

UNIT-I (Mathematical Physics) ⁽¹⁾

Some fundamental quantities in electromagnetics are scalars while others are Vectors. These scalars and Vectors can be functions of position and time. They can be completely described using an appropriate co-ordinate system. There are several co-ordinate systems, the most widely use are Cartesian, Polar co-ordinate, cylindrical and Spherical co-ordinate systems.

(a) Cartesian co-ordinate System: (x, y, z)

An orthogonal system is one in which the co-ordinates are mutually perpendicular. In Cartesian co-ordinate system, x, y, z are the co-ordinates of the point 'P' in Cartesian Space.

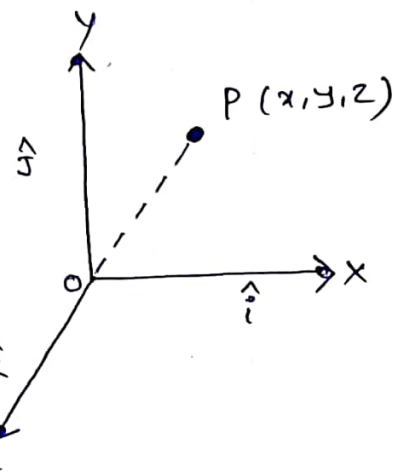
Consider a point 'P' in the space with its co-ordinates as x, y, z .

Draw a line from origin 'O' to the point 'P'. Then

$\vec{OP}$ becomes position Vector.

$$\text{let } \vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\therefore r = |\vec{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$



The ranges of the co-ordinates variables are
i.e. for x, y, z are: $-\infty < x < \infty$
 $-\infty < y < \infty$
 $-\infty < z < \infty$

Ex:- Any body moving in one direction (or)
a car travelling from one point to another
point can be described using Cartesian
co-ordinates.

(b) Polar co-ordinate system: (r, θ)

When a particle is rotating in circular motion
by keeping the radius vector ($\vec{r}$) (or) the
distance of the point 'P' from the origin 'O'
is fixed, it is difficult to study the motion
of a particle using Cartesian co-ordinate
system. Then we use Polar co-ordinate system
to study its motion. In Polar co-ordinate
system (r, θ) are the Polar co-ordinates.

Let a particle be rotating with an angle
' θ ' with x-axis with fixed radius in
a circular path as shown in figure.
Draw a projection P' from 'P'.

Here 'p' is the projection of 'P'

Let x, y be the co-ordinates of P' . Let $\vec{OP'} = \vec{r} = x\hat{i} + y\hat{j}$

From figure, $\cos\theta = \frac{x}{r}$ and

$$\sin\theta = \frac{y}{r}$$

Since $\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta} \quad \text{--- (1)}$

Since $\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta} \quad \text{--- (2)}$

put (1) and (2) in $\vec{r} = x\hat{i} + y\hat{j}$, we get

$$\boxed{\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}} \quad \text{--- (3)}$$

This is the expression for position Vector $\vec{r}$ in terms of r, θ .

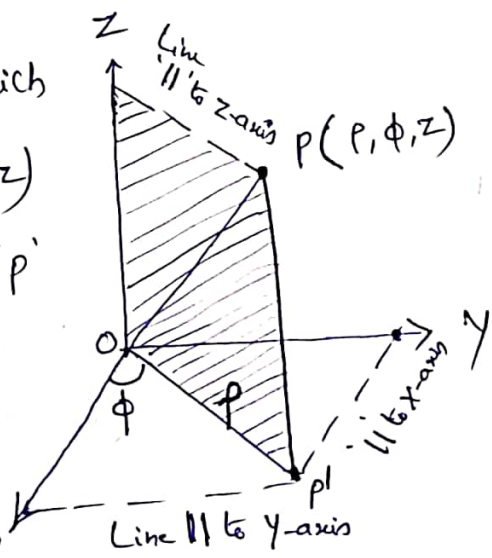
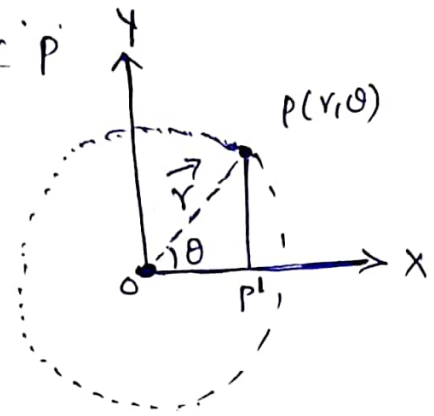
(c) cylindrical co-ordinate system: (ρ, ϕ, z)

Consider a point 'P' which has the co-ordinates (ρ, ϕ, z)

Draw a projection of 'P' on xy plane as shown.

