# UNIT - I

## Definition of Signals & Systems:-

### Signal:-

A function of one (or) more independent variables which contain some information is called Signal.

Ex:- 1) Electric voltage (or) current, such as Radio Signal, TV signal, telephone signal etc.,

2) pressure signal, Sound Signal etc., are Non-electrical Signal

### Systems:-

A System is a set of elements (or) functional blocks that are connected together & produces an output in response to an input Signal.

Ex:- 1) An Audio Amplifier, attenuator, TV set, Transmitter, Receiv etc, are all Systems.

2) Any Machine (or) engine are also Systems.

### Signals & Systems Relationship:-

* Every System has one (or) more i/ps. It is called Excitatio
* Every System has one (or) more o/ps. It is called response
* The i/ps & o/ps of the Systems are always ~~~~ Signa

Fig① shows an Audio Amplifier. It is a System.

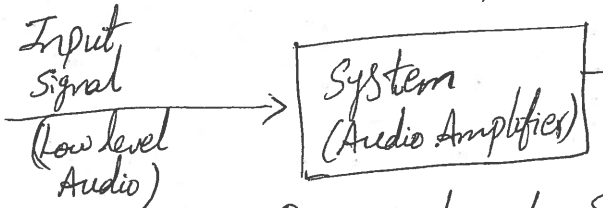



Fig:① Example of Signals & Systems relationship

* Input to the audio amplifier is Low level audio
* Output of the audio amplifier is Amplified Audio.

## Analogy between Vectors & Signals :-

Signals can be represented in terms of orthogonal functions. These orthogonal functions satisfy specific properties.

All the signals are basically vectors. A vectors can be represented in terms of its coordinates.

## orthogonality Concept in vectors :-

Consider the vector f as shown in fig (2). There is another vector x. The dot (scalar) product of vectors f & x is given as

$$f \cdot x = |f||x| \cos\theta$$

Here $\theta$ is an angle between f & x.

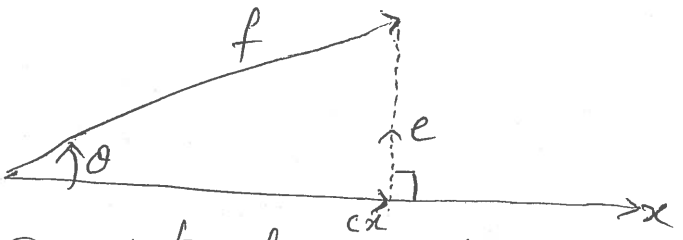



Fig (2) projection of vector along other vectors.

cx is the component of vector f along x.
cx is the projection of f on x.

So f can be expressed as vector addition as

$$f = cx + e$$

'e' is an Error vector. Note that e is minimum only when it is perpindicular to x.

Fig (3) shows 2 other possibilities where e is not perpendicular.

$$f = c_1 x + e_1 = c_2 x + e_2$$

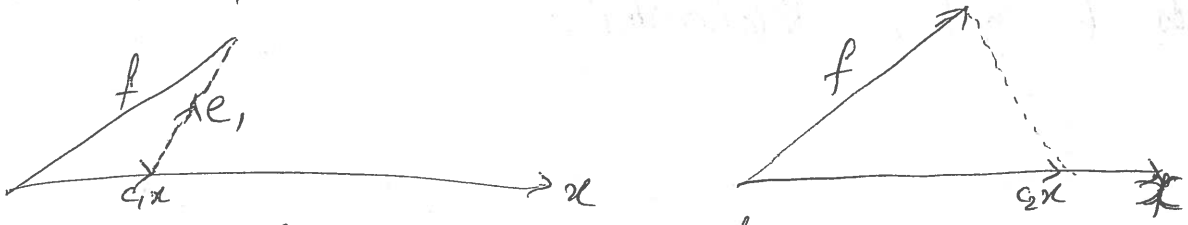



fig (3) $e_1$ & $e_2$ are greater than e.

$c_1$ & $c_2$ are greater than e. Hence e is minimum only when it is perpendicular to x. The component of f along x is cx. It is also given as $|f|\cos\theta$ i.e,

$$c|x| = |f|\cos\theta$$

$$c|x|^2 = |f||x|\cos\theta = f \cdot x$$

$$\therefore C = \frac{1}{|x|^2} f \cdot x$$



Since $x \cdot x = |x|^2$

$$C = \frac{f \cdot x}{x \cdot x}$$

$$C = \frac{f \cdot x}{x \cdot x} .$$

In the above equation $x$ can't be cancelled, since $f \cdot x \neq x \cdot x$ are vector products.

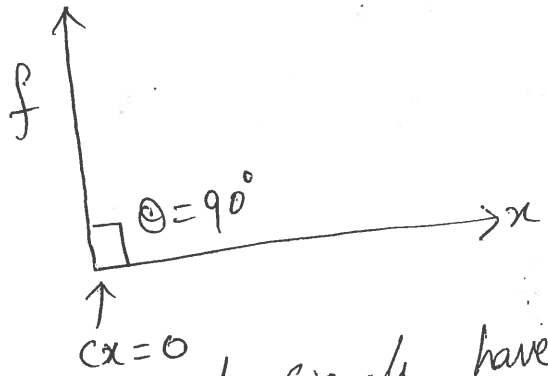



fig:- orthogonal signals have no components along each other

Hence the dot product $f \cdot x$ will be zero.

$$f \cdot x = |f|\,|x| \cos\theta$$

$$f \cdot x = |f|\,|x| \cos 90 = 0 .$$

The vectors $f$ & $x$ are said to be orthogonal if their dot product is zero. In other words, vectors are orthogonal if they are mutually perpendicular.

## Orthogonality in Signals:-

Now let us apply the orthogonality concept of vectors to real signals. Let us consider a signal $f(t)$ to be represented or in terms of $x(t)$ over an interval $t_1$ & $t_2$.

$$f(t) = c\,x(t) + e(t)$$

$$e(t) = f(t) - c\,x(t) \qquad t_1 \le t \le t_2 .$$

$$E_e = \int_{t_1}^{t_2} e^2(t)\,dt . \longrightarrow \text{Energy}$$

$$\overline{e^2(t)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} e^2(t)\,dt = \frac{E_e}{t_2 - t_1} \longrightarrow \text{Mean square value}$$

$$E_c = \int_{t_1}^{t_2} [f(t) - c\,x(t)]^2 \, dt$$

D.w.r.t 'c'

$$\frac{dE_c}{dc} = 0$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} (f(t) - c\,x(t))^2\right] dt = 0$$

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} f^2(t) + c^2 x^2(t) - 2c\,f(t)\,x(t)\right] dt = 0$$

$$2c \int_{t_1}^{t_2} x^2(t)\,dt - 2\int_{t_1}^{t_2} f(t)\,x(t)\,dt = 0$$

$$c = \frac{\int_{t_1}^{t_2} f(t)\,x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}$$

For orthogonality $\int_{t_1}^{t_2} f(t)\,x(t)\,dt = 0$.

If $f(t)$ & $x(t)$ are complex signals, then the orthogonality over the interval $(t_1, t_2)$ is

$$\int_{t_1}^{t_2} f(t)\,x^*(t)\,dt = 0$$

(or)

$$\int_{t_1}^{t_2} f^*(t)\,x(t)\,dt = 0.$$

## Signal Approximation using orthogonal Functions :- (3)

Let us consider the set of Signals which are mutually orthogonal over an interval $(t_1, t_2)$. These Signals can be represents any Signal $f(t)$ as

$$f(t) \cong C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + \cdots\cdots\cdots + C_N x_N(t)$$

$$f(t) = \sum_{n=1}^{N} C_n x_n(t) + e(t)$$

In the above equation any 2 Signals, $x_m(t)$ & $x_n(t)$ are orthogonal over an interval $(t_1, t_2)$ i.e.,

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

In the above equation observe any 2 diff Signals are orthogonal. When $m = n$, it is the same Signal i.e.,

$$\int_{t_1}^{t_2} x_n(t)\, x_n(t)\, dt = \int_{t_1}^{t_2} x_n^2(t)\, dt = E_n$$

This gives energy of the Signal i.e, $E_n$.

$$e(t) = f(t) - \sum_{n=1}^{N} C_n x_n(t)$$

Hence Error Energy will be

$$E_e = \int_{t_1}^{t_2} e^2(t)\, dt = \int_{t_1}^{t_2} \left[ f(t) - \sum_{n=1}^{N} C_n x_n(t) \right]^2 dt$$

Hence $E_e$ is the function of $C_1, C_2, C_3 \cdots\cdots C_N$.

Hence $E_e$ will be minimized w.r.t $C_i$ if

$$\frac{dE_e}{dC_i} = 0$$

$$\frac{d}{dC_i} \left[ \int_{t_1}^{t_2} \left( f(t) - \sum_{n=1}^{N} C_n x_n(t) \right)^2 dt \right] = 0$$

Above Equation will be Executed for $i=1,2,3\cdots N$

$$\frac{d}{dc_i}\left\{\int_{t_1}^{t_2} f^2(t)\,dt - \int_{t_1}^{t_2}\sum_{n=1}^{N} 2c_n f(t)x_n(t)\,dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} c_n^2 x_n^2(t)\,dt\right\} = 0$$

The 2nd & 3rd integration terms will be non zero only when $n=i$. for $n\neq i$, these terms will be constants & their derivatives are Zero. Hence (with $n=i$),

$$\frac{d}{dc_i}\left\{-\int_{t_1}^{t_2} 2c_i f(t)x_i(t)\,dt + \int_{t_1}^{t_2} c_i^2 x_i^2(t)\,dt\right\} = 0$$

$$\therefore\quad -2\int_{t_1}^{t_2} f(t)x_i(t)\,dt + 2c_i\int_{t_1}^{t_2} x_i^2(t)\,dt = 0 \qquad i=1,2,3\cdots N$$

$$c_i = \frac{\int_{t_1}^{t_2} f(t)x_i(t)\,dt}{\int_{t_1}^{t_2} x_i^2(t)\,dt}, \quad i=1,2,3,\cdots N$$

we know that $\int_{t_1}^{t_2} x_i^2(t)\,dt = E_i$, i.e, Energy. Hence above equation becomes

$$c_i = \frac{1}{E_i}\int_{t_1}^{t_2} f(t)x_i(t)\,dt \qquad i=1,2,3,\cdots N.$$

## Mean Square Error:-

The Error Energy is given by

$$E_e = \int_{t_1}^{t_2}\left[f(t) - \sum_{n=1}^{N} c_n x_n(t)\right]^2 dt$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\int_{t_1}^{t_2}\sum_{n=1}^{N} c_n f(t)x_n(t)\,dt + \int_{t_1}^{t_2}\sum_{n=1}^{N} c_n^2 x_n^2(t)\,dt$$

Interchanging the order of Summation & Integration.

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} C_n \int_{t_1}^{t_2} f(t)\,x_n(t)\,dt + \sum_{n=1}^{N} C_n^2 \int_{t_1}^{t_2} x_n^2(t)\,dt$$

$$\int_{t_1}^{t_2} f(t)\,x_n(t)\,dt = C_n E_n$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} C_n C_n E_n + \sum_{n=1}^{N} C_n^2 E_n$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - 2\sum_{n=1}^{N} C_n^2 E_n + \sum_{n=1}^{N} C_n^2 E_n$$

$$E_e = \int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n .$$

The mean Square Error & Error Energy are related as

$$\overline{e^2(t)} = \frac{E_e}{t_2 - t_1} = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{N} C_n^2 E_n\right]$$

In the above equation $C_n^2 E_n$ is always positive. Hence Error Energy $E_e$ can be reduced if No. of terms 'N' used for representation are increased. Ideally $E_e \to 0$ as $N \to \infty$. under this condition, the orthogonal signal set is to be Complete.

Closed (or) Complete set of orthogonal functions:-

The mean Square Error approaches Zero as No. of terms $C_n^2 E_n$ are made infinite. under this condition we can write

$$0 = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_{n=1}^{\infty} C_n^2 E_n\right] \quad \text{with } \overline{e^2(t)} = 0 \text{ as } N = \infty$$

$$\int_{t_1}^{t_2} f^2(t)\,dt = \sum_{n=1}^{\infty} C_n^2 E_n .$$

With N approaching infinity we can write

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t)$$

$x_1(t), x_2(t) \ldots\ldots\ldots x_n(t)$ is a set of mutually orthogonal functions. It is said to be complete (or) closed set if there exists no function $p(t)$ for which

$$\int_{t_1}^{t_2} p(t)\, x_n(t)\, dt = 0 \quad \text{for } n = 1, 2 \ldots\ldots$$

If $p(t)$ exists & above integral is zero, then obviously $p(t)$ must be a member of set $\{x_n(t)\}$.

For the set of mutually orthogonal signals $x_n(t)$ over an interval $(t_1, t_2)$,

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ E_n & \text{if } m = n \end{cases}$$

For this complete set the function $f(t)$ is expressed as

$$f(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + \ldots\ldots$$

$$c_i = \frac{\int_{t_1}^{t_2} f(t)\, x_i(t)\, dt}{\int_{t_1}^{t_2} x_i^2(t)\, dt} =$$

$$c_i = \frac{1}{E_i} \int_{t_1}^{t_2} f(t)\, x_i(t)\, dt.$$

## Orthogonality of Complex Functions :-

Consider that the set of signals $x_1(t), x_2(t), x_3(t) \ldots\ldots$ are complex. Then they are mutually orthogonal if.

$$\int_{t_1}^{t_2} x_m(t)\, x_n^*(t)\, dt = \int_{t_1}^{t_2} x_m^*(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

Then $f(t)$ can be expressed as

$$\therefore f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

where $C_n$ is given in the similar fashion of eq as,

$$C_n = \frac{1}{E_n} \int_{t_1}^{t_2} f(t)\, x_n^*(t)\, dt \qquad n = 1, 2, \ldots\ldots$$

where $E_n$ is given for complex signals as,

$$E_n = \int_{t_1}^{t_2} x_n(t)\, x_n^*(t)\, dt$$

## Orthonormal Signals :-

We know that orthogonality of the real set of signals $x_1(t)$, $x_2(t)$, $x_3(t)$ ---------- $x_N(t)$ over an interval $(t_1, t_2)$ is given as

$$\int_{t_1}^{t_2} x_m(t)\, x_n(t)\, dt = \begin{cases} 0 & \text{for } m \neq n \\ E_n & \text{for } m = n \end{cases}$$

Here ~~as~~ $E_n$ is an Energy of the signal $x(t)$. When Energy $E_n = 1$ for all values of n, then the set is called orthonormal set. Here ~~the~~ note that orthonormal set is always orthogonal, but reverse is not true. An orthogonal set can be converted to orthonormal by dividing $x_n(t)$ by $\sqrt{E_n}$ for all values of n. This process is called Normalization of orthogonal set.

## Trigonometric Fourier Series :-

We know that any function $f(t)$ can be expressed as

$$f(t) = \sum_{n=1}^{\infty} C_n x_n(t)$$

Here $x_n(t)$ represents orthogonal signal set. This equation is called Generalized Fourier Series.

We have seen that the set,

$\{1, \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \ldots\ldots \cos n\omega_0 t, \ldots\ldots \sin\omega_0 t, \sin 2\omega_0 t, \ldots\ldots \sin n\omega_0 t \ldots\ldots\}$

is orthogonal over the period $T_0$. Here $\omega_0$ is called fundamental frequency And $n\omega_0$ is called nth Harmonic. There is DC component of $\cos n\omega_0 t$ at $n = 0$.

i.e., $\cos(0\omega_0 t) = 1$. This signal set consists of Sine & Cosine terms. Hence it is called Trignometric Set.

For this set we can write eq as

$$f(t) = a_0 + a_1\cos\omega_0 t + a_2\cos\omega_0 t + \cdots\cdots + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \cdots\cdots$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin\omega_0 t$$

Values of $a_n$ & $b_n$ can be obtained from eq① i.e,

$$a_n = \frac{\int_t^{t+T_0} f(t)\, x_n(t)\, dt}{\int_t^{t+T_0} x_n^2(t)\, dt}$$

$$a_n = \frac{\int_{t_0}^{t+T_0} f(t)\cos n\omega_0 t\, dt}{\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt}$$

Now Consider denominator of above equation with $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ ie,

$$\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt = \frac{1}{2}\int_t^{t+T_0} (1+\cos 2n\omega_0 t)\, dt$$

$$= \frac{1}{2}\int_t^{t+T_0} dt + \frac{1}{2}\int_t^{t+T_0} \cos 2n\omega_0 t\, dt$$

The 2nd term is integration of full cycles of Cosine waves over one period. It is zero. Hence

$$\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt = \frac{1}{2}(t)_t^{t+T_0} = \frac{T_0}{2}$$

$$a_n = \frac{2}{T_0}\int_t^{t+T_0} f(t)\cos n\omega_0 t\, dt.$$

Eq. $b_n$ can be calculated as

$$b_n = \frac{\int_t^{t+T_0} f(t)\sin n\omega_0 t\,dt}{\int_t^{t+T_0} \sin^2 n\omega_0 t\,dt}$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta).$$

$$\int_t^{t+T_0} \sin^2 n\omega_0 t\,dt = \frac{1}{2}\int_t^{t+T_0}(1-\cos 2n\omega_0 t)\,dt = \frac{1}{2}\int_t^{t+T_0} dt - \frac{1}{2}\int_t^{t+T_0}\cos 2n\omega_0 t\,dt$$

$$\int_t^{t+T_0} \sin^2 n\omega_0 t\,dt = \frac{1}{2}\int_t^{t+T_0} dt = \frac{1}{2}(t)_t^{t+T_0} \equiv \frac{T_0}{2}.$$

$$b_n = \frac{2}{T_0}\int_t^{t+T_0} f(t)\sin n\omega_0 t\,dt.$$

Value of $a_0$ is obtained by putting $n=0$ in eq① i.e.,

$$a_0 = \frac{2}{T_0}\int_t^{t+T_0} f(t)\,dt.$$

We know that $\omega_0 = \frac{2\pi}{T_0}$.

Thus any Periodic Signal $x(t)$ can be expanded in Fourier Series as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t.$$

$$a_0 = \frac{1}{T_0}\int_t^{t+T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\sin n\omega_0 t\,dt.$$

## Classification of Signals :-

The Signals can be classified in to 2 parts depending upon independent variable (time).

A) Continuous time Signals (CT)

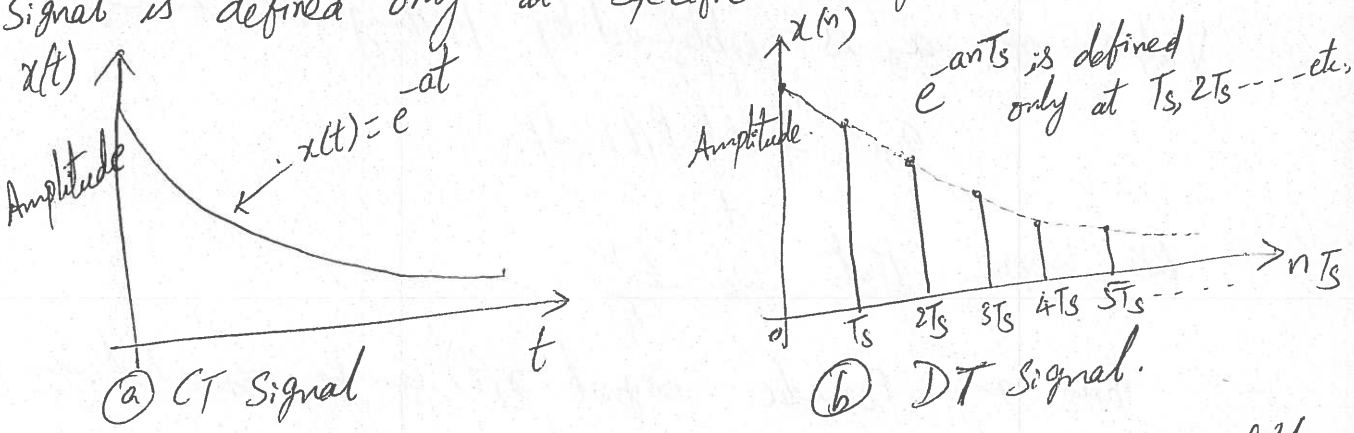
B) Discrete Time Signals (DT).

Both the CT & DT Signals can be classified into following parts:

(i) Periodic & Non-periodic Signals
(ii) Even & odd Signals.
(iii) Energy & Power Signals.
(iv) Deterministic & Random Signals.

## CT & DT Signals :-

A CT Signal is defined continuously w.r.t. time. A DT Signal is defined only at specific (or) regular time instants.




(a) CT Signal (b) DT Signal.

1) Analog Circuits processes CT Signals. Such circuits are op-amps, filters, amplifiers etc.,
2) Digital Circuits process DT Signals. Such circuits are Micro processors, FlipFlops, Counters etc.,

## Analog & Digital Signals :-

(i) When Amplitude of CT Signals varies continuously, it is called Analog Signal. otherwise Amplitude & Time both are continuous for Analog Signal.

(ii) When the amplitude of DT Signal takes only finite values, it is called Digital Signal. Otherwords Amplitude & time both are discrete for Digital Signal.

## Signals:-

Any physical phenomenon that conveys (or) carries some information can be called as a signal. Usually the information carried by signal will be a function of independent variable. The independent variable can be time, temperature, position, pressure, distance, intensity of colours, etc.,

Some signals in our daily life are music, speech, picture, videos etc.,

The value of signal of any specified value of independent variable is called its Amplitude.

A physical quantity that varies with one (or) more independent variable is known as a signal.

A signal is defined as a physical quantity that varies with time, space (or) any other independent variable (or) variables.

## Continuous Time & Discrete Time Signals :-

Continuous Time signals are defined for every value of time t and it is represented by $x(t)$. A continuous Time signal is also called an Analog signals. Most of the signals encountered in practice are continuous signals.

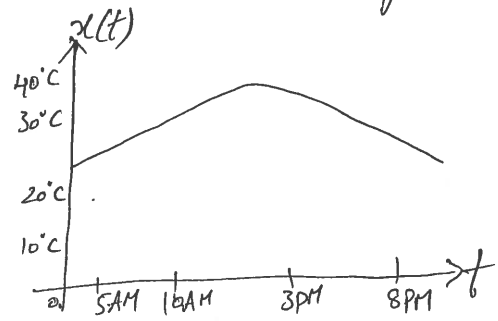



fig 1.1a Temp measurement over an interval of time

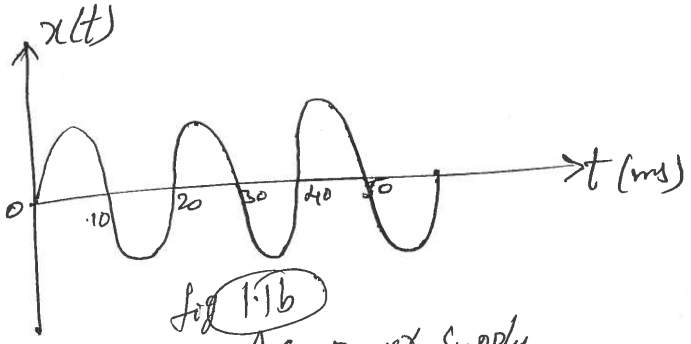



fig 1.1b AC power supply

Discrete Time Signals are defined at discrete instants of time & is represented by $x(n)$, where n is index. Some signals are discrete in nature and some signals may be discrete representation of C.T signals. For example the amount deposited every month in a savings account is discrete. The other type of signal is obtained by sampling a continuous

That is the amplitude of the Continuous time signal is defined at Particular intervals.

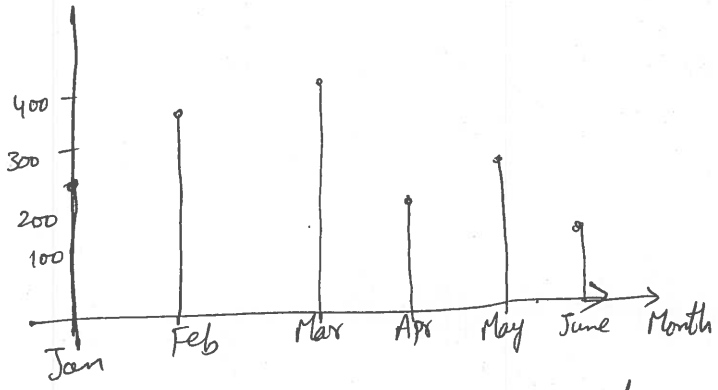



fig (1.2) Discrete Time Signal.

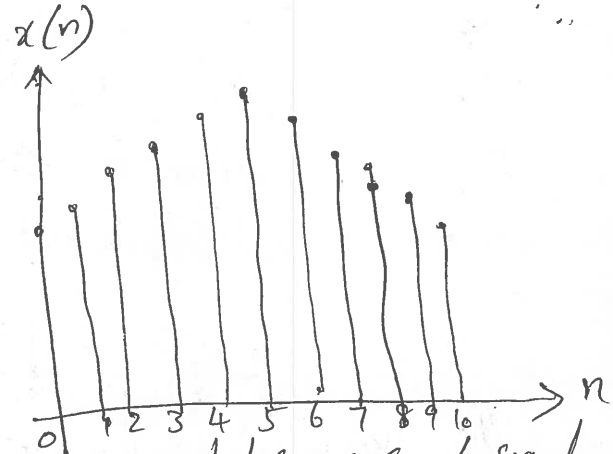
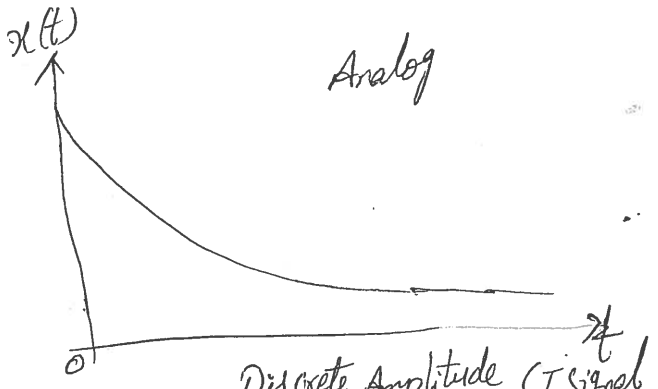


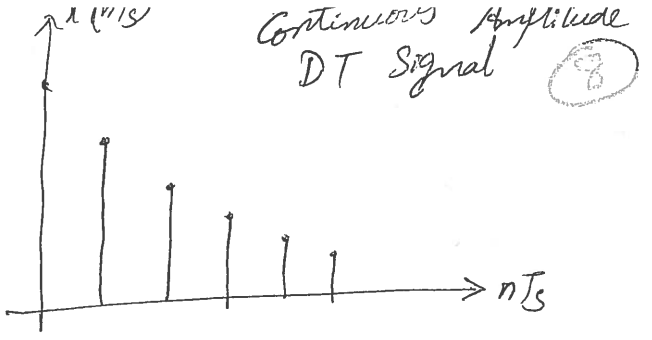

fig (1.3) Sampled sequence of Signal shown in fig 1.1(a)

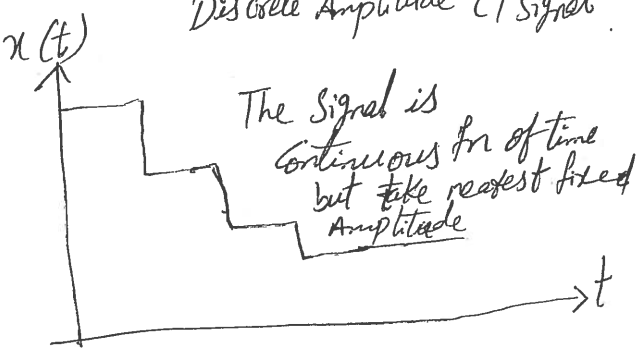
The discrete time signal can be denoted by

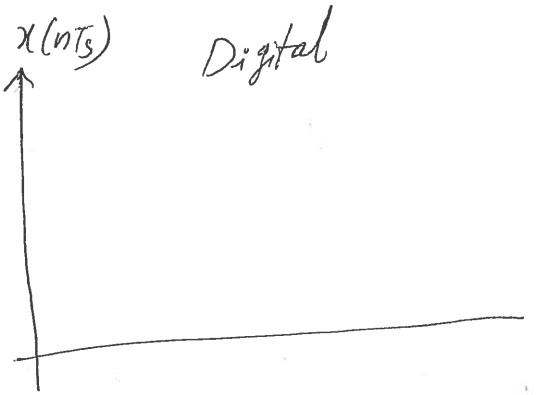
$$x(nT) = x(t)\Big|_{t=nT}$$

T is called Sampling period & n is an integer from $-\infty$ to $\infty$. n → called as time index. We call $x(nT)$ the Sampled sequence of $x(t)$. The instants at which the signal appear are called Sampling instants.



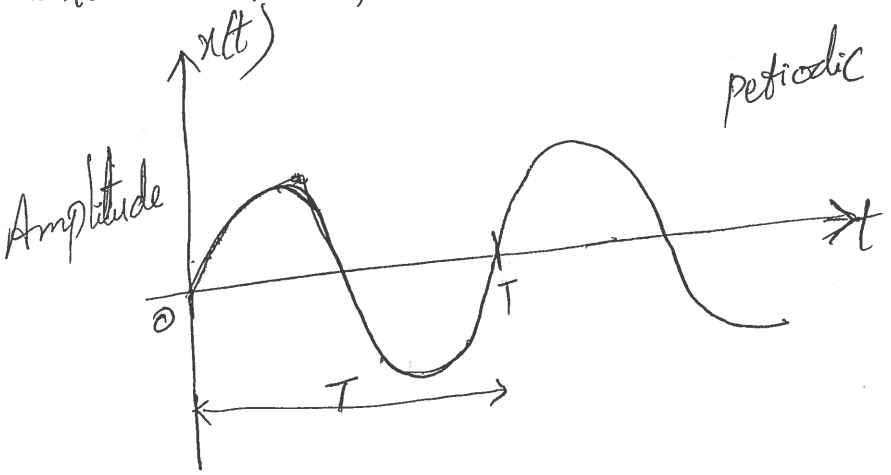




## Periodic & Non Periodic Signals :-

A signal is said to be periodic if it repeats at regular intervals. Non-periodic signals do not repeat at regular intervals.

$x(t) = x(t+T)$ for all t

The smallest value of T that satisfies above condition is known as fundamental period.




Consider sinusoidal signal

$x(t) = A\sin(\omega_0 t + \theta)$

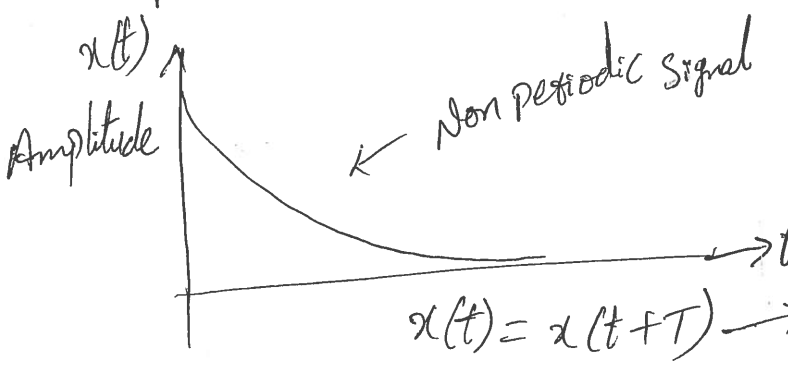
A, $\omega_0$, $\theta$ are amplitude, angular frequency & phase of signal.

$\omega_0 = 2\pi f_0$ $x(t) = x(t+T)$ periodic signal

$x(t) = A\sin(\omega_0 t + \theta)$

$x(t+T) = A\sin[\omega_0(t+T)+\theta)]$

i.e. $\omega_0 T = 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega_0}}$




$x(t) = x(t+T)$ → Condition for periodicity of CT Signal.

Periodicity of Signal $x_1(t) + x_2(t)$ is

$x_1(t) = x_1(t+mT_1)$

$x_2(t) = x_2(t+nT_2)$

Sum of two periodic signals is periodic only if period ratio is rational number.

$x(t)$ will be periodic if

$mT_1 = nT_2 = T_0$.

$\frac{T_1}{T_2} = \frac{n}{m}$ ratio of 2 integers.

## Even & odd Signals :-

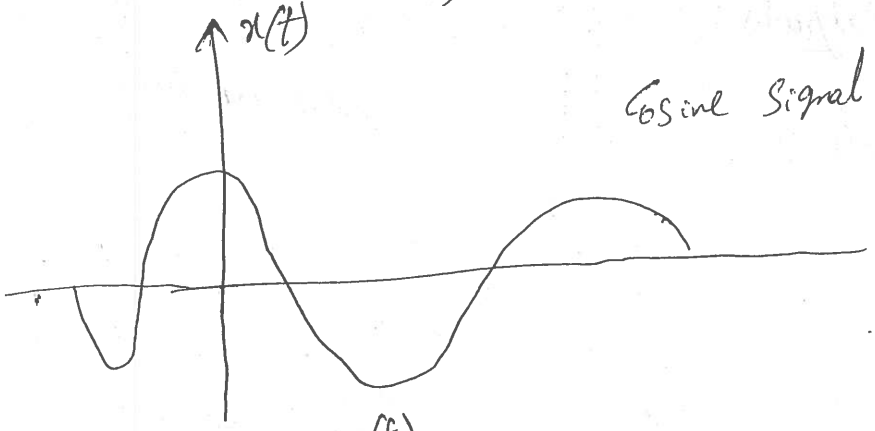
A Signal is said to be even signal if inversion of time axis doesn't change the amplitude

Condition for Signal to be even $x(t) = x(-t)$

Even Signals are also called Symmetric Signals.

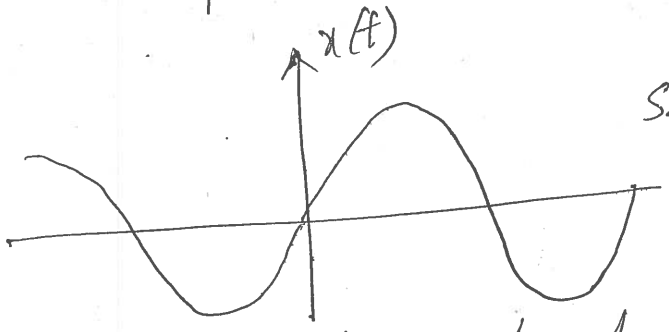
A Signal is said to be odd signal if inversion of time axis also inverts amplitude of the Signal.

Condition for Signal to be odd. $x(t) = -x(-t)$ (or) $x(-t) = -x(t)$




Cosine Signal is an example of even Signal

Since $\cos\theta = \cos(-\theta)$




Sine wave is an example of odd Signal

Since $\sin\theta = -\sin(-\theta)$.

Even (or) odd Symmetry of the Signal have specific harmonic (or) frequency content.

Even (or) odd Symmetry property is used in filter design.

Let the Signal be represented into its even & odd parts as

$$x(t) = x_e(t) + x_o(t)$$

Here $x_e(t)$ is even part of $x(t)$ &

$x_o(t)$ is odd part of $x(t)$

Substitute $-t$ for $t$ in above equation

$$\therefore \quad x(-t) = x_e(-t) + x_o(-t)$$

$$x_e(-t) = x_e(t)$$

$$x_o(-t) = -x_o(t)$$

$$x(-t) = x_e(t) - x_o(t)$$

Add

$$x(t) + x(-t) = 2x_e(t)$$

$$\boxed{x_e(t) = \frac{x(t) + x(-t)}{2}}$$

Subtract

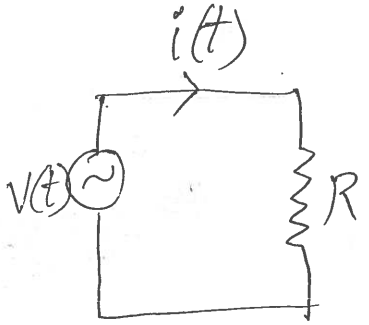
$$x(t) - x(-t) = 2x_o(t)$$

$$\boxed{x_o(t) = \frac{x(t) - x(-t)}{2}}$$

## Energy & power Signal :-

Instantaneous power dissipation

For the circuit shown in fig, the instantaneous power is dissipated in load Resistance 'R' will be given as




$$P(t) = \frac{V^2(t)}{R} = i^2(t)R \longrightarrow (1)$$

Normalized power: It is the power dissipated in $R = 1\Omega$ load.

