

Vector analysis

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UNIT-I

CO-ordinate Systems: There are 4 important co-ordinate systems. They are as shown.

- (a) Cartesian co-ordinate system
- (b) Polar co-ordinate system
- (c) cylindrical co-ordinate system
- (d) Spherical co-ordinate system.

(a) Cartesian Co-ordinate System: (x, y, z)

In Cartesian Co-ordinate System, we use x, y and z -axes. Consider a point

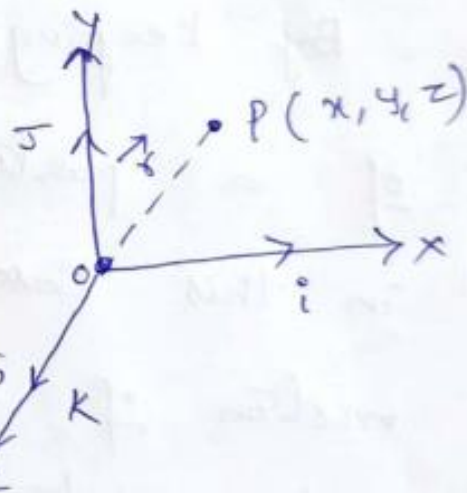
'P' in the space, its Co-ordinates are $P(x, y, z)$. Let $\vec{OP} = \vec{r}$

Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along x, y and z -axes respectively.

$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

This is the magnitude of $|\vec{OP}|$ or $|\vec{r}|$.



For example, if any body is moving in one direction, we can use Cartesian Co-ordinate system to study the motion of the body.

Ex:- ① A car is travelling from one point to another point

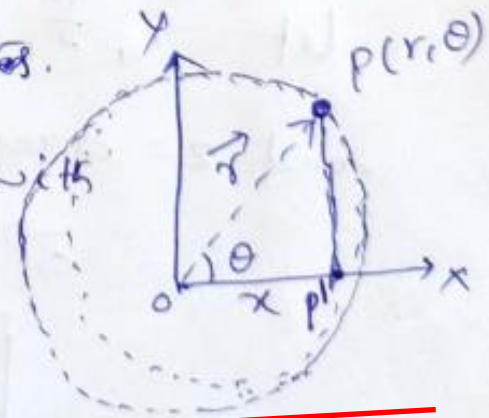
② An insect is moving on a wall.

In these cases, we use this system for their motion.

(b) Polar co-ordinate system: (r, θ)

By keeping the value of ' r ' (∞) the position of a particle is rotating in circular motion, in this case it is difficult to find the motion of a particle using Cartesian co-ordinate system. Then we use polar co-ordinate system. In polar system, we use (r, θ) as polar co-ordinates.

Let the particle be rotating with angle θ with x -axis, i.e. it has fixed radius.



$\therefore$ Its projection is P' . x, y be the co-ordinates of P' .

$\therefore \vec{r}$ can be written as $\vec{r} = x\hat{i} + y\hat{j}$

i.e. the position vector $\vec{r} = x\hat{i} + y\hat{j}$

$$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r \cos\theta}$$

$$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r \sin\theta}$$

$$\therefore \vec{r} = r \cos\theta \hat{i} + r \sin\theta \hat{j}$$

$$\therefore \tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$$

$\therefore$ The position vector $\vec{r}$ can be written in terms of r, θ as $\boxed{\vec{r} = r \cos\theta \hat{i} + r \sin\theta \hat{j}}$

(c) Cylindrical Co-ordinate System: (ρ, ϕ, z)