Let $OP' = \rho$, which is an important measure

in cylindrical co-ordinate system. Draw the parallel lines to all x, y and z -axes as shown.



Let 'p' make an angle ' ϕ ' with x-axis.

Since 'p' makes an angle ' ϕ ', then 'x' component and 'y' component of 'p' can be written as.

$$x = p \cos \phi \text{---(1) and } y = p \sin \phi \text{---(2), } z = z$$

$$\text{Take (1)}^2 + \text{(2)}^2 \Rightarrow x^2 + y^2 = p^2 (\cos^2 \phi + \sin^2 \phi)$$

$$\therefore x^2 + y^2 = p^2 \Rightarrow \boxed{p = \sqrt{x^2 + y^2}}$$

$$\text{Similarly } \tan \phi = \frac{y}{x} \quad (\because \text{(2)/(1)})$$

$$\therefore \boxed{\phi = \tan^{-1} \left(\frac{y}{x} \right)} \quad \text{and } z = z$$

$$\therefore \text{ we have } p = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z$$

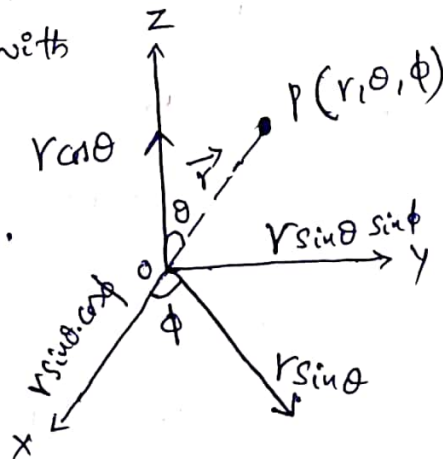
This system is used to study the properties of a cylinder (or) dimensions of a cylinder.

(d) Spherical co-ordinate system (r, θ, ϕ);

Consider a point 'p' with its co-ordinates (r, θ, ϕ)

Draw a vector $\vec{op} = \vec{r}$.

Let $\vec{r}$ makes an angle ' θ ' with z-axis.



$\therefore$ The horizontal Component of ' r ' is $r \cos \theta$ (i.e. along z -axis) and the Vertical Component of ' r ' is $r \sin \theta$ (i.e. in xy plane, since xy plane is $\perp$ to z -axis). $\therefore$ Vertical Component can be drawn.

$\therefore$ The two Components of ' r ' are ' $r \cos \theta$ ', ' $r \sin \theta$ '.

Again, let the Vertical Component ' $r \sin \theta$ ' makes an angle ' ϕ ' with x -axis.

Now again Consider the horizontal and Vertical Components of ' $r \sin \theta$ '.

$\therefore$ The x -component (horizontal) of $r \sin \theta$ is

$$x = r \sin \theta \cos \phi \quad \text{--- (1)}$$

and y -component (Vertical Component) of ' $r \sin \theta$ '

$$y = r \sin \theta \cdot \sin \phi \quad \text{--- (2)}$$

and

$$z = r \cos \theta \quad \text{--- (3)}$$

Take $(1)^2 + (2)^2 + (3)^2 \Rightarrow x^2 + y^2 + z^2 = r^2$

$$\therefore r = \sqrt{x^2 + y^2 + z^2} \quad \text{--- (4)}$$

Take $\frac{(2)}{(1)} \Rightarrow \tan \phi = \frac{y}{x} \Rightarrow \phi = \tan^{-1} \left(\frac{y}{x} \right) \quad \text{--- (5)}$

Take $\frac{①^2 + ②^2}{③^2}$, we get $\frac{r^2 \sin^2 \theta}{r^2 \cos^2 \theta} = \frac{x^2 + y^2}{z^2}$

$$\therefore \tan^2 \theta = \frac{x^2 + y^2}{z^2} \Rightarrow \tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\therefore \boxed{\theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right)} \quad \text{--- ⑥}$$

Ex:- ① Convert Spherical co-ordinates $(4, 30^\circ, 60^\circ)$ to Cartesian co-ordinates.

