

UNIT - III

ELECTROSTATICS - II

- Current density
- Ohm's law
- Poisson's and Laplace equations.
- Boundary conditions of electric field & electrostatic potential.
- Method of images (with one example)
- Energy of a charge distribution and its expression in terms of electric field.

ELECTROSTATICS - II

Current ~~and~~ & current density :

Current: ~~the~~ ~~known~~ ~~that~~ In a metallic conductor, free charges (electrons) are in random motion like gas molecules confined in a container. There is not net directed motion along the conductor. Let the ends of the conductor be connected to a battery i.e., an electric field is maintained within the conductor. Now the field acts on the electrons and gives them a resultant motion in the direction of $-E$ because a free charge in Electric field experiences a force. The flow of electrons constitutes an electric current.

Def: When a charge flows in a conductor from one place to another place, then this flow of charge is called as electric current.

The electric current is measured by the rate of flow of charge. Thus if a net charge q passes through a specified surface area of the conductor in t seconds, then the current is given by $i = (q/t) \rightarrow (1)$

The current is a scalar quantity and units is ampere.

If from (1) $q = 1$ coulomb and $t = 1$ second, then $i = 1$ ampere.

$$\therefore 1 \text{ ampere} = 1 \text{ coulomb} / 1 \text{ second} \rightarrow (2)$$

When the current varies with time then at any instant t , the current is defined as

$$i = dq/dt \rightarrow (3)$$

1 ampere = 6.25×10^{18} electrons per second. It means that in a conductor carrying a current of 1A, $6.25 \times 10^{18} e^-$ enter at one end and equal no. of e^- leave at the other end in one second. The charge flows faster where the conductor is smaller in cross section and slower where the conductor is larger in cross section because the charge must be conserved.

It appears logical to define the direction of current as the direction of motion of free charges. But the difficulty arises in case of liquids and gases where both the positively and negatively charged particles move in opposite directions. As a convention, the direction of current is that in which positive charges will move. In a metallic conductor, +ve charges can't move and the entire transfer of the charge is only due to flow of electrons.

Thus "the conventional current is in opposite direction to the direction of movement of electrons".

Current Density (j):

Current ' i ' is a characteristic of a particular conductor. This is macroscopic quantity like mass, volume, length etc. The current is a scalar quantity. A related microscopic quantity is the current density j . It is a vector quantity and is characteristic of a point inside the conductor rather than conductor as a whole.

Def: The current density is defined as the ratio of the current to the cross-sectional area.

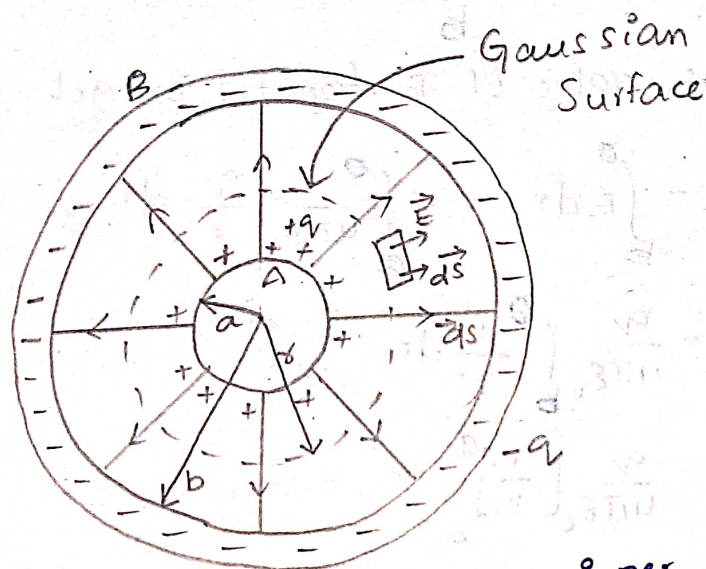
If a current ' i ' in a conductor be distributed uniformly over the area of cross section A of the conductor then

Current density $j = i/A$

The direction of the current density at a point is the direction of the positive charge at that point.

The units for current density - ampere/metre²

Capacitance of a spherical capacitor:



A spherical capacitor consists of an inner metal sphere A of radius 'a' surrounded by an outer metal sphere B of radius 'b' as shown in figure. Suppose the inner sphere A is given a charge $+q$, while the outer sphere B is given a charge $-q$. These two charges produce an electric field E between the space of two spheres. The direction of E will be radially outwards as shown in figure. Now, we consider a concentric spherical Gaussian surface of radius r . The flux Φ_E through the Gaussian surface is given by

$$\Phi_E = \oint E \cdot ds = \oint E ds \cos \theta$$

$$\Phi_E = E \oint ds = EA = E(4\pi r^2)$$

According to Gauss's law,

$$E(4\pi r^2) = q/\epsilon_0$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \rightarrow (1)$$

We know that, $E = -dv/dr$ or $V = -\int E dr$.

i.e the potential difference between the two sphere is equal to the work done in transferring a unit positive charge from outer sphere to inner sphere. Thus

$$V = V_A - V_B = -\int_b^a E dr \longrightarrow (2)$$

Substituting the value of E from (1) we get

$$V = -\int_b^a E dr = -\int_b^a \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} dr.$$

$$V = -\frac{q}{4\pi\epsilon_0} \int_b^a \frac{1}{r^2} dr.$$

$$V = -\frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_b^a.$$

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right] \longrightarrow (3)$$

We know, that

$$C = \frac{q}{V} \longrightarrow (4)$$

Substitute (3) in (4)

$$C = \frac{q}{\frac{q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right]} = \frac{q}{\frac{q}{4\pi\epsilon_0} \left[\frac{b-a}{ab} \right]}$$

$$= \cancel{q} \cdot \frac{4\pi\epsilon_0}{\cancel{q}} \left[\frac{ab}{b-a} \right]$$

$$C = 4\pi\epsilon_0 \left[\frac{ab}{b-a} \right]$$

$$\therefore \boxed{C = \frac{4\pi\epsilon_0 ab}{(b-a)}} \longrightarrow (4)$$

This is the expression for the capacitance of a spherical Condenser.

