat{n} ds$$

$d\mathbf{s}$ is called the area vector.

Now, if $\mathbf{A}$ be a vector at the middle of the element $d\mathbf{s}$ in a direction making an angle θ with $\hat{n}$, then the integral



$$\iint_S \mathbf{A} \cdot d\mathbf{s} = \int_S \mathbf{A} \cdot \hat{n} ds = \int_S (A \cos \theta) ds \longrightarrow (3)$$

is defined as the surface integral or flux of $\mathbf{A}$ across the surface S .

Ex: If ' $\mathbf{A}$ ' denotes the velocity vector of a moving fluid, in which a fixed surface S is drawn, then the surface integral $\int_S \mathbf{A} \cdot d\mathbf{s}$ gives the amount of fluid flowing per unit time normally through the surface S . In case of a closed surface, a positive surface integral means that the fluid is flowing outwards, if negative then inwards.

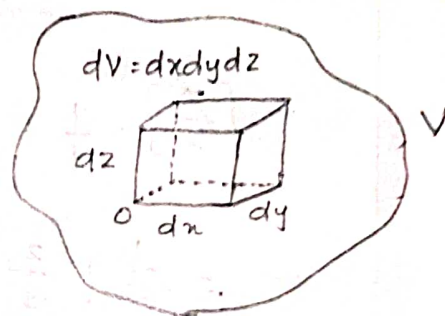
2) $\int_S \mathbf{A} \cdot d\mathbf{s}$ and $\int_S \phi ds$ where ϕ is a scalar function.

Volume integral:

Let $dV = dx dy dz$ denotes the element of volume V . The volume element dV is a scalar. If A be a vector-function inside it, then the integral $\int_V A dV$ is called the volume integral $\iiint_V A dV$ of A over volume V , i.e.,

$$\iiint_V A dV = \int_V A dV = \int_V (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) dx dy dz \longrightarrow (4)$$

In case of scalar function ϕ , the integral $\int_V \phi dV$ represents the volume integral over V .

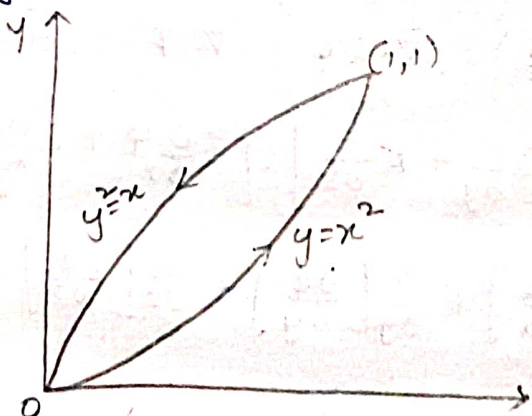


problems:-

1. Evaluate $\oint_C A \cdot d\mathbf{r}$ around the closed curve C defined by $y = x^2$ and $y^2 = x$ with $A = i(x-y) + j(x+y)$?

Sol:- The line integral around the closed surface is given by

$$\begin{aligned} \oint_C A \cdot d\mathbf{r} &= \oint_C [i(x-y) + j(x+y)] \cdot [i dx + j dy] \\ &= \oint_C [(x-y)dx + (x+y)dy] \end{aligned}$$



$$= \left[\int_0^1 (x-x^2)dx + \int_0^1 (x+x^2)2x dx \right] \text{ along } y = x^2$$

$$+ \left[\int_0^1 (y^2-y)2y dy + \int_0^1 (y^2+y)dy \right] \text{ along } x = y^2$$