Hence from (1) we can write

$$P = V^2(t) = i^2(t).$$

Let $V(t)$ (or) $i(t)$ denoted by $x(t)$.

$$P = x^2(t)$$

Thus for current as well as voltage the equation for normalized power is same.

$$\text{Energy } E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ for CT Signals.}$$

$$\text{Power } P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \text{ for CT Signals.}$$

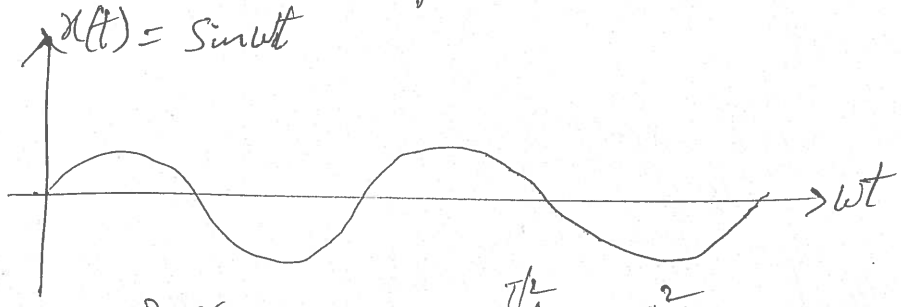
If the signal is periodic, then the period (T) have finite value. Hence there is no need to take limits.

A signal is said to be Energy signal if its total energy is finite and non-zero i.e.,

For Energy signal $0<E<\infty$.

A signal is said to be power signal if its normalized power is non zero & finite i.e.,

For power signal, $0<P<\infty$.




Power $P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\sin\omega t|^2 dt$

Power Signal

~~$P = \lim_{T\to\infty} \left(\frac{1}{\omega}\right) \int_{-T/2}^{T/2} \cos^2\omega t = \lim_{T\to\infty} \frac{1}{T(\omega)} \int_{-T/2}^{T/2}$~~

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \sin^2\omega t\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{1-\cos 2\omega t}{2} dt$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left[ \int_{-T/2}^{T/2} \frac{1}{2} dt - \int_{-T/2}^{T/2} \frac{\cos 2\omega t}{2} dt \right]$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left[ \frac{1}{2}(t)_{-T/2}^{T/2} - \frac{1}{2}\left(\frac{\sin 2\omega t}{2\omega}\right)_{-T/2}^{T/2} \right]$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left[ \frac{1}{2}\left(\frac{T}{2} - \left(-\frac{T}{2}\right)\right) - \frac{1}{4\omega}(\sin\omega T + \sin\omega T) \right].$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left( \frac{T}{2} - \frac{1}{4\omega} 2\sin\omega T \right) = \lim_{T\to\infty} \left( \frac{1}{2} - \frac{\sin\omega T}{2\omega T} \right) = \frac{1}{2}.$$

Most of the periodic signals are power signals.
Most of the Non-periodic signals are Energy signals.

* Energy of the power signal is infinite
* Power of the Energy signal is zero.

## Deterministic & Random Signals :-

A Deterministic Signal can be completely represented by mathematical equation at any time.

Ex:- $x(t) = \cos \omega t$.

A signal which can't be represented by any mathematical equation is called Random signal.

Ex:- Noise generated in electronic components, Transmission channels, cables etc.,

## Elementary Signals :-

Standard signals used for the analysis of systems are

1. unit step signal (or) function
2. Unit impulse function
3. Unit ramp function.
4. Complex Exponential function
5. Sinusoidal function
6. Signum function
7. Sinc function
8. Gaussian function.

## Unit Step Function :-

The unit step signal has an amplitude of '1' for +ve values of independent variable. And it has amplitude of 0 for -ve values of independent variable.

$u(t) = 1$ for $t \geq 0$
$u(t) = 0$ for $t < 0$

$u(n) = 1$ for $n \geq 0$
$= 0$ for $n < 0$

$u(n) = \{0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$

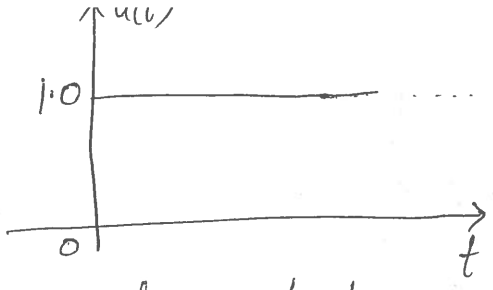
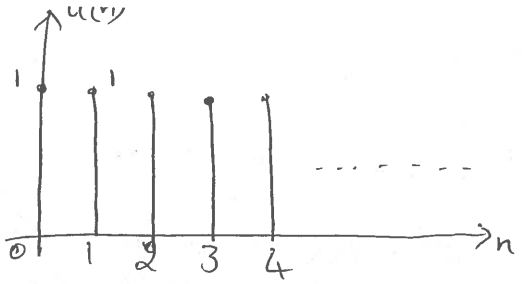



fig:- unit step
(Continuous Time)




(Discrete Time Version of unit Step)

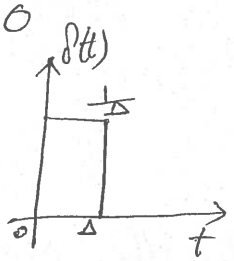
* DT unit step signal is sampled version of CT unit step.
* unit step signal is generated when a DC supply is applied to the circuit. unit step is generated when a switch is closed at t=0

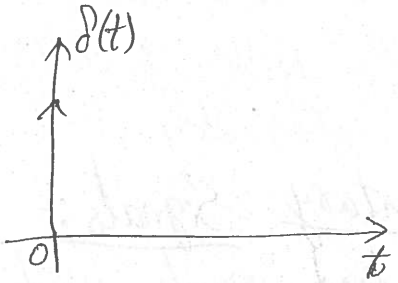
Unit Impulse (or) Dirac Delta Function:-

Area under unit impulse function approaches 1 as its width approaches zero. Thus it has zero value every where except at t=0.

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \quad \text{if } t \to 0$$

$$\delta(t) = 0 \quad \text{for } t \neq 0$$
$$= 1 \quad \text{for } t = 0$$



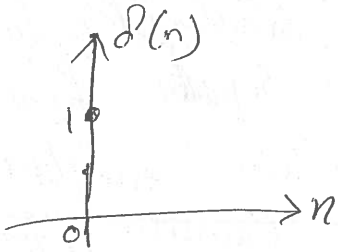

$$\delta(t) = \frac{1}{\Delta - 0} = \frac{1}{\Delta}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{0}^{\Delta} \frac{1}{\Delta}\,dt = \frac{1}{\Delta}(t)_0^{\Delta} = \frac{1}{\Delta}(\Delta - 0) = \frac{\Delta}{\Delta} = 1$$

Amplitude of unit sample is 1 at n=0 & it has zero value at all other values of n.

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$

(or) $\delta(n) = \{0, 0, \underset{\uparrow}{1}, 0, 0, 0\}$




δ(n) is not the sampled version of δ(t). The main difference is area under δ(t)=1, Amplitude of δ(n)=1.

unit Impulse (or) unit sample functions are used to determine the impulse response of the system.

unit sample or unit impulse functions contains all the frequencies from $-\infty$ to $\infty$.

Properties :- (1) $\int_{-\infty}^{\infty} x(t)\,\delta(t) = x(0)$

2) Shifting property :- $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

3) Replication property :- $\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

(4) $\delta(at) = \frac{1}{|a|}\delta(t)$

$x(t) * \delta(t) = x(t)$

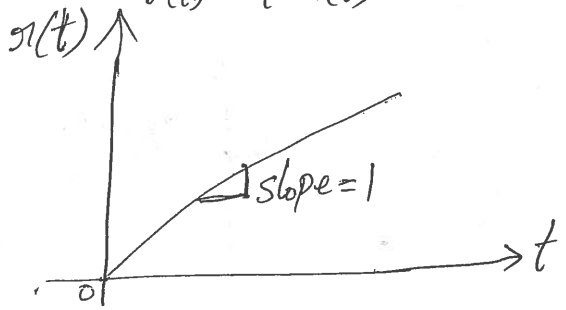
Convolution of any function with delta function leaves that function unchanged.

## Unit Ramp Function :-

It is a linearly growing function for positive values of independent variable.

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

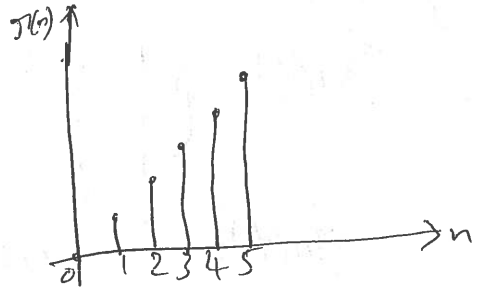
$r(t) = t\,u(t)$




$r(n) = n$ for $n \geq 0$

$= 0$ for $n < 0$

$r(n) = n\,u(n)$.



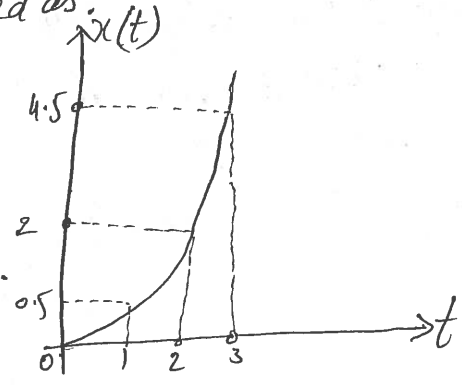

The ramp function indicates linear relationship.
It indicates constant current charging of the capacitor.

## Parabolic Signal :-

The unit parabolic signal is defined as:

$$P(t) = \frac{t^2}{2} \quad \text{for } t \geq 0$$

$= 0$ for $t < 0$




Parabolic fn can be obtained by integrating ramp fn.

$$P(t) = \int r(t) = \int t\,dt = \frac{t^2}{2} \quad (t \geq 0)$$

$r(t) = \frac{dP(t)}{dt}$ Ramp function is derivative of parabolic fn.

## Rectangular pulse function:-

$\Pi(t) = 1$ for $|t| \leq \frac{1}{2}$

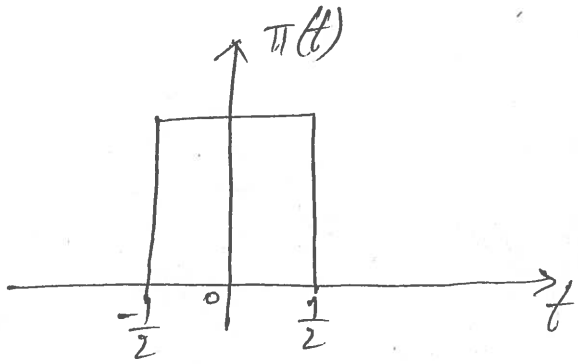
$= 0$ otherwise

(or)

$\Pi(t) = A$, for $|t| \leq \frac{T}{2}$

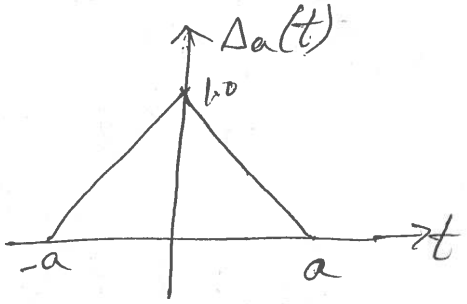
$= 0$ otherwise

$\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$.




## Triangular pulse Function:-

$$\Delta a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| \leq a \\ 0 & |t| > a. \end{cases}$$




## Signum Function:-

$$Sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$Sgn(t) = -1 + 2u(t)$

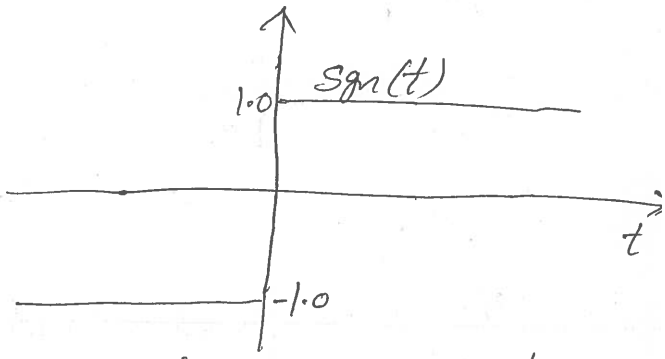


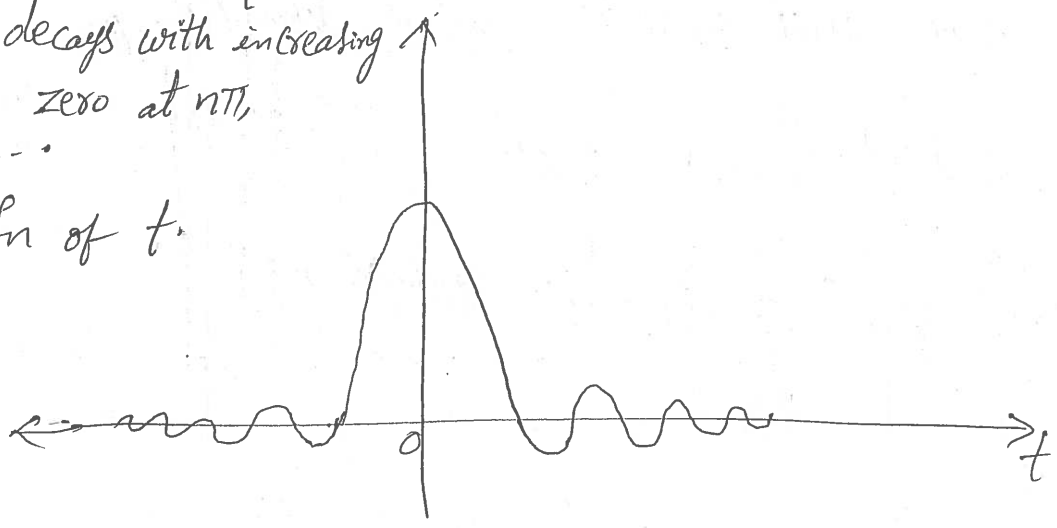

Fig:- Signum Function.

## Sinc function:-

$$Sinc(t) = \frac{\sin t}{t}$$

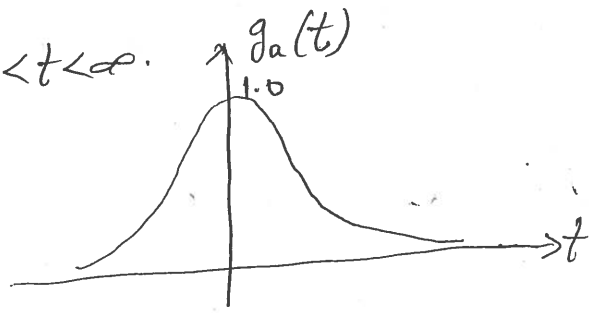
The Sinc function oscillates with period $2\pi$ & decays with increasing t. Its value is zero at $n\pi$, $n = \pm 1, \pm 2 \ldots\ldots$

It is even fn of t.



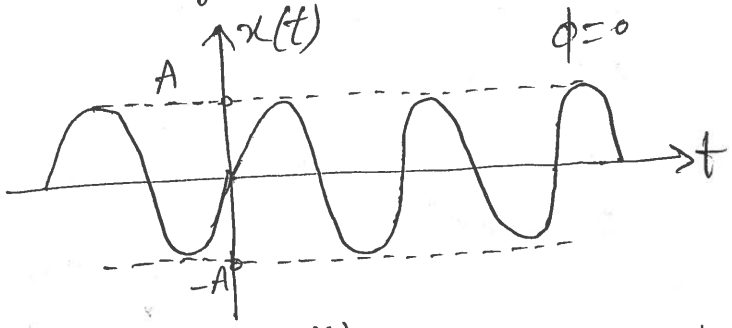
## Gaussian Signal :-

$$g_a(t) = e^{-at^2} \quad -\infty < t < \infty.$$




## Sinusoidal Signal :-

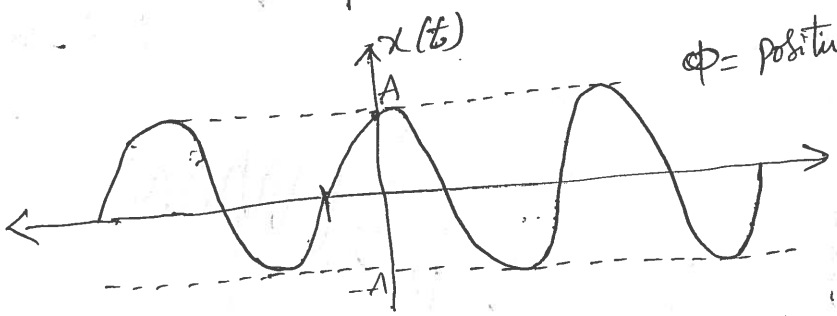
Sinusoidal Signal is defined as $x(t) = A\sin(\omega_0 t + \phi)$

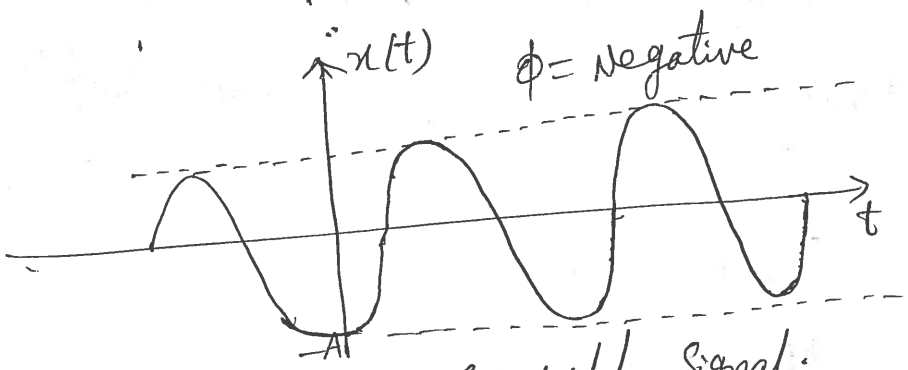



$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ rad/sec

$f_0$ = Frequency in cycles/sec

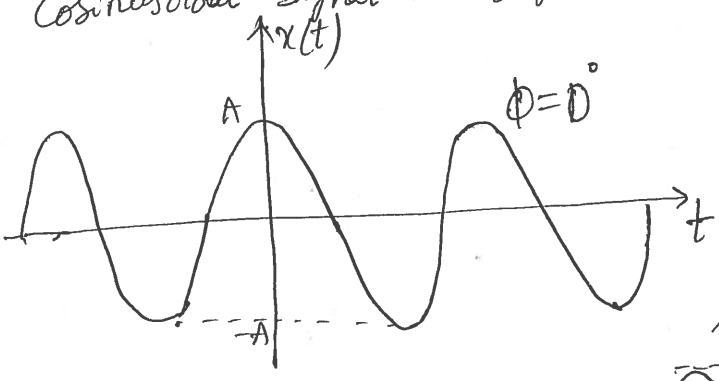
T = Time period in sec

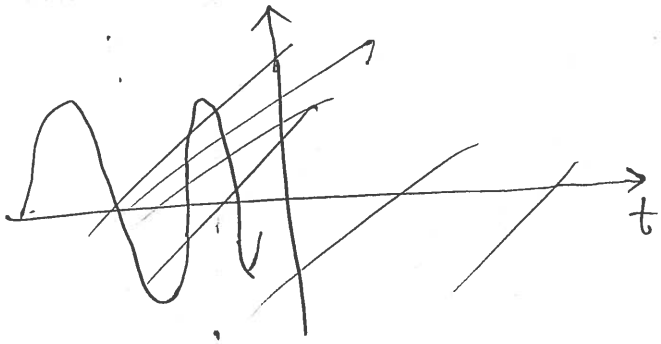


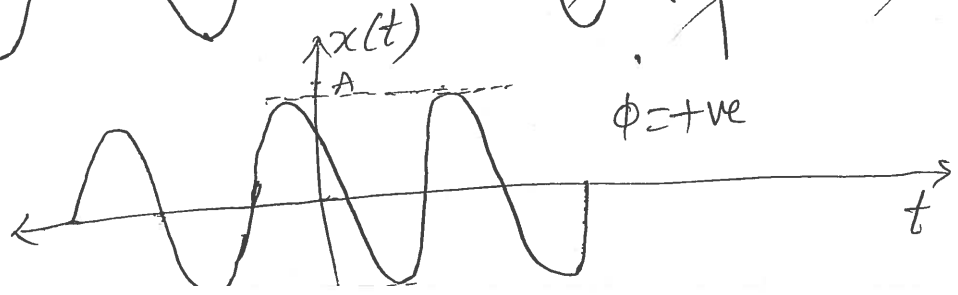



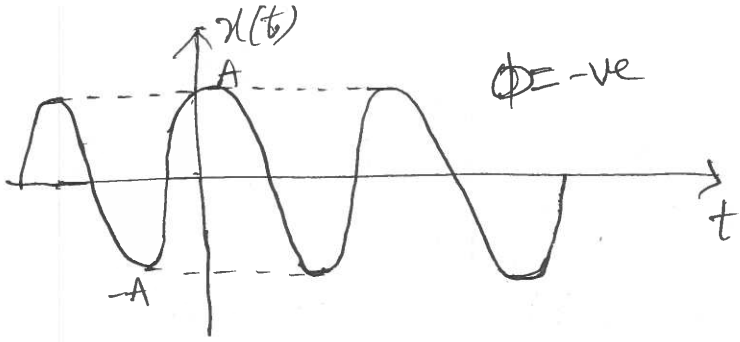

Sinusoidal Signal.

Cosinusoidal Signal is defined as $x(t) = A\cos(\omega_0 t + \phi)$





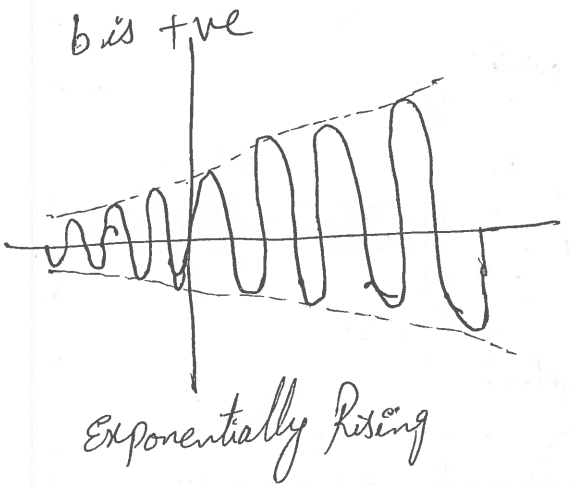

## Complex Exponential Signals :-

$$x(t) = A e^{j\omega_0 t} = A(\cos\omega_0 t + j\sin\omega_0 t) = A\cos\omega_0 t + jA\sin\omega_0 t$$

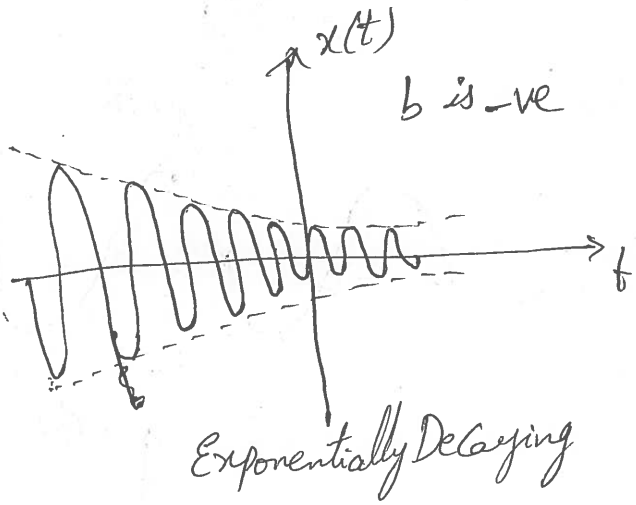
## Exponentially Rising (or) Decaying Sinusoidal Signals :-

$$x(t) = A \cdot e^{bt} \sin\omega_0 t$$

A & b are real Constants.



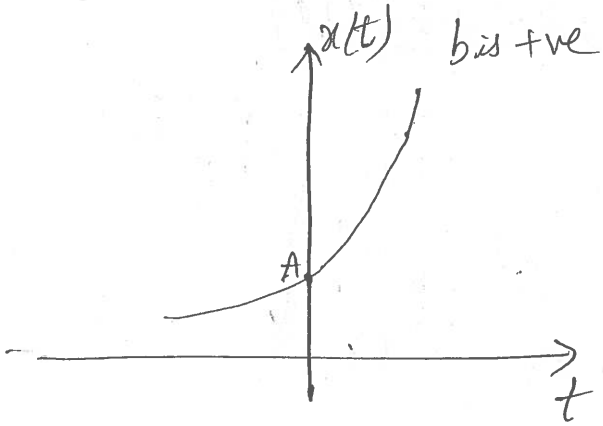

Exponentially Rising

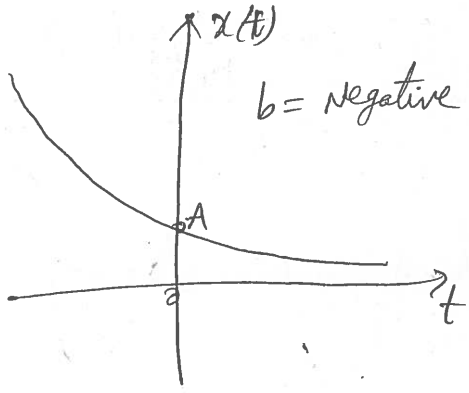



Exponentially Decaying

## Real Exponential Signals :-

$$x(t) = A e^{bt}$$



# Basic operations on Signals

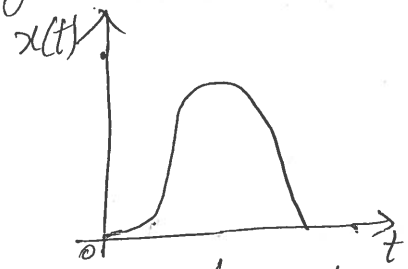
1. Time Shifting
2. Time Reversal
3. Time Scaling
4. Amplitude Scaling
5. Signal Multiplier
6. Signal Addition.

## Time Shifting :

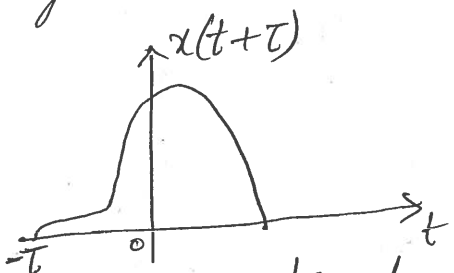
The time shifting of a Signal $x(t)$ may delay (or) advance the Signal in time. Mathematically

$$y(t) = x(t-\tau)$$

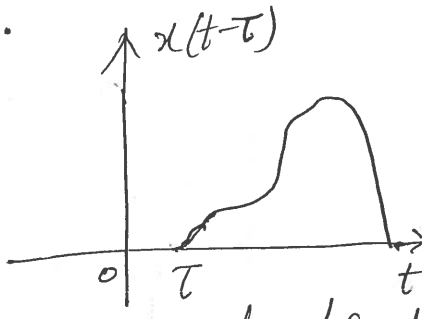
If $\tau$ is +ve then the shifting delays the Signal & if $\tau$ is negative then the shifting advances the Signal.




(a) Original Signal



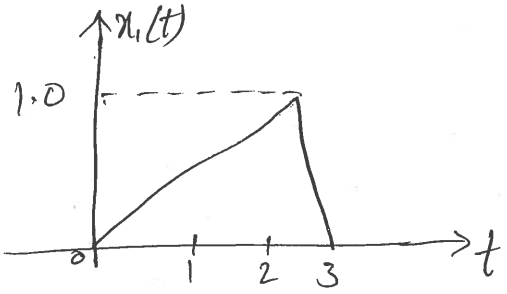

(b) Advanced Signal

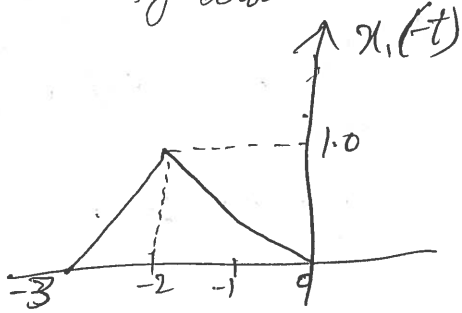


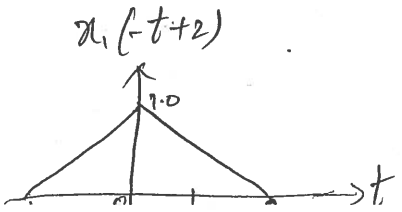

(c) Delayed Signal

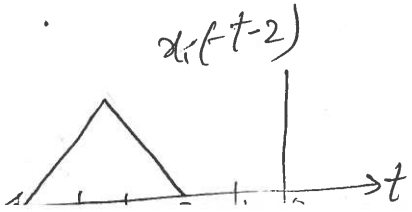
## Time Reversal

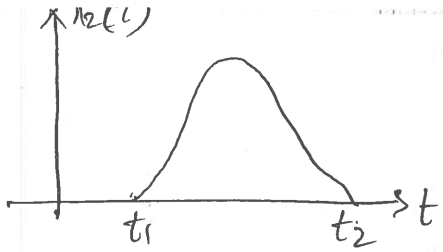
The time reversal of a Signal $x_1(t)$ can be obtained by folding the Signal about $t=0$. It is denoted by $x_1(-t)$. The Signal $x_1(t)$ & its reflection $x_1(-t)$ are shown in fig below.

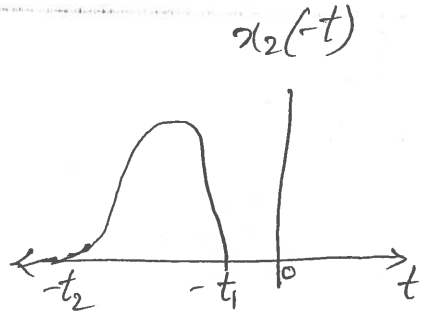


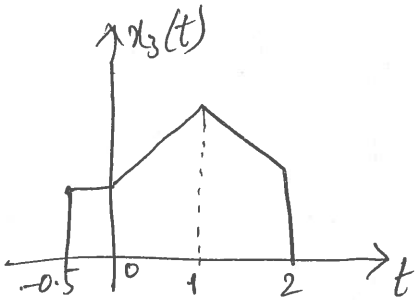



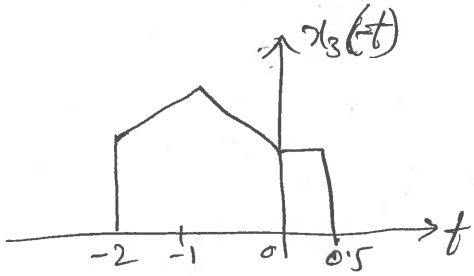





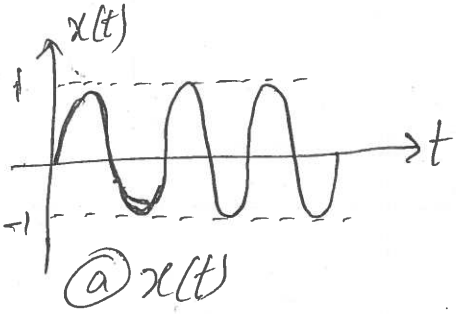

(5) Amplitude Scaling:-

Consider a signal $x(t)$ that is fed to an Amplifier with a gain of 3. The o/p signal of the amplifier can be represented by

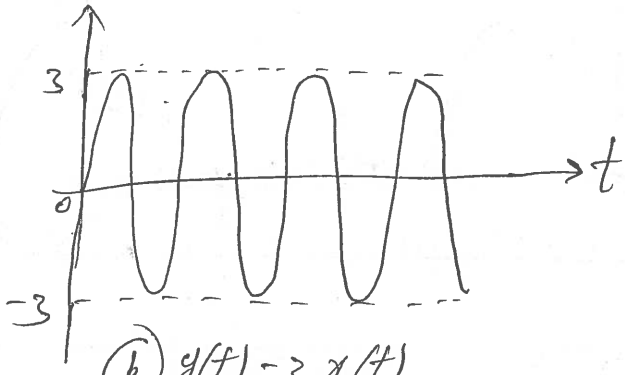
$$y(t) = 3x(t)$$

$$y(t) = A\,x(t).$$

A is the scaling factor




(a) $x(t)$




(b) $y(t) = 3x(t)$

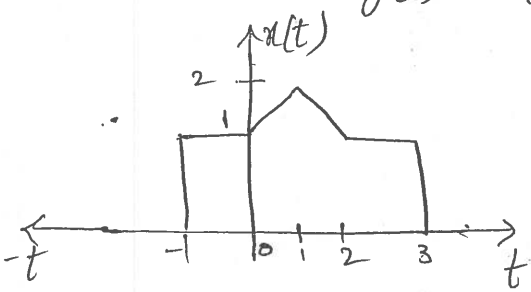
Time Scaling:-

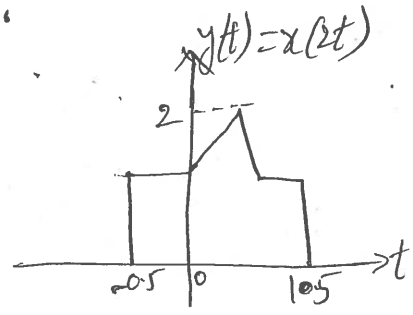
Let us Consider a signal $x(t)$, the time scaling of a signal can be accomplished by replacing $t$ by "$at$" in the signal $x(t)$.

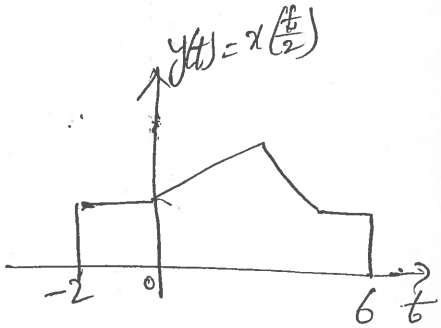
$$y(t) = x(at)$$

If $a = 2$

$$y(t) = x(2t).$$

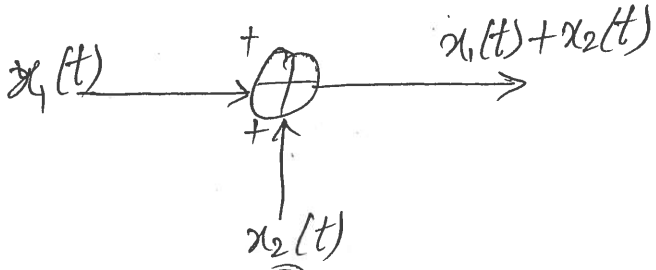


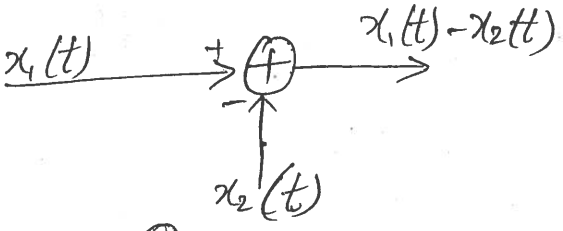



## Signal Addition :-

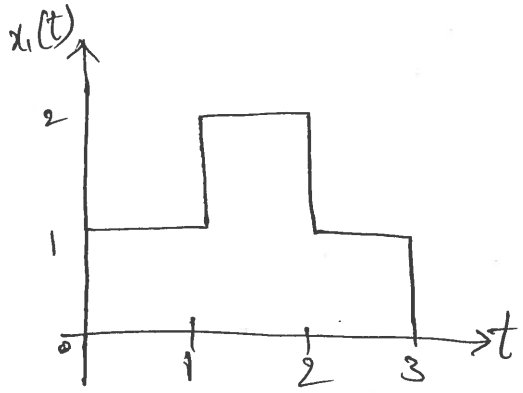
Sum of two continuous time signals can be obtained by adding their values at every instant. Ily Subtraction of 2 signals.