P. For instance the plates are square with sides 1cm long, and they are held 1mm apart, then find the capacitance?

Sol: Given that,

$$\text{side } (a) = 1\text{cm} = 1 \times 10^{-2}\text{m}$$

$$\text{distance, } d = 1\text{mm} = 1 \times 10^{-3}\text{m}$$

$$\text{Capacitance, } C = \frac{\epsilon_0 A}{d}$$

$$\text{Area, } A = a^2 = (10^{-2})^2 = 10^{-4}\text{m}^2$$

$$\therefore C = \frac{8.85 \times 10^{-12} \times 10^{-4}}{1 \times 10^{-3}}$$

$$C = 8.85 \times 10^{-12} \times 10^{-4} \times 10^3$$

$$C = 8.85 \times 10^{-16} \times 10^3$$

$$C = 8.85 \times 10^{-13}\text{ F}$$

Capacitance of isolated conducting sphere.

When the outer sphere is of infinite radius i.e, $b = \infty$, then the sphere left is known as isolated conducting sphere.

Now, from eq'n $C = \frac{4\pi\epsilon_0 ab}{(b-a)}$, we have

$$C = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - \frac{1}{\infty}\right)} = \frac{4\pi\epsilon_0}{\left(\frac{1}{a} - 0\right)}$$

$$\therefore C = 4\pi\epsilon_0 (a) \longrightarrow (5)$$

Eq'n(5) gives the capacitance of an isolated sphere.

Capacitance of cylindrical capacitor:

A cylindrical capacitor consists of two coaxial cylinders of radius a and b with length l as shown in figure. The cylinders are made of metallic material. The inner cylinder is given a charge $+q$, while the outer is given a charge $-q$. In order to calculate the capacitance of this condenser, we ~~have~~ consider a coaxial cylindrical Gaussian surface of radius r .

From Gauss's law,

$$\because \Phi_E = \oint \mathbf{E} \cdot d\mathbf{s} = \oint E ds \cos 0$$

$$\Phi_E = EA)$$

$$\int \mathbf{E} \cdot d\mathbf{s} = \frac{q}{\epsilon_0}$$

$$E(2\pi r l) = \frac{q}{\epsilon_0}$$

$$E = \frac{q}{2\pi \epsilon_0 r l} \rightarrow (1)$$

Now, the potential difference V

between the two cylinders is equal to

the workdone in moving a unit positive charge from the outer cylinder to the inner cylinder i.e.,

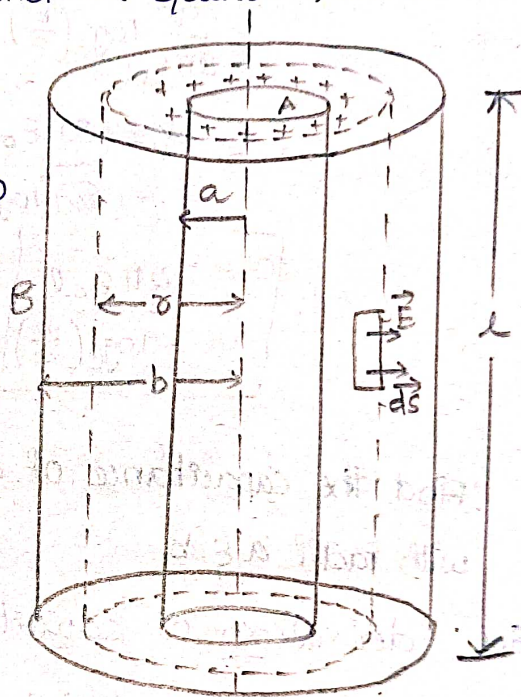
$$V = - \int_b^a E dr \rightarrow (2)$$

$$V = - \int_b^a \frac{q}{2\pi \epsilon_0 r l} dr$$

$$V = - \int_b^a \frac{q}{2\pi \epsilon_0 l} \frac{1}{r} dr$$

$$V = - \frac{q}{2\pi \epsilon_0 l} \int_b^a \frac{1}{r} dr$$

$$V = - \frac{q}{2\pi \epsilon_0 l} [\ln r]_b^a = V = - \frac{q}{2\pi \epsilon_0 l} (\log_e a - \log_e b)$$



$$V = - \frac{q}{2\pi\epsilon_0 l} \left(\log_e \left(\frac{a}{b} \right) \right)$$

$$V = \frac{q}{2\pi\epsilon_0 l} \log_e \left(\frac{b}{a} \right) \rightarrow (3)$$

Since, $C = \frac{q}{V} \rightarrow (4)$

Substitute (3) in eq'n (4), we get

$$C = \frac{q}{\frac{q}{2\pi\epsilon_0 l} \log_e \left(\frac{b}{a} \right)}$$

$$C = q \cdot \frac{2\pi\epsilon_0}{q \left(\log_e \frac{b}{a} \right)}$$

$$C = \frac{2\pi\epsilon_0 l}{\log_e \left(\frac{b}{a} \right)}$$

A coaxial cable consists of a copper core of 1mm radius within an outer metal sheath of 1cm radius separated by an insulating material of dielectric constant 5. What is the capacitance per meter length of the cable in picofarad?

b) How much energy is stored in 10 km length of the cable 10000 V is applied between the core and sheath?

Sol:- Given data,

$$a = 1\text{mm} = 1 \times 10^{-3}\text{m}$$

$$b = 1\text{cm} = 1 \times 10^{-2}\text{m}$$

$$\text{dielectric constant} = 5$$

$$\text{Length, } L = 1\text{m}$$

$$C = ? \text{ in picofarad.}$$

We know that,

$$a) \quad C = \frac{2\pi\epsilon_0 L}{\log_e\left(\frac{b}{a}\right)}$$

$$\epsilon = K \epsilon_0 \Rightarrow K \epsilon_0 = \epsilon$$

$$C = \frac{2\pi\epsilon K L}{\log_e\left(\frac{b}{a}\right)}$$

$$C = \frac{2 \times 3.14 \times 8.85 \times 10^{-12} \times 5 \times 1}{\log_e\left(\frac{1 \times 10^{-2}}{1 \times 10^{-3}}\right)}$$