Sol:- Given $(4, 30^\circ, 60^\circ)$

Here $r = 4$, $\theta = 30^\circ$, $\phi = 60^\circ$

we have $x = r \sin \theta \cdot \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

$$\therefore x = 4 \sin 30^\circ \cdot \cos 60^\circ, y = 4 \sin 30^\circ \cdot \sin 60^\circ, z = 4 \cos 30^\circ$$

$$\therefore x = 4 \times \frac{1}{2} \times \frac{1}{2}, y = 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}, z = 4 \times \frac{\sqrt{3}}{2}$$

$$\therefore x = 1, y = \sqrt{3}, z = 2\sqrt{3}$$

$\therefore$ Cartesian co-ordinates are $(1, \sqrt{3}, 2\sqrt{3})$.

(ii) Convert Cartesian co-ordinates $(1, 2, 3)$ into Spherical co-ordinates.

Sol:- Given $(x, y, z) = (1, 2, 3)$

$$\therefore x=1, y=2, z=3$$

(4)

The Spherical co-ordinates are (r, θ, ϕ)

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow r = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\therefore \boxed{r = \sqrt{14}}$$

$$\theta = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right) \Rightarrow \theta = \tan^{-1} \left(\frac{\sqrt{1^2 + 2^2}}{3} \right)$$

$$\therefore \boxed{\theta = \tan^{-1} \left(\frac{\sqrt{5}}{3} \right)}$$

$$\phi = \tan^{-1} \left(\frac{y}{x} \right) \Rightarrow \phi = \tan^{-1} \left(\frac{2}{1} \right) \Rightarrow \boxed{\phi = \tan^{-1}(2)}$$

$\therefore$ Spherical co-ordinates are $\sqrt{14}, \tan^{-1} \left(\frac{\sqrt{5}}{3} \right), \tan^{-1}(2)$.

(ii) Convert Cartesian co-ordinates $(1, 2, 3)$ to cylindrical co-ordinates.

Sol:- Given $(x, y, z) = (1, 2, 3)$

cylindrical co-ordinates are (ρ, ϕ, z)

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \left(\frac{y}{x} \right), \quad z = z$$

$$\therefore \rho = \sqrt{1^2 + 4} \Rightarrow \boxed{\rho = \sqrt{5}}$$

$$\phi = \tan^{-1} \left(\frac{2}{1} \right) \Rightarrow \boxed{\phi = \tan^{-1}(2)}$$

$$z = z \Rightarrow \boxed{z = 3}$$

$\therefore (\sqrt{5}, \tan^{-1}(2), 3)$ are the cylindrical co-ordinates.

Scalar field: A scalar Quantity which can be specified at every point in a region of space is a scalar field.

Ex:- Electric potential is the example of scalar field.

Vector field: A Vector Quantity which can be specified at every point in a region (or) space is called Vector field.

Ex:- Gravitational force field, Electric field due to a point charge

Gradient: The 'del' operator (∇) plays a very important role in Vector calculus. It is not a Vector.

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

'Del' (∇) is used to define gradient of a scalar field, divergence and curl of a Vector field.

Let $\phi(x, y, z)$ be a scalar function of position of a scalar point of co-ordinates (x, y, z)

Then Gradient of a Scalar function ϕ is defined as $\text{grad}(\phi) = \nabla\phi$

$$\text{i.e. } \nabla\phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \phi$$

$$\therefore \text{Grad } \phi = \nabla\phi = \left(\vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \right) \quad \text{--- (1)}$$

∇ is a Vector operator. When it is operated with a scalar (ϕ), it converts the scalar into a Vector.

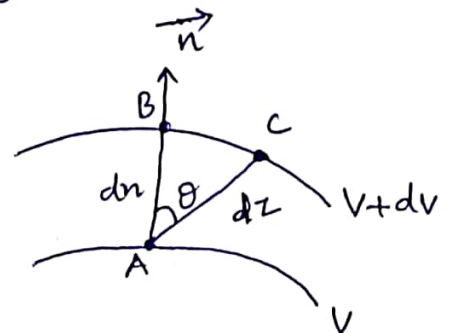
The Vector ($\nabla\phi$) is called Gradient of a scalar.

Physical Significance:

Consider two surfaces V and $V+dv$. Let the magnitude of $\vec{V}$ be constant on these surfaces.

Let dv be the small change in V .