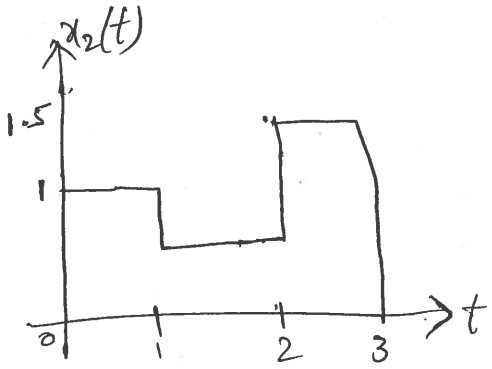


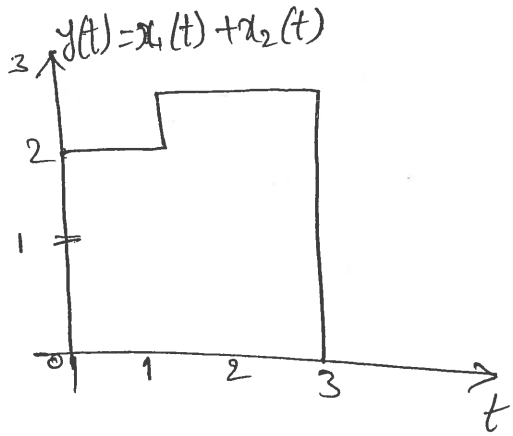

(a) Addition

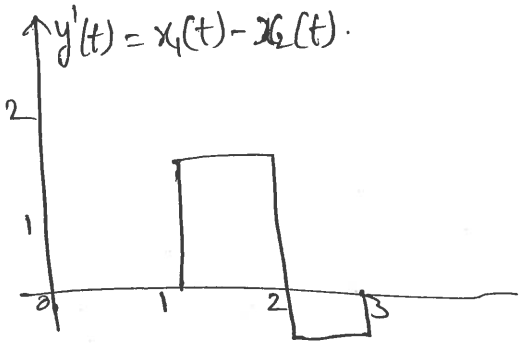



(b) Subtraction.

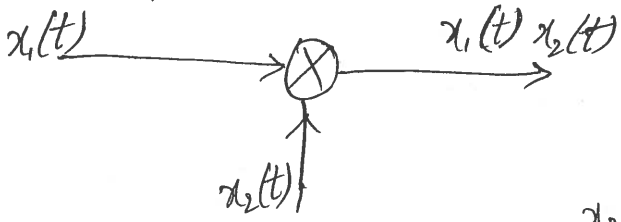


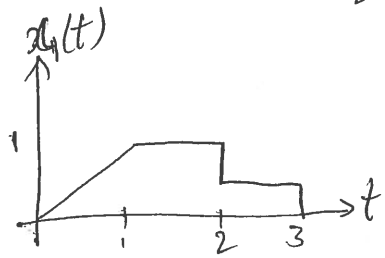



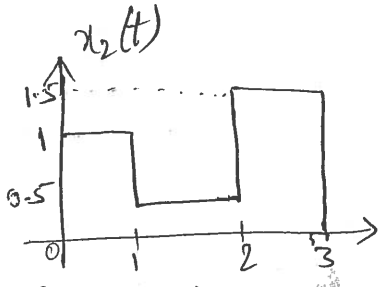




## Signal Multiplication :-

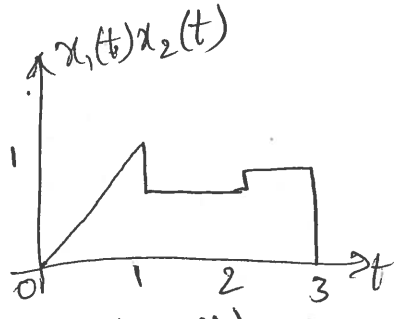



(a) $x_1(t)$




(b) $x_2(t)$




(c) $x_1(t)\cdot x_2(t)$

## Systems:-

A System is defined as an entity that acts on an i/p signal & transforms it into an o/p signal. It is a cause & effect relationship b/w two (or) more signals.

Ex:- Electrical Systems, Mechanical Systems, Biological Systems, Electromechanical systems & so on.

Motor, Amplifier, filter, boiler etc.,

Systems may be Single i/p & Single o/p Systems (or) multi i/p & multi o/p systems.

## Classification of Systems:-

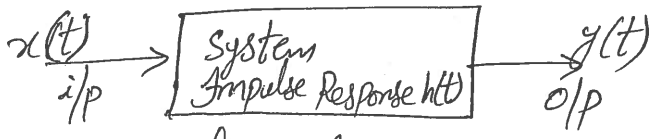



fig:- A system.

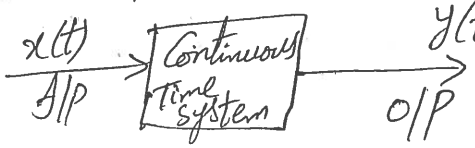
$$y(t) = T\{x(t)\}.$$

I/p x(t) is transformed to y(t).

### ① Continuous time & Discrete Time System:-

A C.T System is one which transforms C.T i/p signals into continuous time o/p signals.

Ex:- Amplifiers, Filters, Integrators & differentiators etc.,




A Discrete Time System is one which which transforms D.T i/p signals into D.T o/p signals.




$$y(n) = T[x(n)].$$

### ② Lumped parameter & Distributed parameter Systems:-

Lumped parameter systems are the systems in which each component is lumped at one point in space. These systems are described by

Distributed parameter systems are the systems in which signals are functions of space as well as time. These systems are described by partial Differential equations.

③ Static & Dynamic Systems :-

A System is said to be static (or) memoryless if the response is due to present i/p alone, i.e., for a static (or) memoryless system, the o/p at any instant time $t$ (or $n$) depends only on the i/p applied at that instant $t$ (or $n$) but not on the past (or) future values of i/p.

Ex:- $y(t) = x^2(t)$

$y(t) = 2x(t)$

$y(n) = 2x^2(n)$

A System is said to be dynamic (or) memory system if the o/p response depends up on past (or) future i/ps.

Ex:- $y(t) = x(t-1) + x(t+2)$, $y(t) = x(2t)$.

$y(t) = \frac{d^2x(t)}{dt^2} + x(t)$

$y(t) = x(t) + x(t+2)$

$y(n) = x(2n)$

$y(n) = x(n) + x(n-2)$.

Any Continuous Time System Described by a differential equation (or) any discrete time system described by a Difference equation is also a Dynamic System.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a Dynamic System

④ Causal & Non-Causal Systems:-

A System is said to be Causal (Non Anticipative) if the o/p of the system at any time 't' depends only on the present & past values of the i/p but not on future i/ps.

Causal systems are real time systems. They are physically realizable. The impulse response of a Causal System is zero for $t$ (or $n$) $< 0$, since

$\delta(t)$ (or $\delta(n)$) exists only at $t$ (or $n$) $= 0$

$h(t) = 0$ for $t < 0$ & $h(n) = 0$ for $n < 0$



Examples

$y(t) = x(t-2) + 2x(t)$

$y(t) = t\,x(t)$

$y(n) = n\,x(n)$

$y(n) = x(n-2) + x(n-1) + x(n)$

A System is said to be Non-Causal (Anticipative) if the o/p of the System at any time $t$ depends on future i/ps. They are not physically realizable.

Ex:- $y(t) = x(t+2) + x(t)$

$y(t) = x^2(t) + t\,x(t+1)$

$y(n) = x(n) + x(2n)$

$y(n) = x^2(n) + 2x(n+2)$

⑤ Linear & Nonlinear Systems:-

A System which obeys the principle of Superposition & principle of homogeneity is called a linear system & a system which doesnot obey the principle of superposition & homogeneity is called Non-linear system.

Homogeneity property means a system which produces an o/p $y(t)$ for an i/p $x(t)$ must produce an o/p $ay(t)$ for an i/p $ax(t)$.

Superposition property means a system which produces an o/p $y_1(t)$ for an i/p $x_1(t)$ & an o/p $y_2(t)$ for an i/p $x_2(t)$ must produce an o/p $y_1(t) + y_2(t)$ for an i/p $x_1(t) + x_2(t)$.

Combining them we can say that a system is linear if an arbitrary i/p $x_1(t)$ produces an o/p $y_1(t)$ & an arbitary i/p $x_2(t)$ produces an o/p $y_2(t)$, then the weighted sum of i/ps $a_1x_1(t) + a_2x_2(t)$; where $a_1$ & $a_2$ are constants, produces an o/p $a_1y_1(t) + a_2y_2(t)$ which is the sum of weighted o/ps. That is

$$T[a_1x_1(t) + a_2x_2(t)] = a_1T[x_1(t)] + a_2T[x_2(t)]$$
$$= a_1y_1(t) + a_2y_2(t)$$

For Discrete time linear Systems

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

$$= a_1 y_1(n) + a_2 y_2(n)$$

Simply we can say that a system is linear if the o/p due to weighted sum of i/ps is equal to the weighted sum of o/ps.

(6) Time Variant & Time Invariant Systems :-

Time Invariance is the property of a system which makes the behaviour of the system independent of the time. This means that the behaviour of the system does not depend on the time at which the i/p is applied.

A System is said to be time Invariant (or shift Invariant) if its Input/output characteristics donot change with time, i.e, if a time shift in the i/p results in a corresponding time shift in the o/p as shown

$x(t) \rightarrow y(t)$

$x(t-T) \rightarrow y(t-T)$

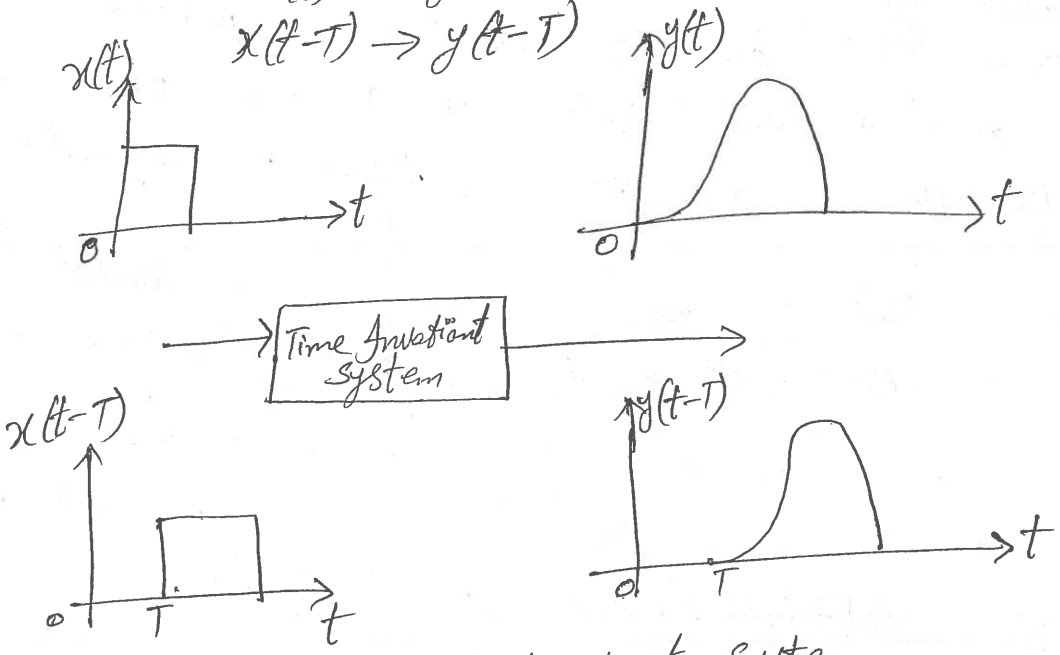



Fig:- Time Invariant System.

A System not satisfying the above requirements is called a Time Varying System (or Shift Varying System).

$$y(t) = T[x(t)]$$

$y(t, T) = T[x(t-T)] = y(t)|_{x(t)=x(t-T)}$ be the o/p ~~data~~ for the delayed i/p $x(t-T)$

$y(t-T) = y(t)|_{t=t-T}$ be the o/p delayed by T .. t.

If $y(t,T) = y(t-T)$

ie, if the delayed o/p is equal to the o/p due to delayed i/p for all possible values of T, then the system is time Invariant.

If $y(t,T) \neq y(t-T)$

ie, If the delayed o/p is not equal to the o/p due to delayed i/p then the system is time Variant.

Ily for D.T. System

$y(n,K) = y(n-K) \longrightarrow$ Time Invariant System.

$y(n,K) \neq y(n-K) \longrightarrow$ Time Variant System.

7) Stable & Unstable Systems :-

A System is said to be Bounded i/p, Bounded o/p (BIBO) stable, if & only if every Bounded i/p produces a Bounded o/p.

Let the i/p Signal $x(t)$ be Bounded (finite) ie,

$$|x(t)| \leq M_x < \infty \text{ for all } t$$

where $M_x$ is a +ve real number.

If $|y(t)| \leq M_y < \infty$.

ie, if $y(t)$ is also bounded, then the System is BIBO stable.

otherwise the system is unstable.

That is we say that a System is unstable even if one bounded i/p produces an unbounded o/p.

The Stability can be found from the impulse response of the System which is nothing but the o/p of the system for a unit impulse i/p.

If the impulse response is absolutely integrable for a Continuous time system (or) absolutely Summable for a D.T System, then the system is stable.

BIBO Stability Criterion :-

The necessary & Sufficient Condition for a System to be BIBO stable is given by the Expression

$$\int |h(t)|\,dt < \infty$$

$h(t)$ is the impulse response of the system. This is called BIBO stability Criterion.

Proof:-

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$ (The i/p and o/p of the system are related by Convolution integral.)

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

$$\left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \leq \int_{-\infty}^{\infty} |x(\tau)|\,|h(t-\tau)|\, d\tau$$

If the i/p $x(t)$ is bounded, ie, there exists a finite number $M_x$ such that,

$$|x(t)| \leq M_x < \infty.$$

If the i/p $x(t)$ is bounded, ie, there exists a finite number $M_x$, such that

$$|x(t)| \leq M_x < \infty$$

$$|y(t)| \leq M_x \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau.$$

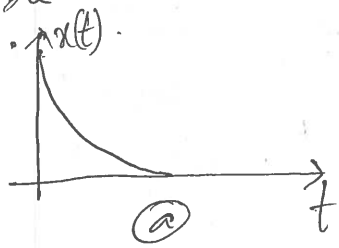
~~$|y(t)| \leq M_x$~~ changing the variables by $m = t-\tau$, the o/p is bounded if

$$\int_{-\infty}^{\infty} |h(m)|\, dm < \infty.$$

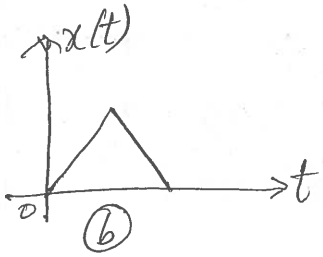
Replacing $m$ by $t$ we have.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty.$$

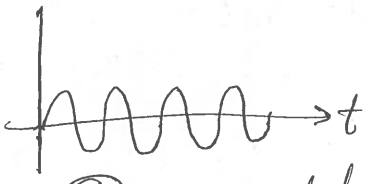
Which is the necessary & sufficient condition for a system to be BIBO stable.



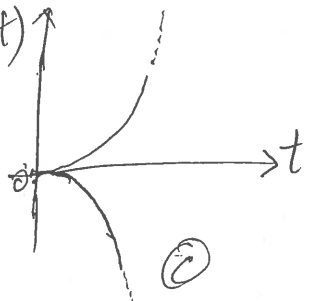
(a)



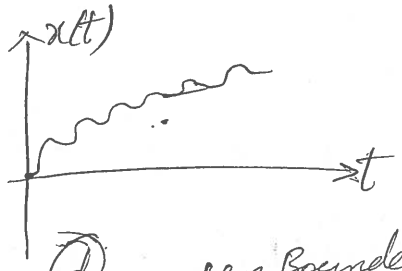
(b)



(d) Bounded.



(c)



(d) Unbounded Signals.

## Invertible & Non Invertible Systems:-

If a system has a unique relationship b/w its i/p $x(t)$ (or $x(n)$) & o/p $y(t)$ (or $x(n)$) the system is known as Invertible.

∴ For an invertible system if $y(t)$ (or $y(n)$) is known $x(t)$ (or $x(n)$) can be found out unambiguously & uniquely.

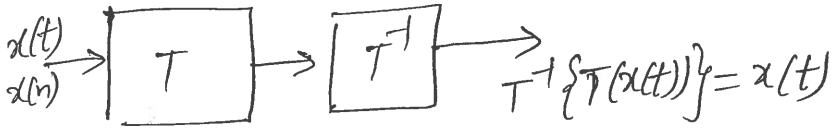
On the other hand if the system doesnot have a unique relationship b/w its i/p & o/p, the system is said to be Non invertible.

For $y(t) = 3x(t)$ the system is said to be invertible.

$y(t) = 2x^2(t)$ the system is said to be Non invertible.

$$x(t) = T^{-1}\left[T[x(t)]\right] \longrightarrow \text{Invertible}$$

(or) $$x(n) = T^{-1}\left[T[x(n)]\right].$$




Invertible System




Non-Invertible System.

Different inputs leads to different outputs, system is invertible

→ $y(t) = 10 + x(t)$

let $x(t) = 2 \Rightarrow y(t) = 10 + 2 = 12$

$x(t) = 21 \Rightarrow y(t) = 10 + 21 = 31$

→ $y(t) = x^2(t)$

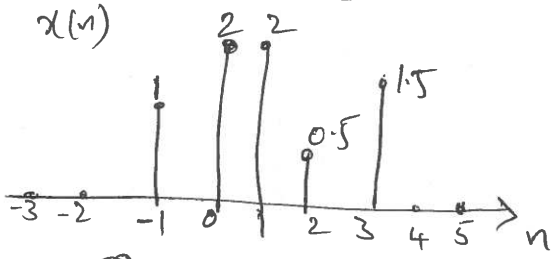
let $x(t) = 5 \Rightarrow y(t) = 25$

$x(t) = -5 \Rightarrow y(t) = 25$

different inputs leads to same outputs, system is non invertible

## Representation of D.T. Signals :-

### ① Graphical representation :-




④ Graphical Representation of a discrete time Signal.

### ② Functional Representation :-

$$x(n) = \begin{cases} 1 & \text{for } n=-1 \\ 2 & \text{for } n=0,1 \\ 0.5 & \text{for } n=2 \\ 1.5 & \text{for } n=3 \\ 0 & \text{otherwise} \end{cases}$$

$x(n) = n^2$

(or)

$x(n) = (n+3)(n-2)$.

### ③ Tabular Representation :-

| n | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| x(n) | 1 | 2 | 2 | 0.5 | 1.5 |

### ④ Sequence Representation :-

$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$

# Concept of Convolution & Correlation :-

Any continuous time signal can be represented as

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\,\delta(t - n\Delta\tau)\cdot\Delta\tau$$

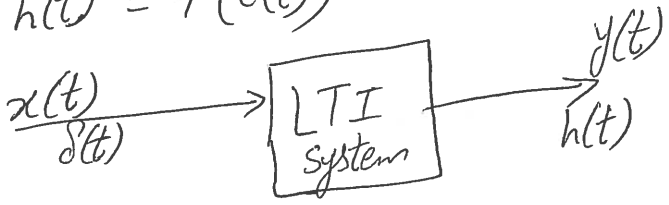
When $\Delta\tau \to 0$, limit $\Delta\tau$ becomes continuous variable $d\tau$ and the summation becomes integral. ∴ representation of $x(t)$ becomes

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

The above equation shows that an Arbitary i/p signal can be represented as a linear combination of the scaled and shifted unit impulse functions. If the response of the system to the impulse is known, then the response for any i/p signal $x(t)$ can be obtained.

Consider an LTI system which is initially relaxed at $t=0$. If the i/p to the system is an impulse then the o/p of the system is denoted by $h(t)$ and is called the impulse response of the system. we can denote impulse response as follows:

$$h(t) = T(\delta(t))$$




we know that any arbitary signal $x(t)$ can be represented as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

System o/p is given by

$$y(t) = T(x(t))$$

$$y(t) = T(x(t)) = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

For a linear system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau$$

$$T(\delta(t-\tau)) = h(t,\tau)$$

$h(t,\tau)$ -- the impulse response of the system due to delay impulse. i.e., $h(t,\tau) = T(\delta(t-\tau))$

For a time invariant system, o/p due to delayed i/p by '$\tau$' seconds is equal to delayed o/p by '$\tau$' seconds. That is

$$h(t,\tau) = h(t-\tau)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This is called the Convolution integral (or) Simply Convolution.

$$y(t) = x(t) * h(t)$$ where $*$ → Convolution operator.

Properties of Convolution:

(1) Commutative property:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau = x_2(t) * x_1$$

(2) Distributive property:-

$$x_1(t) * \left(x_2(t) + x_3(t)\right) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

(3) Associative property:-

$$x_1(t) * \left(x_2(t) * x_3(t)\right) = \left(x_1(t) * x_2(t)\right) * x_3(t)$$

(4) Shift property:-

If $x_1(t) * x_2(t) = z(t)$

then $x_1(t) * x_2(t-T) = z(t-T)$

Ily $x_1(t-T) * x_2(t) = z(t-T)$

(5) Convolution with an impulse:

$$x(t) * \delta(t) = x(t).$$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ if both $x(t)$ & $h(t)$ are non-causal.

$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$ if $x(t)$ is non-causal & $h(t)$ is causal.

$= \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$ if $x(t)$ is causal & $h(t)$ is non-causal.

$= \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau$ if both $x(t)$ & $h(t)$ are causal.

Example:- Find the convolution of the following signals:

(a) $x_1(t) = e^{-2t}u(t)$ $\quad x_2(t) = e^{-4t}u(t)$

(b) $x_1(t) = r(t)$ $\quad x_2(t) = e^{-2t}u(t)$

→ (a) $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$= \int_{0}^{t} e^{-2\tau}\, e^{-4(t-\tau)}\, d\tau$

$= e^{-4t} \int_{0}^{t} e^{2\tau}\, d\tau = e^{-4t}\left(\frac{e^{2\tau}}{2}\right)_0^t = \frac{e^{-4t}}{2}\left(e^{2t}-1\right)$

$x_1(t) * x_2(t) = \left(\frac{e^{-2t} - e^{-4t}}{2}\right)u(t)$

(b) $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} r(\tau)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} r(\tau)\, e^{-2t}\, e^{2\tau}\, u(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} \tau u(\tau)\, e^{-2t}\, e^{2\tau}\, u(t-\tau)\, d\tau = \left(\int_{0}^{t} \tau\, e^{2\tau}\, d\tau\right) e^{-2t}$

$= e^{-2t}\left(\left(\frac{\tau e^{2\tau}}{2}\right)_0^t - \int_0^t \frac{e^{2\tau}}{2}\, d\tau\right) = e^{-2t}\left\{\left(\frac{te^{2t}}{2} - \left(\frac{e^{2\tau}}{4}\right)_0^t\right)\right\} = e^{-2t}\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right)$

$x_1(t) * x_2(t) = \left(\frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4}\right)$

## Signal Comparison: Correlation of functions :-

The signals may be compared on the basis of similarity of waveforms. Quantitatively a comparison may be based upon the amount of the component of one waveform contained in the other waveform. If $x_1(t)$ & $x_2(t)$ are 2 waveforms, then the waveform $x_1(t)$ contains an amount $c_{12}\,x_2(t)$ of that particular waveform $x_2(t)$ in the interval $(t_1, t_2)$ where

$$c_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

The magnitude of the integral in the numerator might be taken as an indication of similarity. If this integral vanishes i.e.,

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

then the 2 signals have no similarity over the interval $(t_1, t_2)$. Such signals are said to be orthogonal over the specified interval $(t_1, t_2)$.

In general we are interested in comparing the 2 signals over the interval $(-\infty, \infty)$. So the test integral becomes,

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$$

However there is a difficulty with this test integral which can be illustrated with the Example of Radar Pulse.

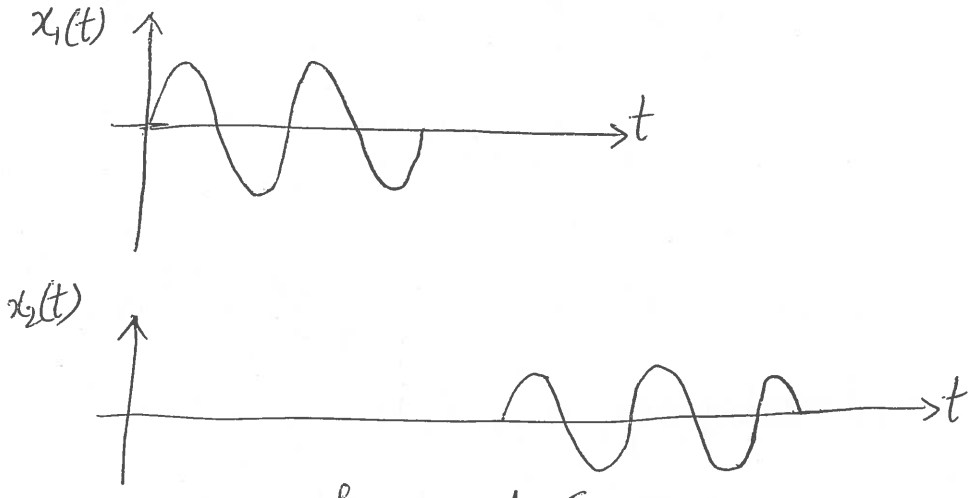



fig:- Signal Comparison

Fig shows a Txd pulse & a Rxd pulse which is delayed by T secs. Obviously 2 waveforms are identical, except that one is delayed with respect to others yet the test integral $\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$ yields zero because the product $x_1(t)\, x_2(t)$ is zero everywhere. This indicates that the 2 waveforms have no measure of similarity which is obviously a wrong conclusion. Hence inorder to search for similarity b/w the 2 waveforms, we must shift one waveform w.r.t the other by various amounts of shifts ~~of one function w.r.t the other~~ and see whether a similarity exists for some amount of shift of one function w.r.t the other.

∴ the test integral is modified as

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt.$$

Where $\tau$ is the searching (or) scanning parameter. This integral is known as the cross & correlation function between $x_1(t)$ & $x_2(t)$ is denoted by $R_{12}(\tau)$.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

(or) $$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

Cross correlation: The cross correlation between two different waveforms or signals is a measure of similarity or match or relatedness or coherence between one signal and time delayed version of another signal. That means cross correlation between two signals indicate how much one signal is related to time delayed version of another signal.

Consider two general complex signals $x_1(t)$ and $x_2(t)$ of finite energy. The cross correlation of these two energy signals denoted by $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2^*(t)\, dt$$

If two signals $x_1(t)$ & $x_2(t)$ are real then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

If $x_1(t)$ and $x_2(t)$ have some similarity, then cross correlation $R_{12}(\tau)$ have some finite value over range of $\tau$. Also if

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0$$

ie if $R_{12}(0) = 0$

then two signals are called orthogonal signals. That is cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) x_1^*(t-\tau)\, dt.$$

In this equation, cross correlation function $R_{21}(\tau)$ is a function of variable $\tau$. The variable $\tau$ is called delay parameter. or scanning parameter or searching parameter.

Cross correlation for two periodic signals $x_1(t)$ and $x_2(t)$ is defined with the help of average form of correlation.

If two periodic signals $x_1(t)$ & $x_2(t)$ have same time period cross correlation is defined as

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)\, x_2^*(t-\tau)\, dt$$

$$\text{or} \quad R_{21}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_2(t)\, x_1^*(t-\tau)\, dt$$

Auto correlation:

It gives the measure of match or similarity or relatedness or coherence between signal & time delayed version. This means that autocorrelation function is a special form of cross correlation function. It is defined as correlation of a function with itself.

autocorrelation of an energy signal $x(t)$ is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt. \quad \text{(or)}$$

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x^*(t)\, dt$$

$\tau$ is delay parameter.

auto correlation for power signals $R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$

## Analogy between Vectors & Signals :-

### Vectors :-

A vector is specified by magnitude & direction. Consider two vectors $V_1$ & $V_2$ as shown in fig(1). The vector $V_1$ can be represented interms of vector $V_2$ in infinite No. of ways by drawing a line from the end of $V_1$ & onto $V_2$. In each representation $V_1$ is represented interms of $V_2$ plus another vector which will be called the error vector.

In fig (a) $V_1$ is approximated by $C_1 V_2$ with an error $V_{e2}$ as

$$V_1 = C_1 V_2 + V_{e1}.$$

In fig (b) $V_1$ is approximated by $C_2 V_2$ with an Error Vector $V_{e2}$ as

$$V_1 = C_2 V_2 + V_{e2}.$$

In fig (c) $V_1$ is approximated by $C_{12} V_2$ with an Error $V_e$ as

$$V_1 = C_{12} V_2 + V_e.$$

Geometrically the Component of a vector $V_1$ along $V_2$ is obtained by drawing a perpendicular from the end of $V_1$ onto $V_2$. In this case Error vector $V_e$ is the minimum. The Component of a vector $V_1$ along another vector $V_2$ is given by $C_{12} V_2$, where $C_2$ is choosen such that the Error Vector $V_e$ is minimum. The magnitude $C_{12}$ is an indication of the similarity of the two vectors.

If $C_{12} = 0$ then vector $V_1$ has no Component along other vector $V_2$ & hence two vectors $V_1$ & $V_2$ are mutually perpendicular. The vectors which are mutually perpendicular to each other they are called orthogonal vectors. orthogonal vectors are thus independent vectors. If the vectors are orthogonal then the Parameter $C_{12}$ is zero.

Component $V_1$ along $V_2 = \dfrac{V_1 \cdot V_2}{V_2} = C_{12} V_2$.

$$\boxed{C_{12} = \frac{V_1 \cdot V_2}{V_2^2} = \frac{V_1 \cdot V_2}{V_2 \cdot V_2}}$$

$$V_1 \cdot V_2 = |V_1|\,|V_2| \cos\theta$$

$$\cos\theta = \frac{C_{12}|V_2|}{|V_1|}$$

$$|V_1| \cos\theta = C_{12} |V_2|$$

$$|V_1|\,|V_2| \cos\theta = C_{12} |V_2|^2$$

$$\boxed{C_{12} |V_2|^2 = V_1 \cdot V_2}$$

When vectors $V_1$ & $V_2$ are orthogonal, their dot product

$$V_1 \cdot V_2 = 0$$

& $C_2 = 0$

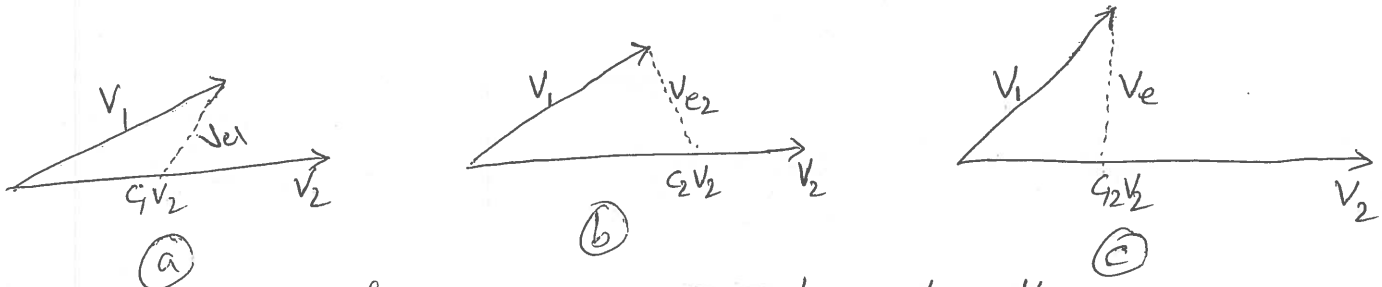



fig:- one vector interms of another.

Signals :-

The concept of vector comparison & orthogonality can be extended to signals. Consider two $x_1(t)$ & $x_2(t)$. we can approximate $x_1(t)$ in terms of $x_2(t)$ over a certain interval $(t_1 < t < t_2)$ as follows:

$$x_1(t) \approx C_2 x_2(t) \quad \text{for } (t_1 < t < t_2)$$

This is similar to $V_1 = C_2 V_2$ in the case of vectors.

Like in vectors, the main criterion in selecting $C_2$ is to minimise the error between the actual function & the approximated function over the interval $(t_1 < t < t_2)$.

The error function $x_e(t)$ is defined as:

$$x_e(t) = x_1(t) - C_2 x_2(t)$$

If we choose to minimise the error $x_e(t)$ over the interval $t_1$ to $t_2$ by minimising the average value of $x_e(t)$ over this interval ie, by minimising.

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_e(t)\, dt \quad \text{ie,} \quad \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (x_1(t) - C_2 x_2(t))\, dt$$

it may give wrong results.

This is because there can be large positive and negative errors present in interval $t_1$ to $t_2$ that may cancel one another in the process of averaging & give the false indication that the error is zero. To overcome this we choose to minimise the mean square of the error instead of the error itself.

By Analogy with the vectors we say that $x_1(t)$ has a component of waveform $x_2(t)$ (i.e., $C_{12}\,x_1(t)$) & this component has a magnitude $C_{12}$. If $C_{12}=0$, the the signal $x_1(t)$ contains no component of the signal $x_2(t)$; & we say that the two functions ~~$x_1(t)$ & $x_2(t)$~~ are orthogonal over the interval $(t_1, t_2)$. So we can say that the two functions $x_1(t)$ & $x_2(t)$ are orthogon over an interval $(t_1, t_2)$ if

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

Thus in case of signals

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}.$$

In both cases for orthogonality $C_{12}=0$

Two vectors $V_1$ & $V_2$ are orthogonal if their dot product is Zero i.e; $V_1 \cdot V_2 = 0$. Two Signals $x_1(t)$ & $x_2(t)$ are orthogonal if the integra of the product of those signals is Zero i.e.,

$$\int_{t_1}^{t_2} x_1(t) x_2(t)\, dt = 0$$

In the Analogy of the vectors & Signals, the dot product of two vectors is analogous to the integral of the product of 2 Signal that is

$$A \cdot B \sim \int_{t_1}^{t_2} x_A(t)\, x_B(t)\, dt.$$

The Square of the length A of vector A is analogous to the integral of the Square of a function, that is

$$A \cdot A = A^2 \sim \int_{t_1}^{t_2} x_A^2(t)\, dt$$

We can easily show that the functions $\sin m\omega_0 t$, $\sin m\omega_0 t$; $\cos m\omega_0 t$, $\cos m\omega_0 t$ & $\sin\omega_0 t$, $\cos m\omega_0 t$ are orthogonal over any interval $(t_0, t_0 + (2\pi/\omega_0))$ fo

Let us designate the average of squared error $(x_e^2(t))$ by $\mathcal{E}$

$$\mathcal{E} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_e^2(t)\,dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(x_1(t) - C_{12}\, x_2(t)\right)^2 dt.$$

The value of $C_{12}$ which minimises $\mathcal{E}$ can be found from

$$\left(\frac{d\mathcal{E}}{dC_{12}}\right) = 0$$

$$\frac{d\mathcal{E}}{dC_{12}} = \frac{d}{dC_{12}} \left( \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left(x_1(t) - C_{12}\, x_2(t)\right)^2 dt \right) = 0$$

$$\frac{d}{dC_{12}} \left\{ \frac{1}{(t_2 - t_1)} \left[ \int_{t_1}^{t_2} \left(x_1^2(t) - 2C_{12} x_1(t) x_2(t) + C_{12}^2 x_2^2(t)\right) dt \right] \right\} = 0$$

Changing the order of integration & differentiation, we get

$$\frac{1}{t_2 - t_1} \left[ \int_{t_1}^{t_2} \frac{d}{dC_{12}} \left[x_1^2(t)\right] dt - 2 \int_{t_1}^{t_2} \frac{d}{dC_{12}} \left(C_{12} x_1(t) x_2(t)\right) dt + \int_{t_1}^{t_2} \frac{d}{dC_{12}} \left(C_{12}^2 x_2^2(t)\right) dt \right] = 0$$

$$\frac{1}{(t_2 - t_1)} \left[ 0 - 2 \int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt + \int_{t_1}^{t_2} 2 C_{12} x_2^2(t)\, dt \right] = 0$$

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt \simeq C_{12} \int_{t_1}^{t_2} x_2^2(t)\, dt.$$

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

For vectors, we had $C_{12} = \frac{V_1 \cdot V_2}{V_2^2}$.

integral values of m & n.