$$C = \frac{277.89 \times 10^{-12}}{\log_e(10^2 \times 10^3)}$$

$$C = \frac{2.77 \times 10^{-10}}{\ln(10)}$$

$$C = \frac{2.7789 \times 10^{-10}}{2.3025}$$

$$C = 1.206 \times 10^{-10} \text{ F}$$

$$C = 120.68 \text{ pF} = 120.6 \times 10^{-12} \text{ F}$$

b) We know that,

$$E = \frac{1}{2} CV^2 \quad (\text{for length } 10 \text{ km})$$

$$\text{Here, } C = 120.68 \times 10^{-12} \times 10 \times 10^3$$

$$= 120.68 \times 10^4 \times 10^{-12}$$

$$= 120.68 \times 10^{-8} \text{ F}$$

$$\therefore E = \frac{1}{2} CV^2$$

$$= \frac{1}{2} \times 120.68 \times 10^{-8} \times (10^4)^2$$

$$= \frac{1}{2} \times 120.68 \times \cancel{10^{-8}} \times \cancel{10^8}$$

$$= \frac{1}{2} \times \frac{60.34}{1}$$

$$= 60.34 \text{ J}$$

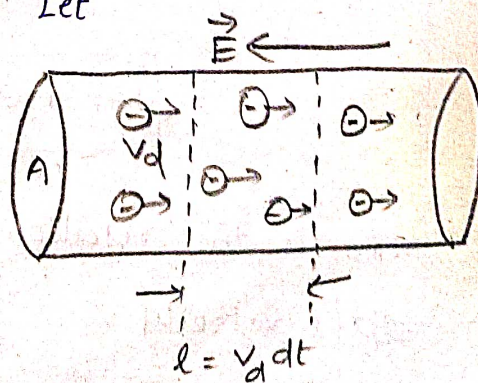
Drift Velocity of charge carriers in a Conductor:

We know that the electrons in a conductor move at random with sufficiently high speed in the vacant spaces between the stationary positive ions. When an electric field is established between the two ends of the conductor, the free electrons experience an electric force. Due to this force, the motion of electrons is accelerated. However, the speed of the electron does not increase continuously. The reason is that the electrons collide ~~each~~ again and again with positive ions of the conductor. As a result, they lose their energy which, of course, appears in the form of heat. Thus the field does not give an accelerating motion to the electrons but it simply gives them a small constant velocity along the conductor which is superposed on the random motion of the electrons. This constant velocity of the electrons is called as drift velocity. The drift velocity is very small in comparison to the random motion of the electrons.

Relation between current density and drift velocity:

As shown in below figure, let an electric field is maintained between the two ends of a conductor towards the left. The electrons move towards the right. Let the drift velocity of the electrons be v_d . Suppose there are n charge carriers per unit volume and each charge carrier has a charge e . In time dt , the electron advances a distance ' l ' which is given by

$$l = v_d dt \longrightarrow (*)$$



Now, calculate the number of electrons crossing the length l of the conductor in time dt . This will be equal to the no. of electrons contained in a volume A i.e., $Av_d dt$.

$\therefore$ No. of electrons = Volume $\times$ no. of electrons per unit volume.

$$= Av_d dt \times n$$

Hence charge crossing in time dt

= no. of electrons $\times$ charge on the electron

$$= Av_d dt \times ne$$

$\therefore$ current, $i = \frac{\text{charge crossing in time } dt}{\text{time } dt}$

$$i = \frac{Av_d dt ne}{dt} = Av_d ne$$

$$\therefore \boxed{i = neAv_d} \longrightarrow (2)$$

This is the relationship between Electric current & drift velocity.

$\therefore$ Current density is given by

$$j = \frac{i}{A} = \frac{neAv_d}{A}$$

$$\therefore \boxed{j = nev_d} \longrightarrow (3)$$

This is the relationship between current density & drift velocity.

A silver wire 1mm in diameter carries a charge of 90C in one hour and 15 min. Silver contains 5.8×10^{22} free electrons per cm^3 . Calculate

(a) The current in the wire in ampere.

(b) Drift velocity

Sol:- Given data,

$$\text{diameter} = 1\text{mm} = 1 \times 10^{-3} \text{ m}$$

$$\text{radius} = \frac{10^{-3}}{2} \text{ m}$$

$$\text{charge, } q = 90\text{C}$$

$$\text{time} = 1\text{hour} \& 15\text{min} = 3600 + 900 = 4500 \text{ sec}$$

$$e = 1.60 \times 10^{-19} \text{ C}$$

$$\text{a) } i = \frac{q}{t}$$

$$= \frac{90}{4500}$$

$$= 0.02 \text{ A}$$

b) Drift velocity,

$$j = \frac{i}{A}$$

$$j = \frac{0.02}{\pi r^2}$$

$$j = \frac{0.02}{3.14 \left(\frac{10^{-3}}{2}\right)^2}$$

$$j = \frac{0.02}{3.14 \times 10^{-6}} \times 4$$

$$j = \frac{0.08}{3.14} \times 10^6$$

$$j = 0.0254 \times 10^6 \text{ A/m}^2$$

$$j = 2.55 \times 10^4 \text{ A/m}^2$$

$$\text{drift velocity} = \frac{j}{ne}$$

$$v_d = \frac{2.55 \times 10^4}{[5.8 \times 10^{22} / (10^{-2})^3] \times 1.6 \times 10^{-19}}$$

$$v_d = \frac{2.55 \times 10^4 \times 10^{-6}}{5.8 \times 10^{22} \times 1.6 \times 10^{-19}}$$

$$v_d = \frac{2.55 \times 10^2}{5.8 \times 1.6 \times 10^3}$$

$$v_d = \frac{2.55 \times 10^{-5}}{5.8 \times 1.6}$$

$$v_d = \frac{2.55 \times 10^{-5}}{9.28} = 0.274 \times 10^{-5} = 2.74 \times 10^{-6} \text{ m/s}$$

There are 8.4×10^{22} free electrons per cm^3 in copper.
 Calculate the drift velocity of electrons in copper wire
 of 1 mm^2 cross section. when the current in the wire
 is 0.21 A

Sol:

Given data,

$$n = \frac{8.4 \times 10^{22}}{\text{cm}^3} = \frac{8.4 \times 10^{22}}{(10^{-2})^3} = 8.4 \times 10^{28} \text{ e}$$

$$A = 1 \text{ mm}^2 = 1 \times (10^{-3})^2 = 10^{-6} \text{ m}^2$$

$$\text{Current, } i = 0.21 \text{ A}$$

We know, $i = neAv_d$

$$v_d = \frac{i}{neA}$$

$$v_d = \frac{0.21}{8.4 \times 10^{28} \times 1.6 \times 10^{-19} \times 10^{-6}}$$

$$v_d = \frac{0.21}{8.4 \times 1.6 \times 10^{-3}} = \frac{0.21 \times 10^3}{13.44} = 15.625 \text{ m/s}$$

$$V_d = 0.015 \times 10^{-3}$$

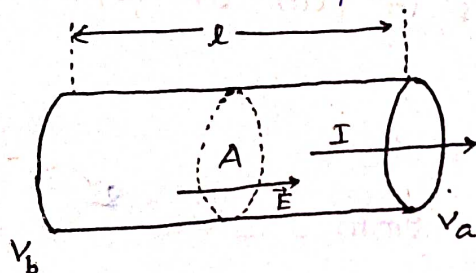
$$v_d = 1.5 \times 10^{-5} \text{ m/s}$$

Resistance and Ohm's law:

Current density and an electric field are established in a conductor whenever a potential difference is maintained across the conductor. In some materials, the current density is proportional to the electric field;

$$\boxed{J = \sigma E}$$

where the constant of proportionality σ is called the conductivity of the conductor. The above eqn is said to be Ohm's law.