Let A be point on surface V and B, C be the points on the surface $V+dv$.



The rate of change of function on the point A

along $\vec{AB}$ is $\frac{dV}{dn}$

ly along $\vec{AC}$, the rate of change of function = $\frac{dV}{dz}$.

From figure, $\cos \theta = \frac{dn}{dz} \Rightarrow dn = dz \cos \theta$.

$$\therefore \frac{dv}{dn} = \frac{dv}{dz \cos \theta} \quad (\because dn = dz \cos \theta)$$

$$\Rightarrow \frac{dv}{dz} = \frac{dv}{dn} \cos \theta \quad \text{--- (2)}$$

It is obvious from the equation, that when $\theta = 0$,

$\frac{dv}{dz}$ is maximum.

$$\therefore \text{when } \theta = 0, \quad \frac{dv}{dz} = \frac{dv}{dn}$$

$\therefore$ The maximum rate of increase of a scalar function 'V' at any point is given in magnitude & direction by Vector $\left(\frac{dv}{dn}\right) \hat{n}$, where $\hat{n} \rightarrow$ unit normal Vector at that point

The Vector is defined as gradient of scalar field at that point and can be written as

$$\text{grad } V = \frac{dv}{dn} \hat{n} \quad \text{--- (3)}$$

Divergence: The dot product of an operator $\text{del } (\nabla)$ with a Vector $\vec{A}$ is called divergence

The divergence is a scalar.

Let $\vec{A}$ be a Vector function with co-ordinates (x, y, z) in a region of space.

The divergence of $\vec{A}$ is given by

$$\nabla \cdot \vec{A} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (A_x \vec{i} + A_y \vec{j} + A_z \vec{k})$$

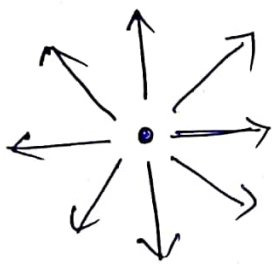
$$\therefore \nabla \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

$$\therefore \operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{--- (1)}$$

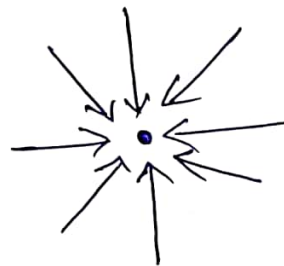
physical significance:

The divergence of a Vector $\vec{V}$ is a measure of how much a Vector $\vec{V}$ spreads out from the point.

The Vector may have positive divergence (arrows pointed out) and negative divergence (arrows pointed in) as shown.



(a) Positive divergence



(b) Divergence is negative.

A point of positive divergence is a source.

A // " -ve divergence // Sink (drain)

If $\text{div } \vec{A} = 0$, then $\vec{A}$ is called Solenoidal Vector

∴ If $\nabla \cdot \vec{A} = 0$, then $\vec{A}$ = Solenoid (or) incompressible.

Curl:- The cross product of an operator ∇ with a Vector $\vec{A}$ is called as curl. 'Curl' is a Vector.

Let $\vec{A}$ be a Vector with co-ordinates (x, y, z) in a region of space.

The curl of $\vec{A}$ is given by $(\nabla \times \vec{A})$.

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \hat{j} \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \\ &\quad + \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad \text{--- (1)} \end{aligned}$$

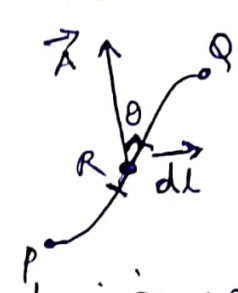
physical Significance: The curl of a Vector function $\vec{V}$ ($\nabla \times \vec{V}$) is a measure of how much the Vector $\vec{V}$ rotates around that point.

When $\text{curl } \vec{V} = 0$, it means that no rotation is attached with Vector $\vec{V}$, whereas as $\text{curl } \vec{V} \neq 0$ it means that rotation is attached with Vector $\vec{V}$ as shown in figure.

Integral Calculus

Line Integral:

The integration of a Vector along a curve is called its line integral. Consider a curve PQ , drawn between two points P and Q in Vector field as shown. Let ' dl ' be the small element of the curve PQ at a point ' R '. Let ' A ' be a Vector drawn from ' R ' making an angle ' θ ' with the direction of dl .



$$\text{Then } A \cdot dl = A dl \cos \theta = dl (A \cos \theta) \quad \text{--- (1)}$$

This means that the value of $A \cdot dl$ at any point of curve is equal to product of small element ' dl ' and the component of $A \cos \theta$ of ' A ' along the direction dl .