## Orthogonal Signal Space:-

Any signal $x(t)$ can be expressed as a sum of its components along a set of n mutually orthogonal functions if these functions form a complete set.

## Approximation of a function by a set of mutually orthogonal Functions

Consider a set of n functions $g_1(t), g_2(t) \cdots\cdots g_n(t)$ which are orthogonal to one another over an interval $t_1$ to $t_2$, that is

$$\int_{t_1}^{t_2} g_j(t)\, g_k(t)\, dt = 0 \quad j \neq k$$

& let $\int_{t_1}^{t_2} g_j^2(t)\, dt = K_j$

Let an arbitary signal $x(t)$ be approximated over an interval $(t_1, t_2)$ by a linear combination of these n mutually orthogonal signals.

$$x(t) \cong C_1 g_1(t) + C_2 g_2(t) + \cdots\cdots + C_k g_k(t) + \cdots\cdots + C_n g_n(t)$$

$$x(t) = \sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation we must find the proper values of Constants $C_1, C_2, \cdots C_n$ such that $\varepsilon$, the mean square of error $x_e(t)$ is minimised.

By definition,

$$x_e(t) = x(t) - \sum_{r=1}^{n} C_r g_r(t)$$

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[ x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt.$$

This eq shows that $\varepsilon$ is a function of $C_1, C_2, \cdots C_n$ & to minimise $\varepsilon$, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots\cdots = \frac{\partial \varepsilon}{\partial C_j} = \cdots\cdots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

Consider the equation

$$\frac{\partial \varepsilon}{\partial C_j} = 0.$$

Since $(t_2 - t_1)$ is a Constant, we can Write

$$\frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2}\left[x(t) - \sum_{r=1}^{n} C_r g_r(t)\right]^2 dt\right\} = 0$$

$$\frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2}\left[x^2(t) - 2\sum_{r=1}^{n} x(t) C_r g_r(t) + \sum_{r=1}^{n} C_r^2 g_r^2(t)\right] dt\right\} = 0$$

In this Expansion, all the terms arising due to the Cross product of the orthogonal functions are zero by virtue of orthogonality, that is all the terms of the from $\int g_j(t) g_k(t)\, dt$ are zero. By the derivative w.r.t $C_j$ of all the terms that donot Contain $C_j$ are zero, that is

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2} x^2(t)\, dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r^2 g_r^2(t) dt = \frac{\partial}{\partial C_j}\int_{t_1}^{t_2} C_r x_r(t)\, g_r(t)\, dt = 0$$

This leaves only two non-zero terms in the equation $\frac{\partial \varepsilon}{\partial C_j} = 0$ as follows:

$$\frac{\partial}{\partial C_j}\int_{t_1}^{t_2}\left(-2 C_j x(t) g_j(t) + C_j^2 g_j^2(t)\right) dt = 0$$

changing the order of differentiation & integration we get

$$-2\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt = 2 C_j \int_{t_1}^{t_2} g_j^2(t)\, dt$$

$$C_j = \frac{\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} = \frac{1}{K_j}\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt$$

Evaluation of Mean Square Error:-

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left(x(t) - \sum_{r=1}^{n} C_r g_r(t)\right)^2 dt = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t) dt + \sum_{r=1}^{n} C_r^2 \int_{t_1}^{t_2} g_r^2(t)\, dt - 2\sum_{r=1}^{n} C_r \int_{t_1}^{t_2} x(t) g_r(t) dt\right]$$

But from the equation for $C_j$ we have $\int_{t_1}^{t_2} x(t)\, g_r(t)\, dt = C_r \int_{t_1}^{t_2} g_r^2(t)\, dt = C_r K_r$

$$\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t) dt + \sum_{r=1}^{n} C_r^2 K_r - 2\sum_{r=1}^{n} C_r^2 K_r\right] = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t) dt - \sum_{r=1}^{n} C_r^2 K_r\right]$$

(1) Show that the following signals are orthogonal over an interval (0,1); (2)

$$x_1(t) = 2$$

$$x_2(t) = \sqrt{3}(1-2t)$$

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0.$$

$$= \int_0^1 2\left(\sqrt{3}(1-2t)\right) dt = 2\sqrt{3}\int_0^1 dt - 4\sqrt{3}\int_0^1 t\, dt = 2\sqrt{3}\,(t)_0^1 - 4\sqrt{3}\left(\frac{t^2}{2}\right)_0^1$$

$$= 2\sqrt{3}\,(1-0) - 4\sqrt{3}\left(\frac{1}{2}-0\right) = 0.$$

So $x_1(t)$ & $x_2(t)$ are orthogonal.

(2) Show that the functions $\sin n\omega_0 t$ & $\cos m\omega_0 t$ are orthogonal over any interval $\{t_0 \; (t_0 + \frac{2\pi}{\omega_0})\}$ for integral values of $n$ & $m$.

$$I = \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \cos m\omega_0 t\, dt = 0$$

$$I = \int_{t_0}^{t_0+2\pi/\omega_0} \frac{1}{2}\left(\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t\right) dt.$$

$$I = \frac{1}{2}\left(-\frac{\cos(n+m)\omega_0 t}{n+m} - \frac{\cos(n-m)\omega_0 t}{n-m}\right)_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$I = -\frac{1}{2}\left(\frac{\cos(n+m)\omega_0 t}{n+m} + \frac{\cos(n-m)\omega_0 t}{n-m}\right)_{t_0}^{t_0+(2\pi/\omega_0)}$$

$$I = \frac{-1}{2}\left(\left(\frac{\cos(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \cos(n+m)\omega_0 t_0}{n+m}\right) + \left(\frac{\cos(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \cos(n-m)\omega_0 t_0}{n-m}\right)\right)$$

$$I = \frac{-1}{2}\left(\frac{\cos(n+m)(\omega_0 t_0 + 2\pi) - \cos(n+m)\omega_0 t_0}{n+m} + \frac{\cos(n-m)(\omega_0 t_0 + 2\pi) - \cos(n-m)\omega_0 t_0}{n-m}\right)$$

$$I = \frac{-1}{2}\left(\frac{\cos(n+m)\omega_0 t_0 - \cos(n+m)\omega_0 t_0}{n+m} + \frac{\cos(n-m)\omega_0 t_0 - \cos(n-m)\omega_0 t_0}{n-m}\right)$$

$$I = 0.$$

Ex:(3) Prove that the Complex exponential Signals are orthogonals

$$x_1(t) = e^{jn\omega_0 t} \qquad x_2(t) = e^{jm\omega_0 t}$$

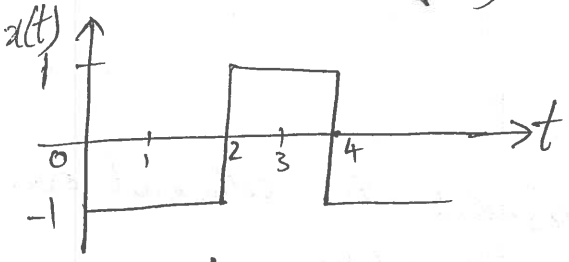
$$\int_{t_0}^{t_0+(2\pi/\omega_0)} x_1(t)\, x_2^*(t)\, dt = 0.$$

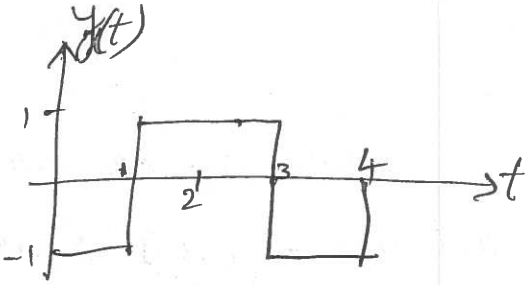
(4) If $x_p(t) = \frac{1}{\sqrt{T}}(\cos p\omega_0 t + \sin p\omega_0 t)$ & $x_q(t) = \frac{1}{\sqrt{T}}(\cos q\omega_0 t + \sin q\omega_0 t)$.

$T = \frac{2\pi}{\omega_0}$ are orthogonal over the period ~~$2\pi$~~ (0 to T).

$$\Rightarrow \int_0^T x_p(t)\, x_q(t)\, dt = 0.$$

(5) Prove that the signals $x(t)$ & $y(t)$ shown in fig are orthogonal over the interval $(0,4)$.




$$x(t) = \begin{cases} -1 & \text{for } 0<t<2 \\ 1 & \text{for } 2<t<4 \end{cases}$$

$$y(t) = -1 \quad 0<t<1$$
$$= 1 \quad 1<t<3$$
$$= -1 \quad 3<t<4.$$

$$\int_0^4 x(t)\, y(t)\, dt = 0$$

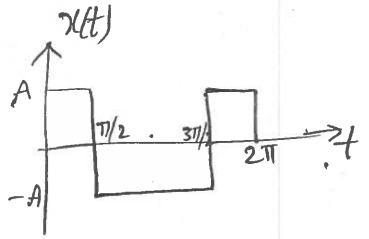
(6) A rectangular function is defined as $x(t) = \begin{cases} A & \text{for } 0<t<\frac{\pi}{2} \\ -A & \text{for } \frac{\pi}{2}<t<\frac{3\pi}{2} \\ A & \text{for } \frac{3\pi}{2}<t<2\pi \end{cases}$

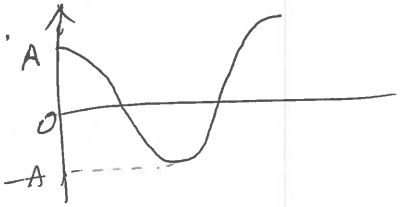
Approximate the above function by $A\cos t$ between the intervals $(0, 2\pi)$ such that the mean square Error is minimum.

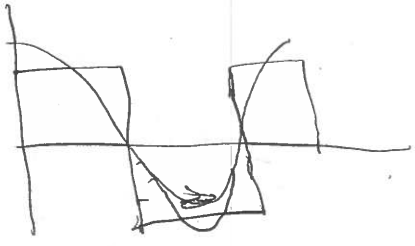
$$\Rightarrow x(t) \simeq C_{12} A\cos t$$

$$C_{12} = \frac{\int_0^{2\pi} x(t)\, A\cos t\, dt}{\int_0^{2\pi} (A\cos t)^2 dt} = \frac{4}{\pi}.$$

$$x(t) = \frac{4}{\pi} A\cos t.$$



## Orthogonality in Complex functions:-

If $x_1(t)$ & $x_2(t)$ are Complex functions of real variable t, $x_1(t)$ can be approximated by $C_{12}x_2(t)$ over an interval $(t_1, t_2)$

$$x_1(t) \cong C_{12}\, x_2(t)$$

The optimum value of $C_{12}$ to minimise the mean square Error ~~its~~ magnitude is given by

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt}{\int_{t_1}^{t_2} x_2(t)\, x_2^*(t)\, dt}$$

where $x_2^*(t)$ is a Complex Conjugate of $x_2(t)$.

For the Complex functions to be orthogonal over the interval $(t_1, t_2)$

$$C_{12} = 0 \cdot \int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt = \int_{t_1}^{t_2} x_1^*(t)\, x_2(t)\, dt = 0$$

For a set of Complex functions $\{g_n(t)\}$ $(n = 1, 2, \ldots)$ mutually orthogonal over the interval $(t_1, t_2)$.

$$\int_{t_1}^{t_2} g_m(t)\, g_n^*(t)\, dt = \begin{cases} 0 & \text{if } m \neq n \\ K_m & \text{if } m = n. \end{cases}$$

If this set of functions is complete, then any function $x(t)$ can be Expressed as:

$$x(t) = C_1 g_1(t) + C_2 g_2(t) + \cdots + C_n g_n(t) + \cdots$$

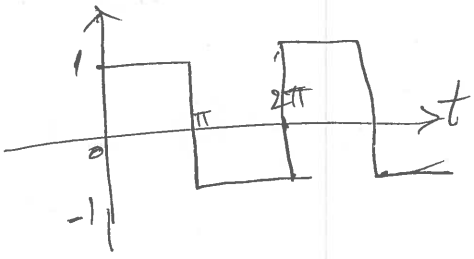
$$C_n = \frac{1}{K_n} \int_{t_1}^{t_2} x(t)\, g_n^*(t)\, dt.$$

$$K_n = \int_{t_1}^{t_2} g_n^*(t)\, dt.$$

① A Rectangular function is defined by

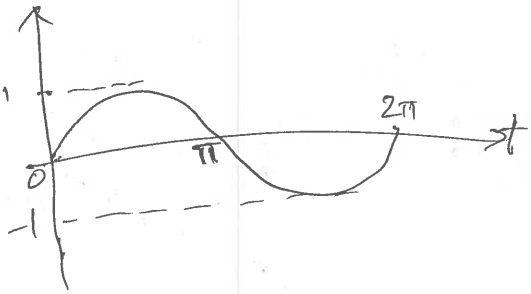
$$x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$$

Approximate the above rectangular function by a single sinusoid $\sin t$ over the interval $(0, 2\pi)$ such that the mean square error is minimum. Evaluate the mean square error in this approximation. Also show what happens when more No of sinusoids is used for approximation.




$\rightsquigarrow$ $x(t) \simeq C_2 \sin t.$

$$C_2 = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$




$$x_1(t) = x(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ -1 & \text{for } \pi < t < 2\pi \end{cases}$$

$$C_2 = \frac{\int_0^{2\pi} x(t) \sin t\, dt}{\int_0^{2\pi} \sin^2(t)\, dt} = \frac{\int_0^{\pi} 1 \times \sin t\, dt + \int_{\pi}^{2\pi} -1 \times \sin t\, dt}{\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right) dt}$$

$$C_2 = \frac{(\cos t)_0^{\pi} - (-\cos t)_{\pi}^{2\pi}}{\int \frac{1}{2} dt - \frac{1}{2}\int \cos 2t\, dt} = \frac{-\left((\cos\pi - \cos 0) - (-\cos 2\pi + \cos\pi)\right)}{\frac{1}{2}(2\pi) - \frac{1}{2}\left(\frac{\sin 2t}{2}\right)}$$

$$C_2 = \frac{-(-1-1) + (1-(-1))}{\pi - \frac{1}{4}(\sin 4\pi - \sin 0)} = \frac{2+2}{\pi} = \frac{4}{\pi}$$

$$x(t) = \frac{4}{\pi} \sin t.$$

Wkt $\varepsilon = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2} x^2(t)\, dt - \int_{t_1}^{t_2} C_2^2 \sin^2(t)\, dt\right] = 1 - \frac{8}{\pi^2} = 0.189$

Ex:- Show that the Complex Exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic only if $\omega_0/2\pi$ is a Rational number.

⇒

$$x(n) = e^{j\omega_0 n}$$

$x(n+N) = x(n)$ → Condition for periodicity.

$$x(n+N) = e^{j\omega_0(n+N)} = e^{j\omega_0 n} e^{j\omega_0 N}$$

$$e^{j\omega_0 n} e^{j\omega_0 N} = e^{j\omega_0 n}$$

$$e^{j\omega_0 N} = 1$$

$$e^{j\omega_0 N} = e^{j2\pi K}$$

$$\omega_0 N = 2\pi K$$

$$\frac{\omega_0}{2\pi} = \frac{K}{N} = \text{Rational number.}$$

This shows that the Complex Exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number.

Ex:- obtain the condition for discrete time Sinusoidal Signal to be periodic.

⇒ Let $x(n) = A\sin(\omega_0 n + \theta)$

If $x(n+N) = x(n)$ for all $n$ then the signal is periodic

$$x(n+N) = A\sin(\omega_0(n+N)+\theta) = A\sin(\omega_0 n + \theta + \omega_0 N)$$

$x(n) = x(n+N)$ if $\omega_0 N = 2\pi m$. That is there must be an integer $m$ such that

$$\omega_0 = \frac{2\pi m}{N} = 2\pi\left(\frac{m}{N}\right)$$

$$N = 2\pi\left(\frac{m}{\omega_0}\right)$$

From the above eq we find that for the discrete time Signal to be Periodic, the fundamental frequency $\omega_0$ must be a rational multiple of $2\pi$. The smallest value of +ve integer $N$ for some integer $m$, is the fundamental period.

Ex:- Determine whether the following discrete-time Signals are periodic (or) not. If periodic, determine the fundamental period.

(a) $\sin(0.02\pi n)$ (b) $\sin(5\pi n)$ (c) $\cos 4n$ (d) $\sin\frac{2\pi n}{3} + \cos\frac{2\pi n}{5}$

(e) $\cos\left(\frac{n}{6}\right)\cos\left(\frac{n\pi}{6}\right)$ (f) $\cos\left(\frac{\pi}{2} + 0.3n\right)$ (g) $x(n) = 1 + e^{j2\pi n/3} - e^{j4\pi n/7}$

Ans (a) $x(n) = \sin(0.02\pi n)$

$$\omega_0 = 0.02\pi \Rightarrow 2\pi f = 0.02\pi \quad f = \frac{0.02\pi}{2\pi} = 0.01 = \frac{1}{100} = \frac{K}{N}$$

$f$ is expressed as a Ratio of 2 integers. with $K=1$ & $N=100$.
Hence the signal is periodic with fundamental period $N=100$

(b) $\sin 5\pi n$

$\omega_0 = 5\pi \qquad f = \frac{5\pi}{2\pi} = \frac{5}{2} = \frac{K}{N}$ (or) $\omega_0 N = 2\pi m$

$N = \frac{2\pi m}{\omega_0} = \frac{2\pi m}{5\pi} = \frac{2}{5}m$

$f$ is a ratio of 2 integers with $K=5$ & $N=2$. Hence it is rational.
∴ the given signal is periodic with fundamental period $N=2$.

(c) $\cos 4n$

$\omega_0 = 4 \qquad f_0 = \frac{4}{2\pi} = \frac{2}{\pi}$ is not a rational. ∴ $x(n)$ is Aperiodic

(d) $x(n) = \sin\frac{2\pi n}{3} + \cos\frac{2\pi n}{5}$

$\omega_{01} = 2\pi f_1 = \frac{2\pi}{3} \Rightarrow f_1 = \frac{1}{3} = \frac{K_1}{N_1} \Rightarrow N_1 = 3$

$\omega_{02} = 2\pi f_2 n = \frac{2\pi}{5}$ (or) $f_2 = \frac{1}{5} \Rightarrow N_2 = 5$.

$\frac{N_1}{N_2} = \frac{3}{5}$ is a Ratio of 2 integers. $N = LCM\ of\ (3,5) = 15$.

(e) $x(n) = \cos\left(\frac{n}{6}\right)\cos\left(\frac{n\pi}{6}\right)$

$2\pi f_1 = \frac{1}{6} \Rightarrow f_1 = \frac{1}{12\pi} \longrightarrow$ Not a Rational $\longrightarrow$ Non Periodic

$2\pi f_2 = \frac{\pi}{6} \Rightarrow f_2 = \frac{1}{12} \longrightarrow$ Rational $\longrightarrow$ Periodic

Product of Non-Periodic & Periodic Signals is Non-Periodic.

(f) $x(n) = \cos\left(\frac{\pi}{2} + 0.3n\right)$

$x(n) = \cos(2\pi f n + \theta)$

$2\pi f n = 0.3n$ & phase shift $\theta = \frac{\pi}{2}$

$f = \frac{0.3}{2\pi} = \frac{3}{20\pi}$ which is not a Rational.

Hence Non-periodic.

(g) $x(n) = 1 + e^{j2\pi n/3} - e^{j4\pi n/7} = e^{j2\pi n} + e^{j2\pi n/3} - e^{j4\pi n/7}$

$e^{j2\pi n} = 1 \qquad \omega_{01} = 2\pi \qquad f_{0F} = \frac{2\pi}{2\pi} = \frac{1}{1} \Rightarrow N_1 = 1 \quad \left(f_0 = \frac{1}{1} = \frac{K_1}{N_1}\right)$

$e^{j2\pi n/3} \longrightarrow \omega_{02} = \frac{2\pi}{3} \qquad f_{02} = \frac{1}{3} = \frac{K_2}{N_2} \Rightarrow N_2 = 3$

$e^{-j4\pi n/7} \longrightarrow \omega_{03} = +\frac{4\pi}{7} \qquad f_{03} = +\frac{2}{7} = \frac{K_3}{N_3} \Rightarrow N_3 = 7$

$\frac{N_1}{N_2} = \frac{1}{3}$ & $\frac{N_1}{N_3} = \frac{1}{7} = \frac{21}{7}$ Rational number. LCM of $(N_1, N_2, N_3) = 21 = N$

Ex:- Find the even & odd components of the following signals:

(a) $x(t) = e^{j2t}$ (b) $x(t) = \cos\left(\omega_0 t + \frac{\pi}{3}\right)$ (c) $x(t) = (1+t^2+t^3)\cos^2 10t$.

=> (a) $x(t) = e^{j2t} = x_e(t) + x_o(t)$

$x(-t) = e^{-j2t}$

$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right) = \frac{1}{2}\left(e^{j2t} + e^{-j2t}\right) = \cos 2t.$

$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) = \frac{1}{2}\left(e^{j2t} - e^{-j2t}\right) = j\sin 2t.$

(b) $x(t) = \cos\left(\omega_0 t + \frac{\pi}{3}\right) = x_e(t) + x_o(t)$

$x(-t) = \cos\left(-\omega_0 t + \frac{\pi}{3}\right) = \cos\left(\omega_0 t - \frac{\pi}{3}\right)$

$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right) = \frac{1}{2}\left(\cos\left(\omega_0 t + \frac{\pi}{3}\right) + \cos\left(\omega_0 t - \frac{\pi}{3}\right)\right)$

$= \cos\omega_0 t \cos\frac{\pi}{3} = \frac{1}{2}\cos\omega_0 t.$

$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) = \frac{1}{2}\left(\cos\left(\omega_0 t + \frac{\pi}{3}\right) - \cos\left(\omega_0 t - \frac{\pi}{3}\right)\right)$

$= -\sin\omega_0 t \sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}\sin\omega_0 t.$

(c) $x(t) = (1+t^2+t^3)\cos^2 10t$

$x(-t) = (1+t^2-t^3)\cos^2 10t$

$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right) = \frac{1}{2}\left((1+t^2+t^3)\cos^2 10t + (1+t^2-t^3)\cos^2 10t\right)$

$x_e(t) = \frac{1}{2}\left(2(1+t^2)\cos^2 10t\right) = (1+t^2)\cos^2 10t.$

$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) = \frac{1}{2}\left((1+t^2+t^3)\cos^2 10t - (1+t^2-t^3)\cos^2 10t\right)$

$= \frac{1}{2}\left(2t^3\cos^2 10t\right) = t^3\cos^2 10t.$

Ex:- Find whether the following signals are even (or) odd.

(a) $x(t) = e^{-3t}$ (b) $x(t) = u(t+2)$ (c) $x(t) = 3e^{j4\pi t}$

=> (a) $x(t) = e^{-3t}$

$x(-t) = e^{3t}$

$-x(t) = -e^{-3t}.$

Since $x(t) \neq x(-t)$ and $x(-t) \neq -x(t)$, the given signal is neither even signal nor odd signal.

(b) $x(t) = u(t+2)$

$x(-t) = u(-t+2)$

$-x(t) = u(t+2)$.

$u(t+2) = \begin{cases} 1 & \text{for } t \geq -2 \\ 0 & \text{for } t < -2 \end{cases}$

$u(-t+2) = \begin{cases} 1 & \text{for } t \leq 2 \\ 0 & \text{for } t > 2 \end{cases}$

$-u(t+2) = \begin{cases} -1 & \text{for } t \geq -2 \\ 0 & \text{for } t < 2 \end{cases}$

$\therefore$ The given signal is neither even nor odd signal.

(c) $x(t) = 3e^{j4\pi t}$

$x(-t) = 3e^{-j4\pi t}$

$-x(t) = -3e^{j4\pi t}$

Since $x(-t) \neq x(t)$ & $x(-t) \neq -x(t)$ the signal is neither even (nor) odd signal.

Ex- Find the even & odd components of the following signals

(a) $x(n) = \{-3, 1, \underset{\uparrow}{2}, -4, 2\}$ (b) $x(n) = \{-2, 5, \underset{\uparrow}{1}, -3\}$

(c) $x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$ (d) $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$

Sol- (a) $x(n) = \{-3, 1, \underset{\uparrow}{2}, -4, 2\}$

$n = -2, -1, 0, 1, 2$

$x_e(n) = \frac{1}{2}(x(n) + x(-n))$

$x_e(0) = \frac{1}{2}(x(0) + x(-0)) = \frac{1}{2}(2+2) = 2$

$x_e(1) = \frac{1}{2}(x(1) + x(-1)) = \frac{1}{2}(-4+1) = -1.5$

$x_e(2) = \frac{1}{2}(x(2) + x(-2)) = \frac{1}{2}(2-3) = -0.5$

$x_e(n) = \{-0.5, -1.5, \underset{\uparrow}{2}, -1.5, -0.5\}$.

$x_o(n) = \frac{1}{2}(x(n) - x(-n))$

$x_o(0) = \frac{1}{2}(x(0) - x(-0)) = \frac{1}{2}(2-2) = 0$

$x_o(1) = \frac{1}{2}(x(1) - x(-1)) = \frac{1}{2}(-4-1) = -2.5$

$x_o(2) = \frac{1}{2}\left(x(2) - x(-2)\right) = \frac{1}{2}\left(2 - (-3)\right) = 2.5$

$x_o(n) = \{-2.5,\ 2.5,\ \underset{\uparrow}{0},\ -2.5,\ 2.5\}$

(b) $x(n) = \{-2.5,\ \underset{\uparrow}{1},\ -3\}$. $n = -2, -1, 0, 1$

$x_e(n) = \frac{1}{2}\left(x(n) + x(-n)\right)$

$x_e(0) = \frac{1}{2}\left(x(0) + x(-0)\right) = \frac{1}{2}(1+1) = 1$

$x_e(1) = \frac{1}{2}\left(x(1) + x(-1)\right) = \frac{1}{2}(-3+5) = 1$

$x_e(2) = \frac{1}{2}\left(x(2) + x(-2)\right) = \frac{1}{2}(0-2) = -1$

$x_e(n) = \{-1,\ 1,\ \underset{\uparrow}{1},\ 1,\ -1\}$

$x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$

$x_o(0) = \frac{1}{2}\left(x(0) - x(-0)\right) = \frac{1}{2}(1-1) = 0$

$x_o(1) = \frac{1}{2}\left(x(1) - x(-1)\right) = \frac{1}{2}(-3-5) = -4$

$x_o(2) = \frac{1}{2}\left(x(2) - x(-2)\right) = \frac{1}{2}(0+2) = 1$

$x_o(n) = \{-1,\ 4,\ \underset{\uparrow}{0},\ -4,\ 1\}$

(c) $x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$

$x_e(n) = \frac{1}{2}\{x(n) + x(-n)\} = \frac{1}{2}\{1, 2, 3, 4, 5+5, 4, 3, 2, 1\} =$ ~~$\{0.5, 1, 1.5, 2, 0$~~

$x_e(n) = \{0.5, 1, 1.5, 2, \underset{\uparrow}{5}, 2, 1.5, 1, 0.5\}$

$x_o(n) = \frac{1}{2}\{-1, -2, -3, -4, 0, 4, 3, 2, 1\} = \{-0.5, -1, -1.5, -2, \underset{\uparrow}{0}, 2, 1.5, 1, 0.5\}$

(d) $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$

$n = -4, -3, -2, -1, 0$

$x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$ $\quad x(-n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$.

$x_e(n) = \frac{1}{2}\{2.5, 2, 1.5, 1, 1, 1, 1.5, 2, 2.5\}$.

$x_o(n) = \frac{1}{2}(5, 4, 3, 2, 0, -2, -3, -4, -5)$

$x_o(n) = (2.5, 2, 1.5, 1, 0, -1, -1.5, -2, -2.5)$

Ex:- Find whether the Signal

$$x(t) = \begin{cases} t-2 & -2 \le t \le 0 \\ 2-t & 0 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

is energy Signal (or) Power Signal. Also find the energy & power of the signal.

⇝ The given Signal is non-periodic finite duration Signal. So it has finite energy and zero average power. So it is an energy signal.

Energy of the Signal $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$= \int_{-2}^{0} (t-2)^2 dt + \int_{0}^{2} (2-t)^2 dt$$

$$= \int_{-2}^{0} (t^2 - 4t + 4) dt + \int_{0}^{2} (4 + t^2 - 4t) dt$$

$$= \left(\frac{t^3}{3} - \frac{4t^2}{2} + 4t\right)_{-2}^{0} + \left(4t + \frac{t^3}{3} - \frac{4t^2}{2}\right)_{0}^{2}$$

$E = \frac{64}{3}$ Joules.

Power of the Signal $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

$$P = \lim_{T \to \infty} \frac{1}{2T} \left[ \int_{-2}^{0} (t-2)^2 dt + \int_{0}^{2} (2-t)^2 dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left(\frac{64}{3}\right) = 0$$

Since Energy is finite & Power is Zero, it is an Energy Signal.

Ex:- Find which of the following Signals are, Energy Signals, Power Signals, neither energy (nor) power Signals:

(a) $\left(\frac{1}{2}\right)^n u(n)$ (b) $u(n) - u(n-6)$ (c) $n u(n)$. (d) $\cos\left(\frac{\pi}{4} n\right)$

(a) $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Energy of the Signal $E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$ (or) $\sum_{n=-\infty}^{\infty} |x(n)|^2$

$$E = \sum_{n=0}^{\infty} \left|\left(\frac{1}{2}\right)^n\right|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \text{ Joules.}$$

Power of the Signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |x(n)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n = \lim_{N\to\infty} \frac{1}{2N+1} \left(\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\left(\frac{1}{4}\right)}\right)$$

$$P = \frac{1}{\infty} = 0$$

The energy is finite & Power is Zero. $x(n)$ is an Energy Sig

(b) $x(n) = u(n) - u(n-6)$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \sum_{n=-N}^{N} |u(n) - u(n-6)|^2$$

$$E = \sum_{n=0}^{5} 1 = 6 \text{ Joules}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (u(n) - u(n-6))^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{5} 1 = 0.$$

Energy is finite & Power is Zero. Signal is an Energy Sign

(c) $x(n) = n u(n)$.

Energy $E = \lim_{N\to\infty} \sum_{n=-N}^{N} n^2 u(n) = \lim_{N\to\infty} \sum_{n=0}^{N} n^2 = \sum_{n=0}^{\infty} n^2 = \infty$

Power of the Signal $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} n^2 = \infty$

Signal is neither Energy (nor) Power Signal.

(d) $x(n) = \cos \frac{\pi}{4} n$

Energy $E = \lim_{N\to\infty} \sum_{n=-N}^{N} \cos^2 \frac{\pi}{4} n = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1+\cos\frac{\pi}{2}n}{2}$

$$E = \frac{1}{2} \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1+\cos\tfrac{\pi}{2}n\right) = \frac{1}{2} \lim_{N\to\infty} \sum_{n=-N}^{N} 1 = \frac{1}{2} \lim_{N\to\infty} (2N+1) = \infty$$

Power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-1}^{N} \cos^2 \frac{\pi}{4} n = \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} (2N+1) = \frac{1}{2}.$

Power is finite & Energy is $\infty$, so signal is a Power Sign

Ex:- Find whether the following Systems are dynamic (or) not:

(a) $y(t) = x(t-3)$ (b) $y(t) = x(2t)$ (c) $y(n) = x(n+2)$ (d) $y(n) = x^2(n)$

(e) $y(n) = x(n-2) + x(n)$.

Ans (a) $y(t) = x(t-3)$.

o/p depends on past value of input. ∴ System is dynamic.

(b) $y(t) = x(2t)$

$t=0 \quad y(0) = x(0), \quad y(1) = x(2)$

The o/p depends on future value of i/p. The System is dynamic.

(c) $y(n) = x(n+2)$

The o/p depends on future value of i/p. The System is dynamic.

(d) $y(n) = x^2(n)$.

$y(0) = x^2(0), \quad y(1) = x^2(1) \quad y(2) = x^2(2)$

The o/p depends on present value of i/p alone. The system is static.

(e) $y(n) = x(n-2) + x(n)$

The system is described by a Difference Equation. ∴ System is dynamic.

Ex:- Check whether the following Systems are Causal (or) not.

(a) $y(t) = x^2(t) + x(t-4)$ (b) $y(t) = x(2-t) + x(t-4)$ (c) $y(n) = x(2n)$

(a) $y(t) = x^2(t) + x(t-4)$

$t=-2 \quad y(-2) = x^2(-2) + x(-6)$

$t=0 \quad y(0) = x^2(0) + x(-4)$

$t=2 \quad y(2) = x^2(2) + x(-2)$

For all values of 't' o/p depends on present & past values of i/p. System is Causal.

(b) $y(t) = x(2-t) + x(t-4)$

$t=-1, \; y(-1) = x(3) + x(-5)$ For Some values of 't' the o/p depends on the

$t=0, \; y(0) = x(2) + x(-4)$ future input. ∴ the System is Non-Causal.

$t=1, \; y(1) = x(1) + x(-3)$

(c) $y(n) = x(2n)$

$n=-2 \quad y(-2) = x(-2\times 2) = x(-4)$

$n=0 \quad y(0) = x(0)$

$n=2 \quad y(2) = x(2\times 2) = x(4)$

For positive values of n the output depends on the future values of the i/p. The system is non causal.

Ex:- check whether the following systems are linear (or) not.

(a) $y(t) = x(t^2)$ (b) $y(t) = 2x^2(t)$ (c) $y(n) = n^2 x(n)$

(d) $y(n) = 2x(n) + 4$ (e) $y(n) = x(n) + \frac{1}{2x(n-2)}$

Ans (a) $y_1(t) = x(t^2)$

Let an i/p $x_1(t)$ produce an o/p $y_1(t)$

$$y_1(t) = x_1(t^2)$$

i/p $x_2(t)$ produce an o/p $y_2(t)$

$$y_2(t) = x_2(t^2)$$

The linear combination of above equations can be written as

$$\underbrace{a\,y_1(t) + b_2 y_2(t)}_{\text{weighted sum of o/ps}} = \underbrace{a x_1(t^2) + b\,x_2(t^2)}_{\text{weighted sum of i/ps}}$$

∴ The system is linear

(b) $y(t) = 2x^2(t)$.

$y(t) = T(x(t)) = 2x^2(t)$

$y_1(t) = T(x_1(t)) = 2x_1^2(t)$

$y_2(t) = T(x_2(t)) = 2x_2^2(t)$

$ay_1(t) + by_2(t) = a(2x_1^2(t)) + b(2x_2^2(t)) = 2(ax_1^2(t) + bx_2^2(t))$

The o/p due to weighted sum of i/ps is

$y_3(t) = T(ax_1(t) + bx_2(t)) = 2(ax_1(t) + bx_2(t))^2$

$y_3(t) \neq ay_1(t) + by_2(t)$.