Considering a segment of straight wire of uniform cross-sectional area A and length l as shown in figure. A potential difference $V_b - V_a$ is maintained across the wire, creating in the wire an electric field and a current. If the field is assumed to be uniform, the potential difference is related to the field through the following eqn.

$$\Delta V = E l$$

$$J = \sigma E \text{ (ohm's law)}$$

$$J = \sigma \frac{\Delta V}{l} \quad (\because E = \frac{\Delta V}{l})$$

$$\frac{I}{A} = \sigma \frac{\Delta V}{l} \quad (\because J = I/A)$$

$$\Delta V = \frac{l I}{A \sigma}$$

$$\Delta V = IR \text{ (where } R \text{ is resistance)}$$

Resistivity (ρ) is inverse of conductivity (σ).

$$\text{i.e., } \rho = \frac{1}{\sigma}$$

$$\boxed{R = \frac{\rho L}{A}}$$

$$(\because R = \frac{1}{\sigma})$$

$$\sigma = \frac{J}{E}$$

$$\rho = \frac{E}{J} \quad (\because \rho = \frac{1}{\sigma})$$

Calculate the drift speed of the electrons when 1A of current exists in a copper wire of cross section 2mm^2 . The no. of free electrons in 1cm^3 of copper is 8.5×10^{22} .

Sol:

Given data,

$$I = 1\text{A}$$

$$A = 2\text{mm}^2 = 2 \times (10^{-3})^2 = 2 \times 10^{-6}\text{m}^2$$

$$n = 8.5 \times 10^{22}/\text{cm}^3 = \frac{8.5 \times 10^{22}}{(10^{-2})^3}$$

$$n = 8.5 \times 10^{22} \times 10^6 = 8.5 \times 10^{28}\text{m}^{-3}$$

we know,

$$i = neAv_d$$

$$v_d = \frac{i}{neA}$$

$$v_d = \frac{1}{8.5 \times 10^{28} \times 10^6 \times 1.6 \times 10^{-19} \times 2 \times 10^{-6}}$$

$$v_d = \frac{1}{8.5 \times 1.6 \times 2 \times 10^3}$$

$$v_d = \frac{1}{27.2 \times 10^3}$$

$$v_d = 0.036 \times 10^{-3}$$

$$v_d = 3.6 \times 10^{-5}\text{m/s}$$

Poisson's Equation and Laplace's Equation:

The electric field can be written as gradient of a scalar potential

$$\mathbf{E} = -\nabla V$$

The divergence of $\mathbf{E}$, & curl of $\mathbf{E}$,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V,$$

The divergence of $\mathbf{E}$ is the Laplacian of V .

from Gauss's law,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \longrightarrow (1)$$

This eq'n (1) is known as Poisson's equation.

Laplace's equation:

In regions where there is no charge, so $\rho = 0$, Poisson's equation reduces to Laplace's equation.

$$\nabla^2 V = 0 \longrightarrow (2)$$

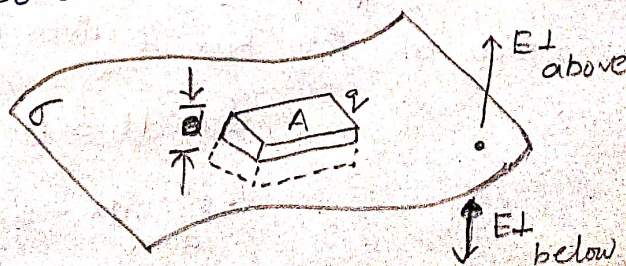
$$\& \quad \nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0$$

This shows that curl of gradient is always zero.

Boundary conditions of electric field & electrostatic potential:

Let us consider a surface with a surface charge density σ , on the surface may be curved.

Consider a small area A and build a small pill box that extends a small distance above and below the surface as shown in figure.



From Gauss's law,

The flux through the pill box is given by

$$\oint \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\epsilon_0} q,$$

Where, q , is the charge enclosed by the pill box,

The normal component of the electric field over the box of area A is given by

$$\therefore \oint \mathbf{E} \cdot d\mathbf{s}$$

$$= \oint_{\text{above}} \mathbf{E}^\perp \cdot d\mathbf{s} - \oint_{\text{below}} \mathbf{E}^\perp \cdot d\mathbf{s} = \frac{q}{\epsilon_0}$$

$$E_{\text{above}}^\perp \oint d\mathbf{s} - E_{\text{below}}^\perp \oint d\mathbf{s} = \frac{q}{\epsilon_0}$$

$$E_{\text{above}}^\perp A - E_{\text{below}}^\perp A = \frac{q}{\epsilon_0}$$

We know,

$$\sigma = \frac{dq}{ds}$$

$$\sigma ds = dq$$

$$\int \sigma \cdot ds = \int dq$$

$$\therefore \sigma A = q$$

$$E_{\text{above}}^\perp - E_{\text{below}}^\perp = \frac{\sigma}{\epsilon_0} \longrightarrow (1)$$

The normal (or) perpendicular component of electric field has discontinuity of $\frac{\sigma}{\epsilon_0}$ as we cross the surface.

The sides of pill box contributes nothing to the flux in the limit as the thickness goes to zero.

