

Gauss's law: * Chapter II

The flux through any surface enclosing the charge is $\frac{q}{\epsilon_0}$. where q is point charge.

$$\oint E \cdot da = \frac{q}{\epsilon_0} \quad \text{--- (1)}$$

Suppose, if we have a bunch of charges instead of single charge, according to the principle of superposition, the total field $E = \sum_{i=1}^n E_i$

$\therefore$ The flux through the surface that encloses them all

$$\therefore \oint E \cdot da = \sum_{i=1}^n \left(\frac{1}{\epsilon_0} q_i \right) \quad \text{--- (2)}$$

$\therefore$ For any closed surface $\oint E \cdot da = \frac{1}{\epsilon_0} Q_{\text{enc}}$

where Q_{enc} is the total charge enclosed within the surface. This is quantitative statement of Gauss's law.

* Q_{enc} represents in terms of the charge density ρ .
we have $Q_{\text{enc}} = \int_V \rho \, d\tau$

$\therefore$ Integral form of Gauss's law can convert into differential form as follow:

$$\int_V (\nabla \cdot E) \, d\tau = \int_V \left(\frac{\rho}{\epsilon_0} \right) \, d\tau \quad \left[\because \oint_S E \cdot da = \int_V \nabla \cdot E \, d\tau \right]$$

for any volume the integral must be equal

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad \text{--- (3)}$$

This is the differential form of Gauss's law.

Applications of Gauss's law:

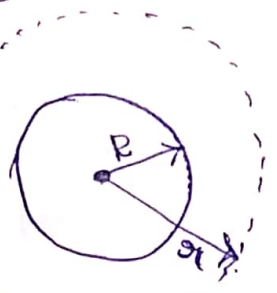
1. Find the field outside uniformly charged solid sphere of radius R and total charge q .

A. Imagine a spherical surface at radius $r > R$. This is called a Gaussian surface.

Gauss's law says that

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} q_{\text{enc}} \quad \text{--- (1)}$$

In this case $q_{\text{enc}} = q$, where $\mathbf{E}$ is buried ~~the~~ inside the surface integral. Electric field $\mathbf{E}$ certainly radially outward.



$$\therefore \oint_S \mathbf{E} \cdot d\mathbf{a} = \int_S |\mathbf{E}| \cdot d\mathbf{a} \quad \text{--- (2)}$$

Gaussian surface a concentric sphere.

The magnitude of $\mathbf{E}$ is constant over the Gaussian surface, so it comes outside integral

$$\int_S |\mathbf{E}| d\mathbf{a} = |\mathbf{E}| \int_S d\mathbf{a} = |\mathbf{E}| 4\pi r^2$$

$$\therefore |\mathbf{E}| 4\pi r^2 = \frac{q}{\epsilon_0}$$

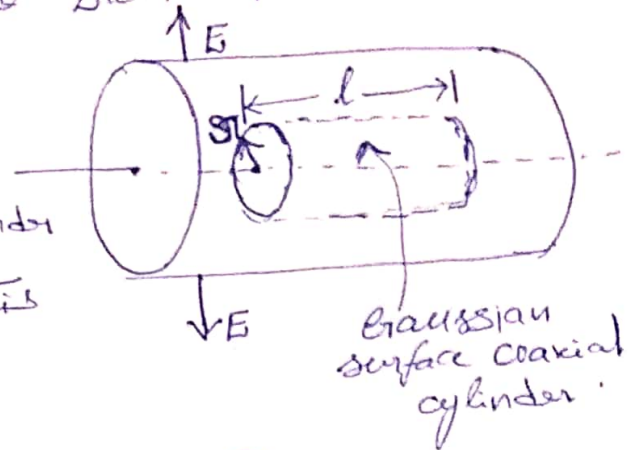
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}}$$

Therefore, we conclude that the field outside the sphere is exactly the same as it would have been if all the charges had been concentrated at the center.

example ② A long cylinder carries a charge density that is proportional to the distance from the axis $\rho = kr$ for some constant k . Find the Electric field inside the cylinder

Sol:

Consider the Gaussian cylinder length l and radius r' for this surface, Gauss's law