(c) $y(n) = n^2 x(n)$

$y_1(n) = n^2 \cdot x_1(n)$

$y_2(n) = n^2 x_2(n)$

$y_3(n) = T(a x_1(n) + b x_2(n)) = n^2 (a x_1(n) + b x_2(n)) = n^2 a x_1(n) + n^2 b x_2($

$a y_1(n) + b y_2(n) = a n^2 x_1(n) + b n^2 x_2(n)$

$y_3(n) = a y_1(n) + b y_2(n)$.

Weighted sum of o/ps equal to the o/p due to weighted sum of i/ps. Super position principle is satisfied. The system is Linear.

(d) $y(n) = 2x(n) + 4$

$y_1(n) = 2x_1(n) + 4$

$y_2(n) = 2x_2(n) + 4$

$a y_1(n) + b y_2(n) = a(2x_1(n) + 4) + b(2x_2(n) + 4) = 2a x_1(n) + 2b x_2(n) + 4a + 4$

$y_3(n) = T(a x_1(n) + b x_2(n)) = 2(a x_1(n) + b x_2(n)) + 4$

$y_3(n) = 2a x_1(n) + 2b x_2(n) + 4$.

$y_3(n) \neq a y_1(n) + b y_2(n) \longrightarrow$ Non-Linear System

(e) $y(n) = x(n) + \frac{1}{2x(n-2)}$

$y_1(n) = x_1(n) + \frac{1}{2x_1(n-2)}$

$y_2(n) = x_2(n) + \frac{1}{2x_2(n-2)}$

$a y_1(n) + b y_2(n) = a\left(x_1(n) + \frac{1}{2x_1(n-2)}\right) + b\left(x_2(n) + \frac{1}{2x_2(n-2)}\right)$

$y_3(n) = T(a x_1(n) + b x_2(n)) = (a x_1(n) + b x_2(n)) + \frac{1}{2(a x_1(n-2) + b x_2(n-2))}$

$y_3(n) \neq a y_1(n) + b y_2(n)$

$\therefore$ System is Non Linear.

Ex- Determine whether the following systems are time invariant or not: (a) $y(t) = t^2 x(t)$ (b) $y(t) = e^{2x(t)}$ (c) $y(n) = x(\frac{n}{2})$ (d) $y(n) = x^2(n-2)$.

~> (a) $y(t) = t^2 x(t)$

$y(t,\tau) = T(x(t-\tau)) = t^2 x(t-\tau)$ → o/p due to i/p delayed by $\tau$ sec

$y(t-\tau) = (t-\tau)^2 x(t-\tau)$ → o/p delayed by T sec

$y(t,\tau) \neq y(t-\tau)$

System is Time variant.

(b) $y(t) = e^{2x(t)}$

$y(t,\tau) = T(x(t-\tau)) = e^{2x(t-\tau)}$

$y(t-\tau) = e^{2x(t-\tau)}$

$y(t,\tau) = y(t-\tau)$

System is Time invariant.

(c) $y(n) = x(\frac{n}{2})$ $y(n,K) = T(x(n-K))$

$y(n,K) = x(\frac{n}{2} - K)$

$y(n-K) = x(\frac{n-K}{2})$

$y(n,K) \neq y(n-K)$

System is Time variant.

(d) $y(n) = x^2(n-2)$

$y(n,K) = T(x(n-K)) = x^2(n-2-K)$

$y(n-K) = x^2(n-K-2)$

$y(n,K) = y(n-K)$

System is Time invariant.

Ex:- Find whether the following systems are stable (or) not

(a) $y(t)=e^{x(t)}$ (b) $y(t)=(t+5)u(t)$ (c) $h(t)=(2+e^{-3t})u(t)$ (d) $h(t)=\frac{1}{RC}e^{-t/RC}u($ (e) $h(n)=a^n \quad 0<n<11$

(a) $y(t)=e^{x(t)} \quad |x(t)|\le 8$

Here the i/p is bounded, $|x(t)|\le 8$

$\therefore$ for stability o/p must be bounded.

$y(t)$ becomes $e^{-8}\le y(t)\le e^{8}$.

Hence $y(t)$ is also bounded. $\therefore$ the system is stable.

(b) $y(t)=(t+5)u(t)$

$y(t)=t+5$ for $t\ge 0$

As $t\to\infty$ $y(t)\to\infty$.

o/p grows without any bound & hence the given system is unstable.

(c) $h(t)=(2+e^{-3t})u(t)$

For a system to be stable

$\int_{-\infty}^{\infty}|h(t)|dt<\infty$

$\int_{-\infty}^{\infty}(2+e^{-3t})u(t)\,dt = \int_{0}^{\infty}(2+e^{-3t})dt=\left(2t+\frac{e^{-3t}}{-3}\right)_0^{\infty}=\infty.$

The impulse Response is absolutely not integrable, since $\int_{-\infty}^{\infty}|h(t)|dt=\infty$, the system is unstable.

(d) $h(t)=\frac{1}{RC}e^{-t/RC}u(t)$

$\int_{-\infty}^{\infty}|h(t)|dt=\int_{-\infty}^{\infty}\left|\frac{1}{RC}e^{-t/RC}u(t)\right|dt=\int_{0}^{\infty}\frac{1}{RC}e^{-t/RC}dt=\frac{1}{RC}(-RC)\left(e^{-t/RC}\right)_0^{\infty}$

$=-\left(e^{-\infty}-e^{0}\right)=-(0-1)=1$

As $\int_{-\infty}^{\infty}|h(t)|dt=1<\infty$ the system is stable

(e) $h(n)=a^n$ for $0<n<11$

$\sum_{n=-\infty}^{\infty}|h(n)|<\infty$ for a system to be stable.

$\therefore \sum_{n=-\infty}^{\infty}|a^n| = \sum_{n=0}^{11}a^n=\frac{1-a^{12}}{1-a}$. If $a$ is finite, the result is finite. So System is stable.

① Evaluate the following integrals:

ⓐ $\int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt$ ⓑ $\int_{0}^{\infty} t^2\delta(t-6)\,dt$ ⓒ $\int_{0}^{3}\delta(t)\sin 5\pi t\,dt$.

ⓓ $\int_{-\infty}^{\infty}\delta(t+2)\,e^{-2t}\,dt$.

ⓐ $\int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt$

$\delta(t-5) = 1$ for $t=5$

$= 0$ elsewhere

$\therefore \int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt = \left[e^{-at^2}\right]_{t=5} = e^{-25a}$

ⓑ $\int_{0}^{\infty} t^2\delta(t-6)\,dt = (t^2)_{t=6} = 36$

ⓒ $\int_{0}^{3}\sin 5\pi t\,\delta(t)\,dt = (\sin 5\pi t)_{t=0} = 0$.

ⓓ $\int_{-\infty}^{\infty}\delta(t+2)\,e^{-2t}\,dt = (e^{-2t})_{t=-2} = e^{4}$.

② Find the following summations

ⓐ $\sum_{n=-\infty}^{\infty} e^{3n}\delta(n-3)$ ⓑ $\sum_{n=-\infty}^{\infty}\delta(n-2)\cos 3n$ ⓒ $\sum_{n=-\infty}^{\infty} n^2\delta(n+4)$

ⓓ $\sum_{n=-\infty}^{\infty}\delta(n-2)\,e^{n^2}$

ⓐ $\sum_{n=-\infty}^{\infty} e^{3n}\delta(n-3)$

$\delta(n-3) = 1$ for $n=3$

$0$ elsewhere

$\sum_{n=-\infty}^{\infty} e^{3n}\delta(n-3) = (e^{3n})_{n=3} = e^{9}$

ⓑ $\sum_{n=-\infty}^{\infty}\delta(n-2)\cos 3n = (\cos 3n)_{n=6} = \cos 6$

ⓒ $\sum_{n=-\infty}^{\infty} n^2\delta(n+4) = (n^2)_{n=-4} = 16$

(d) $\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^n = (e^n)_{n=2} = e^2 = e^4$.

3) Prove the properties of impulse function.

> ① $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$

let $x(t)$ be Continuous Signal, at $t=0$, the value of $x(t)$ at $t=0$ is $x(0)$ ∴ The impulse $\delta(t)$ exists only at $t=0$.
∴ the integration of $x(t)\,\delta(t)$ from $t=-\infty$ to $\infty$ has a value only at $t=0$.

$$\therefore \int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} x(0)\,\delta(t)\,dt = x(0)\int_{-\infty}^{\infty} \delta(t)\,dt = x(0).$$

② $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$.

Let $x(t)$ be Continuous at $t=t_0$. The value of $x(t)$ at $t=t_0$ is $x(t_0)$. $\delta(t-t_0)$ is an impulse function that exists only at $t=t_0$

$$\therefore x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

③ $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$.

$\delta(t-t_0)$ exists only for $t=t_0$, and equal to Zero for all other value of $t$. ∴ the integration of the product term $x(t)\,\delta(t-t_0)$ from $-\infty$ to $\infty$ has a value only at $t=t_0$.

$$\therefore \int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = x(t_0)$$

④ $\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t)$

Let $x(\tau)$ be Continuous at $\tau=t$. Let the value of $x(\tau)$ at $\tau=t$ be $x(t)$. we know that $\delta(t-\tau)=1$ only at $\tau=t$. for all other $\tau$ it is Zero. So the integration of the product term $x(\tau)\,\delta(t-\tau)$ has a value only at $\tau=t$. $\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = (x(\tau))_{\tau=t} = x(t)$

(5) $\delta(at) = \frac{1}{|a|}\delta(t)$ Scaling Property.

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt \quad \text{for } a>0$$

$$at = \tau$$

$$t = \frac{\tau}{a} \qquad dt = \frac{d\tau}{a}$$

If $a>0$ $\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau$

$$= \frac{1}{a}\left(x\left(\frac{\tau}{a}\right)\right)_{\tau=0} = \frac{1}{a}x(0)$$

If $a<0$ $\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{-a}x(0)$

$$\therefore \int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{|a|}x(0) \longrightarrow (A)$$

(00)

$$x(0) = \int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt$$

$$\frac{1}{|a|}x(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} x(t)\frac{1}{|a|}\delta(t)\,dt$$

$$\frac{1}{|a|}x(0) = \int_{-\infty}^{\infty} x(t)\frac{1}{|a|}\delta(t)\,dt \longrightarrow (B)$$

Compare (A) & (B) $\delta(at) = \frac{1}{|a|}\delta(t)$.

(6) $\delta(t) = \delta(-t)$

Impulse function is an even function.

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$a=-1$

$$\delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$$

(4) Show that the Complex Exponential Signal $x(t) = e^{j\omega_0 t}$ is periodic with period $2\pi/\omega_0$.

$\therefore x(t) = e^{j\omega_0 t}$

$x(t+T) = x(t) \rightarrow$ Condition for periodicity

$$e^{j\omega_0(t+T)} = e^{j\omega_0 t}$$

$$e^{j\omega_0 t} e^{j\omega_0 T} = e^{j\omega_0 t}$$

$$e^{j\omega_0 T} = 1$$

$$e^{j\omega_0 T} = e^{j2n\pi}$$

$$\omega_0 T = 2n\pi \Rightarrow T = \frac{2n\pi}{\omega_0}.$$

Thus the fundamental period i.e., the smallest positive $T$ of $x(t)$ is given by $T = \frac{2\pi}{\omega_0}$.

(5) Show that the sinusoidal signal $x(t) = \sin(\omega_0 t + \theta)$ is periodic with period $\frac{2\pi}{\omega_0}$.

$x(t) = \sin(\omega_0 t + \theta)$

$x(t) = x(t+T)$

$x(t+T) = \sin(\omega_0 t + \omega_0 T + \theta) = \sin(\omega_0 t + \theta + \omega_0 T)$

$\omega_0 T = 2n\pi$ for above signal to be periodic

$\omega_0 = \frac{2n\pi}{T}$ (or) $T = \frac{2\pi n}{\omega_0}$.

This shows that the sinusoidal signal $x(t) = \sin(\omega_0 t + \theta)$ is periodic with fundamental period $T = \frac{2\pi}{\omega_0}$.

(6) Let $x_1(t)$ & $x_2(t)$ be periodic signals with fundamental periods $T_1$ and $T_2$ respectively. Under what conditions is the sum ~~$x(t)+x(t)$~~ $x(t) = x_1(t) + x_2(t)$ is periodic and what is the fundamental period of $x(t)$ if it is periodic?

$\Rightarrow$ $x_1(t) = x_1(t+T_1) = x_1(t+mT_1)$, $m$ is a +ve integer.

$x_2(t) = x_2(t+T_2) = x_2(t+KT_2)$, $K$ is a +ve integer.

Given $x(t) = x_1(t) + x_2(t)$

$= x_1(t+mT_1) + x_2(t+KT_2)$

For $x(t)$ to be periodic with period $T$, we require

$x(t) = x(t+T) = x_1(t+T) + x_2(t+T)$

$= x_1(t+mT_1) + x_2(t+KT_2)$

$\therefore mT_1 = KT_2 = T$.

$T_1/T_2 = \frac{K}{m}$ = rational number.

The fundamental period of $T$ of $x(t)$ is LCM of $T_1$ & $T_2$.

$\therefore T = mT_1 = KT_2$

So we can say that the sum of 2 periodic signals is period only if the ratio of their respective periods is a rational numb Then the fundamental period is the LCM of the respective fundame Periods.

(7) Examine whether the following signals are periodic (or) not? If periodic determine the fundamental period?

(a) $\sin 12\pi t$ (b) $e^{-j4\pi t}$ (c) $\sin \pi t\, u(t)$ (d) $e^{-|t|}$ (e) $\cos 2t + \sin \sqrt{3}t$

(f) $3\sin 200\pi t + 4\cos 100t$ (g) $\sin(10t+1) - 2\cos(5t-2)$ (h) $\sin 10\pi t + \cos 20\pi t$

(i) $je^{j6t}$ (j) $3u(t) + 2\sin 2t$. (K) $6e^{j\left(4t+\frac{\pi}{3}\right)} + 8e^{j\left(3\pi t+\frac{\pi}{4}\right)}$

(a) $x(t) = \sin 12\pi t$

Compare with $\sin \omega t$ (or) $\sin \omega t$.

$$\boxed{T = \frac{2\pi}{\omega}}$$

$$\omega = 12\pi \Rightarrow 2\pi f = 12\pi \Rightarrow \frac{2\pi}{T} = 12\pi \Rightarrow T = \frac{1}{6}.$$

$T$ is a Ratio of 2 integers, $x(t)$ is periodic with fundamental period $T = 1/6$ sec.

(b) $x(t) = e^{j4\pi t}$.

Compare with $e^{j\omega t}$.

$$\omega = 4\pi \Rightarrow \frac{2\pi}{T} = 4\pi \Rightarrow T = \frac{1}{2}.$$

Periodic with $T = \frac{1}{2}$ sec.

(c) $x(t) = \sin \pi t \, u(t)$.

$\sin \pi t$ is periodic with period $\omega = \pi$

$$\frac{2\pi}{T} = \pi$$

$T = 2$ seconds.

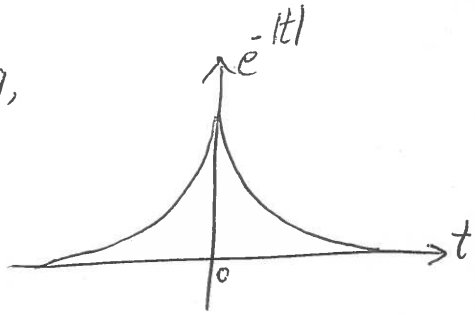
$u(t)$ Exists only b/w $t = 0$ to $t = \infty$. Hence it is not periodic.

The product of periodic & a non-periodic signal is not periodic

(d) $x(t) = e^{-|t|}$

Plot of $x(t) = e^{-|t|}$ versus $t$ is shown in fig,

It is not a periodic signal, because it doesnot repeat at all.




(e) $x(t) = \cos 2t + \sin \sqrt{3} t$.

$$T_1 = \frac{2\pi}{2} = \pi.$$

$$T_2 = \frac{2\pi}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

$$\frac{T_1}{T_2} = \frac{\pi}{2\pi/\sqrt{3}} = \frac{\sqrt{3}}{2}$$ is not a Ratio of 2 integers (not a rational)

$\therefore$ $x(t)$ is Aperiodic

(f) $x(t) = 3\sin 200\pi t + 4\cos 100t$.

$$T_1 = \frac{2\pi}{200\pi} = \frac{1}{100}$$

$$T_2 = \frac{2\pi}{100} = \frac{\pi}{50}.$$

$$\frac{T_1}{T_2} = \frac{\frac{1}{100}}{\frac{\pi}{50}} = \frac{1}{2\pi}$$ is not a Ratio of 2 integers. Hence x(t

is non-periodic.

(g) $x(t) = \sin(10t+1) - 2\cos(5t-2)$

$$x(t) = x_1(t) + x_2(t)$$

$x_1(t) = \sin(10t+1)$ $\quad x_2(t) = \cos(5t-2)$

Compare with $x_1(t) = \sin(\omega_1 t + \theta_1)$

$$\omega_1 = 10 \Rightarrow 2\pi f_1 = 10 \Rightarrow f_1 = \frac{10}{2\pi} = \frac{5}{\pi}$$

$$T_1 = \frac{\pi}{5}\text{ sec}$$

Ily $T_2 = \frac{2\pi}{5}$ sec.

Ratio of $T_1/T_2 = \frac{\pi/5}{2\pi/5} = \frac{1}{2}$ Ratio of 2 integers. (Rational nu

Fundamental period $T = 2T_1 = T_2 = \frac{2\pi}{5}$ sec.

(h) $x(t) = \sin 10\pi t + \cos 20\pi t \Rightarrow \omega_1 = 10\pi$ & $\omega_2 = 20\pi$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{10\pi} = \frac{1}{5}$$

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{20\pi} = \frac{1}{10}.$$

$$\frac{T_1}{T_2} = \frac{\frac{1}{5}}{\frac{1}{10}} = \frac{1}{5} \times \frac{10}{1} = 2$$ Ratio of 2 integers.

$T = T_1 = 2T_2 \Rightarrow$ [illegible].

$T = \frac{1}{5}$ sec is Fundamental period of $x(t)$.

(i) $je^{j6t}$

$\omega = 6 \Rightarrow 2\pi f = 6 \quad (or) \quad f = \frac{3}{\pi}$

$T = \frac{\pi}{3}$.

T is not a Rational number. $x(t)$ is not periodic

(j) $x(t) = 3u(t) + 2\sin 2t$.

$u(t)$ is an Aperiodic

$T_2 = \frac{2\pi}{2} = \pi \Rightarrow \sin 2t$ is periodic with $T = \pi$ sec.

Sum of an Aperiodic & a periodic Signal is aperiodic.

(K) $x(t) = 6e^{j(4t + \frac{\pi}{3})} + 8e^{j(3\pi t + \frac{\pi}{4})}$

$\omega_1 = 4 \Rightarrow 2\pi f_1 = 4 \quad (or) \quad f_1 = \frac{2}{\pi}$.

$T_1 = \frac{\pi}{2}$.

$\omega_2 = 3\pi, \quad T_2 = \frac{2}{3}$.

$\frac{T_1}{T_2} = 3\pi$ is not a Rational, $\therefore$ Signal is not periodic.

# Unit-2

## Fourier Series

**Introduction:-** A Signal is said to be a continuous signal if it is available at all instants of time. A real time naturally available signal is in the form of time domain. However the analysis of the signal is far more convenient in the frequency Domain. There are 3 important classes of transformation methods available for continuous time systems. They are:

(1) Fourier Series (2) Fourier transform (3) Laplace Transform.

out of these 3 methods the fourier series is applicable only to periodic signals i.e., signals which repeat periodically over $-\infty < t < \infty$. Not all periodic signals can be represented by Fourier Series.

The representation of signals over a certain interval of time interms of the linear combination of orthogonal functions is called fourier series. The Fourier analysis is also sometimes called the Harmonic Analysis. Fourier Series is applicable only for periodic signals. It can't be applied to non-periodic signals.

Three important classes of Fourier series methods are available.

(1) Trignometric Form
(2) Cosine Form
(3) Exponential Form.

In the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are Exponential functions, then it is called Exponential Fourier Series

If the orthogonal functions are trignometric functions, then it is called Trignometric fourier series.

## Existence of Fourier Series:-

The conditions under which a periodic signal can be represented by a Fourier series are known as Dirchelets conditions named after the mathematician Dirichlet who 1st found them.

In each period:

(1) The function $x(t)$ must be a single valued function.
(2) The function $x(t)$ has only a finite No. of maxima & minima

(3) The function $x(t)$ has a finite No. of discontinuities

(4) The function $x(t)$ is absolutely integrable over one period that is $\int_0^T |x(t)|\,dt < \infty$

## Representation of a waveform using trigonometric fourier series

Trigonometric Fourier series is the representation of any unknown signal by Cosine & Sine terms.

The Complete set of orthogonal functions is $1, \cos\omega_0 t, \cos 2\omega_0 t$ ----- $\sin\omega_0 t, \sin 2\omega_0 t$ ----- etc.,

Any arbitary function $x(t)$ can be Expressed by the sum of its Components along this set of mutually orthogonal functions

$$x(t) = a_0 + a_1\cos\omega_0 t + a_2\cos 2\omega_0 t + \cdots\cdots + b_1\sin\omega_0 t + b_2\sin 2\omega_0 t + \cdots$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad t_0 \le t \le t_0 + \frac{2\pi}{\omega_0}$$

$$T = \frac{2\pi}{\omega_0}.$$

$$a_n = \frac{\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt}{\int_{t_0}^{t_0+T} \cos^2 n\omega_0 t\,dt} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\,dt}{\int_{t_0}^{t_0+T} \sin^2 n\omega_0 t\,dt} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\,dt$$

&

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt.$$

## Fourier Series Representation of Periodic Signals:-

It is mainly used for representation of periodic waveform. Periodic Signal is nothing but it is a waveform which is repetitive in nature. Any periodic Signal can be Expressed as linear weighted sum of Harmonically related Sinusoidal Signals. This leads to representation by an of Series called the Fourier Series.

$$f(t) = f(t+T)$$

The Fourier Series can be used to perform frequency domain analysis of Signals, in which we can study the various frequency components present in the signal, magnitude & phase characteristics of Signal etc.,

The plot of magnitude (vs) frequency is called a Magnitude Spectrum & the plot of Phase (vs) frequency is called as phase spectrum.

The Signal representation in the frequency Domain is called as Line Spectrum.

An Aperiodic Signal can be viewed as periodic with $T \to \infty$

### Trigonometric Fourier Series :-

The trigonometric form of Fourier Series of a periodic Signal $x(t)$ with time period "T" can be defined as follows:

~~$x(t) = a_0 + \sum_{n=1}^{K}$~~

$$x(t) = a_0 + \sum_{n=1}^{K} a_n \cos n\omega_0 t + \sum_{n=1}^{K} b_n \sin n\omega_0 t$$

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots\cdots\cdots + a_K \cos K\omega_0 t +$$
$$b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots\cdots\cdots + b_K \sin K\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{K} a_n \cos n\omega_0 t + \sum_{n=1}^{K} b_n \sin n\omega_0 t.$$

Where $a_0, a_1, a_2, a_3 \cdots\cdots a_K$ & $b_0, b_1 \cdots\cdots b_K$ are Constants & $\omega_0$ is the fundamental frequency.

For the signal $x(t)$ to be periodic, it must satisfy the condition

$$x(t) = x(t+T)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 n(t+T) + b_n \sin\omega_0 n(t+T)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 n\left(t+\frac{2\pi}{\omega_0}\right) + b_n \sin\omega_0 n\left(t+\frac{2\pi}{\omega_0}\right)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} a_n \cos(\omega_0 nt + 2\pi n) + b_n \sin(\omega_0 nt + 2\pi n)$$

$$x(t+T) = a_0 + \sum_{n=1}^{K} a_n \cos\omega_0 nt + b_n \sin\omega_0 nt$$

The series of sine & cosine terms of frequencies $0, \omega_0, 2\omega_0 \text{----} K\omega_0$ is known as $x(t+T) = x(t)$.

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Trigonometric form of Fourier Series & can be written as:

$a_n$ & $b_n$ are constants; the coefficient $a_0$ is called the dc Compone
$a_1 \cos\omega_0 t + b_1 \sin\omega_0 t$ be the 1st Harmonic, $a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$ the 2nd Harmonic and so on.

The constant $b_0 = 0$ because $\sin n\omega_0 t = 0$ for $n = 0$.

Evaluation of Fourier coefficients of the Trigonometric Fourier Series:—

$a_0, a_1, a_2 \text{----} a_n, b_0, b_1 \text{-----} b_n$ are called Fourier coefficients.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) dt$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n \int \cos n\omega_0 t\,dt + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t\,dt$$

We know that

$$\int_{t_0}^{t_0+T} \cos n\omega_0 t\,dt = \int_{t_0}^{t_0+T} \sin n\omega_0 t\,dt = 0.$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt}$$



To evaluate $a_n$ & $b_n$ we can use the following results:

$$\int_{t_0}^{t_0+T} \cos m\omega_0 t \cos m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \neq 0 \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin m\omega_0 t \sin m\omega_0 t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \neq 0 \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin m\omega_0 t \cos m\omega_0 t\,dt = 0 \quad \text{for all } m \text{ \& } n$$

To find $a_n$, multiply the equation ~~for~~ $x(t)$ by $\cos m\omega_0 t$ & integrate over one period.

$$\int_{t_0}^{t_0+T} x(t)\cos m\omega_0 t\,dt = a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t\,dt + \sum_{n=1} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t\,dt + \sum_{n=1} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t\,dt$$

The 1st & 3rd term in the R.H.S become 0 & second integral is equal to $\frac{T}{2}$ when $m=n$.

$$\int_{t_0}^{t_0+T} x(t)\cos m\omega_0 t\,dt = a_m \frac{T}{2}$$

$$a_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos m\omega_0 t\,dt$$

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt}$$

To find $b_n$ multiply both sides of $x(t)$ by $\sin m\omega_0 t$ & integrate over one period.

$$\int_{t_0}^{t_0+T} x(t)\sin m\omega_0 t\,dt = a_0\int_{t_0}^{t_0+T}\sin m\omega_0 t\,dt + \sum_{n=1}^{\infty} a_n\int_{t_0}^{t_0+T}\cos n\omega_0 t\sin m\omega_0 t\,dt + \sum_{n=1}^{\infty} b_n\int_{t_0}^{t_0+T}\sin n\omega_0 t\sin m\omega_0 t\,dt$$

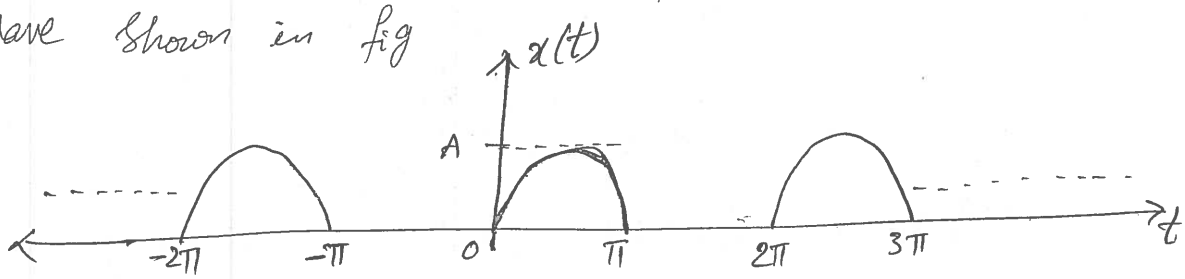
The 1st & 2nd integrals in the above equation are zero & the third integral is equal to $\frac{T}{2}$ when $m=n$.

$$\int_{t_0}^{t_0+T} x(t)\sin m\omega_0 t\,dt = b_m\frac{T}{2}$$

$$b_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin m\omega_0 t\,dt$$

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t\,dt}$$

Ex:- Find the Fourier Series Expansion of the half wave rectified sine wave shown in fig




$$x(t) = \begin{cases} A\sin\omega t = A\sin\frac{2\pi}{2\pi}t = A\sin t & 0\le t\le\pi \\ 0 & \pi\le t\le 2\pi. \end{cases}$$

Now the fundamental period $T=2\pi$.

Fundamental frequency $\omega_0 = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$t_0 = 0 \quad t_0+T = 0+2\pi = 2\pi.$

$$a_0 = \frac{1}{T}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} A\sin t\,dt = \frac{A}{2\pi}\left(-\cos t\right)_0^{\pi} = \frac{A}{2\pi}(1+1) = \frac{2A}{2\pi}$$

$$a_0 = A/\pi$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\cos nt\,dt$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} A\sin t\cos nt\,dt = \frac{A}{\pi}\int_0^{\pi}\sin t\cos nt\,dt$$

$$a_n = \frac{A}{2\pi}\int_0^{\pi}\left(\sin(1+n)t + \sin(1-n)t\right)dt$$

$$a_n = \frac{A}{2\pi}\left[\frac{-\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$a_n = -\frac{A}{2\pi}\left[\frac{\cos(1+n)\pi - \cos 0}{1+n} + \frac{\cos(1-n)\pi - \cos 0}{1-n}\right]$$

$$a_n = -\frac{A}{2\pi}\left\{\left[\frac{(-1)^{n+1}-1}{1+n}\right] + \left[\frac{(-1)^{n-1}-1}{1-n}\right]\right\}.$$

For $n$ odd, $a_n = 0$

For $n$ even, $a_n = \frac{-2A}{\pi(n^2-1)}$ for $n$ even

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} A\sin t\sin nt\,dt = \frac{A}{\pi}\int_0^{\pi}\sin t\sin nt\,dt$$

$$b_n = \frac{A}{2\pi}\int_0^{\pi}\left[\cos(n-1)t - \cos(n+1)t\right]dt = \frac{A}{2\pi}\left(\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right)_0^{\pi}$$

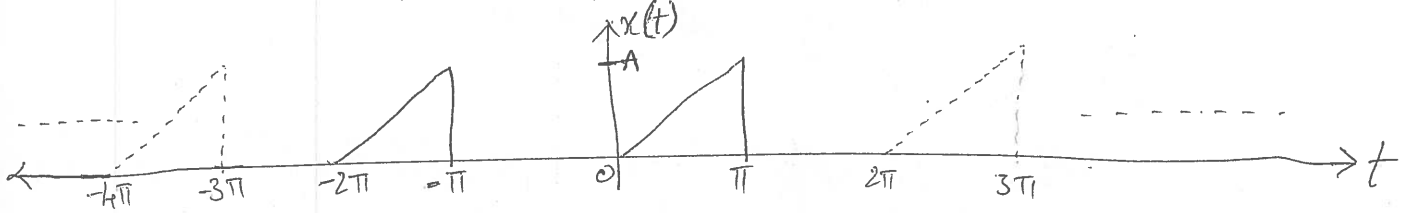
$b_n = 0$ for all values of $n$ except for $n=1$

For $n=1$ $\quad b_1 = \frac{A}{2\pi}\cdot b_1 = \frac{A}{2}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega_0 t + b_n\sin n\omega_0 t = a_0 + b_1\sin t + \sum_{n=1}^{\infty} a_n\cos nt$$

$$x(t) = \frac{A}{\pi} + \frac{A}{2}\sin t - \sum_{n=\text{even}}^{\infty}\frac{2A}{\pi(n^2-1)}\cos nt.$$

Ex:- Obtain the trigonometric fourier series for the waveform shown in fig




$t_0 = 0 \quad t_0 + T = 2\pi$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$x(t) = \begin{cases} \frac{A}{\pi} t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi \le t \le 2\pi. \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} \frac{A}{\pi} t\,dt = \frac{A}{2\pi^2}\left(\frac{t^2}{2}\right)_0^{\pi} = \frac{A}{4}.$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt. = \frac{2}{2\pi}\int_0^{\pi} \frac{A}{\pi} t\cos nt\,dt = \frac{A}{\pi^2}\int_0^{\pi} t\cos nt\,dt$$

$$a_n = \frac{A}{\pi^2 n^2}(\cos n\pi - \cos 0)$$

$$a_n = \begin{cases} -\left(\frac{2A}{\pi^2 n^2}\right) & \text{for } n\text{-odd}. \\ 0 & \text{for } n\text{-even}. \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} \frac{A}{\pi} t\sin nt\,dt$$

$$b_n = \frac{A}{\pi^2}\int_0^{\pi} t\sin nt\,dt = \frac{A}{\pi^2}\left\{\left(\frac{t(-\cos nt)}{n}\right)_0^{\pi} - \int_0^{\pi} \frac{-\cos nt}{n}\,dt\right\}$$

$$b_n = \frac{A}{\pi^2}\left\{-\pi\frac{\cos n\pi}{n} + \left(\frac{\sin nt}{n^2}\right)_0^{\pi}\right\} = \frac{-A}{n\pi}\cos n\pi = \frac{A}{n\pi}(-1)^{n+1}$$

$$b_n = \begin{cases} A/n\pi & \text{for odd } n \\ -\left(\frac{A}{n\pi}\right) & \text{for even } n. \end{cases}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = \frac{A}{4} - \frac{2A}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{\cos nt}{n^2} + \frac{A}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$$

$$x(t) = \frac{A}{4} - \frac{2A}{\pi^2}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right) + \frac{A}{\pi}\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right)$$

## Cosine Representation (Alternate form of the trigonometric representation)



$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2+b_n^2}\left(\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin n\omega_0 t\right)$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n\left(\cos\theta_n \cos n\omega_0 t - \sin\theta_n \sin n\omega_0 t\right)$$

$A_0 = a_0$

$A_n = \sqrt{a_n^2+b_n^2}$, $\cos\theta_n = \dfrac{a_n}{\sqrt{a_n^2+b_n^2}}$, $\sin\theta_n = \dfrac{-b_n}{\sqrt{a_n^2+b_n^2}}$

$\theta_n = -\tan^{-1}\left(\dfrac{b_n}{a_n}\right)$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n).$$

This is the cosine representation of $x(t)$ which contains the sinusoids of frequencies upto $2\omega_0$ ----------.

$A_0 \rightarrow$ DC Component

$A_n \cos(n\omega_0 t + \theta_n) \rightarrow$ The $n^{th}$ harmonic component of $x(t)$

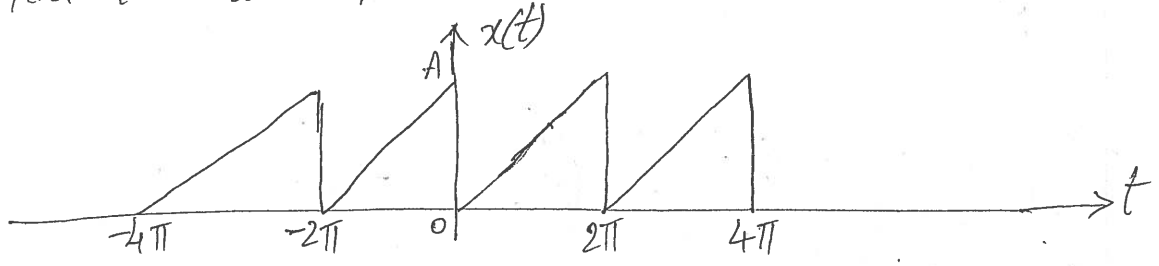
$A_1 \cos(\omega_0 t + \theta_1)$ is called the fundamental frequency component.

$A_n \rightarrow$ Amplitude coefficients (or) Harmonic Amplitudes (or) Spectral Amplitudes of the Fourier series.

$\theta_n \rightarrow$ phase coefficients (or) phase angles. (or) Spectral phase of the Fourier series.

Cosine form is also called the Harmonic form Fourier series (or) polar form Fourier series.

Ex:- Find the cosine Fourier series for the wave form shown in fig.