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ \Rightarrow dy &= 2x dx \end{aligned}$$

$$\begin{aligned} \frac{dx}{dy} &= 2y \\ dx &= 2y dy \end{aligned}$$

$$= \int_0^1 (x-x^2) dx + 2 \int_0^1 (x^2+x^3) dx + 2 \int_1^0 (y^3-y) dy + \int_1^0 (y^2+y) dy$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 + 2 \left[\frac{x^3}{3} + \frac{x^4}{4} \right]_0^1 + 2 \left[\frac{y^4}{4} - \frac{y^3}{3} \right]_1^0 + \left[\frac{y^3}{3} + \frac{y^2}{2} \right]_1^0$$

$$= \left[\frac{x^2}{2} \right]_0^1 - \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{2x^3}{3} \right]_0^1 + \left[\frac{x^4}{2} \right]_0^1 + \left[\frac{y^4}{2} \right]_1^0 - \left[\frac{2y^3}{3} \right]_1^0 + \left[\frac{y^3}{3} \right]_1^0 + \left[\frac{y^2}{2} \right]_1^0$$

$$= \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{2}{3} - \frac{1}{2}$$

$$= \frac{2}{3}$$

Ex. Evaluate $\int \nabla^2 \phi$ where ϕ is a scalar function.

Laplacian operator: (∇^2) .

def: The divergence of the gradient of a scalar function is called the Laplacian operator.

In rectangular coordinates

$$\text{div}(\text{grad } \phi) = \nabla \cdot \nabla \phi = \nabla^2 \phi$$

$$\nabla^2 \phi = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$= \frac{\partial}{\partial x} \left[\frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[\frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[\frac{\partial \phi}{\partial z} \right]$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\nabla^2 \phi = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \phi$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

This equation is called the Laplacian equation.

STOKE'S THEOREM:-

Theorem:- This theorem states that the line integral of a vector field $\vec{A}$ around a closed curve is equal to the surface integral of the curl of $\vec{A}$ taken over the surface S surrounded by the closed curve i.e.,

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S \text{curl } \vec{A} \cdot d\vec{s} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{s}.$$

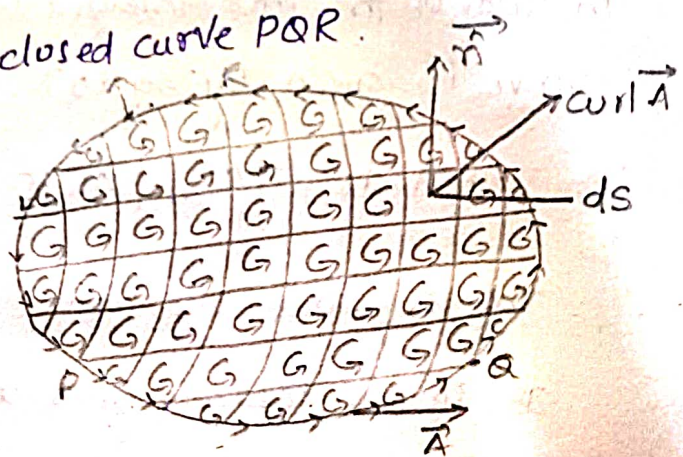
(or)
This states that the flux of the curl of a vector field $\vec{A}$ over any surface of any shape is equal to the line integral of the vector field $\vec{A}$ over the boundary of that surface. We must note that the sense of line integral is related with the positive normal to the surface.

$$\iint_S \text{curl } \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{r} \longrightarrow (1)$$

Proof:- Consider a surface S enclosed in a vector field $\vec{A}$. The boundary of the surface S is closed curve PQR .

The line integral $\vec{A}$ around the curve PQR traced counter-clockwise is

$$\oint_C \vec{A} \cdot d\vec{r}.$$



Let the entire surface be divided into a large number of square loops. Let the area enclosed by each infinitesimally small loop be ds . Suppose $\hat{n}$ be a unit positive outward normal upon ds . The vector area of the element is

$$\hat{n} ds = d\vec{s}$$

We know that the curl of a vector field at any point is the maximum line integral of the vector computed per unit area along the boundary of an infinitesimal area at that point. So the line integral of A around the boundary of the area ds is

$$\boxed{\text{curl } A \cdot dS}$$

This applies to all surface elements. Hence the sum of the line integrals of A around the boundaries of all the area elements is given by

$$\boxed{\iint_S \text{curl } A \cdot dS} \longrightarrow (2)$$

This is the surface integral of A .