$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \quad \text{--- (1)}$$

The enclosed charge is

$$\begin{aligned} Q_{\text{enc}} &= \int \rho \, d\tau = \int (kr') (r' dr' d\phi dz) \\ &= 2\pi k l \int_0^{r'} r'^2 dr' \\ &= \frac{2}{3} \pi k l r^3 \quad \text{--- (2)} \end{aligned}$$

The Electric field ' $\mathbf{E}$ ' must point radially outward so the curved portion of the Gaussian cylinder we have

$$\begin{aligned} \int \mathbf{E} \cdot d\mathbf{a} &= \int |\mathbf{E}| \cdot d\mathbf{a} \\ &= |\mathbf{E}| \int d\mathbf{a} \\ &= |\mathbf{E}| 2\pi r l \quad \text{--- (3)} \end{aligned}$$

At the two ends contribution becomes zero, where $\mathbf{E}$ is $\perp$ to $d\mathbf{a}$.

$$\begin{aligned} \therefore |\mathbf{E}| 2\pi r l &= \frac{1}{\epsilon_0} \frac{2}{3} \pi k l r^3 \\ \therefore \mathbf{E} &= \frac{1}{3\epsilon_0} k r^2 \hat{r} \end{aligned}$$

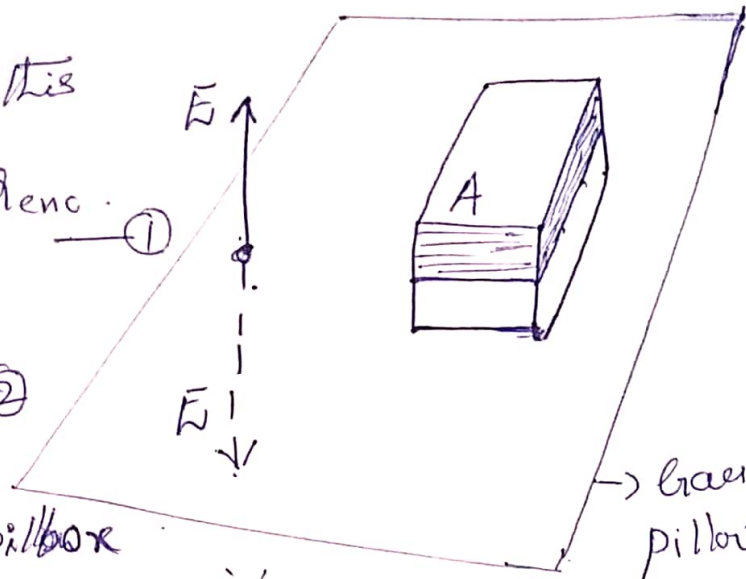
③ An infinite plane carries a uniform surface charge σ . Find its electric field.

Sol: Gauss's law to this

$$\text{surface} \int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} q_{\text{enc}} \quad \text{--- ①}$$

$$q_{\text{enc}} = \sigma A \quad \text{--- ②}$$

Where 'A' is the area of the lid of the pillbox



As we know that Electric field $\vec{E}$ points away from the plane. So the top and bottom surfaces yields

extending equal distance above and below

$$\int \vec{E} \cdot d\vec{a} = 2A|E| \quad \text{--- ③}$$

Here also sides contributing is zero.

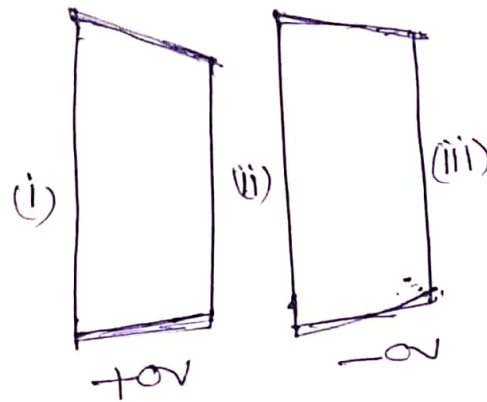
$$\therefore 2A|E| = \frac{\sigma A}{\epsilon_0} \quad \text{--- ③} \quad [\text{From ② \& ③}]$$

$$\therefore E = \frac{\sigma}{2\epsilon_0} \hat{n}$$

Where $\hat{n}$ is a unit vector pointing away from the surface.