5 $x(t) = \frac{A}{2\pi} t$, for $0 \le t \le 2\pi$

$t_0 = 0, \quad t_0 + T = 2\pi \qquad T = 2\pi$

Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$

$$a_0 = \frac{1}{T}\int_0^{2\pi} \frac{A}{2\pi} t \, dt = \frac{A}{2\pi T}\int_0^{2\pi} t \, dt = \frac{A}{2\pi \times 2\pi}\left(\frac{t^2}{2}\right)_0^{2\pi}$$

$$a_0 = \frac{A}{2\times 4\pi^2}\left(4\pi^2 - 0\right) = \frac{A}{2}.$$

$$a_n = \frac{2}{T}\int_{t_0}^{t+T} x(t)\cos n\omega_0 t \, dt = \frac{2}{2\pi}\int_0^{2\pi} \frac{A}{2\pi} t \cos n\omega_0 t \, dt$$

$\int uv\,dx = u\int v\,dx - \int u' \int v\,dx\,dx.$

$$a_n = \frac{2A}{2\pi\times 2\pi}\int_0^{2\pi} t\cos n\omega_0 t \, dt = \frac{A}{2\pi^2}\left[\left\{\frac{t\sin nt}{n}\right\}_0^{2\pi} - \int_0^{2\pi}\frac{\sin nt}{n}\,dt\right]$$

$$a_n = \frac{A}{2\pi^2}\left[\left\{\frac{t\sin nt}{n}\right\}_0^{2\pi} - \int_0^{2\pi}\frac{\sin nt}{n}\,dt\right]$$

$$a_n = \frac{A}{2\pi^2}\left[(0-0) + \frac{1}{n}\left(\frac{\cos nt}{n}\right)_0^{2\pi}\right] = \frac{A}{2\pi^2}\left(\frac{1}{n^2}(1-1)\right) = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin n\omega_0 t \, dt = \frac{2}{2\pi}\int_0^{2\pi}\frac{At}{2\pi}\sin nt \, dt$$

$$b_n = \frac{2A}{4\pi^2}\int t\sin nt \, dt = \frac{A}{2\pi^2}\left(\left(\frac{t(-\cos nt)}{n}\right)_0^{2\pi} - \int_0^{2\pi}\left(\frac{-\cos nt}{n}\right)dt\right)$$

$$b_n = \frac{A}{2\pi^2}\left(\left(-\frac{t\cos nt}{n}\right)_0^{2\pi} + \frac{1}{n}\left(\frac{\sin nt}{n}\right)_0^{2\pi}\right) = \frac{A}{2\pi^2}\left(\frac{-1}{n}(2\pi - 0) + \frac{1}{n^2}(0-0)\right) = \frac{-A}{\pi n}.$$

$\therefore a_0 = \frac{A}{2} \qquad a_n = 0 \qquad b_n = \frac{-A}{\pi n}$

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left( \frac{-A}{n\pi} \sin nt \right) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n} \qquad 0 \le t \le 2\pi$$

$$A_0 = a_0 = \frac{A}{2}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0^2 + \left(\frac{-A}{n\pi}\right)^2} = \frac{A}{n\pi}$$

$$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) = -\tan^{-1}\left(\frac{-A/n\pi}{0}\right) = +\frac{\pi}{2}$$

$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \cos\left(nt + \frac{\pi}{2}\right)$ is the cosine representation.

Wave Symmetry:

If the periodic signal $x(t)$ has some type of symmetry, then some of the trigonometric fourier series coefficients may become zero & calculation of coefficients becomes simple.

These are following four types of symmetry a function $x(t)$ can have:

① Even Symmetry ② odd Symmetry ③ Half wave Symmetry

④ Quarter wave Symmetry.

If $x(t)$ has Even Symmetry (also called mirror Symmetry) then $b_n = 0$ and only $a_0$ & $a_n$ are to be evaluated.

If $x(t)$ has odd symmetry (also called Rotation Symmetry), then $a_0$ & $a_n$ both are equal to zero ($a_0 = 0$ & $a_n = 0$) only $b_n$ is to be evaluated.

If $x(t)$ has half wave Symmetry, then $a_0 = 0$ & only odd harmonics Exist.

If $x(t)$ has Quarter wave Symmetry, then $a_0 = 0$ & only either $a_n$ (or) only $b_n$ Exists, that too only for odd values of $n$.

$x(t) = x(-t) \rightarrow$ Even function $\qquad x_e(t) = \frac{x(t) + x(-t)}{2}$

$x(t) = -x(-t) \rightarrow$ odd function. $\qquad x_o(t) = \frac{x(t) - x(-t)}{2}$.

We choose the interval of integration from $-\frac{T}{2}$ to $\frac{T}{2}$ instead of from $t_0$ to $t_0 + T$.

odd function × odd function = Even Function.
Even Function × Even function = Even Function.
odd function × Even Function = odd function.

For Any Even function $x_e(t)$

$$\int_{-\frac{T}{2}}^{T/2} x_e(t)\,dt = 2\int_{0}^{T/2} x_e(t)\,dt.$$

For Any odd function $x_o(t)$

$$\int_{-T/2}^{T/2} x_o(t)\,dt = 0.$$

Based on these relationships we can easily calculate Fourier Series coefficients.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt$$

① For Even Symmetry $x_o(t)=0$ $\left(x(t) = x_e(t)+x_o(t) = x_e(t)\right)$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x_e(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} x_e(t)\cos n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{T/2} x(t)\cos n\omega_0 t\,dt$$

Ily.

$$a_0 = \frac{2}{T}\int_{0}^{T/2} x(t)\,dt \quad \& \quad b_n = 0.$$

② For odd Symmetry $x_e(t)=0$ $x(t) = x_o(t)$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{2}{T}\left(2\int_{-T/2}^{T/2} x_o(t)\sin n\omega_0 t\,dt\right)$$

$$b_n = \frac{2}{T}\left(2\int_{0}^{T/2} x_o(t)\sin n\omega_0 t\,dt\right) = \frac{4}{T}\int_{0}^{T/2} x(t)\sin n\omega_0 t\,dt.$$

(3) For Half wave Symmetry

$$x(t) = -x\left(t \pm \frac{T}{2}\right)$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt \quad (n\text{-odd}) \;\&\; a_n = 0 \quad (n\text{-even})$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\,dt \quad (n\text{-odd}) \;\&\; b_n = 0 \quad (n\text{-even}).$$

(4) For Quatter wave Symmetry

$x(t) = x(-t)$ (or) $x(t) = -x(-t)$ and also $x(t) = -x\left(t \pm \frac{T}{2}\right)$

Case 1:- $x(t) = -x(-t)$ & $x(t) = -x\left(t \pm \frac{T}{2}\right)$

$a_0 = 0$, $a_n = 0$ $\quad b_n = \frac{8}{T}\int_0^{T/4} x(t)\sin n\omega_0 t\,dt$ ($n$ is odd)

Case 2:- $x(t) = x(-t)$ & $x(t) = -x\left(t \pm \frac{T}{2}\right)$

$a_0 = 0$, $b_n = 0$ (when $n$ is even) $\quad a_n = \frac{8}{T}\int_0^{T/4} x(t)\cos n\omega_0 t\,dt.$

Exponential Fourier Series:-

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2}\left(e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right)$$

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{jn\omega_0 t} e^{j\theta_n}\right) + \sum_{n=1}^{\infty} \frac{A_n}{2}\left(e^{-jn\omega_0 t} e^{j\theta_n}\right)$$

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} e^{j\theta_n}\right) e^{j\omega_0 n t} + \sum_{K=-1}^{\infty}\left(\frac{A_K}{2} e^{j\theta_K}\right) e^{jK\omega_0 t}$$

$A_n = A_K$; $\quad -\theta_n = \theta_K \quad n > 0$

Let us define

$$C_0 = A_0 \qquad C_n = \frac{A_n}{2} e^{j\theta_n} \quad n > 0$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j\theta_n} e^{j\omega_0 n t} + \sum_{n=-1}^{\infty} \frac{A_n}{2} e^{j\theta_n} e^{jn\omega_0 t}$$

Synthesis Equation

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

$C_0 = A_0$

$C_n = \frac{A_n}{2} e^{j\theta_n}$

## Determination of the coefficients of Exponential Fourier Series:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T}$$

$$\boxed{C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt} \longrightarrow \text{Analysis Equation.}$$

$$C_0 = A_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt.$$

The magnitude line spectrum is always even fn of n
Phase line spectrum is always an odd function of n.

## Trigonometric Fourier Series from Exponential Fourier Series:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$a_0 = C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = j(C_n - C_{-n}).$$

## Exponential Fourier Series from Trigonometric Fourier Series:-

$$C_0 = a_0$$

$$C_n = \frac{1}{2}(a_n - jb_n)$$

$$C_{-n} = \frac{1}{2}(a_n + jb_n).$$

## Cosine Fourier Series from Exponential Fourier Series:-

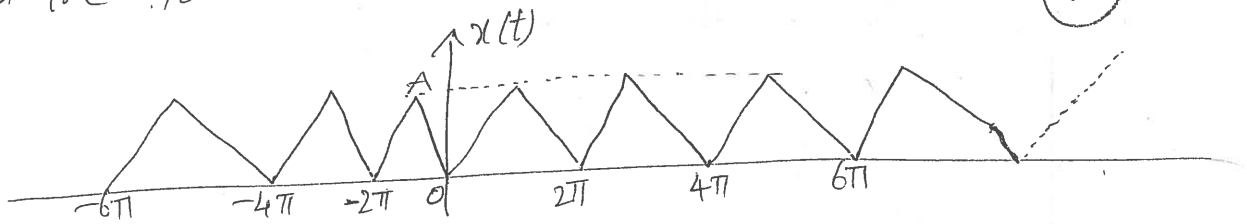
$$A_0 = a_0 = C_0$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{(C_n + C_{-n})^2 + (j(C_n - C_{-n}))^2} = \sqrt{4 C_n C_{-n}}$$

$$A_n = 2|C_n|$$

$$A_0 = C_0 \quad \& \quad A_n = 2|C_n|$$

1) Find the Fourier Series Expansion for the waveform




$T = 2\pi$

$$x(t) = \begin{cases} \frac{A}{\pi}t & 0 \le t \le \pi \\ -\frac{A}{\pi}t & \pi \le t \le 2\pi \end{cases}$$

The given wave form has even symmetry because $x(t) = x(-t)$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt, \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt \text{ & } b_n = 0.$$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{2\pi}\int_0^{\pi} \frac{A}{\pi}t\,dt = \frac{A}{\pi^2}\int_0^{\pi} t\,dt = \frac{A}{\pi^2}\left(\frac{t^2}{2}\right)_0^{\pi}$$

$$a_0 = \frac{A}{\pi^2}\left(\frac{\pi^2 - 0}{2}\right) = \frac{A}{2}.$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt = \frac{4}{2\pi}\int_0^{\pi} \frac{A}{\pi}t\cos nt\,dt = \frac{2A}{\pi^2}\left(\left(\frac{t\sin nt}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}\,dt\right)$$

$$a_n = \frac{2A}{\pi^2 n^2}(\cos n\pi - \cos 0) = \frac{2A}{n^2\pi^2}((-1)^n - 1)$$

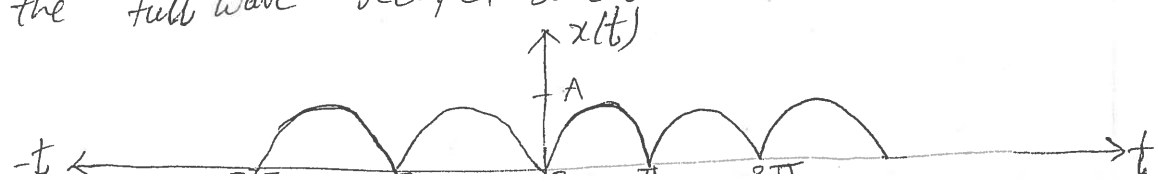
$$a_n = \begin{cases} -\frac{4A}{n^2\pi^2} & \text{for odd } n. \\ 0 & \text{for even } n. \end{cases}$$

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$x(t) = \frac{A}{2} + \sum_{n=\text{odd}} \frac{-4A}{n^2\pi^2}\cos nt = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=\text{odd}}\frac{\cos nt}{n^2}$$

2) Find the Exponential Fourier Series & plot the frequency spectrum for the full wave rectified Sine wave



Fourier Spectrum:-

The Fourier Spectrum of a periodic Signal $x(t)$ is a plot of its Fourier coefficients verses frequency $\omega$. It is in two parts:

(a) Amplitude Spectrum
(b) phase Spectrum.

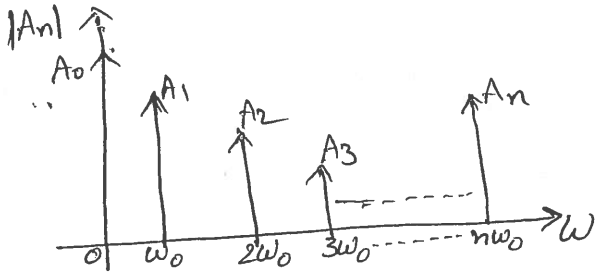
The plot of amplitude of x Fourier series coefficients (Vs) frequency is known as the amplitude spectra and the plot of the phase of Fourier coefficients Vs frequency is known as the phase spectra.

The two plots together are known as Fourier Frequency Spectra of $x(t)$. This type of representation is ~~known as~~ also called Frequency Domain representation. The Fourier Spectrum Exists only at discrete frequencies $n\omega_0$, where $n=0, 1, 2----$. Hence it is know as Discrete spectrum (or) line Spectrum. The envelope of the Spectrum depends only upon the Pulse shape, but not upon the period of repetetion.

Trigonometric ~~Re~~ representation of a periodic signal $x(t)$ contains both Cosine & Sine terms with +ve & -ve amplitude coefficients ($a_n$ & $b_n$) but with no phase angles.

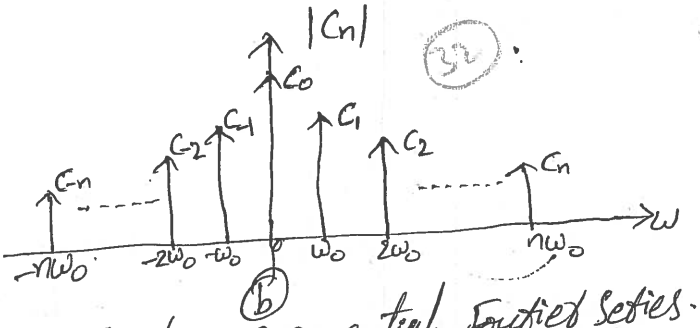
The Cosine representation of a periodic signal contains only +ve amplitude coefficients with phase angle $\theta_n$. $\therefore$ we can draw amplitude Spectra ($A_n$ Vs $\omega$) and phase Spectra ($\theta_n$ Vs $\omega$). Since in this representation Fourier coefficients Exist only for +ve frequencies, this spectra is called Single sided Spectra.

The Exponential representation of a periodic signal $x(t)$ contains amplitude coefficients $c_n$ which are complex. Hence they can be represented by magnitude & phase. $\therefore$ we can plot two spectra; The Magnitude Spectrum $|c_n|$ (Vs) $\omega$ and phase spectrum $\angle c_n$ Vs $\omega$. The spectra can be plotted for both +ve & -ve frequencies. Hence it is called a 2-sided spectra.




(a)

Complex Frequency Spectrum for Trigonometric Fourier Series




(b)

Complex Exponential Fourier Series.

Fig(a) represents the Spectrum of a Trigonometric Fourier series extending from 0 to ∞, producing a one-sided Spectrum as no -ve frequencies Exist here.

Fig(b) represents the spectrum of a Complex Exponential Fourier Series extending from -∞ to ∞, producing a two sided Spectrum.

The amplitude Spectrum of the Exponential Fourier series is Symmetrical about the vertical axis. This is true for all periodic functions.

If $C_n$ is a general Complex number then

$$C_n = |C_n| e^{j\theta_n}$$

$$C_{-n} = |C_n| e^{-j\theta_n}$$

$$|C_n| = |C_{-n}|.$$

When $x(t)$ is real $C_{-n} = C_n^*$, the Complex Conjugate of $C_n$.

The Magnitude spectrum Exhibits even Symmetry & phase spectrum Exhibits odd Symmetry.

(1) Obtain the Exponential Fourier Series for the waveform Shown in fig. Also Draw the frequency Spectrum.

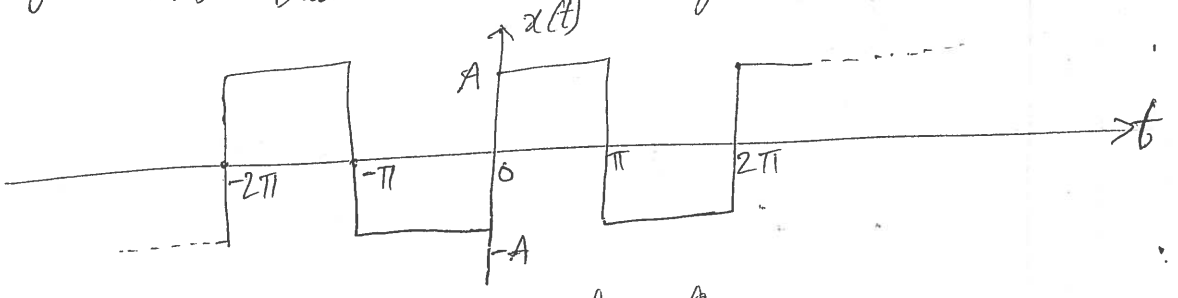



fig:- Waveform

$x(t) = A$ for $0 \le t \le \pi$

$= -A$ $\pi \le t \le 2\pi$

$t_0 = 0$ $t_0 + T = 2\pi$ $T = 2\pi$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Exponential Fourier Series

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t} \quad (\text{or}) \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\left(\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} -A\,dt\right)$$

$$C_0 = \frac{A}{2\pi}\left((t)_0^{\pi} - (t)_{\pi}^{2\pi}\right) = 0$$

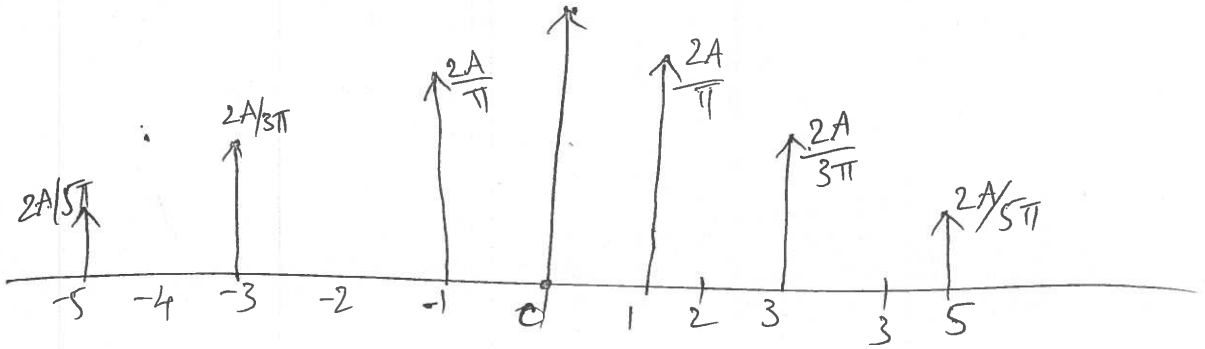
$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{2\pi}\left(\int_0^{\pi} A e^{-jnt}\,dt + \int_{\pi}^{2\pi} -A e^{-jnt}\,dt\right)$$

$C_n = -\frac{j2A}{n\pi}$ for $n$ odd

$= 0$ for $n$ even.

$$x(t) = 0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -\frac{j2A}{n\pi} e^{jnt} \quad \text{for } n\text{-odd.}$$

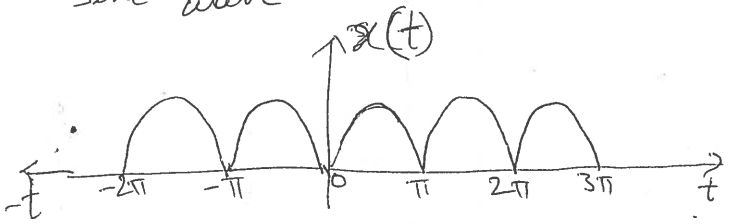
$C_0 = 0$ $C_1 = C_{-1} = \frac{2A}{\pi}$, $C_3 = C_{-3} = \frac{2A}{3\pi}$, $C_5 = C_{-5} = \frac{2A}{5\pi}$



(2) Ans. $x(t) = A\sin\omega t$

(33) Find the Exponential Fourier series & plot the Frequency Spectrum for the full wave rectified Sine wave

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1.$$




$x(t) = A\sin t \quad 0 \le t \le \pi.$

$t_0 = 0$

$t_0 + T = 0 + \pi = \pi$

Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{\pi}\int_0^{\pi} A\sin t\, e^{-j2nt}$$

$$C_n = \frac{A}{\pi}\int_0^{\pi} \sin t\, e^{-j2nt}\, dt = \frac{A}{\pi}\int_0^{\pi} \sin t\, e^{-j2nt}\, dt$$

$$C_n = \frac{A}{\pi}\int_0^{\pi}\left(\frac{e^{jt} - e^{-jt}}{2j}\right)\left(e^{-j2nt}\right)dt = \frac{A}{j2\pi}\left(\int_0^{\pi} e^{j(1-2n)t} - e^{-j(1+2n)t}\right)dt$$

$$C_n = \frac{A}{j2\pi}\left(\frac{e^{j(1-2n)t}}{j(1-2n)} - \frac{e^{-j(1+2n)t}}{-j(1+2n)}\right)_0^{\pi} = \frac{A}{j2\pi}\left[\frac{e^{j(1-2n)\pi} - e^{0}}{j(1-2n)} - \frac{e^{-j(1+2n)\pi} - e^{0}}{-j(1+2n)}\right]$$

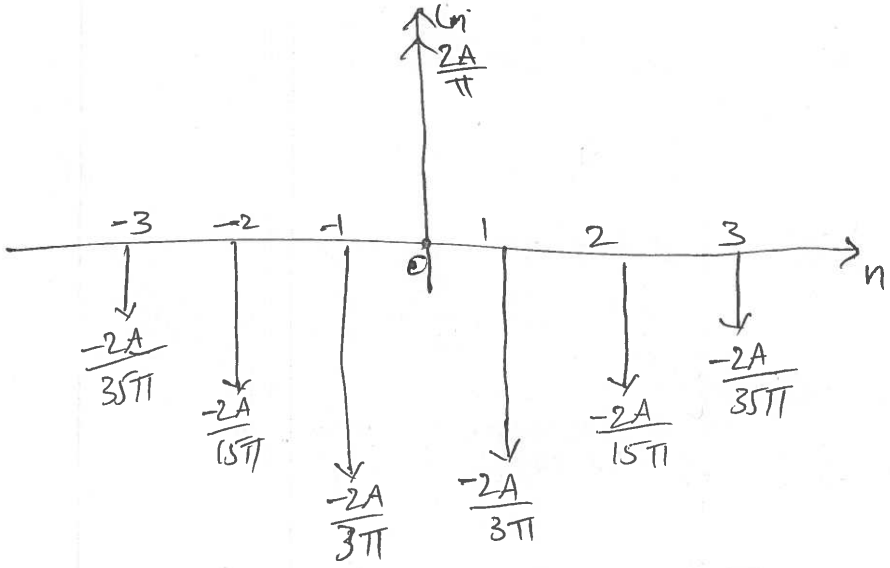
$$C_n = \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_0^{T} x(t)\,dt = \frac{1}{\pi}\int_0^{\pi} A\sin t\, dt = \frac{A}{\pi}(-\cos t)_0^{\pi} = \frac{2A}{\pi}.$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2nt} = \frac{2A}{\pi} + \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{e^{j2nt}}{1-4n^2}\right)$$

$$C_0 = \frac{2A}{\pi}$$

$$C_1 = C_{-1} = \frac{-2A}{3\pi} \qquad C_2 = C_{-2} = \frac{-2A}{15\pi} \qquad C_3 = C_{-3} = \frac{-2A}{35\pi}$$


Cn
2A/π
-3
-2
-1
0
1
2
3
n
-2A/35π
-2A/15π
-2A/3π
-2A/3π
-2A/15π
-2A/35π

# Properties of Continuous time Fourier Series :-

## (1) Linearity :-

If $x_1(t) \xleftrightarrow{F.S} C_n$ and $x_2(t) \xleftrightarrow{F.S} D_n$ } $C_n$ & $D_n$ are Fourier Series coefficients

then $A x_1(t) + B x_2(t) \xleftrightarrow{FS} A C_n + B D_n$.

**Proof :-**

$$FS\{A x_1(t) + B x_2(t)\} = \frac{1}{T}\int_{t_0}^{t_0+T} (A x_1(t) + B x_2(t)) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\left\{ A\int_{t_0}^{t_0+T} x_1(t) e^{-jn\omega_0 t} dt + B\int_{t_0}^{t_0+T} x_2(t) e^{-jn\omega_0 t} dt \right\}$$

$$= \frac{A}{T}\int_{t_0}^{t_0+T} x_1(t) e^{-jn\omega_0 t} dt + \frac{B}{T}\int_{t_0}^{t_0+T} x_2(t) e^{-jn\omega_0 t} dt$$

$$= A C_n + B D_n.$$

$$A x_1(t) + B x_2(t) \xleftrightarrow{FS} A C_n + B D_n.$$

## (2) Time shifting :-

$$x(t) \xleftrightarrow{FS} C_n$$

$$x(t-t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} C_n$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x(t-t_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 (t-t_0)} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} e^{-jn\omega_0 t_0}$$

$$x(t-t_0) = \sum_{n=-\infty}^{\infty} \left(C_n e^{-jn\omega_0 t_0}\right) e^{jn\omega_0 t} = FS^{-1}\left(C_n e^{-jn\omega_0 t_0}\right)$$

$$FS\{x(t-t_0)\} = C_n e^{-jn\omega_0 t_0}.$$

## (3) Time Reversal :-

$$x(t) \xleftrightarrow{FS} C_n$$

$$x(-t) \longleftrightarrow C_{-n}$$

**Proof :-** $x(t) = \sum C_n e^{jn\omega_0 t}$ =

$$x(-t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0(-t)}$$

Substitute $n = -p$ in the R.H.S

$$x(-t) = \sum_{p=-\infty}^{-\infty} C_{-p} e^{-jp\omega_0(-t)} = \sum_{p=-\infty}^{\infty} C_{-p} e^{jp\omega_0 t}$$

p with n

$$x(-t) = \sum_{n=-\infty}^{\infty} C_{-n} e^{jn\omega_0 t} = FS^{-1}(C_{-n})$$

$$FS(x(-t)) \longleftrightarrow C_{-n}$$

(4) Time Scaling:-

$$x(t) \xleftrightarrow{FS} C_n$$

$$x(\alpha t) \xleftrightarrow{FS} C_n \quad \text{with } \omega_0 \longrightarrow \alpha\omega_0.$$

Proof:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 \alpha t} = \sum_{n=-\infty}^{\infty} C_n e^{jn(\alpha\omega_0 t)} = FS^{-1}(C_n)$$

where $\omega_0 \longrightarrow \alpha\omega_0$.

$$x(\alpha t) \xleftrightarrow{FS} C_n$$

(5) Time Differentiation:-

$$x(t) \xleftrightarrow{FS} C_n$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS} jn\omega_0 C_n.$$

proof:-

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\frac{d}{dt}x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}(jn\omega_0) = \sum_{n=-\infty}^{\infty} (jn\omega_0 C_n) e^{jn\omega_0 t} = FS^{-1}(jn\omega_0 C_n)$$

$$FS\left\{\frac{d}{dt}x(t)\right\} = jn\omega_0 C_n$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS} jn\omega_0 C_n.$$

(6) Time Integration:-

$$x(t) \xleftrightarrow{FS} C_n$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FS} \frac{c_n}{jn\omega_0} \quad (\text{if } c_0 = 0).$$

**Proof:-**

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} \left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0\tau}\right] d\tau$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{t} e^{jn\omega_0\tau}\,d\tau = \sum_{n=-\infty}^{\infty} c_n \left(\frac{e^{jn\omega_0\tau}}{jn\omega_0}\right)_{-\infty}^{t}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{c_n}{jn\omega_0}\right) e^{jn\omega_0 t} = FS^{-1}\left(\frac{c_n}{jn\omega_0}\right)$$

$$\therefore \int_{-\infty}^{\infty} x(\tau)\,d\tau \xleftrightarrow{FS} \frac{c_n}{jn\omega_0}, \quad c_0 = 0 \ldots$$

⑦ **Conjugation & Conjugate Symmetry Property:-**

$$x(t) \xleftrightarrow{FS} c_n$$

$$x^*(t) \xleftrightarrow{FS} c_n^* \quad (\text{for complex } x(t))$$

$$c_{-n} = c_n^* \quad (\text{for real } x(t))$$

**Proof:-**

$$FS(x^*(t)) = \frac{1}{T}\int_{t_0}^{t_0+T} x^*(t)\, e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{t_0}^{t_0+T} \left(x(t)\, e^{jn\omega_0 t}\,dt\right)^*$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} \left(x(t)\, e^{-j(-n)\omega_0 t}\,dt\right)^* = \left(c_{-n}\right)^*$$

$$x^*(t) \xleftrightarrow{FS} c_{-n}^*.$$

⑧ **Parseval's Relation (or) Theorem (or) Property:-**

$$x_1(t) \xleftrightarrow{FS} c_n \quad (\text{for complex } x_1(t))$$

$$x_2(t) \xleftrightarrow{FS} D_n \quad (\text{for complex } x_2(t))$$

then the parsevals relation states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t)\, dt = \sum_{n=-\infty}^{\infty} C_n D_n^*$$

and parsevals identity states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \text{if } x_1(t) = x_2(t) = x(t).$$

Proof:-

$$LH.S = \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2^*(t) dt = \frac{1}{T}\int_{t_0}^{t_0+T} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right) \cdot x_2^*(t) dt$$

Interchanging the order of integration & Summation

$$= \sum_{n=-\infty}^{\infty} C_n \left(\frac{1}{T}\int_{t_0}^{t_0+T} x_2^*(t)\, e^{jn\omega_0 t} dt\right) = \sum_{n=-\infty}^{\infty} C_n \left(\frac{1}{T}\int_{t_0}^{t_0+T} x_2(t)\, e^{-jn\omega_0 t} dt\right)^*$$

$$= \sum_{n=-\infty}^{\infty} C_n D_n^*$$

Parsevals identity

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, x^*(t) dt = \sum_{n=-\infty}^{\infty} C_n C_n^*$$

$|x(t)|^2 = x(t)\, x^*(t)$ & $|C_n|^2 = C_n C_n^*$

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2. \quad \text{proved.}$$

Convolution Theorem:- $x_1(t) \xleftrightarrow{FS} C_n$ & $x_2(t) \xleftrightarrow{FS} D_n$

$x_1(t) * x_2(t) \xleftrightarrow{FS} T C_n D_n$.

$$FS\{x_1(t) * x_2(t)\} = \frac{1}{T}\int_{t_0}^{t_0+T} (x_1(t) * x_2(t)) e^{jn\omega_0 t} dt = \frac{1}{T}\int_0^T x_1(t) * x_2(t)\, e^{-jn\omega_0 t} dt$$

$$= T(C_n)(D_n)$$

$x_1(t) * x_2(t) = \int_0^T x_1(\tau)\, x_2(t-\tau) d\tau$ (or) $\int_0^T x_1(t-\tau)\, x_2(\tau) d\tau$

$$x_1(t) * x_2(t) = \int_0^T x_1(t-\tau)\, x_2(\tau)\, d\tau$$

$$FS\{x_1(t) * x_2(t)\} = \frac{1}{T}\int_0^T \int_0^T x_1(\tau)\, x_2(t-\tau)\, e^{-jn\omega_0 t}\, dt.$$

Interchanging the order of integration, we get

$$FS\{x_1(t) * x_2(t)\} = \frac{1}{T}\int_0^T x_1(\tau) \int_0^T x_2(t-\tau)\, e^{-jn\omega_0 t}\, dt$$

$t - \tau = t'$ in RHS we have $dt = dt'$.

$$FS\{x_1(t) * x_2(t)\} = \frac{1}{T}\int_0^T x_1(\tau)\left(\int_{-\tau}^{T-\tau} x_2(t')\, e^{-jn\omega_0(t'+\tau)}\, dt'\right) d\tau$$

$$= T\left\{\left(\frac{1}{T}\int_0^T x_1(\tau) e^{-jn\omega_0\tau}\, d\tau\right)\left(\frac{1}{T}\int_{-\tau}^{T-\tau} x_2(t') e^{-jn\omega_0 t'}\right)\right\}$$

$$FS\{x_1(t) * x_2(t)\} = T(C_n D_n).$$

Modulation (or) Multiplication property :-

$$x_1(t)\, x_2(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} C_l D_{n-l}$$

Proof :-

$$FS\{x_1(t)\, x_2(t)\} = \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, x_2(t)\, e^{-jn\omega_0 t}\, dt$$

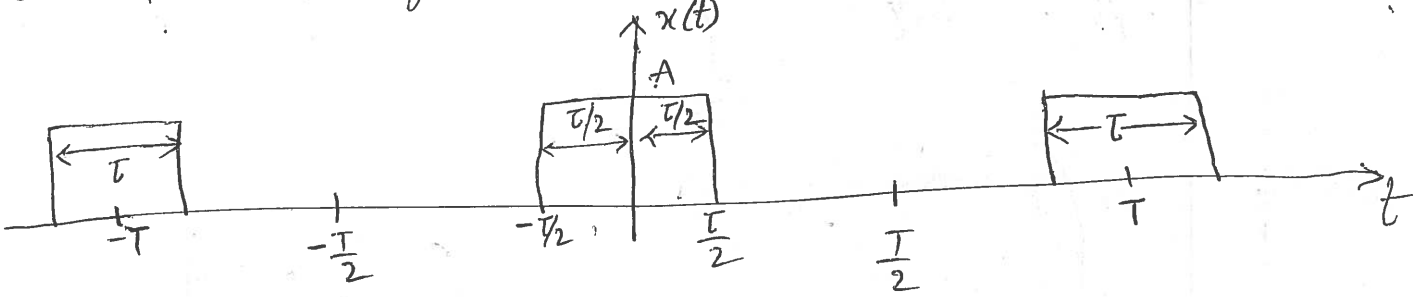
$$= \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\left(\sum_{l=-\infty}^{\infty} C_l e^{jl\omega_0 t}\right) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\left(\sum_{l=-\infty}^{\infty} C_l e^{-j(n-l)\omega_0 t}\, dt\right)$$

Interchange the order of Integ & Summation

$$FS\{x_1(t)\, x_2(t)\} = \sum_{l=-\infty}^{\infty} C_l\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\, e^{-j(n-l)\omega_0 t}\, dt\right) = \sum_{l=-\infty}^{\infty} C_l D_{n-l}.$$

① For the periodic gate function shown in fig.
(a) Find the trignometric fourier series
(b) Derive the Exponential Fourier Series
(c) Find the Exponential Fourier series directly
(d) plot the magnitude & phase spectra.




$$\Rightarrow \quad x(t) = A \qquad -\frac{\tau}{2} \le t \le \frac{\tau}{2}$$

$$= 0 \qquad \text{else}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$t_0 = -T/2 \qquad t_0 + T = \frac{T}{2}.$$

$$\omega_0 = \frac{2\pi}{T}.$$

(a) $$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A\tau}{T}.$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_{-\tau/2}^{\tau/2} A\cos n\omega_0 t\,dt = \frac{2A}{T}\tau\,\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)$$

$$b_n = 0.$$

$$x(t) = \frac{A\tau}{T} + \sum_{n=1}^{\infty}\frac{2A}{T}\tau\,\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)\cos n\omega_0 t.$$

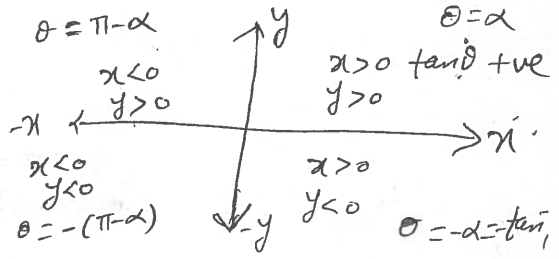
(b) $$c_0 = a_0 = \frac{A\tau}{T}$$

$$c_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2}\left(\frac{2A}{T}\tau\,\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)\right) = \frac{A}{T}\tau\,\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)$$

$$x(t) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{jn\omega_0 t} = \frac{A\tau}{T} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{A}{T}\tau\,\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)e^{jn\omega_0 t}$$

$$x(t) = \frac{A\tau}{T} + \frac{A}{T}\tau\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\text{Sinc}\left(n\omega_0\frac{\tau}{2}\right)e^{jn\omega_0 t}$$

(c) $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$




$$C_0 = a_0 = \frac{A\tau}{T}$$

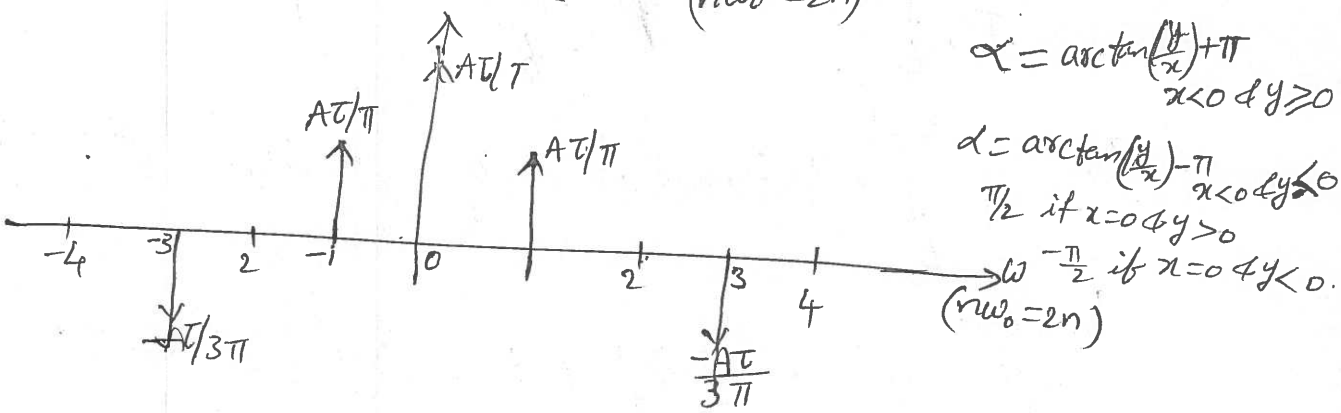
$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, e^{-jn\omega_0 t}\, dt = \frac{A}{T}\left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right)_{-\tau/2}^{\tau/2}$$

$$C_n = \frac{A}{T}\left(\frac{e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}}{-jn\omega_0}\right) = \frac{2A}{n\omega_0 T}\left(\frac{e^{jn\omega_0(\tau/2)} - e^{-jn\omega_0(\tau/2)}}{2j}\right)$$

$$C_n = \frac{A}{T}\left(\frac{\sin n\omega_0(\tau/2)}{n\omega_0(\tau/2)}\right)\tau = \frac{A}{T}\tau\, \mathrm{Sinc}(n\omega_0\tau/2)$$

$$x(t) = \frac{A}{T}\tau + \sum_{n=-\infty}^{\infty} \frac{A\tau}{T}\, \mathrm{Sinc}\left(n\omega_0\frac{\tau}{2}\right) e^{jn\omega_0 t}.$$

(d) Amplitude & Phase Spectra:- $(n\omega_0 = 2n)$

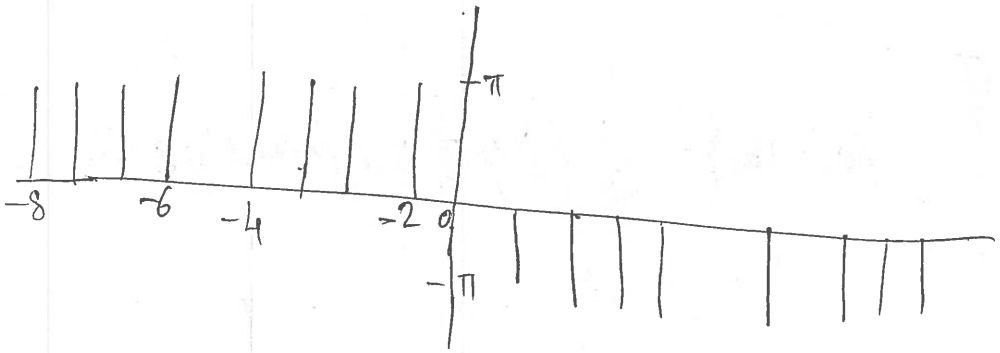



$\alpha = \arctan\left(\frac{y}{x}\right) + \pi$ $x<0$ & $y \geq 0$

$\alpha = \arctan\left(\frac{y}{x}\right) - \pi$ $x<0$ & $y<0$

$\pi/2$ if $x=0$ & $y>0$

$-\frac{\pi}{2}$ if $x=0$ & $y<0$.