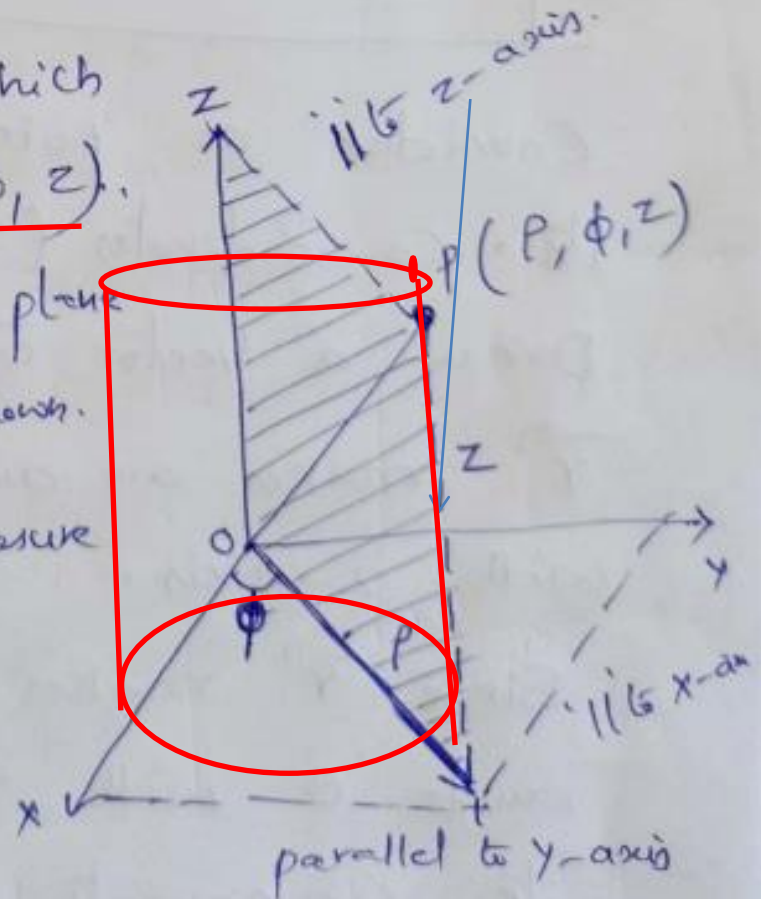
Consider a point 'P' which has the co-ordinates (ρ, ϕ, z) .

The projection 'P' on xy plane

can be represented as shown.

One of the important measurement that use in this system is 'P'.

Draw the parallel lines as shown to all x, y and z-axes.



Since $\vec{p}$ makes an angle ϕ with x -axis

$\therefore$ The component of $\vec{p}$ along x -axis, i.e.

x -component of $\vec{p}$ is $x = p \cos \phi$

y -component of $\vec{p}$ is $y = p \sin \phi$

$$\therefore x^2 + y^2 = p^2 \Rightarrow \boxed{p = \sqrt{x^2 + y^2}}$$

$$\text{Similarly } \tan \phi = \frac{y}{x} \Rightarrow \boxed{\phi = \tan^{-1} (y/x)}$$

$$\boxed{z = z}$$

This system is used to study the properties of a cylinder (or) dimensions of a cylinder.

(d) Spherical Co-ordinate System (r, θ, ϕ)

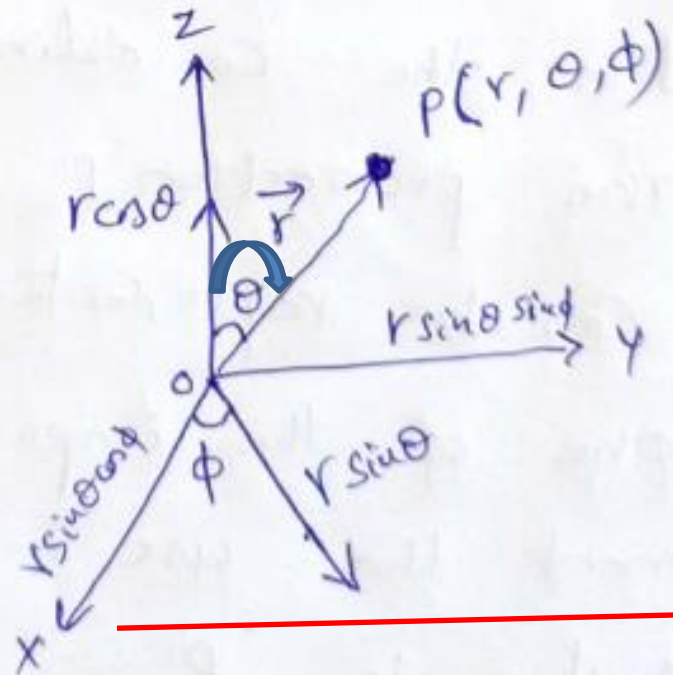
Consider a point P with its Co-ordinates (r, θ, ϕ)

Draw a Vector $\vec{OP} = \vec{r}$

$\vec{r}$ makes an angle θ with z -axis.