④

- ④. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm \sigma$ as shown in fig. Find the field in each of the three regions.
- (i) to the left of both (ii) between them (iii) to the right of both.



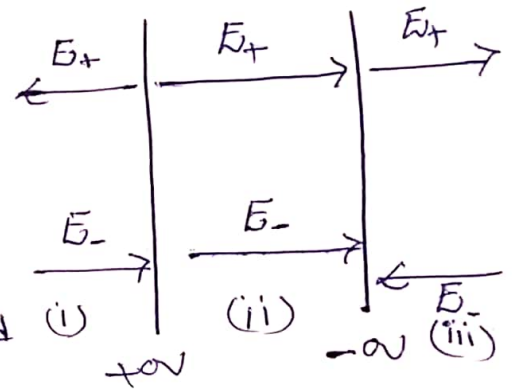
Solution:

* The left plate produces the field $\frac{\sigma}{2\epsilon_0}$ which points away from it.

* The right plate being -vely charged, produces a field $\frac{\sigma}{2\epsilon_0}$ which points towards it.

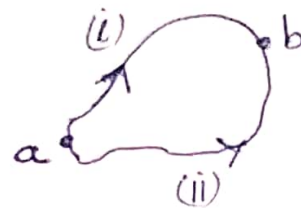
Two fields cancel in region (i) & (iii) and the conspire in region (ii)

The field b/w the plates is $\frac{\sigma}{\epsilon_0}$, and points to the right.



Electric potential:

* The line integral ^{of E} is independent of path, it is same for all paths.



$$\therefore V(a) \equiv - \int_0^a \mathbf{E} \cdot d\mathbf{l} \quad \text{--- (1)}$$

0 is the standard reference point. potential 'V' depends only on the point a . This is called the Electric potential:

* The potential difference b/w two points a and b is

$$\begin{aligned} V(b) - V(a) &= - \int_0^b \mathbf{E} \cdot d\mathbf{l} + \int_0^a \mathbf{E} \cdot d\mathbf{l} \\ &= - \int_0^b \mathbf{E} \cdot d\mathbf{l} - \int_a^0 \mathbf{E} \cdot d\mathbf{l} \end{aligned}$$

$$\therefore V(b) - V(a) = - \int_a^b \mathbf{E} \cdot d\mathbf{l} \quad \text{--- (2)}$$

According to fundamental theorem of gradients states that

$$V(b) - V(a) = \int_a^b (\nabla V) \cdot d\mathbf{l} \quad \text{--- (3)}$$

$$\therefore \int_a^b (\nabla V) \cdot d\mathbf{l} = - \int_a^b \mathbf{E} \cdot d\mathbf{l} \quad \text{--- (4) [From 2 & 3]}$$

This is true for any points a and b , then the integrals must be equal.

$$\therefore \boxed{\mathbf{E} = -\nabla V} \quad \text{--- (5)}$$

The electric field is equal to the gradient of a scalar potential.

Important points about potential (A)

* 1) The surface over which the potential is constant is called equipotential.

2) $E = -\nabla V$, where E = vector quantity, but V = scalar quantity.

'E' vector have three components E_x, E_y, E_z .

if $\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$, $\frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}$, $\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$, then $\nabla \times E = 0$.

(3) The potential at infinity is 'zero' used as reference. $V(\infty) = 0$. The point where the potential is zero.

(4) potential obeys the superposition principle.

$$V = V_1 + V_2 + \dots$$

The potential at any point is the sum of the potentials due to all the source charges separately. This is ordinary sum, not a vector sum.

(5) The units of potential $N\cdot m / \text{Coulomb}$ (or) $\text{joules} / \text{Coulomb}$. This $\text{joules} / \text{Coulomb}$ is called Volt.

problem: Find the potential inside and outside a spherical shell of radius R that carries uniform surface charge. Set the reference of point at infinity.