The tangential ^{component} of a electric field over the thin rectangular loop of pill box is given by

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\oint_{\text{above}} \mathbf{E}^\parallel \cdot d\mathbf{l} - \oint_{\text{below}} \mathbf{E}^\parallel \cdot d\mathbf{l} = 0$$

$$E''_{\text{above}} \oint dl - E''_{\text{below}} \oint dl = 0$$

$$E''_{\text{above}} L - E''_{\text{below}} L = 0$$

$$E''_{\text{above}} L = E''_{\text{below}} L$$

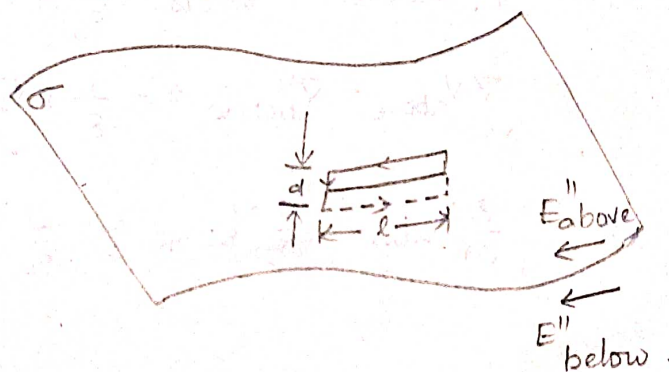
$$E''_{\text{above}} = E''_{\text{below}} \longrightarrow (2)$$

Where E'' stands for the components of E parallel to the surface. The boundary conditions on E can be combined into a single

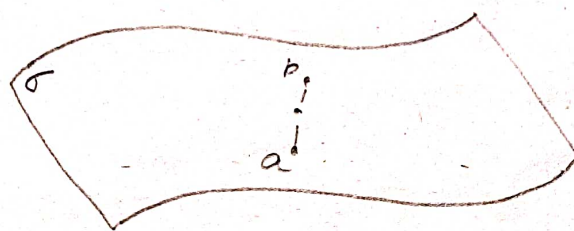
formula

$$E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n} \longrightarrow (3)$$

Where $\hat{n}$ is the unit vector perpendicular to the surface.



Boundary conditions of Electrostatic potential:



The potential is continuous across any boundary, since

$$V_{\text{above}} - V_{\text{below}} = - \int_a^b E \cdot dl$$

as the path length shrinks to zero, so

$$V_{\text{above}} - V_{\text{below}} = 0$$

$$V_{\text{above}} = V_{\text{below}}$$

The normal component of electric field is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary. The electric potential is continuous across any boundary

from eq'n (3)

$$E_{\text{above}} - E_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}.$$

Since $E = -\nabla V$,

$$-\nabla V_{\text{above}} + \nabla V_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}.$$

$$-\left[\nabla V_{\text{above}} - \nabla V_{\text{below}}\right] = \frac{\sigma}{\epsilon_0} \hat{n}.$$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0} \sigma \hat{n}.$$

$$\Rightarrow \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0} \sigma.$$

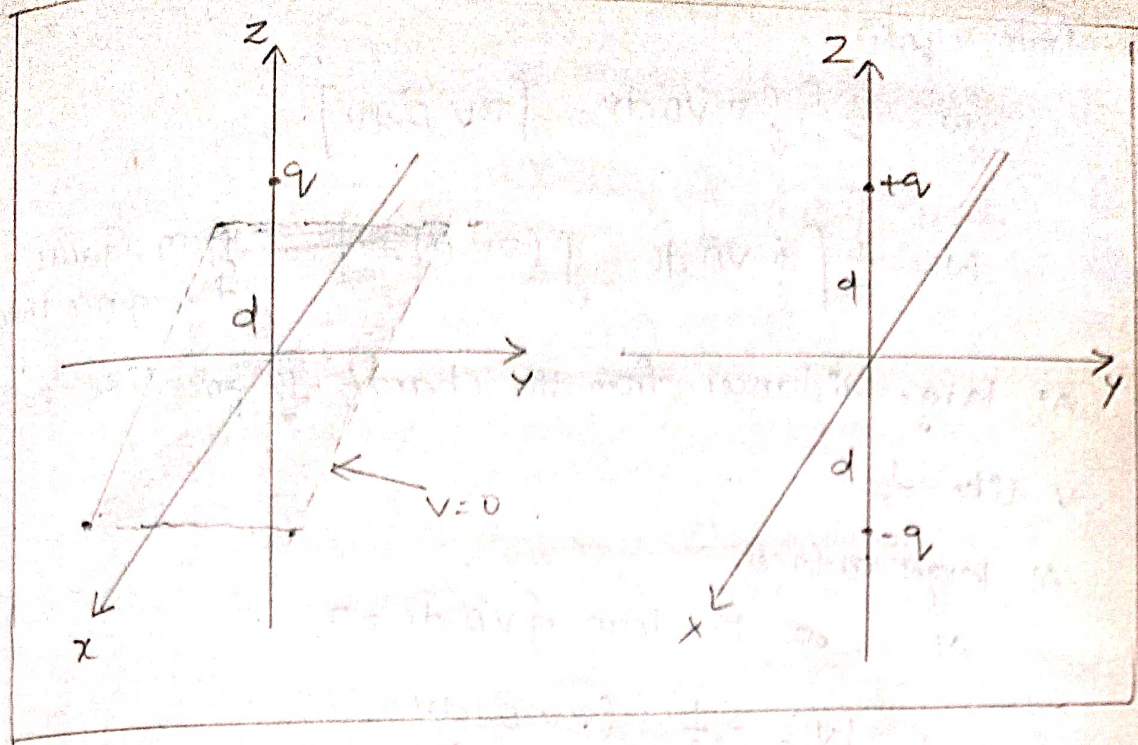
Method of images:

Method of Image of charges:

If the charge configuration placed above the infinite grounded conducting plate then we apply the concept of method of images. It states that if any charge configuration placed above infinite grounded perfect conducting plate then it is replaced by charge configuration itself and its image and equipotential surface.

A point charge ' q ' is placed at a distance d above a grounded conducting infinite plane sheet as shown in below figure.

Find the potential above the plate.



Boundary conditions:

At $z=0$, $V(0) = 0$

At $z=\infty$, $V(\infty) = 0$

$$V = \left[\frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{1}{4\pi\epsilon_0} \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}} \right]$$

distance b/w two point

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} \right] \quad \left(\because \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \right)$$

- * The image charge is just opposite sign of given point charge.
- * The image charge should be on the other side of the boundary.