Since $\vec{r}$ makes an angle θ with z -axis

its component along z -axis becomes ' $r \cos \theta$ ' which is parallel component. And the



perpendicular component is ' $r \sin \theta$ ' as shown

Since ' z ' is normal to x, y plane, its $\perp$ to

Component can be drawn as shown in fig.

This Vertical Component ' $r \sin \theta$ ' makes an angle ' ϕ ' with x -axis.

Now,

$x = r \sin \theta \cos \phi$, i.e. the one component of $r \sin \theta$ along x -axis.

$y = r \sin \theta \sin \phi$, i.e. another component of $r \sin \theta$ along y -axis.

$$z = r \cos \theta$$

$$\therefore x^2 + y^2 + z^2 = r^2 \Rightarrow$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore x^2 + y^2 + z^2 = r^2 \Rightarrow \boxed{r = \sqrt{x^2 + y^2 + z^2}}$$

$$\therefore \tan \theta = \frac{\sqrt{x^2 + y^2}}{z} \Rightarrow \boxed{\theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right)}$$

$$\text{and } \tan \phi = \frac{y}{x} \text{ and } \boxed{\phi = \tan^{-1} \frac{y}{x}}$$

Ex:- (i) Conversion of Spherical co-ordinates $(4, 30^\circ, 60^\circ)$
to Cartesian co-ordinates.

Sol:- Given $(4, 30^\circ, 60^\circ)$, i.e. $r=4$, $\theta=30^\circ$,
 $\phi=60^\circ$.

But we have, $x = r \sin \theta \cos \phi$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$\therefore$ Cartesian co-ordinates can be written as

$$x = 4 \sin 30^\circ \cos 60^\circ \Rightarrow x = 4 \times \frac{1}{2} \times \frac{1}{2} \Rightarrow \boxed{x=1}$$

$$\text{Take, } y = 4 \times \sin 30^\circ \times \sin 60^\circ \Rightarrow y = 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} \Rightarrow \boxed{y=\sqrt{3}}$$

$$\text{and } z = r \cos \theta \Rightarrow z = 4 \times \cos 30^\circ \Rightarrow z = 4 \times \frac{\sqrt{3}}{2} \Rightarrow \boxed{z=2\sqrt{3}}$$

$$\therefore (x, y, z) = (1, \sqrt{3}, 2\sqrt{3})$$

$\therefore$ Cartesian Co-ordinates are $(1, \sqrt{3}, 2\sqrt{3})$

(2) Cartesian co-ordinates into Spherical co-ordinates

Convert $(1, 2, 3)$ into Spherical co-ordinates (r, θ, ϕ)

$$\text{We have } r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r = \sqrt{1^2 + 2^2 + 3^2}$$

$$\therefore r = \sqrt{1+4+9} \Rightarrow \boxed{r = \sqrt{14}}$$

and We have $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \phi = \tan^{-1}\left(\frac{2}{1}\right) \Rightarrow \boxed{\phi = \tan^{-1}(2)}$$

and We have $\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$

$$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{1+4}}{3}\right) \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{\sqrt{5}}{3}\right)}$$

$$\therefore \text{Spherical coordinates becomes } \left(\sqrt{14}, \tan^{-1}\left(\frac{\sqrt{5}}{3}\right), \tan^{-1}(2)\right)$$

(iii) Conversion of Cartesian co-ordinates to cylindrical co-ordinates.

Convert $(1, 2, 3)$ into cylindrical co-ordinates.

Sol:- We have $\rho = \sqrt{x^2 + y^2}$
 $\phi = \tan^{-1}(y/x)$
 $z = z$

but given $x=1$, $y=2$, $z=3$

$$\therefore \rho = \sqrt{1^2 + 2^2} \Rightarrow \boxed{\rho = \sqrt{5}}$$

$$\phi = \tan^{-1}\left(\frac{2}{1}\right) \Rightarrow \boxed{\phi = \tan^{-1}(2)}$$

$$z = 3 \Rightarrow \boxed{z = 3}$$

$\therefore$ The cylindrical co-ordinates are $(\sqrt{5}, \tan^{-1}(2), 3)$

Thank you very much