Sol: Case (1): The Electric field outside according to Gauss's law

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \quad \text{--- (1)}$$



Where q is the total charge, the field inside is zero. for outside point ($r > R$).

$$\begin{aligned} V(r) &= - \int_0^r E \cdot dl = - \frac{1}{4\pi\epsilon_0} \int_0^r \frac{q}{r'^2} dr' \\ &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r'} \right]_0^r \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{r}. \end{aligned}$$

$$\int \frac{1}{r'^2} dr' = -\frac{1}{r'}$$

Case (2) potential inside the sphere ($r < R$), the integral break into two parts as follows.

$$\begin{aligned} V(r) &= - \frac{1}{4\pi\epsilon_0} \int_0^R \frac{q}{r'^2} dr' - \int_R^r (0) dr'. \quad \text{(inside charge } q=0) \\ &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r'} \right]_0^R + 0 \\ &= \frac{q}{4\pi\epsilon_0 R}. \end{aligned}$$

Note: potential is not zero inside the shell, even though the field is constant.

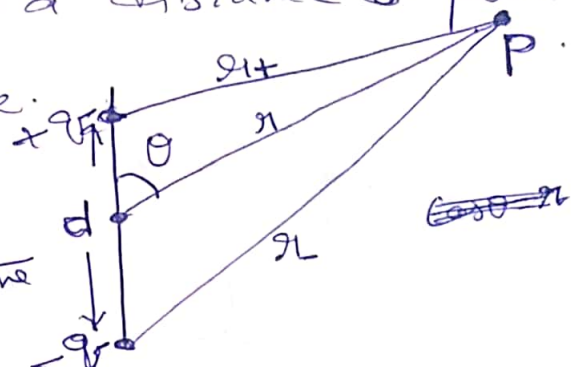
Electric dipole:

(5)

Electric dipole:-

It consists of two equal and opposite charges ($\pm q$) separated by a distance d . potential at a point from the dipole.

* Let us consider r_+ is the distance from $+q$ and r_- is the distance from $-q$ respectively.



Now, the potential from the centre of dipole is given by
$$V(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_+} - \frac{q}{r_-} \right) \quad \text{--- (1)}$$

* According to law of cosines

$$r_{\pm}^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp r d \cos \theta$$
$$= r^2 \left(1 \mp \frac{d}{r} \cos \theta + \frac{d^2}{4r^2} \right)$$

$$r^2 = a^2 + b^2 + 2ab \cos \theta$$

if $r \gg d$, the third term is negligible, and the binomial expansion

$$\Rightarrow \frac{1}{r_{\pm}^2} = \frac{1}{r^2} \left(1 \mp \frac{d}{r} \cos \theta \right)^{-1/2}$$

$$\Rightarrow \frac{1}{r_{\pm}} = \frac{1}{r} \left(1 \mp \frac{d}{r} \cos \theta \right)^{-1/2}$$
$$= \frac{1}{r} \left(1 \pm \frac{d}{2r} \cos \theta \right)$$

$$\Rightarrow \frac{1}{r_+} - \frac{1}{r_-} \approx \frac{d \cos \theta}{r^2} \quad \text{--- (2)}$$

[From (1) & (2)]

and hence

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q d \cos \theta}{r^2} \quad \text{--- (3)}$$

potential varies more rapidly with $\frac{1}{r^2}$ values.
eq (3) represents electric potential due to dipole at a point P.

' $q \cdot d$ ' represents the dipole moment. $p = q \cdot d$

$\therefore$ eq (3) becomes $V_{\text{dip}}(r) = \frac{1}{4\pi\epsilon_0} \frac{p \cdot \hat{r}}{r^2}$ — (4)

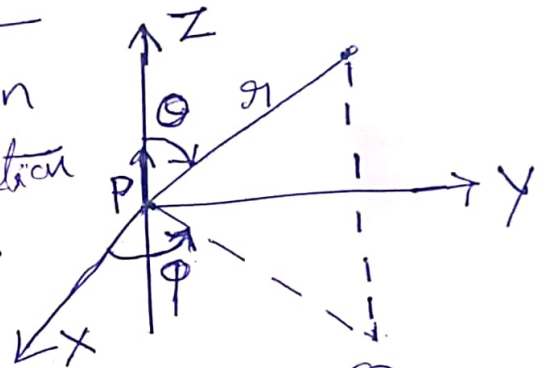
The dipole moment is determined by the Geometry size, shape and density of the charge distribution.