The value of $A \cdot dl$ for complete curve ' PQ ' can be obtained by integrating the equation (1),

$$\text{Hence } \int_P^Q A \cdot dl = \int_P^Q dl (A \cos \theta) \quad \text{--- (2)}$$

Integral $\int_P^Q A \cdot dl$ is defined as the line integral of ' A ' along the curve ' PQ '.

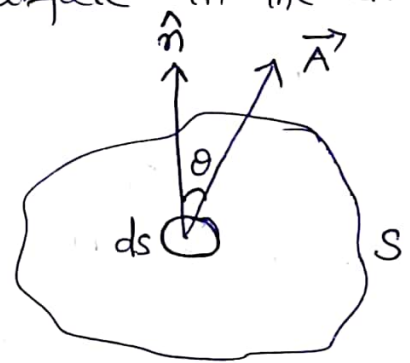
Example: If ' $\vec{A}$ ' represents the force acting on a particle moving along a curve from ' P ' to ' Q ', then the line integral $\int_P^Q \vec{A} \cdot d\vec{r}$ represents the total work done by a force during the motion of a particle over its entire path from ' P ' to ' Q '.

Surface integral:

Consider a simple surface ' S ' in a Vector field bounded by a curve as shown. Let ' ds ' be an infinitesimally small element of the surface.

This surface element of area ' ds ' can be represented by area Vector ' $d\vec{s}$ '. Let $\hat{n}$ be a unit Vector drawn outward the surface in the direction of ' $d\vec{s}$ ', then $d\vec{s} = \hat{n} ds$

Let ' $\vec{A}$ ' be a Vector at the middle of ' ds ' element making an angle ' θ ' with $\hat{n}$.



Now, the scalar product $\vec{A} \cdot d\vec{s} = \vec{A} \cdot \hat{n} ds$
 $= A ds \cos \theta$

This is called flux of Vector field ' $\vec{A}$ ' across the elemental (differential) area Vector ' $d\vec{s}$ '.

The total flux of the Vector field across the entire surface area ' S ' is given by

(PTO)

$$\iint_S \mathbf{A} \cdot d\mathbf{s} = \iint_S \mathbf{A} \cdot \hat{\mathbf{n}} d\mathbf{s} = \iint_S A \cos\theta d\mathbf{s} \quad .$$

This is defined as Surface integral.

Ex:- Let 'V' denotes Velocity Vector of a moving fluid. Consider a surface 'S' in the fluid.

Now, the Surface integral, i.e. $\iint_S \mathbf{V} \cdot d\mathbf{s}$

It represents the amount of fluid flowing per unit time normally to the surface.

The Surface integral is taken as '+ve' when the fluid flows outside the closed surface and '-ve' when the fluid flows into the closed surface.

Volume integral:

Consider a closed surface in space enclosing a Volume 'V'. Let 'A' be a Vector at a point in a small element of Volume 'dv', then the integral $\iiint_V \mathbf{A} dv$ is called Volume integral of Vector 'A' over the surface.

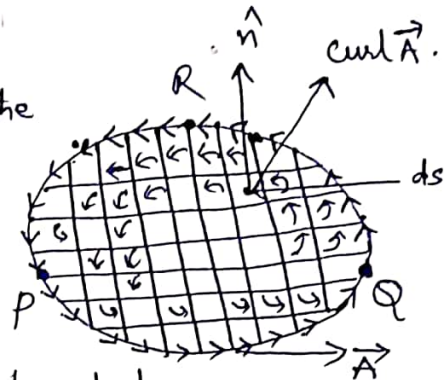
Stoke's Theorem

The line integral of a Vector field 'A' along a closed curve is equal to the surface integral of the curl of a Vector 'A' taken over the surface 'S' surrounded by closed curve.

$$\text{i.e. } \oint \vec{A} \cdot d\vec{r} = \iint_S \text{curl } \vec{A} \cdot d\vec{s} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{s}$$

Proof:- Consider a surface 'S' enclosed in a Vector field $\vec{A}$. The boundary of the surface is closed curve PQR.