# Fourier Transforms

## Introduction:-

We have studied that Fourier Series is applicable for an analysis of periodic Signal. Any periodic waveform can be very well represented using fourier Series. We can either use Trigonometric (or) Exponential form of Fourier Series. For Analysis of periodic Signal, it is necessary to Consider one period of the wave form.

For Aperiodic Signal the time period is ideally $\infty$. That means we may Consider an aperiodic Signal is repeating after the $\infty$ time for its Analysis. Hence fourier transform is a tool for an Analysis of a aperiodic Signal.

## Developing Fourier Transform from Fourier Series:-

The Fourier series can be represented in the Exponential form as follows:

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt \qquad \omega_0 = \frac{2\pi}{T}$$

Let $x(t)$ be a Non-periodic function & $x_T(t)$ be periodic with and let their relation is given by

$$x(t) = \lim_{T\to\infty} x_T(t)$$

$$TC_n = \int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t}\, dt$$

Let $n\omega_0 = \omega$ at $T\to\infty$. As $T\to\infty$ we have $\omega_0 = \frac{2\pi}{T} \to 0$ & the discrete spectrum (Fourier spectrum) becomes Continuous & the Summation becomes integral & $x_T(t) \to x(t)$

Thus as $T \to \infty$,

$$TC_n = \lim_{T\to\infty} \int_{-T/2}^{T/2} x_T(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \left( \lim_{T\to\infty} x_T(t) \right) e^{-j\omega t}\, dt$$

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

$X(\omega)$ represents the frequency spectrum of $x(t)$ & is called the Spectral density function. $X(\omega)$ is called the fourier Transform (or) the fourier integral of $x(t)$.

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t} \qquad \left( TC_n = X(\omega) \right)$$

$$x_T(t) = \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T}\, e^{jn\omega_0 t}$$

$$x_T(t) = \sum \frac{X(\omega)}{\frac{2\pi}{\omega_0}}\, e^{jn\omega_0 t}$$

$$x_T(t) = \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi}\, e^{jn\omega_0 t} (\omega_0)$$

$$x(t) = \lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \left( \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi}\, e^{jn\omega_0 t} (\omega_0) \right)$$

As $T \to \infty$ $\omega_0 = \frac{2\pi}{T} \to$ becomes infinitely small & may be represented by ~~$X(\omega)\, x(t)$~~ $d\omega$. Also the summation becomes integration.

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi}\, e^{jn\omega_0 t} (\omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega .$$

F. $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = F\{x(t)\}$$

$$x(t) = F^{-1}\{X(\omega)\}.$$

(or)

$$x(t) \longleftrightarrow X(\omega).$$

Magnitude & phase Representation of Fourier Transform:-

Magnitude & phase representation of the F.T is the tool used to analyse the transformed Signal.

In general $X(\omega)$ is a Complex valued function of $\omega$. $\therefore X(\omega)$ can be written as:

$$X(\omega) = X_R(\omega) + jX_I(\omega).$$

$X_R(\omega)$ is the real part of $X(\omega)$

$X_I(\omega)$ is the imaginary part of $X(\omega)$

The magnitude of $X(\omega)$ is given by

$$|X(\omega)| = \sqrt{(X_R(\omega))^2 + (X_I(\omega))^2}$$

The phase of $X(\omega)$ is given by

$$\angle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right)$$

The plot of $|X(\omega)|$ (VS) $\omega$ is known as "Amplitude Spectrum" and the plot of $\angle X(\omega)$ (VS) $\omega$ is known as "phase Spectrum". The Amplitude & phase spectrum together is called Frequency Spectrum.

## Existence of Fourier Transforms:-

The Fourier Transform doesnot Exist for all Aperiodic functions. The Conditions for a function $x(t)$ to have Fourier transform is called Dirchlets Conditions, are:

(1) $x(t)$ is absolutely integrable over the interval $-\infty$ to $\infty$, that is

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty.$$

(2) $x(t)$ has a finite No. of discontinuities in every finite time interval. Further each of these discontinuities must be finite.

(3) $x(t)$ has a finite No. of maxima & minima in every finite time interval.

Almost all the signals that we come across in physical problems satisfy all the above conditions Except possibly the absolute integrable Condition.

Dirchlets Condition is a sufficient but not necessary Condition.

## Fourier Transforms of Standard Signals:-

### Impulse function:-

$$\delta(t) = 1 \quad \text{for } t=0$$
$$0 \quad \text{for } t\neq 0$$

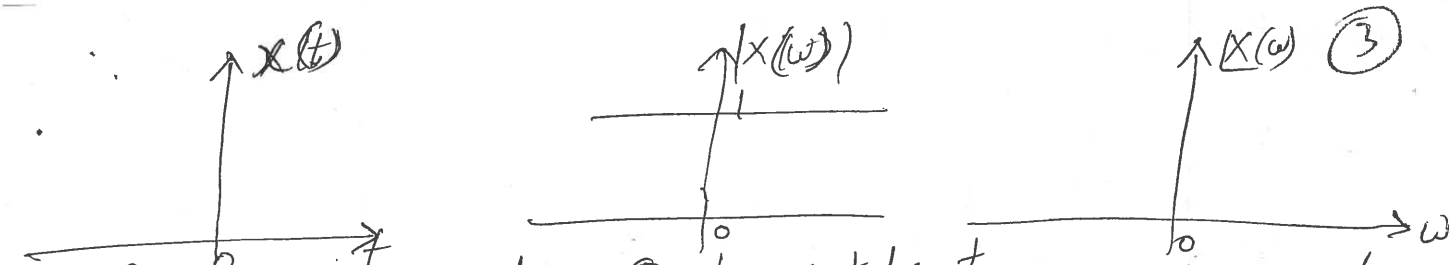
Then $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\,dt$

$$X(\omega) = e^{-j\omega t}\Big|_{t=0} = 1.$$

$$F\{\delta(t)\} = 1 \quad (\text{or}) \quad \delta(t) \longleftrightarrow 1$$

Hence the Fourier transform of a unit impulse function is unity.

$$|X(\omega)| = 1 \text{ for all } \omega$$
$$\angle X(\omega) = 0 \text{ for all } \omega$$




(a) I/P signal-unit impulse (b) Its magnitude spectrum (c) phase spectrum

Ily $F\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-j\omega t}\, dt = e^{-j\omega t_0}$

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}.$$

(2) Single sided Real Exponential function $e^{-at}u(t)$:-

$$x(t) = e^{-at}\, u(t)$$

$u(t) = 1$ for $t \geq 0$

$= 0$ for $t < 0$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{0}^{\infty} e^{-t(a+j\omega)}\, dt = \left(\frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\right)_0^{\infty}$$

$$X(\omega) = \frac{-1}{(a+j\omega)}\left(e^{-\infty} - e^{0}\right) = \frac{1}{a+j\omega}.$$

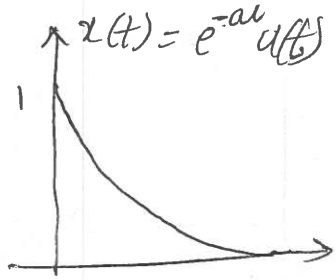
$$F\{e^{-at}u(t)\} = \frac{1}{a+j\omega} \qquad \boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}}$$

$$X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2}$$

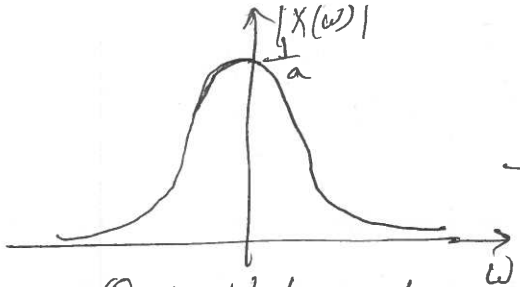
$$X(\omega) = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \sqrt{\frac{(a^2+\omega^2)}{(a^2+\omega^2)^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

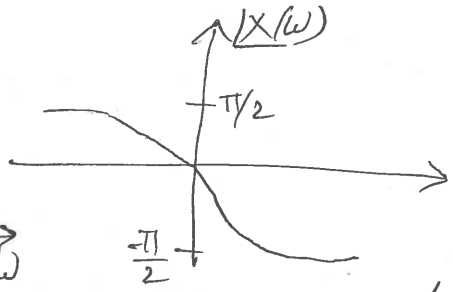
$$\angle X(\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right) \text{ for all } \omega.$$




(a) one sided Exponential




(b) Amplitude Spectrum




(c) phase spectrum.

Double sided Exponential function $e^{-a|t|}$ :-

$$x(t) = e^{-a|t|}$$

$$x(t) = e^{-a|t|} = \begin{cases} e^{-a(-t)} = e^{at} & \text{for } t \le 0 \\ e^{-at} & \text{for } t > 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

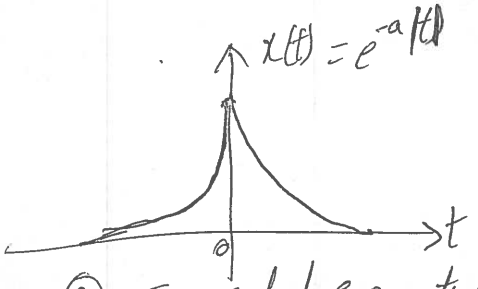
$$X(\omega) = \int_{-\infty}^{0} e^{t(a-j\omega)}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$\Rightarrow X(\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

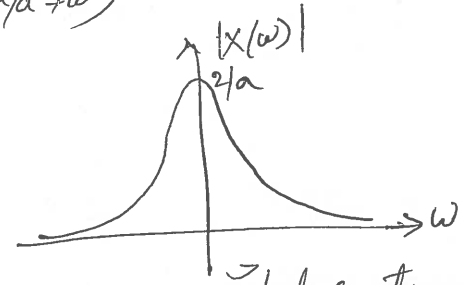
$$F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}$$

$$|X(\omega)| = \frac{2a}{a^2+\omega^2} \quad \forall \omega$$

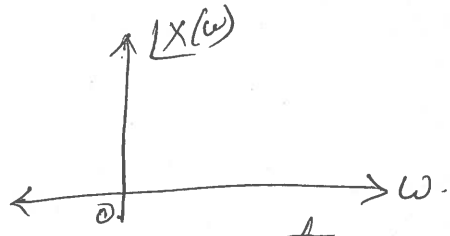
$$\angle X(\omega) = \tan^{-1}\left(\frac{0}{2a/a^2+\omega^2}\right) = \tan^{-1}(0) = 0 \quad \forall \omega$$




(a) Two sided Exponential




(b) Magnitude Spectrum




(c) phase spectrum.

Complex Exponential function $e^{j\omega_0 t}$ :-

To find the fourier transform of complex Exponential function $e^{j\omega_0 t}$, Consider finding the inverse Fourier transform of $\delta(\omega-\omega_0)$.

$\therefore X(\omega) = \delta(\omega - \omega_0)$

$$x(t) = F^{-1}\{X(\omega)\} = F^{-1}\{\delta(\omega-\omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$F^{-1}\{\delta(\omega-\omega_0)\} = \frac{e^{j\omega_0 t}}{2\pi} \quad (\text{or}) \quad F^{-1}\{2\pi\delta(\omega-\omega_0)\} = e^{j\omega_0 t}$$

$$F\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0).$$

(or)

$$F\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$F\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t}\, dt = \left(\frac{e^{j(\omega_0-\omega)t}}{j(\omega_0-\omega)}\right)_{-\infty}^{\infty}$$

Constant Amplitude (1):-

Consider $X(\omega) = \delta(\omega)$

$$\delta(\omega) = \begin{cases} 1 & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0 \end{cases}$$

$$x(t) = F^{-1}\{X(\omega)\} = F^{-1}(\delta(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}(e^{j\omega t})_{\omega=0} = \frac{1}{2\pi}(e^0) = \frac{1}{2\pi}(1) = \frac{1}{2\pi}$$

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \qquad F^{-1}\{2\pi\delta(\omega)\} = 1$$

$$F\{1\} = 2\pi\delta(\omega)$$

$$F\{A\} = 2\pi A\delta(\omega).$$

Alternate method:-

Find the Inverse Fourier Transform of $X(\omega) = \delta(\omega) = 1$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$F^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi}\left(e^{j\omega t}\right)\Big|_{\omega=0} = \frac{1}{2\pi}(e^0) = \frac{1}{2\pi}$$

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}$$

$$F^{-1}\{2\pi\delta(\omega)\} = 1$$

$$2\pi\delta(\omega) = F\{1\}.$$

$$F\{1\} = 2\pi\delta(\omega)$$

Signum Function Sgn(t):-

Signum function is denoted by sgn(t) & is defined by

$$Sgn(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$$

This function is not absolutely integrable. So we can't find its Fourier Transform.

∴ Let us Consider the signal (or) function $x(t) = e^{-a|t|}Sgn(t)$ & substitute lt $a \to 0$ to obtain the above Sgn(t).

$$x(t) = Sgn(t) = \lim_{a\to 0} e^{-a|t|}Sgn(t) = \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right).$$

$$X(\omega) = F\{Sgn(t)\} = \int_{-\infty}^{\infty} \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)e^{-j\omega t}\,dt$$

$$= \lim_{a\to 0}\int_{-\infty}^{\infty}\left(e^{-at}u(t) - e^{at}u(-t)\right)e^{-j\omega t}\,dt$$

$$= \lim_{a\to 0}\left\{\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)\,dt - \int_{-\infty}^{\infty} e^{at}u(-t)e^{-j\omega t}\,dt\right\}$$

$$= \lim_{a\to 0}\left\{\int_{0}^{\infty} e^{-(at+j\omega t)}\,dt - \int_{-\infty}^{0} e^{(at-j\omega t)}\,dt\right\} = \lim_{a\to 0}\left\{\int_{0}^{\infty} e^{-(at+j\omega)t}\,dt - \int_{0}^{\infty} e^{-(a-j\omega)t}\,dt\right\}$$

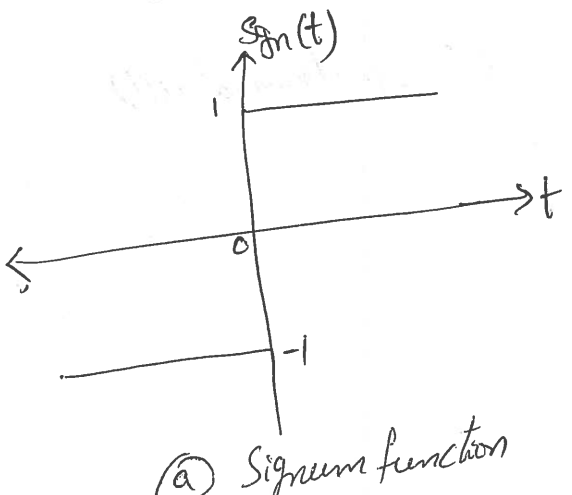
$$\therefore X(\omega) = \lim_{a\to 0}\left(\left(\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right)_0^{\infty} - \left(\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right)_0^{\infty}\right)$$

$$X(\omega) = \lim_{a\to 0}\left\{\left(\frac{e^{\infty} - e^{0}}{-(a+j\omega)}\right) - \left(\frac{e^{-\infty} - e^{0}}{-(a-j\omega)}\right)\right\} = \lim_{a\to 0}\left\{\frac{1}{a+j\omega} + \frac{1}{-(a-j\omega)}\right\}$$

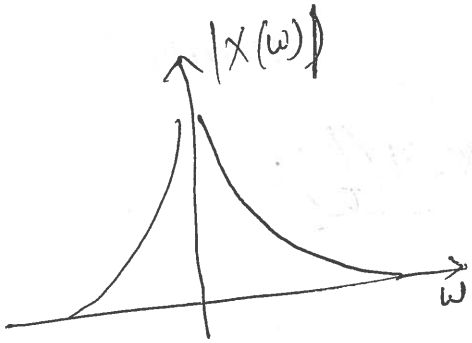
$$X(\omega) = \lim_{a\to 0}\left\{\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right\} = \lim_{a\to 0}\left\{\frac{a-j\omega-(a+j\omega)}{a^2+\omega^2}\right\}$$

$$X(\omega) = \lim_{a\to 0}\left\{\frac{-2j\omega}{a^2+\omega^2}\right) = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

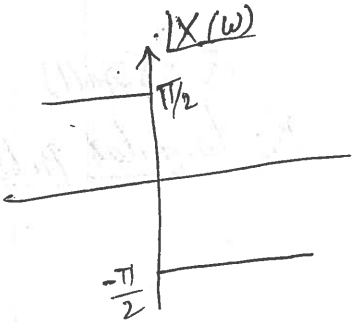
$$F\{Sgn(t)\} = \frac{2}{j\omega}$$




(a) Signum function




(b) Amplitude Spectrum




(c) Phase Spectrum

Unit step function:-

unit step function is defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Since the unit step is not absolutely integrable, we can't directly find its Fourier transform. So express unit step function in terms of Signum function. So $u(t) = \frac{1}{2} + \frac{1}{2}Sgn(t)$

$$x(t) = u(t) = \frac{1}{2}(1 + Sgn(t))$$

$$X(\omega) = F(u(t)) = F\left\{\frac{1}{2}(1 + Sgn(t))\right\}$$

$$X(\omega) = \frac{1}{2}\{F(1) + F(Sgn(t))\}$$

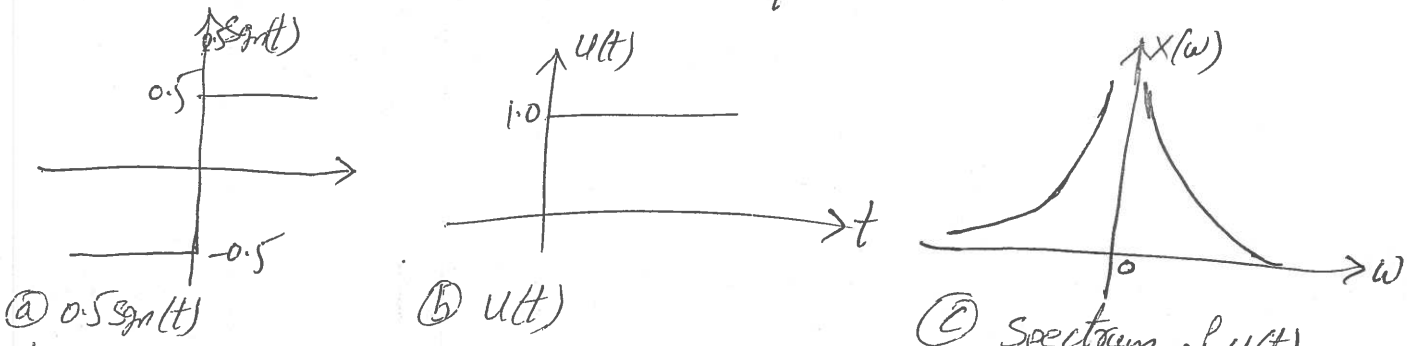
$$F(1) = 2\pi\delta(\omega)$$

$$F\{Sgn(t)\} = \frac{2}{j\omega}$$

$$F\{u(t)\} = \frac{1}{2}\left(2\pi\delta(\omega) + \frac{2}{j\omega}\right) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$\therefore |X(\omega)| = \infty$ at $\omega = 0$ & is equal to 0 at $\omega = -\infty$ & $\omega = \infty$.




ⓐ $0.5\,Sgn(t)$ ⓑ $u(t)$ ⓒ Spectrum of $u(t)$.

Rectangular Pulse $\Pi\left(\frac{t}{\tau}\right)$:-

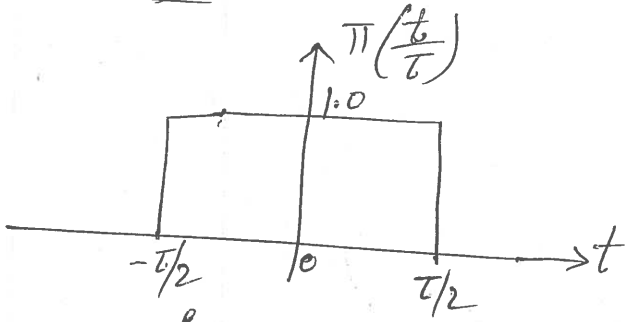



fig:- Rectangular pulse (or) unit gate function.

$$x(t) = rect\left(\frac{t}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

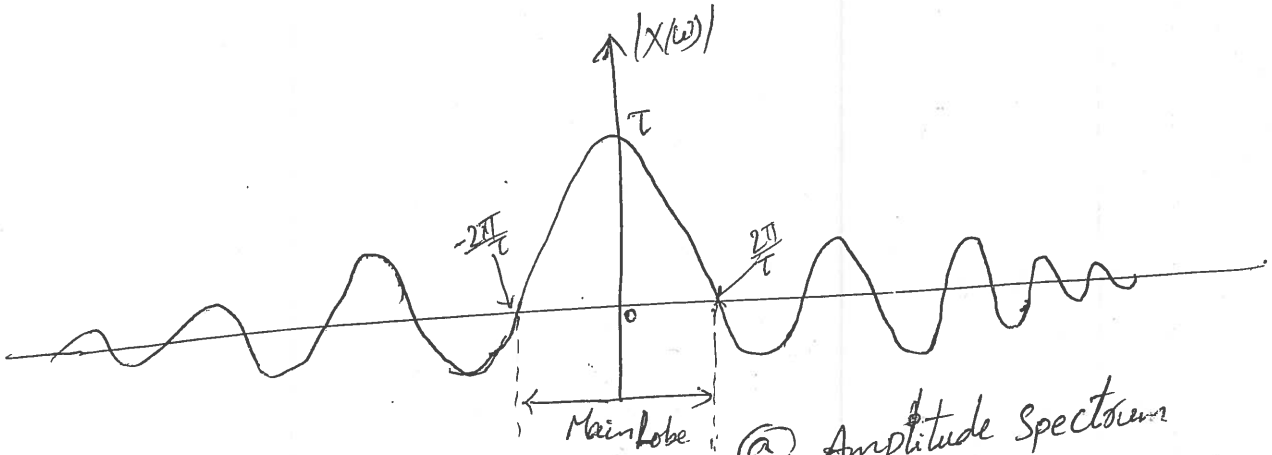
$$X(\omega) = F\left\{\Pi\left(\frac{t}{\tau}\right)\right\} = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} (1)\, e^{-j\omega t}\, dt = \left(\frac{e^{-j\omega t}}{-j\omega}\right)_{-\tau/2}^{\tau/2} = \frac{e^{-j\omega\tau/2} - e^{+j\omega\tau/2}}{-j\omega}$$

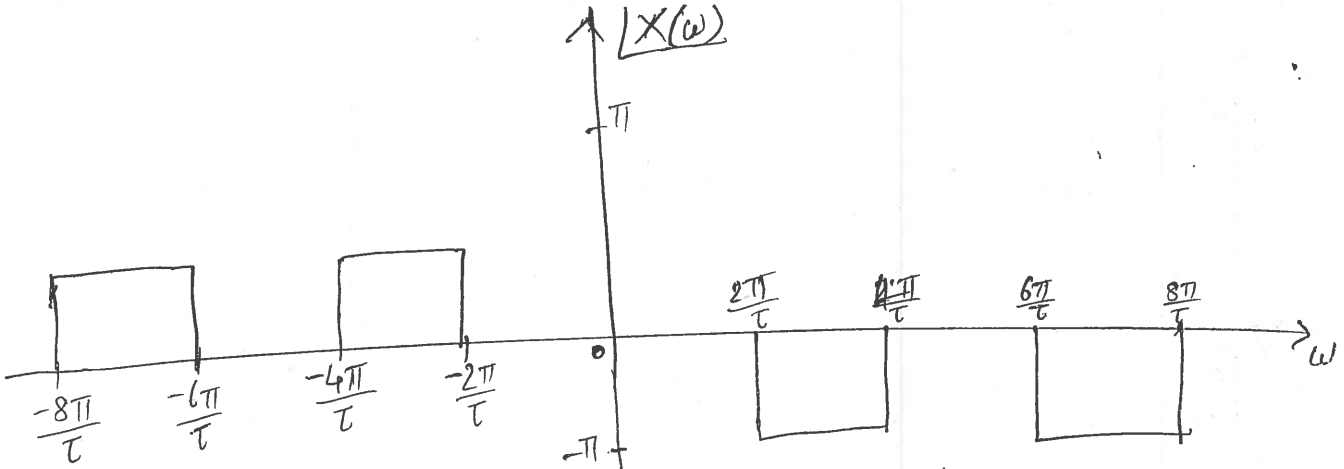
$$X(\omega) = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} = \frac{2\sin\omega(\tau/2)}{\omega}$$

$$= \frac{2\sin\frac{\omega\tau}{2}}{\omega} = \frac{2\sin\frac{\omega\tau}{2}}{\frac{2}{\tau}\cdot\frac{\omega\tau}{2}} = \tau\left(\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}\right) = \tau\,\mathrm{Sinc}\frac{\omega\tau}{2}$$

$$F\left\{\mathrm{rect}\left(\frac{t}{\tau}\right)\right\} = \pi\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\mathrm{Sinc}\left(\frac{\omega\tau}{2}\right)$$




ⓐ Amplitude Spectrum




ⓑ phase Spectrum.

At $\omega = 0$ $\mathrm{Sinc}\left(\frac{\omega\tau}{2}\right) = 1$. $\therefore |X(\omega)|$ at $\omega = 0$ is equal to $\tau$.

At $\frac{\omega\tau}{2} = \pm n\pi$, i.e., at

$$\omega = \pm\frac{2n\pi}{\tau} \quad n = 1, 2 \text{ ------} \quad \mathrm{Sinc}\left(\frac{\omega\tau}{2}\right) = 0$$

phase Spectrum is $\angle X(\omega) = 0$ if $\mathrm{Sinc}\left(\frac{\omega\tau}{2}\right) > 0$

$= \pm\pi$ if $\mathrm{Sinc}\left(\frac{\omega\tau}{2}\right) < 0$

Triangular pulse $\Delta\left(\frac{t}{\tau}\right)$ :-

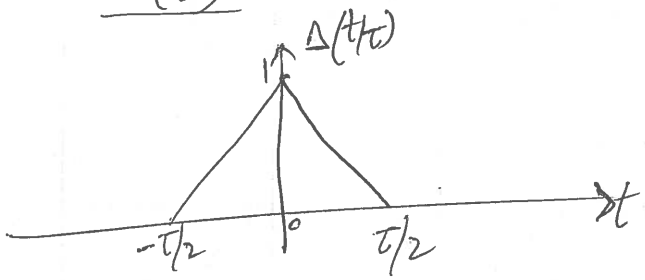



fig:- Triangular function.

$$x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2|t|}{\tau} & \text{for } |t| < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = F\{x(t)\} = F\left\{\Delta\left(\frac{t}{\tau}\right)\right\} = \int_{-\infty}^{\infty} \Delta\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\tau/2}^{0} \left(1 + \frac{2t}{\tau}\right) e^{-j\omega t}\, dt + \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{j\omega t}\, dt + \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{0}^{\tau/2} e^{j\omega t}\, dt - \int_{0}^{\tau/2} \frac{2t}{\tau} e^{j\omega t}\, dt + \int_{0}^{\tau/2} e^{-j\omega t}\, dt - \int_{0}^{\tau/2} \frac{2t}{\tau} e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{0}^{\tau/2} \left(e^{j\omega t} + e^{-j\omega t}\right) dt - \frac{2}{\tau} \int_{0}^{\tau/2} t\left(e^{j\omega t} + e^{-j\omega t}\right) dt$$

$$X(\omega) = \int_{0}^{\tau/2} 2\cos\omega t\, dt - \frac{2}{\tau} \int_{0}^{\tau/2} 2t\cos\omega t\, dt$$

$$X(\omega) = 2\left(\frac{\sin\omega t}{\omega}\right)_0^{\tau/2} - \frac{4}{\tau}\left[\left(t\frac{\sin\omega t}{\omega}\right)_0^{\tau/2} - \int_0^{\tau/2} \frac{\sin\omega t}{\omega}\, dt\right]$$

$$X(\omega) = 2\left(\frac{\sin\omega t}{\omega}\right)_0^{\tau/2} - \frac{4}{\tau}\left(\left(t\frac{\sin\omega t}{\omega}\right)_0^{\tau/2} + \left(\frac{\cos\omega t}{\omega^2}\right)_0^{\tau/2}\right)$$

$$= \frac{2}{\omega}\left(\frac{\sin\omega t}{\omega}\right)_0 = \frac{2}{\omega}\left(\sin\frac{\omega\tau}{2}\right) - \frac{4}{\omega\tau}\left(\frac{\tau}{2}\sin\frac{\omega\tau}{2}\right) - \frac{4}{\omega^2\tau}\left(\cos\frac{\omega\tau}{2} - 1\right)$$

$$= \frac{4}{\omega^2\tau}\left(1 - \cos\frac{\omega\tau}{2}\right) = \frac{4}{\omega^2\tau}\left(2\sin^2\frac{\omega\tau}{4}\right)$$

$$= \frac{8}{\omega^2\tau}\left(\frac{\omega\tau}{4}\right)^2 \frac{\sin^2(\omega\tau/4)}{(\omega\tau/4)^2} = \frac{\tau}{2}\,\mathrm{sinc}^2\frac{\omega\tau}{4}$$

$$F\left\{\Delta\left(\frac{t}{\tau}\right)\right\} = \frac{\tau}{2}\,\mathrm{sinc}^2\frac{\omega\tau}{4}.$$

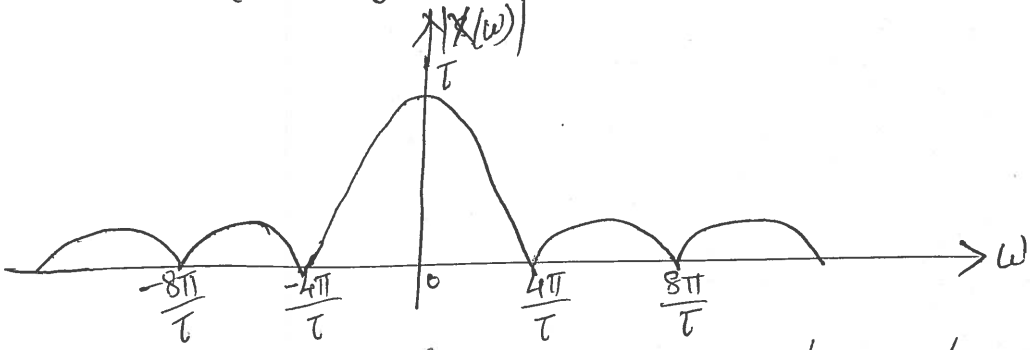



fig:- Amplitude spectrum of a $\Delta^{le}$ pulse.