Dipole moment is vector quantity.

Electric field due to dipole:

Let us consider 'p' is at origin and the points in the z-direction then the potential at (r, θ) is

$$V_{\text{dip}}(r, \theta) = \frac{p \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2} \quad \text{--- (5)}$$



If we take negative gradient of V to obtain Electric field

$$E_r = -\frac{\partial V}{\partial r} =$$

$$= \frac{-p \cos \theta}{4\pi\epsilon_0} \left(-\frac{2}{r^3} \right)$$

$$E_r = \frac{2p \cos \theta}{4\pi\epsilon_0 r^3},$$

and $E_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} = \frac{p \sin \theta}{4\pi\epsilon_0 r^3}$

$$E_\phi = -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} = 0$$

$$\left[\frac{\partial}{\partial r} \left(\frac{1}{r^2} \right) \right] = \frac{\partial}{\partial r} (r^{-2}) = -2r^{-2-1} = -\frac{2}{r^3}$$

$$\therefore E_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \quad \text{--- (7)}$$

Where 'p' lies along z-direction.

⑥

Note:

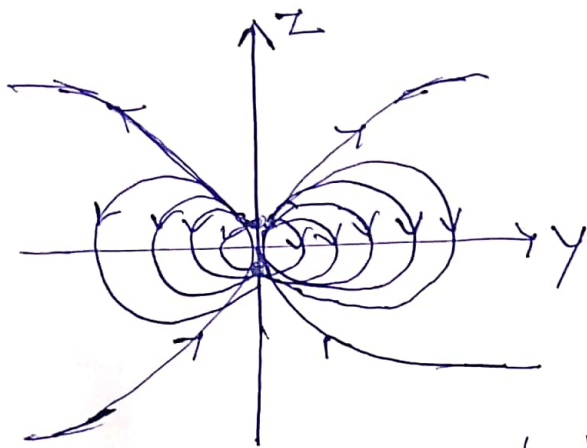
① The Monofield $E = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$, goes as the inverse square, $\left(\text{falls} \right)$

② dipole field falls off the inverse of the cube

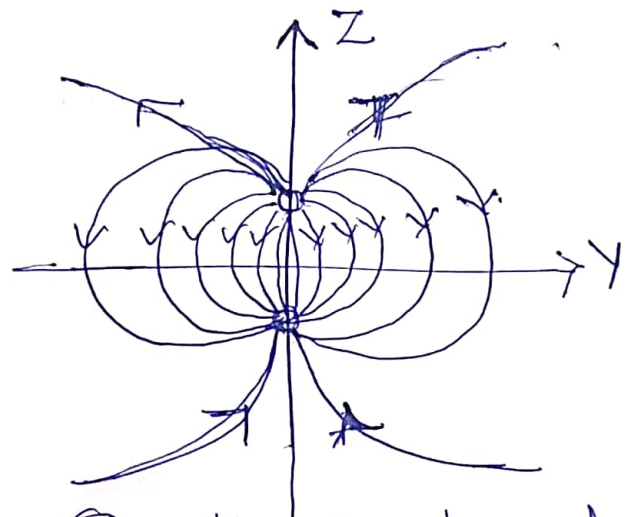
ie $E = \frac{P}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$ ⑦

Smiley

Quadrupole, octopole falls ~~in~~ of $\frac{1}{r^4}$, $\frac{1}{r^5}$ respectively.



① Field of pure dipole



② Field of a physical dipole.

* If $r \gg d$ eq ⑦ represents valid approximation to the field of a physical dipole.

* This regime can be reached either by going to larger r or by squeezing the charges very close to each other.