It is clear that from the figure, that the line integrals along the common sides of the continuous elements mutually cancel because they are traversed in opposite directions. Now the sides of the element which lie in the periphery of the surface (i.e., in the closed curve) contribute the line integral. The sum of the line integrals on the boundary line of the curve is given by eq. (2). This is also given by eq. (1). Hence

$$\oint_C A \cdot d\vec{r} = \iint_S \text{curl } A \cdot d\vec{S} = \iint_S (\nabla \times A) \cdot d\vec{S} \longrightarrow (3)$$

This is Stokes's theorem.

Stokes's theorem gives a method to convert a surface integral into a line integral and vice versa. When $\text{curl } A$ is zero, the line integral of A over the closed path is zero and the field is conservative.

Gauss' Theorem:

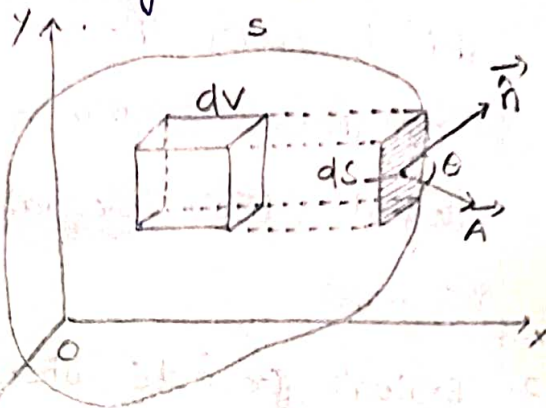
Theorem: This theorem states that the surface integral of the normal component of vector A taken over a closed surface S is equal to the volume integral of the divergence of vector A over the volume V enclosed by surface S , i.e.,

$$\oint_S A \cdot ds = \iiint_V \text{div } A \, dV = \iiint_V (\nabla \cdot A) \, dV.$$

Proof:-

Consider a closed surface S of any arbitrary shape drawn in a vector field A . Let the surface encloses a volume V .

Now the whole volume may be assumed to be divided into a very large number of cubical volume elements adjoining to each other.



Consider a small cubical volume element as shown in figure. We know that $\text{div } A$ represents the ~~flux~~ amount of flux diverging per unit volume and hence the flux diverging from the element of volume dV will be $\text{div } A \, dV$. So the total flux coming out from the entire volume is given by

$$\iiint_V \text{div } A \, dV \longrightarrow (1).$$

Now we consider a small element of area ds on the surface S . Let $\hat{n}$ represents the unit vector drawn ^{normal} from area ds . It should be remembered that outward drawn normal on a surface is taken as positive. If the field vector A and outward normal $\hat{n}$ are at an angle θ , then the component of A along $\hat{n}$ is

$$A \cos \theta = A \cdot \hat{n}$$

the flux of A through the surface element ds is given by

$$(A \cdot \hat{n}) \, ds = A \cdot ds$$

($\because$ the flux is defined as the product of normal component of vector & surface area)

So the flux through the entire surface S is given by

$$\iint_S \mathbf{A} \cdot d\mathbf{S} \longrightarrow (2).$$

This must be equal to the total flux diverging from the whole volume V enclosed by surface S . Hence from (1) & (2)

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \text{div } \mathbf{A} \, dV \longrightarrow (3).$$

$$\Rightarrow \iint_S (\mathbf{A} \cdot \hat{n}) \, dS = \iiint_V (\nabla \cdot \mathbf{A}) \, dV.$$

This is Gauss theorem of divergence.

This theorem provides a method for connecting volume integrals (triple integrals) to surface integrals (double integrals).

2. Evaluate $\iint_S \sigma \cdot \hat{n} \, dS$ where S is a closed surface.

Sol. Given, $\iint_S \sigma \cdot \hat{n} \, dS$.