The line integral $\vec{A}$ around the curve PQR traced counter clockwise is $\oint \vec{A} \cdot d\vec{r}$ — ①



Let the entire surface be divided into no. of small loops of each area ds . Let $\hat{n}$ be a unit vector drawn out ward normal upon ds .

The Vector area of the element is $\hat{n} ds = d\vec{s}$.

Since the curl of a Vector at any point is the maximum line integral of the Vector per unit area along the boundary of infinitesimal area at that point, so the line integral of $\vec{A}$ around the boundary of area ds is (becomes) $\text{curl } \vec{A} \cdot d\vec{s}$

It applies to all surface elements.

Hence, the sum of the line integrals of $\vec{A}$ around the boundaries of all area elements is given by

$$\iint_S \text{curl } \vec{A} \cdot d\vec{s} \text{ — ②}$$

From the figure, it is clear that the line integrals along the common sides of continuous elements

mutually cancel because they are traversed in opposite directions. only the sides of element which lie in the periphery of the surface (in closed curve) contribute to the line integral.

The sum of the line integrals on the boundary line of the curve is given by equation (2). It is also given by equation (1).

$$\text{Hence } \oint \mathbf{A} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{A} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}.$$

Significance: Stokes's theorem gives a method to convert a surface integral into a line integral and vice-versa. When $\text{curl } \mathbf{A} = 0$, the line integral of $\mathbf{A}$ over the closed path is zero.

* Gauss - Divergence Theorem:

Statement:

The Surface integral of a Vector field $\mathbf{A}$ over a closed surface is equal to the Volume integral of the divergence of a Vector field $\mathbf{A}$ over the volume V enclosed by the closed surface.

$$\text{i.e. } \iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V \text{div } \mathbf{A} \cdot d\mathbf{v} = \iiint_V (\nabla \cdot \mathbf{A}) d\mathbf{v}$$

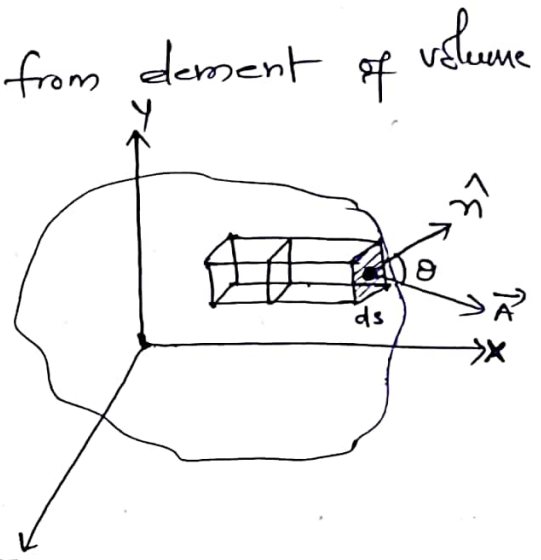
This theorem provides a method for connecting volume integrals to surface integrals.

proof:- Consider a closed surface 'S' of any arbitrary shape in a Vector field 'A'. Let 'V' be the volume of the surface. Let the whole volume may be assumed to be divided into a large no. of cubical volume elements.

We know that $\text{div } A$ represents the amount of flux diverging per unit volume.

Hence, the flux diverging from element of volume $dv = \text{div } A \cdot dv$

$\therefore$ The total flux coming out of the entire volume 'V' is given by $\iiint_V \text{div } A \cdot dv$ — (1)



Consider a small element of area 'ds' on the surface 'S' as shown. Let $\hat{n}$ be the unit Vector drawn normal to area 'ds'. If the Vector field (or) Vector A and out ward normal $\hat{n}$ are at an angle ' θ ', then the component of A along $\hat{n}$ is

$$A \cos \theta = A \cdot \hat{n}$$

The flux of 'A' through the surface element 'ds' is given by $(A \cdot \hat{n}) ds = A \cdot ds$

Since, the flux is defined as the product of

normal Component of Vector and surface area, (10)

The total flux through entire surface (S) is given by

$$\iint_S \mathbf{A} \cdot d\mathbf{s} \quad \text{--- (2)}$$

This must be equal to total flux diverging from the whole volume 'V' enclosed by the surface S.

Hence, from (1) & (2), we get

$$\iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_V \text{div } \mathbf{A} \cdot dV \quad \text{--- (3)}$$

This is Gauss theorem of divergence.

It may also be written as

$$\iint_S (\mathbf{A} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \mathbf{A}) dV$$