Energy density in Electrostatic fields:

Energy stored ~~in~~ ^{during} the configuration (potential energy) of point charges by the amount of work is represented as

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(r_i) \quad \text{--- (1)}$$

where $r_i = r_1, r_2, \dots$
 $V(r_i) = \text{potential at point } r_i$

For a volume charge density ρ , then

$$W = \frac{1}{2} \int \rho V d\tau \quad \text{--- (2)}$$

According to Gauss's law

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\Rightarrow \rho = \epsilon_0 \nabla \cdot \mathbf{E} \quad \text{--- (3)}$$

Substitute eq (3) in eq (2)

$$W = \frac{1}{2} \int \epsilon_0 \nabla \cdot \mathbf{E} V d\tau$$

$$W = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V d\tau \quad \text{--- (4)}$$

Integrating by parts to transfer the derivative from $\mathbf{E}$ to V .

$$W = \frac{\epsilon_0}{2} \left[- \int_V \mathbf{E} \cdot (\nabla V) d\tau + \oint V \mathbf{E} \cdot d\mathbf{a} \right]$$

$$(\because \nabla V = -\mathbf{E})$$

$$W = \frac{\epsilon_0}{2} \left[\int \mathbf{E}^2 d\tau + \oint V \mathbf{E} \cdot d\mathbf{a} \right] \quad \text{--- (5)}$$

eq (5) gives the correct energy W , whatever volume ~~you use~~ you used, volume integral goes up, then surface integral goes down, as you take larger

(7)

* When surface integral goes to zero, eq (6) becomes

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau \quad \text{--- (6)}$$

* The energy stored in the field with a density

$$\text{Energy per unit time} = \frac{1}{2} \epsilon_0 E^2 \quad \text{--- (7)}$$

* In electrostatics, it is stored in the form of charge, with density $\frac{1}{2} \rho V$.

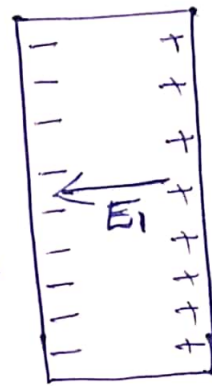
* Electrostatic energy is quadratic in the fields it does not obey a superposition principle. The energy of a compound system is not sum of all the energies of its parts. It is considered separately.

$$\begin{aligned} W_{\text{Total}} &= \frac{\epsilon_0}{2} \int E^2 d\tau \\ &= \frac{\epsilon_0}{2} \int (E_1 + E_2)^2 d\tau \\ &= \frac{\epsilon_0}{2} \int (E_1^2 + E_2^2 + 2E_1 E_2) d\tau \\ W_{\text{Total}} &= W_1 + W_2 + \epsilon_0 \int E_1 E_2 d\tau \end{aligned}$$

Note: If you double the charge everywhere, you quadruple the total Energy.

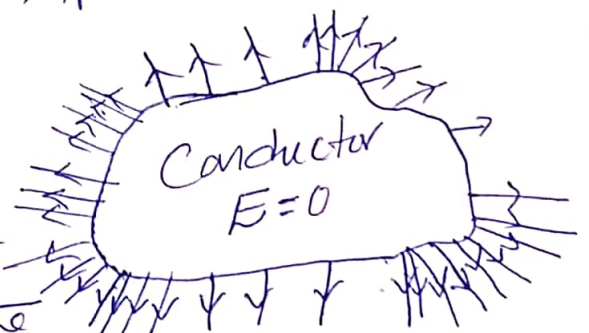
Fields inside perfect conductors:

* When you put a conductor into an external electric field E_0 as shown in fig (a)



Initially the field will drive positive (free) charges to right and -ve free charges (electrons) to the left. When they come to edge of the material, the charges pile up +ve on right side and -ve left side. Therefore electric field produced b/w the edges of the conductor due to induced charges as E_1 shown in fig (a). The direction of E_1 is opposite to the applied field E_0 . The induced charges tends to cancel the applied electric field. Charge will continue to flow until the cancellation is complete. The field inside the conductor is precisely zero. Charge density $\rho = 0$ inside of conductor since $\nabla \cdot E = \rho / \epsilon_0$, if $E = 0$, then $\rho = 0$.

The charge on a conductor flows to the surface, Because of their mutual repulsion, the charge naturally spread out as much as possible, but



for all of them to go to the surface and reside it. As waste of interior space. All the charges on the surface, and this is true regardless of the size (or shape) of the conductor.