By Gauss theorem,

$$\iint_S \sigma \cdot \hat{n} \, dS = \iiint_V \text{div } \sigma \, dV.$$

$$= \iiint_V \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (xi + yj + zk) \, dV.$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) \, dV.$$

$$= \iiint_V (3 + 1 + 1) \, dV.$$

$$= \iiint_V (3) \, dV.$$

$$= 3 \iiint_V dV$$

$$= 3V.$$

Where V is the volume enclosed by surface S .

the laplacian operator:

Let ϕ be a scalar potential function. If ϕ represents an electric field E , then

$$E = -\nabla\phi = -\text{grad}\phi \longrightarrow (1)$$

According to Gauss's law in electrostatics the divergence of an electric field emerging from a closed surface is equal to $\frac{1}{\epsilon_0}$ times the charge density in the surface i.e.

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \longrightarrow (2)$$

Here ϵ is the permittivity of the vacuum.

By substituting eq'n (1) in (2) we get

$$-\nabla \cdot \nabla\phi = \frac{\rho}{\epsilon_0} \longrightarrow (3)$$

But $\nabla \cdot \nabla\phi = \text{div}(\text{grad}\phi)$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \left(i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z} \right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi \longrightarrow (4)$$

$$\nabla \cdot \nabla\phi = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \Rightarrow$$

Thus $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as laplacian operator (or) del squared.

Poisson's and laplacian Equations:

By substituting eq'n (4) in eq'n (3) we obtain

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \longrightarrow (5)$$

The above eq'n is known as poisson's equation. By this equation we can calculate "the electric potential at a point due to a charge distribution."

In a dielectric medium the above equation becomes

$$\nabla^2 \phi = \frac{-\rho}{k\epsilon_0} \quad (\because k \text{ is dielectric constant})$$

In a charge free region, volume charge density $\rho = 0$, then the eqn becomes, $\nabla^2 = 0$

This equation is known as Laplace's equation. By solving this, one can calculate the potential in an electric field, which is devoid of the source charge.

In spherical polar coordinates, poisson's equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial \phi}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial \phi}{\partial \theta} \right] + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \theta^2} = \frac{\rho}{k\epsilon_0}$$

poisson's eqn in cylindrical polar coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \phi}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\rho}{k\epsilon_0}$$

Questions:

1. How many kinds of fields are there?

A. There are two kinds of fields. They are

1. scalar field 2. vector field.

2. If $\phi(x, y, z)$ is a scalar field, is $i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$ a vector field?

A. We know that a scalar field $\phi(x, y, z)$ evaluated at a particular point is independent of the coordinates of that point. obviously, scalar field is invariant with respect to the transformation of coordinates. Its gradient, which is a vector field, must also be invariant with respect to the transformation. Now on applying coordinate transformation

the vector $i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$ does not retain its form

applying a coordinate transformations. Thus it cannot be a vector field corresponding to given scalar field $\phi(x, y, z)$.

3. What are lamellar fields?

A. certain vector fields can be expressed a gradient of a scalar field ϕ and such fields is known as lamellar fields.

4. What is name of the theorem which provides a method of reducing volume integrals to surface integrals?

A. Gauss' theorem of divergence.

5. In two dimensions, the divergence transforms as a Scalar.

6. The curl of a vector function A is Vector.

7. The divergence of a curl is always Zero.

8. The curl of a gradient is always Zero.

9. $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is known as Laplacian operator.

10. Which one of the following is an example of scalar field:

- a) gravitational field. c) Electric potential [c]
b) electric field. d) Magnetic field.

11. The maximum rate of change of the scalar function is known as the [a]

a) gradient of a scalar field

b) divergence of a scalar field.

c) curl of a vector field

d) none of above.

12. A vector, A which satisfies the condition, $\nabla \cdot A = 0$, is called [b]

- a) scalar b) solenoidal c) tensor d) none of above

13. A vector, A which satisfies the condition, $\nabla \times A = 0$, is called [b]

- a) scalar b) Irrotational c) vector d) All the above