Energy of a charge distribution and its expression in terms of electric field:

If the charge is distributed in a volume

$$W = \frac{1}{2} \int \rho v dv$$

$$\left[\because \rho = \frac{dq}{dv} \Rightarrow dq = \rho dv \Rightarrow q = \int \rho dv \right]$$

$$W = \frac{1}{2} \int_V (\nabla \cdot \vec{D}) v dv \longrightarrow (1)$$

$$\therefore \nabla \cdot \nabla (\vec{D}) = \nabla v \cdot \vec{D} + v (\nabla \cdot \vec{D})$$

$$v (\nabla \cdot \vec{D}) = \nabla v \cdot \vec{D} - \nabla v \cdot \vec{D}$$

From eq'n (1)

$$W = \frac{1}{2} \left[\int_V \nabla \cdot V \vec{D} dv - \int_V \nabla V \cdot \vec{D} dv \right]$$

$$W = \frac{1}{2} \left[\oint_S V \vec{D} \cdot d\vec{s} - \int_V (\nabla V \cdot \vec{D}) dv \right] \quad \because \text{from Gauss divergence theorem}$$

At large distances from the charge D goes like $\frac{1}{r^2}$ and V like $\frac{1}{r}$.

At larger surfaces,

At $r = \infty$, the term $\oint V \vec{D} \cdot d\vec{s} = 0$.

$$W = -\frac{1}{2} \int_V (\nabla V \cdot \vec{D}) dv$$

$$W = \frac{1}{2} \int_V (-\nabla V \cdot \vec{D}) dv$$

$$W = \frac{1}{2} \int_V (\vec{E} \cdot \epsilon_0 \vec{E}) dv$$

$$W = \frac{1}{2} \epsilon_0 \int_V E^2 dv$$

Induced charge density:

$$E_{z=0} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \epsilon_0 E_{z=0}$$

$$\sigma = \epsilon_0 (-\nabla V)$$

$$\sigma = \epsilon_0 \left[-\frac{\partial V}{\partial z} \right]_{z=0}$$

$$\text{Since, } V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z} \left[\frac{q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \right]$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \frac{\partial}{\partial z} \left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \left(-\frac{1}{2}\right) \frac{1}{\sqrt{x^2+y^2+(z-d)^2}^{3/2}} \frac{\partial}{\partial z} (x^2+y^2+(z-d)^2) - \left[-\frac{1}{2}\right] \frac{1}{\sqrt{x^2+y^2+(z+d)^2}^{3/2}} \frac{\partial}{\partial z} (x^2+y^2+(z+d)^2) \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \left[-\frac{1}{2}\right] \frac{1}{\sqrt{x^2+y^2+(z-d)^2}^{3/2}} \cdot 2(z-d)(1) - \left[-\frac{1}{2}\right] \frac{1}{\sqrt{x^2+y^2+(z+d)^2}^{3/2}} \cdot 2(z+d)(1) \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \frac{-(z-d)}{\sqrt{x^2+y^2+(z-d)^2}^{3/2}} - \frac{-(z+d)}{\sqrt{x^2+y^2+(z+d)^2}^{3/2}} \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \frac{z+d}{\sqrt{x^2+y^2+(z+d)^2}^{3/2}} - \frac{(z-d)}{\sqrt{x^2+y^2+(z-d)^2}^{3/2}} \right\}$$

$$\left[\frac{\partial V}{\partial z}\right]_{z=0} = \frac{q_v}{4\pi\epsilon_0} \left\{ \frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} - \frac{(-d)}{\sqrt{x^2+y^2+(-d)^2}^{3/2}} \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} + \frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right\}$$

$$= \frac{q_v}{4\pi\epsilon_0} \left\{ \frac{2d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right\}$$

$$= \frac{q_v}{2\pi\epsilon_0} \left\{ \frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right\}$$

Therefore, $\sigma = \epsilon_0 \left[-\frac{\partial V}{\partial z} \right]_{z=0}$.

$$= \epsilon_0 \left[-\frac{q_v}{2\pi\epsilon_0} \left\{ \frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right\} \right]$$

$$= -\frac{q_v\epsilon_0}{2\pi\epsilon_0} \left[\frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right]$$

$$\sigma = -\frac{q_v}{2\pi} \left[\frac{d}{\sqrt{x^2+y^2+d^2}^{3/2}} \right]$$

Induced charge (Q):

As we know, $\sigma = \frac{dq}{da}$

$$dq = \sigma da$$

$$\int dq = \sigma da$$

$$Q = \int \sigma da$$

$$Q = \int \sigma dx dy$$

$$Q = \int \sigma r dr d\theta$$

$$Q = \int_0^\infty \int_0^{2\pi} \frac{-q d}{2\pi (x^2 + y^2 + d^2)^{3/2}} r dr d\theta \quad \left[\because \sigma = \frac{-q d}{2\pi (x^2 + y^2 + d^2)^{3/2}} \right]$$

$$= \int_0^{2\pi} d\theta \int_0^\infty \frac{-q d}{2\pi (x^2 + y^2 + d^2)^{3/2}} r dr$$

$$= [0]_0^{2\pi} - \frac{q d}{2\pi} \int_0^\infty \frac{r dr}{(x^2 + y^2 + d^2)^{3/2}}$$

$$= 2\pi \cdot \frac{q d}{2\pi} \int_0^\infty \frac{-r dr}{(x^2 + y^2 + d^2)^{3/2}}$$

$$= q d \int_0^\infty \frac{-r dr}{(x^2 + y^2 + d^2)^{3/2}}$$

here $x^2 = x^2 + y^2$

Let $u = x^2 + y^2 + d^2$

$$= q d \int_0^\infty \frac{-r dr}{(r^2 + d^2)^{3/2}} \quad \text{Let } u = \frac{1}{r^2 + d^2}$$

$$= q d \int du$$

$$\frac{du}{dr} = \frac{-1}{2} \frac{1}{(r^2 + d^2)^{3/2}}$$

$$= q d [u]_0^\infty$$

$$du = \frac{-r}{(r^2 + d^2)^{3/2}} dr$$

$$Q = qd \left[\frac{1}{\sqrt{x^2 + d^2}} \right]_0^\infty$$

$$= qd \left[\frac{1}{\sqrt{\infty^2 + d^2}} - \frac{1}{\sqrt{0^2 + d^2}} \right]$$

$$= qd \left[0 - \frac{1}{(d)} \right]$$

$$= qd \left[-\frac{1}{d} \right]$$

$$= -\frac{qd}{d}$$

$$\therefore Q = -q$$

Force and Energy:

$$F = \frac{1}{4\pi\epsilon_0} \frac{q(-q)}{(2d)^2}$$

$$\left[\because \text{Coulomb's force } F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \right]$$

$$= \frac{-q^2}{4\pi\epsilon_0 (4d^2)} = \frac{-q^2}{16\pi\epsilon_0 d^2}$$

Work done,

$$W = \int F \cdot dl$$

$$= \int_0^d \frac{-q^2}{16\pi\epsilon_0 d^2} \cdot dl$$

$$= \frac{-q^2}{16\pi\epsilon_0} \int_0^d \frac{1}{d^2} dl$$

$$= \frac{-q^2}{16\pi\epsilon_0} \left[-\frac{1}{d} \right]_0^d$$

$$= \frac{q^2}{16\pi\epsilon_0} \left[\frac{1}{d} - \frac{1}{\infty} \right]$$

$$W = \frac{q^2}{16\pi\epsilon_0} \left[\frac{1}{d} - \frac{1}{\infty} \right] = \frac{q^2}{16\pi\epsilon_0 d}$$

$$\begin{aligned} \int \frac{1}{x^2} &= \int x^{-2} dx \\ &= \frac{x^{-2+1}}{-2+1} \\ &= -\frac{1}{x} \end{aligned}$$

Q. calculate the resistance of an aluminium wire of length 50 cm and cross-sectional area 2.0 mm^2 . The resistivity of Aluminium is $\rho = 2.6 \times 10^{-8} \Omega \cdot \text{m}$.

Sol:- Given data,

$$\text{Length, } l = 50 \text{ cm} = 50 \times 10^{-2} \text{ m}$$

$$\text{Area, } A = 2.0 \text{ mm}^2 = 2.0 \times (10^{-3})^2 = 2 \times 10^{-6} \text{ m}^2$$

$$\text{Resistivity, } \rho = 2.6 \times 10^{-8} \Omega \cdot \text{m}$$

$$\text{We know, } R = \frac{\rho l}{A}$$

$$R = \frac{2.6 \times 10^{-8} \times 50 \times 10^{-2}}{2 \times 10^{-6}}$$

$$R = 2.6 \times 25 \times 10^{-4}$$

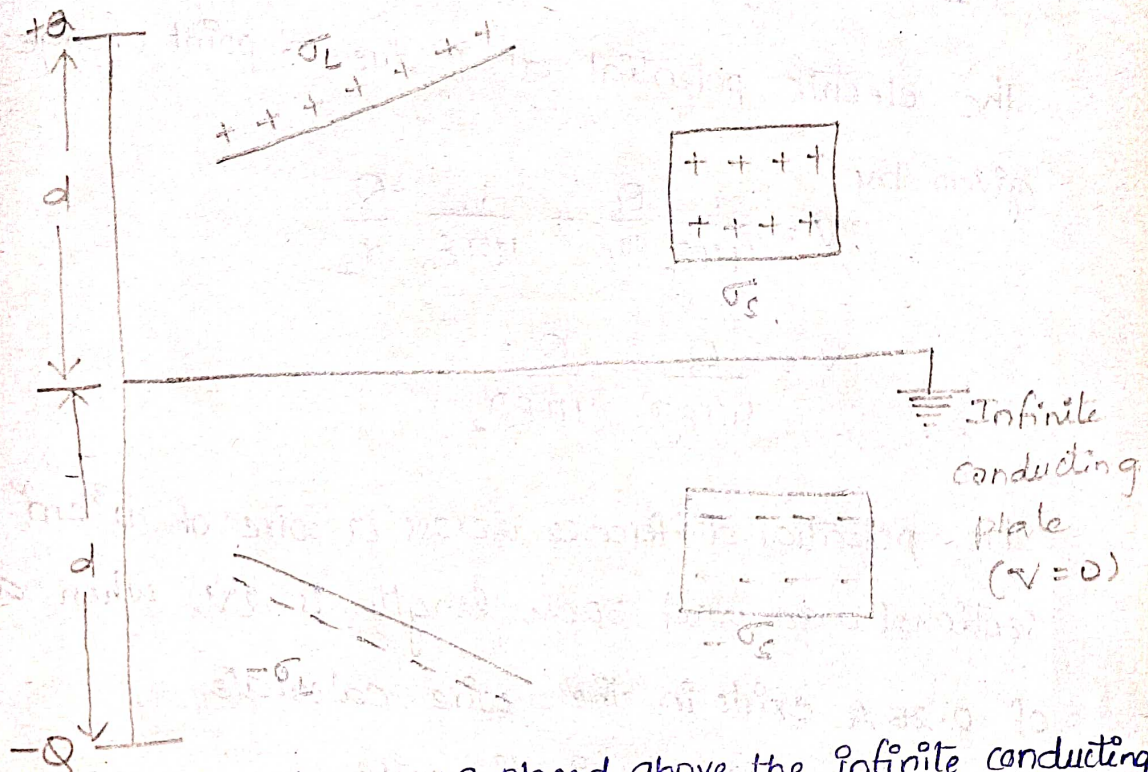
$$R = 65 \times 10^{-4}$$

$$R = 6.5 \times 10^{-3} \Omega$$

the electric field and electric potential of a point charge placed above the grounded conducting plane:

(or)

the electric field intensity and electric potential due to point charge placed about grounded conducting plate:



Consider a point charge $+Q$ placed above the infinite conducting plane which is grounded. The coordinates of point charge is $(0, 0, h)$.

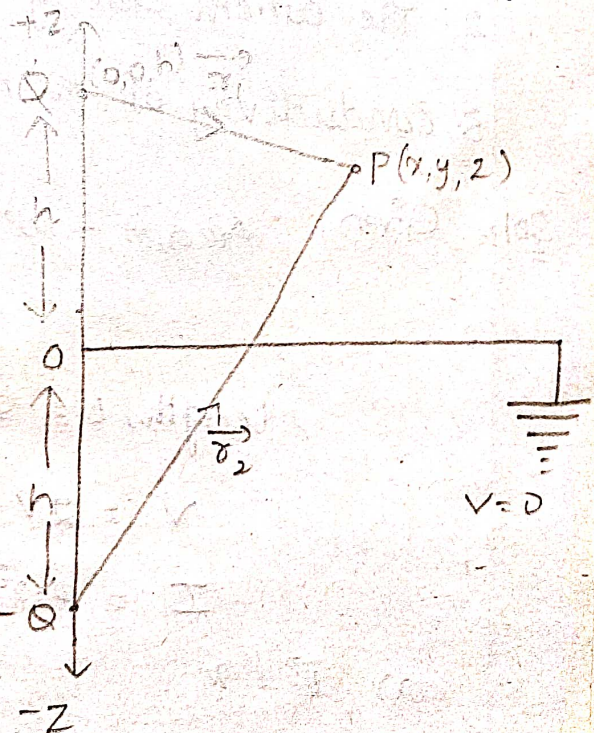
The electric field intensity at point P is given by

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_1^2} \hat{a}_{r_1} + \frac{1}{4\pi\epsilon_0} \frac{-Q}{r_2^2} \hat{a}_{r_2}$$

$$[\because \hat{a}_1 = \frac{\vec{r}_1}{|\vec{r}_1|}]$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_1^2} \frac{\vec{r}_1}{|\vec{r}_1|} + \frac{1}{4\pi\epsilon_0} \frac{(-Q)}{r_2^2} \frac{\vec{r}_2}{|\vec{r}_2|}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_1^3} \vec{r}_1 - \frac{1}{4\pi\epsilon_0} \frac{Q}{r_2^3} \vec{r}_2$$



$$\vec{r}_1 = \vec{OP} = (x-0, y-0, z-h)$$

$$\vec{r}_1 = x\hat{a}_x + y\hat{a}_y + (z-h)\hat{a}_z$$

$$|\vec{r}_1| = \sqrt{x^2 + y^2 + (z-h)^2}$$

$$|\vec{r}_2| = \sqrt{x^2 + y^2 + (z+h)^2}$$

The electric potential at P due to point charge is

given by

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_1} + \frac{1}{4\pi\epsilon_0} \frac{-Q}{r_2}$$

$$V = \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$

The potential difference across a wire of 10^3 cm^2 cross sectional area and 50cm length is 2V, when a current of 0.25A exists in the wire. calculate

1. The field strength in the wire

2. The current density &

3. conductivity of the metal

Sol:- Given, Area, $A = 10^3 \text{ cm}^2 = 10^3 \times (10^{-2})^2 \text{ m}^2$

$$= 10^3 \times 10^{-4} \text{ m}^2 = 10^{-1} \text{ m}^2$$

$$\text{length, } L = 50 \text{ cm} = 50 \times 10^{-2} \text{ m}$$

$$V = 2 \text{ V}$$

$$I = 0.25 \text{ A}$$

a) $\vec{E} = ?$

b) $J = ?$

c) $\sigma = ?$

a) field strength, $E = ?$

$$\Delta V = E \cdot d$$

$$E = \frac{V}{d} = \frac{2}{50 \times 10^{-2}} = \frac{200}{50} = 4 \text{ V/m}$$

b) current density, $J = ?$

$$J = \frac{\text{current}}{\text{Area}} = \frac{I}{A} = \frac{0.25}{10^{-7} \text{ m}} = \frac{2.5 \times 10^1}{10^{-7}} = 2.5 \times 10^7 \text{ A/m}^2$$

$$J = 2.5 \times 10^6 \text{ A/m}^2$$

c) ~~$R = \frac{E}{J}$~~

~~$R = \frac{E}{J}$~~

conductivity, $\sigma = ?$

$$\text{We know, } V = IR \Rightarrow R = \frac{V}{I} = \frac{2}{0.25} = 8 \Omega$$

$$\text{As, Resistivity } \rho = \frac{RA}{l} \quad (\because R = \frac{\rho l}{A})$$

$$\rho = \frac{8 \times 10^{-7}}{50 \times 10^{-2}} = \frac{8 \times 10^{-7} \times 10^2}{50} = 0.16 \times 10^{-5}$$

$$\rho = 1.6 \times 10^{-6} \Omega \cdot \text{m}$$

$$\text{We know, } \rho = \frac{1}{\sigma}$$

$$\sigma = \frac{1}{1.6 \times 10^{-6}} = 6.25 \times 10^5 \frac{\text{C}}{\text{m}} \text{ (or)} \Omega^{-1} \text{m}^{-1}$$

(or)

$$\rho = \frac{E}{J}$$

$$\rho = \frac{4}{2.5 \times 10^6}$$

$$\rho = 1.6 \times 10^{-6} \Omega \cdot \text{m}$$

$$\rho = \frac{1}{\sigma}$$

$$\sigma = \frac{1}{1.6 \times 10^{-6}} = 6.25 \times 10^5 \frac{\text{C}}{\text{m}} \text{ (or)} \Omega^{-1} \text{m}^{-1}$$

2. Find the resistance of 1 cm of copper when drawn into a wire of diameter 0.33 mm. Specific Resistance of copper $1.59 \times 10^{-6} \Omega\text{-cm}$.

Sol: Given that,

Resistance $R = ?$

$$\text{Volume, } V = 1 \text{ cm}^3 = 1 \times (10^{-2})^3 \text{ m}^3 = 10^{-6} \text{ m}^3.$$

$$\text{Diameter, } D = 0.33 \text{ mm}$$

$$\text{radius, } r = \frac{D}{2} = \frac{0.33 \text{ mm}}{2} = \frac{0.33}{2} \times 10^{-3} \text{ m} = 0.16 \times 10^{-3} \text{ m}$$

$$\text{Resistivity, } \rho = 1.59 \times 10^{-6} \Omega\text{-cm} = 1.59 \times 10^{-6} \times 10^{-2} \Omega\text{m}.$$

$$\text{We know, } R = \frac{\rho L}{A}.$$

$$R = \frac{\rho V}{\frac{A}{A}} \quad (\because V = L \cdot A) \\ L = \frac{V}{A}.$$

$$R = \frac{\rho V}{A} \cdot \frac{1}{A} = \frac{\rho V}{A^2} = \frac{\rho V}{(\pi r^2)^2}.$$

$$R = \frac{(1.59 \times 10^{-8})(10^{-6})}{[3.14 \times (0.16 \times 10^{-3})^2]^2}.$$

$$R = \frac{1.59 \times 10^{-14}}{(3.14 \times 2.56 \times 10^{-8})^2}.$$

$$R = \frac{1.59 \times 10^{-14}}{9.85 \times 65536 \times 10^{-16}}.$$

$$R = 0.0246 \times 10^{-14} \times 10^{16}$$

$$R = 0.0246 \times 10^2$$

$$\boxed{R = 2.46 \Omega}$$