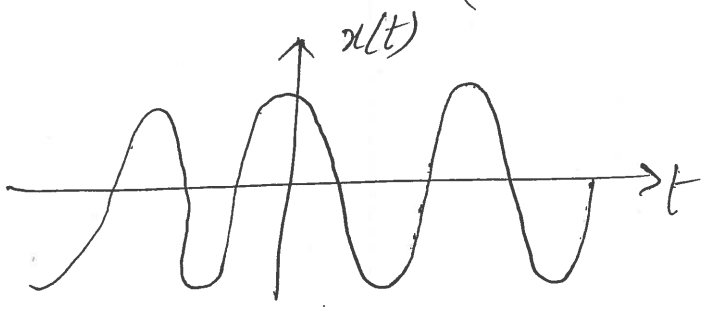
Cosine wave $\cos\omega_0 t$:-

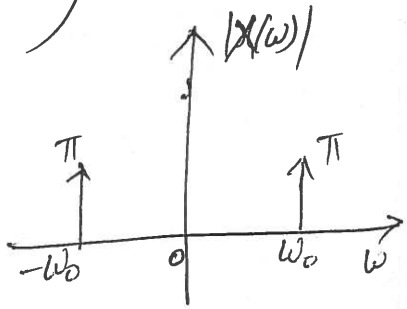
$$x(t) = \cos\omega_0 t$$

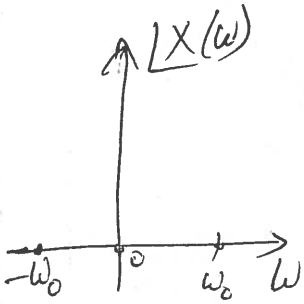
$$X(\omega) = F\{x(t)\} = F\{\cos\omega_0 t\} = F\left\{\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right\}$$

$$X(\omega) = \frac{1}{2}\left\{F(e^{j\omega_0 t}) + F(e^{-j\omega_0 t})\right\} = \frac{1}{2}\left(2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right)$$

$$X(\omega) = \pi\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right).$$



## Sine wave $\sin\omega_0 t$ :-

$$x(t) = \sin\omega_0 t$$

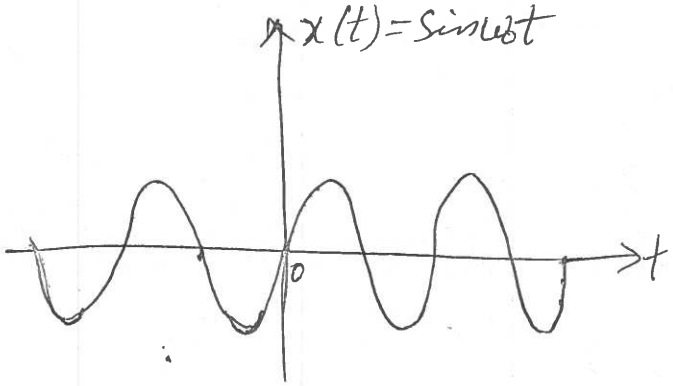
$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

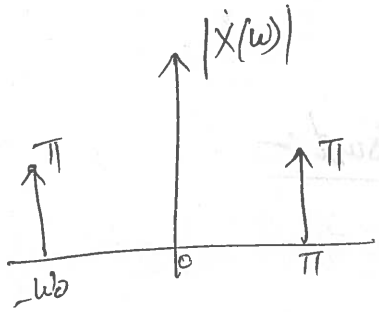
$$F\{x(t)\} = F\left\{\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right\} = \frac{1}{2j}\left\{F(e^{j\omega_0 t}) - F(e^{-j\omega_0 t})\right\}$$

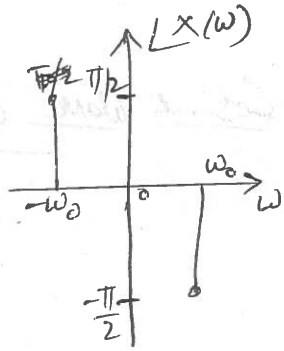
$$F\{x(t)\} = \frac{1}{2j}\left\{2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right\}$$

$$F\{x(t)\} = -j\pi\left\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right\}$$

$$F\{\sin\omega_0 t\} = -j\pi\left\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right\}$$



## Properties of Continuous Time Fourier Transform:-

### 1) Linearity:-

The linearity property states that the fourier transform of a weighted sum of 2 signals is equal to the weighted sum of their individual Fourier transforms:

If $x_1(t) \longleftrightarrow X_1(\omega)$ & $x_2(t) \longleftrightarrow X_2(\omega)$

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(\omega) + b X_2(\omega)$$

where a & b are constants.

Proof:-

$$F(a x_1(t) + b x_2(t)) = \int_{-\infty}^{\infty} (a x_1(t) + b x_2(t)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} a x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} b x_2(t) e^{-j\omega t} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$F(a x_1(t) + b x_2(t)) = a X_1(\omega) + b X_2(\omega)$$

$$a x_1(t) + b x_2(t) \longleftrightarrow a X_1(\omega) + b X_2(\omega).$$

### (2) Time Shifting Property:-

The time shifting property states that if a signal $x(t)$ is shifted by $t_0$ seconds, the spectrum is modified by a linear shift of slope $-\omega t_0$ i.e.,

$$x(t) \longleftrightarrow X(\omega)$$

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega).$$

Proof:-

$$F\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt$$

$t - t_0 = p \quad t = t_0 + p$

$$= \int_{-\infty}^{\infty} x(p) e^{-j\omega(t_0 + p)} dp = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p) e^{-j\omega p} dp$$

$$F\{x(t - t_0)\} = e^{-j\omega t_0} X(\omega).$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

Ily $x(t+t_0) \longleftrightarrow e^{+j\omega t_0} X(\omega)$

$$|e^{-j\omega t_0} X(\omega)| = |X(\omega)|.$$

$$\underline{|e^{-j\omega t_0} X(\omega)} = e^{-j\omega t_0} + \underline{|X(\omega)} = \underline{|-\omega t_0} + \underline{|X(\omega)}$$

This shows that shifting a function by $t_0$ results in multiplying its Fourier transform by $e^{-j\omega t_0}$. Thus there is no change in magnitude spectrum but the phase spectrum is linearly shifted.

③ Frequency Shifting property:- (Multiplication by an Exponential).

⇝ Frequency shifting property states that the multiplication of a time domain signal $x(t)$ by $e^{j\omega_0 t}$ results in frequency spectrum shifted by $\omega_0$ i.e,

$$x(t) \longleftrightarrow X(\omega)$$

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0).$$

Proof:-

$$F\{e^{j\omega_0 t} x(t)\} = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} x(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= X(\omega - \omega_0)$$

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$$

Ily $e^{-j\omega_0 t} x(t) \longleftrightarrow X(\omega + \omega_0)$

④ Time Reversal property:-

$$x(t) \longleftrightarrow X(\omega)$$

$$x(-t) \longleftrightarrow X(-\omega)$$

Proof:-

$$F\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

$$-t = p \qquad -dt = dp$$

$t=-\infty \quad p=\infty$

$t=\infty \quad p=-\infty$

$$F\{x(t)\} = -\int_{+\infty}^{-\infty} x(p)\, e^{j\omega p}\, dp = \int_{-\infty}^{+\infty} x(p)\, e^{-j(-\omega)p}\, dp$$

$$F\{x(-t)\} = X(-\omega).$$

(5) Time Scaling:-

$$F\{x(at)\} \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof:-

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

$at = p \qquad t = \frac{p}{a} \qquad dt = \frac{dp}{a}$

$$= \int_{-\infty}^{\infty} x(p)\, e^{-j\omega \frac{p}{a}}\, \frac{dp}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j\frac{\omega}{a}p}\, dp$$

$$F\{x(at)\} = \frac{1}{a} \int_{-\infty}^{\infty} x(p)\, e^{-j\left(\frac{\omega}{a}\right)p}\, dp = \frac{1}{a} X\left(\frac{\omega}{a}\right) \quad (a>0)$$

When $a<0$

$$F\{x(at)\} = \frac{1}{-a} \int_{-\infty}^{\infty} x(p)\, e^{j\frac{\omega}{a}p}\, dp = \frac{1}{-a} \int_{-\infty}^{\infty} x(p)\, e^{-j\left(\frac{-\omega}{a}\right)p}\, dp$$

$$F\{x(at)\} = \frac{-1}{a} X\left(\frac{-\omega}{a}\right) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

(6) Differentiation in Time Domain:-

$$x(t) \longleftrightarrow X(\omega)$$

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)$$

Proof:-

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}(e^{j\omega t})\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{e^{j\omega t}}{j}(j\omega)\,dt$$

$$= \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,dt$$

$$= j\omega\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,dt\right)$$

$$= j\omega F^{-1}\{X(\omega)\}$$

$$F\left\{\frac{d}{dt}x(t)\right\} = j\omega X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega)$$

(7) Differentiation in Frequency Domain:-

$$x(t) \longleftrightarrow X(\omega)$$

$$t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(\omega)$$

Proof:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}(e^{-j\omega t})\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}(-jt)\,dt$$

$$\frac{j}{t}\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$j\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} t\,x(t)\,e^{-j\omega t}\,dt$$

$$j\frac{d}{d\omega}X(\omega) = F\{t\,x(t)\}$$

$$\boxed{t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(\omega)}$$

⑧ Time Integration :-

$$x(t) \longleftrightarrow X(\omega)$$

then $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{j\omega} X(j\omega)$ if $X(0)=0$

Proof:- If $X(0)=0$ this property can be easily proved by using integration by parts as in the case of the differentiation property.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\omega\, d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\frac{e^{j\omega\tau}}{j\omega}\right)_{-\infty}^{t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega}\left(e^{j\omega t} - 0\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega}\, e^{j\omega t}\, d\omega = F^{-1}\left\{\frac{X(\omega)}{j\omega}\right\}$$

$$F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega}.$$

If $X(0)\neq 0$ then $x(t)$ is not an Energy function & the fourier transform of $\int_{-\infty}^{t} x(\tau)\,d\tau$ includes an impulse function that is

$$F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = \frac{1}{j\omega}X(\omega) + \pi X(0)\,\delta(\omega).$$

⑨ Convolution Theorem:-

$$x_1(t) \longleftrightarrow X_1(\omega) \quad \& \quad x_2(t) \longleftrightarrow X_2(\omega)$$

$$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

(or) $F\{x_1(t) * x_2(t)\} = X_1(\omega) X_2(\omega)$.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$F\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) e^{-j\omega t}\, d\tau\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left( \int_{-\infty}^{\infty} x_2(t-\tau) e^{-j\omega t}\, dt \right) d\tau$$

$t - \tau = P \qquad t = \tau + P \qquad dt = dP.$

$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty} x_2(P)\, e^{-j\omega(\tau+P)}\, dP\, d\tau = \int_{-\infty}^{\infty} x_1(\tau) \left( \int_{-\infty}^{\infty} x_2(P) e^{-j\omega P} dP \right) e^{-j\omega \cdot} d$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} x_2(P)\, e^{-j\omega P} dP$$

$$F\{x_1(t) * x_2(t)\} = X_1(\omega)\, X_2(\omega)$$

⑩ Multiplication property :-

$$F\{x_1(t)\, x_2(t)\} = \frac{1}{2\pi} X_1(\omega) * X_2(\omega).$$

Proof :-

$$F\{x_1(t)\, x_2(t)\} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} X_1(\gamma)\, e^{j\gamma t} d\lambda \right) x_2(t)\, e^{j\omega t} dt$$

$$F\{x_1(t)\, x_2(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \left( \int_{-\infty}^{\infty} x_2(t)\, e^{j\lambda t}\, e^{-j\omega t} dt \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \left( \int_{-\infty}^{\infty} x_2(t)\, e^{-jt(\omega-\lambda)} \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$$

$$F\{x_1(t)\, x_2(t)\} = \frac{1}{2\pi} X_1(\omega) * X_2(\omega).$$

(or) $x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

(or) $2\pi\, x_1(t)\, x_2(t) \longleftrightarrow X_1(\omega) * X_2(\omega)$.

(or) $x_1(t)\, x_2(t) \longleftrightarrow X_1(f) * X_2(f)$.

(11) Duality (or) Symmetry property:-

$x(t) \longleftrightarrow X(\omega)$

$X(t) \longleftrightarrow 2\pi x(-\omega)$

Proof:-

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega(-t)}\, d\omega$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$-t = p$

$$2\pi x(-p) = \int_{-\infty}^{+\infty} X(\omega)\, e^{-j\omega p}\, d\omega$$

p is a Dummy variable, replace p with t

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging t with $\omega$

~~$2\pi x(t) \longleftrightarrow F(X(t))$~~

~~$x(t) \longleftrightarrow$~~ $2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi x(-\omega) = F\{X(t)\}$$

$X(t) \longleftrightarrow 2\pi x(-\omega)$.

(12) Modulation property:-

$x(t) \longleftrightarrow X(\omega)$

$x(t)\cos\omega_c t \longleftrightarrow \frac{1}{2}\left(X(\omega-\omega_c) + X(\omega+\omega_c)\right)$

Proof:- $x(t)\cos\omega_c t = x(t)\left\{\frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t}\right\}$

$= \frac{1}{2}\{x(t)\, e^{j\omega_c t} + x(t)\, e^{-j\omega_c t}\}$

$$F\{x(t)\cos\omega_c t\} = \frac{1}{2} F\{x(t)e^{j\omega_c t} + x(t)e^{-j\omega_c t}\}$$

$$= \frac{1}{2}\left[F\{x(t)e^{j\omega_c t}\} + F\{x(t)e^{-j\omega_c t}\}\right]$$

$$F\{x(t)\cos\omega_c t\} = \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$$

$$x(t)\cos\omega_c t \longleftrightarrow \frac{1}{2}\left(X(\omega-\omega_c) + X(\omega+\omega_c)\right).$$

Ily $x(t)\sin\omega_c t \longleftrightarrow \frac{1}{2j}\left(X(\omega-\omega_c) - X(\omega+\omega_c)\right)$

3) Conjugation Property:-

$$x(t) \longleftrightarrow X(\omega)$$

$$x^*(t) \longleftrightarrow X^*(-\omega)$$

Proof:-

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$F\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \left(\int_{-\infty}^{\infty} x(t)e^{+j\omega t}dt\right)^*$$

$$F\{x^*(t)\} = \left(\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt\right)^* = \left(X(-\omega)\right)^* = X^*(-\omega).$$

14) Auto correlation:-

$$x(t) \longleftrightarrow X(\omega)$$

$$R(\tau) \longleftrightarrow |X(\omega)|^2$$

Proof:-

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)dt$$

$$F\{R(\tau)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\,x^*(t-\tau)dt\;e^{-j\omega\tau}d\tau$$

$$F\{R(\tau)\} = \int_{-\infty}^{\infty} x(t)\left(\int_{-\infty}^{\infty} x^*(p)\cdot e^{-j\omega(t-p)}dp\right)dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\int_{-\infty}^{\infty} x^*(p)e^{j\omega p}dp \quad (t-\tau=p)$$

$$F\{R(\tau)\} = X(\omega)\left(\int_{-\infty}^{\infty} x(p)e^{-j\omega p}dp\right)^* = X(\omega)\,X^*(\omega) = |X(\omega)|^2.$$

(15) Perseval's Relation (or) Theorem:-

⇒ $x_1(t) \longleftrightarrow X_1(\omega)$ & $x_2(t) \longleftrightarrow X_2(\omega)$

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega .$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x_1(t) = x_2(t) = x(t)$.

Proof:-

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, e^{j\omega t}\, d\omega \right\} x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \left\{ \int_{-\infty}^{\infty} x_2^*(t) e^{j\omega t} dt \right\} d$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \left\{ \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t} dt \right\}^* d\omega .$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega) \left(X_2(\omega)\right)^* d\omega = RHS .$$

$$\boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega}$$

Parseval's Identity:- $x_1(t) = x_2(t) = x(t)$ then the energy of the signal is given by.

$$E = \int_{-\infty}^{\infty} x(t)\, x^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega) d\omega$$

$|x(t)|^2 = x(t)\, x^*(t)$ & $|X(\omega)|^2 = X(\omega) X^*(\omega)$. we get

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega .$$

16) Area under the curve :-

(12)

① The area under a function $x(t)$ is equal to the value of its fourier transform $X(\omega)$ at $\omega=0$ ie.,

$$x(t) \longleftrightarrow X(\omega).$$

$$\int_{-\infty}^{\infty} x(t)\,dt = \frac{1}{2\pi} X(0)$$

② The area under the Fourier transform $X(\omega)$ of a function $x(t)$ is equal to the value of the function $x(t)$ at $t=0$,

$$\int_{-\infty}^{\infty} X(\omega)\,d\omega = x(0).$$

17) Fourier Transform of Complex & Real Functions :-

If the signal $x(t)$ is Complex, it can be represented as:

$$x(t) = x_R(t) + jx_I(t)$$

$$F(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \left(x_R(t) + jx_I(t)\right)\left(\cos\omega t - j\sin\omega t\right)dt$$

$$= \int_{-\infty}^{\infty} \left(x_R(t)\cos\omega t + x_I(t)\sin\omega t\right)dt + j\int_{-\infty}^{\infty} \left(x_I(t)\cos\omega t - x_R(t)\sin\omega t\right)dt$$

$$F\{x(t)\} = X_R(\omega) + jX_I(\omega)$$

$$X_R(\omega) = \int_{-\infty}^{\infty} \left(x_R(t)\cos\omega t + x_I(t)\sin\omega t\right)dt$$

$$X_I(\omega) = \int_{-\infty}^{\infty} \left(x_I(t)\cos\omega t - x_R(t)\sin\omega t\right)dt$$

By

$$x_R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(X_R(\omega)\cos\omega t - X_I(\omega)\sin\omega t\right)d\omega$$

$$x_I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(X_R(\omega)\sin\omega t + X_I(\omega)\cos\omega t\right)d\omega$$

Case 1:- If $x(t)$ is real then $x_I(t)=0$ & $X(\omega)=X^*(\omega)$

$$X_R(\omega) = \int_{-\infty}^{\infty} x(t)\cos\omega t\,dt \quad \& \quad X_I(\omega) = -\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt$$

Case 2:- $x(t)$ is even & real.

$$x(t) = x_e(t)$$

$$X_R(\omega) = \int_{-\infty}^{\infty} x_e(t)\cos\omega t\,dt = 2\int_{0}^{\infty} x_e(t)\cos\omega t\,dt$$

$$X_I(\omega) = 0$$

$$X(\omega) = 2\int_{0}^{\infty} x_e(t)\cos\omega t\,dt$$

Case 3:- $x(t)$ is odd & real.

$$x(t) = x_o(t)$$

$$X_R(\omega) = 0$$

$$X_I(\omega) = jX(\omega) = -j\int_{-\infty}^{\infty} x_o(t)\sin\omega t = -j2\int_{0}^{\infty} x_o(t)\sin\omega t$$

For a Non Symmetrical function

$$F\{x(t)\} = X(\omega) = X_R(\omega) + jX_I(\omega)$$

$$= \int_{-\infty}^{\infty} x_e(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} x_o(t)\sin\omega t\,dt$$

$$= X_e(\omega) + X_o(\omega).$$

## Fourier Transform of a Periodic Signal:-

The periodic signals or functions can be analysed using Fourier series and that non-periodic functions can be analysed using Fourier transform. But we can find the Fourier Transform of a periodic function also. This means that the FT can be used as a universal mathematical tool in the analysis of both periodic & Non-periodic wave forms over the entire interval. Fourier Transform of periodic functions may be found using the concept of impulse function.

We know that using fourier series, any periodic signal can be represented as a sum of complex exponentials. ∴ we can represent a periodic signal using the fourier integral. Let us consider a periodic signal $x(t)$ with period $T$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Fourier transform of $x(t)$ is:

$$\therefore X(\omega) = F\{x(t)\} = F\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right] = \sum_{n=-\infty}^{\infty} C_n F\{e^{jn\omega_0 t}\}.$$

Using the Frequency Shifting theorem, we have

$$F\{1 \times e^{jn\omega_0 t}\} = F[1]\Big|_{\omega=\omega-n\omega_0} = 2\pi\delta(\omega)\Big|_{\omega=\omega-n\omega_0} = 2\pi\delta(\omega-n\omega_0)$$

$$\therefore X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega-n\omega_0).$$

Where $C_n$s are the fourier coefficients associated with $x(t)$ & are given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt.$$

Thus the F.T of a periodic function consists of a train of equally spaced impulses. These impulses are located at the harmonic frequencies of the signal & the strength (area) of each impulse is given as $2\pi C_n$.

Merits:- ① The original time function can be uniquely recovered from it.

② Convolution integrals can be uniquely recovered from it.

③ The F.T is the most useful tool for Analysing Signals involved in Communication Systems because here the amplitude & phase characteristics are readily known.

Limitations:-

① It is less powerful than the Laplace transform because in the Fourier transform, the damping factor $\sigma = 0$, but in the Laplace transform, the damping factor $\sigma$ is finite.

② There are many functions for which the Laplace transform Exists, but the fourier transform does not Exist. In fact the F.T is a Special case of Laplace transform with $s = j\omega$, that is fourier transform is the L.T evaluated along the imaginary axis of the s-plane.

## Sampling Theorem:- Introduction:-

The process of Converting a continuous time signal into a discrete time signal is called Sampling. After sampling the signal is defined at discrete instants of time and the time interval between 2 successive sampling instants is called Sampling period (or) Sampling interval. In the process of sampling one of the important factors that we have to consider is the sampling rate must be kept sufficiently high so that the original signal can be reconstructed from its samples.

### Sampling:-

The Sampling operation can be represented by a fictitious switch shown in fig(1). The switch is closed for a very short interval of time $\tau$ (ideally $\tau=0$), once every $T$ sec during which the signal is available at the o/p. $\therefore$ the i/p is $x(t)$, then the o/p $x_s(t)$ is $x(nT_s)$, $n=0,\pm1,\pm2\ldots\ldots$ & $x(nT)$ is called the Sampled sequence of $x(t)$, where $T_s$ is called the sampling period (or) Sampling Interval. It is the time interval between successive Samples & the sampling frequency is given by $f_s=\frac{1}{T_s}$ Hzs. Although through a mechanical switch is shown in fig(1), in actual practice an electronic switch may be used.

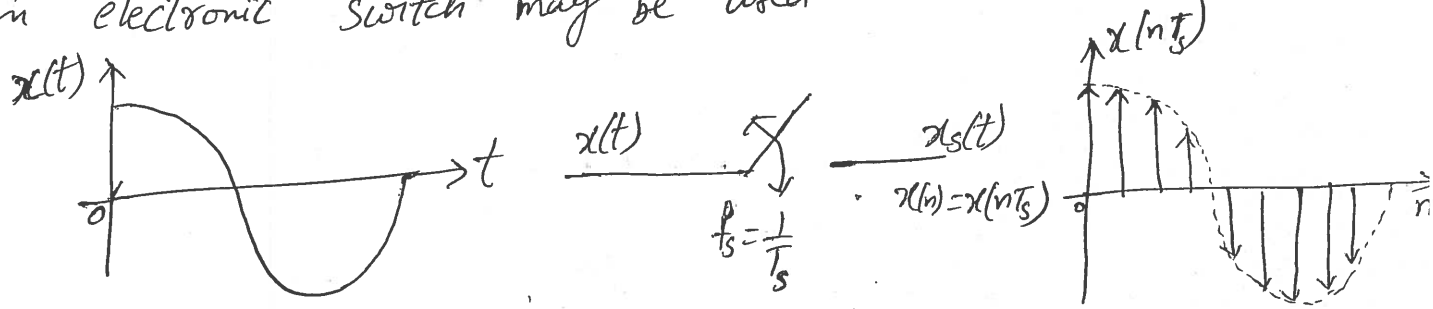



Fig:- Sampling operation.

### Sampling Theorem:-

The Sampling theorem is one of the most useful theorems since it applies to digital Communication systems. The Sampling theorem states that A band limited signal $x(t)$ with $X(\omega)=0$ for $|\omega|\geq\omega_m$ (i.e, $X(f)=0$ fo

$f \geq f_m$) can be represented in to & uniquely determined from its Sample $x(nT_s)$ if the sampling frequency $f_s \geq 2f_m$, where $f_m$ is the highest frequency Component present in it. That is for Signal recovery, the Sampling frequency must be atleast twice the highest frequency present in the Signal.

This theorem is known as uniform Sampling theorem since it pertains to the specification of a given Signal by its samples at uniform intervals of $\frac{1}{2f_m}$ sec.

It is also known as Lowpass Sampling theorem because it applies to lowpass signals i.e., signals for which $X(f)=0$ for all frequencies such that $|f| \geq f_m$, where $f_m$ is some finite frequency.

"Sampling Theorem" states that a band limited Continuous time signal can be completely represented in its Sampled version & recovered back to the original form, if the sampling frequency $f_s$ is atleast greater than (or) equal to twice of $f_m$.

$$f_s \geq 2f_m.$$

Proof:-

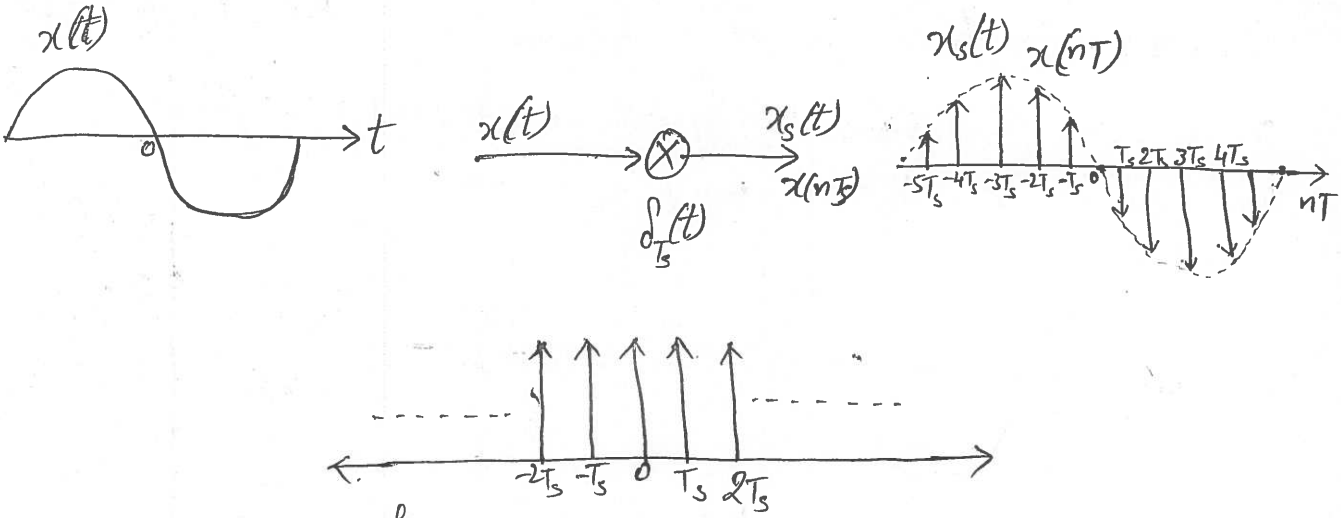



fig:(2) Sampling operation.

$x(t)$ is a continuous time band limited Signal to be sampled which has no Spectral Components above $f_m$ Cycles per second. That means $X(\omega)$, the F.T of $x(t)$ is 0 for $\omega > \omega_m$. $\delta_{T_s}(t)$ is an impulse train which Samples at a rate of $f_s$ Hz & $x_s(t)$ is the Sampled Signal.

$T_s$ is the sampling period & $f_s = \frac{1}{nT_s}$ is the sampling frequency.

$x_s(t)$ is the product of signal $x(t)$ & $\delta_{T_s}(t)$. It is a sequence of impulses located at regular intervals of $T_s$ sec, and having the strength equal to the values of $x(t)$ at the corresponding instants.

$$x_s(t) = x(t)\,\delta_{T_s}(t) \quad \text{where } \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The Exponential form of Fourier series of $\delta_{T_s}(t)$ is

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$$

$$\text{where } C_n = \frac{1}{T_s}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_s t}\, dt$$

$$C_n = \frac{1}{T_s}\left(e^{-jn\omega_s t}\right)_{t=0} = \frac{1}{T_s}$$

$$\therefore \delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{jn\omega_s t} = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$x_s(t) = x(t)\,\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s)$$

$$F\{x_s(t)\} = \sum_{n=-\infty}^{\infty} F\left(x(t)\,\delta(t - nT_s)\right) = \sum_{n=-\infty}^{\infty} F(x(t)) \ast F\{\delta(t - nT_s)\} \times \frac{1}{2\pi}$$

$$F\{x_s(t)\} = X(\omega)$$

$$F\{x_s(t)\} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega) \ast F\{\delta(t - nT_s)\}.$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega) \ast \left(\frac{2\pi}{T_s}\,\delta(\omega - n\omega_s)\right)$$

$$= \frac{1}{2\pi} f_s (2\pi) \sum_{n=-\infty}^{\infty} X(\omega) \ast \delta(\omega - n\omega_s)$$

$$= f_s\left(\sum_{n=-\infty}^{\infty} X(\omega) \ast \delta(\omega - n\omega_s)\right)$$

$$X_s(\omega) = f_s \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) = f_s \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

(or) $x_s(t) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$

(or)

$$x_s(t) = x(t)\,\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}$$

$$F\{x_s(t)\} = F\left\{\frac{1}{T_s}\sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t}\right\}$$

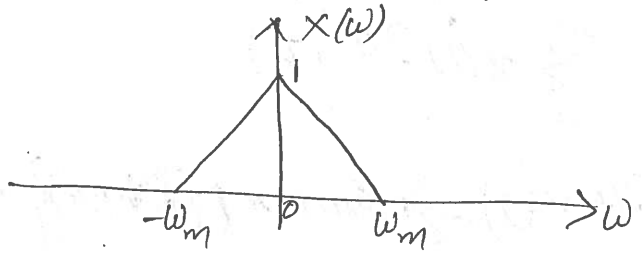
$$X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F\left(x(t)\, e^{jn\omega_s t}\right) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$X_s(\omega) = f_s \sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi}{T_s} n\right)$$

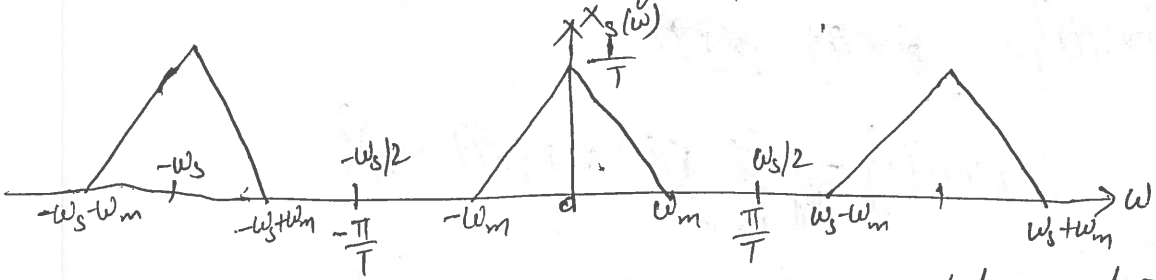
(or)

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

where $X(\omega)$ (or) $X(f)$ is the spectrum of the i/p signal & $X_s(\omega)$ (or) $X_s(f)$ is the spectrum of the sampled signal.



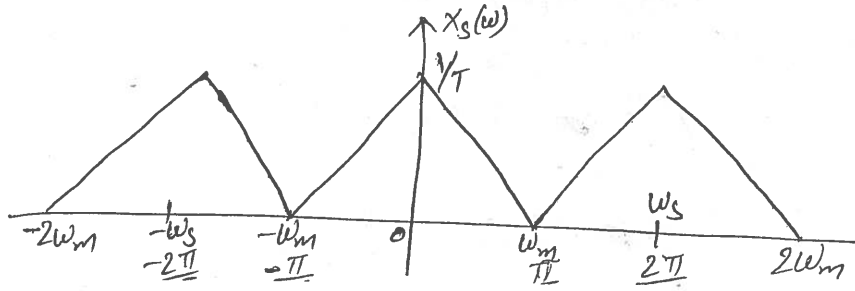

(a) Frequency spectrum of $x(t)$




(b) Frequency spectrum of Sampled signal $x_s(t)$ for $\left(\frac{\pi}{T}\right) > \omega_m$,

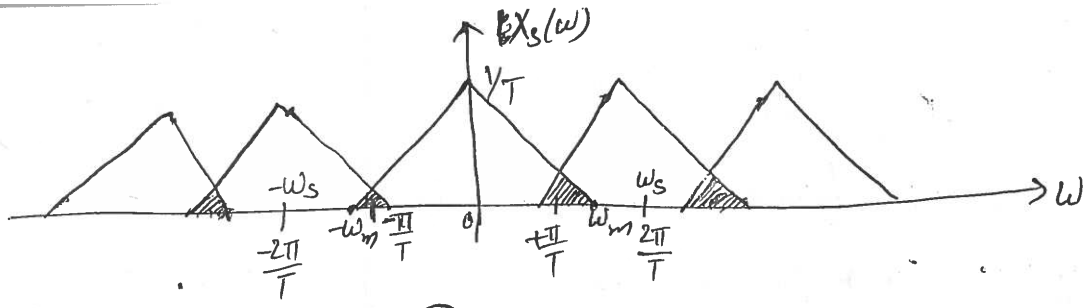
~~$\frac{\pi}{T} < \omega_m$~~ & ~~$\frac{\pi}{T} > \omega_m$~~ i.e., $\frac{\omega_s}{2} > \omega_m$, $2\omega_m < \omega_s$

$\omega_s > 2\omega_m$.




(c) $\omega_m = \frac{\pi}{T} = \frac{\omega_s}{2}$

$\omega_s = 2\omega_m$.




(d) $\frac{\pi}{T} < \omega_m$ $\quad \frac{\omega_s}{2} < \omega_m$ $\quad \& \omega_s < 2\omega_m$.

Thus the Fourier transform of Sampled Signal is given by an ∞ Sum of Shifted replicas of the fourier transform of the original Signal.

The Signal $x(t)$ is band limited to $f_m$. The term $X\left(\omega - \frac{2\pi n}{T}\right)$ is the shifting of $X(\omega)$ from $\omega = 0$ to $\omega = \frac{2\pi n}{T}$. Hence $X_s(\omega)$ is the Sum of shifted replicas of $\frac{1}{T}X(\omega)$ Centering at $\left(\frac{2\pi}{T}\right)n$, $n = 0, \pm 1, \pm 2 \cdots$ fig (3) shows the plot of $X(\omega)$ & $X_s(\omega)$ for various values of $\frac{\pi}{T}$. It shows that fig 3(b) & (c) if $\left(\frac{\pi}{T}\right) \geq \omega_m$, the replicas will not overlap & as a result the frequency Spectrum of $T X_s(\omega)$ in the frequency range $\left(\frac{-\pi}{T}, \frac{\pi}{T}\right)$ is identical to $X(\omega)$. $X(\omega)$ can be recovered from $X_s(\omega)$ by passing it through a lowpass filter which has sharp Cutoff at $\omega = \frac{\pi}{T}$ (or with Bandwidth $B$, where $\omega_m \leq B \leq \omega_s - \omega_m$). If $\left(\frac{\pi}{T}\right) < \omega_m$ (fig 3(d)) the successive frequency Spectra will overlap and the original signal Can't be recovered from the sampled signal.

∴ we can say that for Signal recovery,

$\omega_s - \omega_m \geq \omega_m$, i.e., $\omega_s \geq 2\omega_m$.

(or)

$f_s - f_m \geq f_m$ i.e., $f_s \geq 2f_m$.

$\frac{\pi}{T} \geq \omega_m$, i.e., $\pi f_s \geq 2\pi f_m$ i.e., $f_s \geq 2f_m$

$\frac{1}{f_s} \leq \frac{1}{2f_m}$ i.e., $T \leq \frac{1}{2f_m}$

So we can conclude that if sampling interval T is small ($< \frac{1}{2f_m}$), $X(\omega)$ can be recovered from $X_s(\omega)$, but if T becomes larger than $\frac{1}{2f_m}$, then there is an overlap between successive cycles & $X(\omega)$ can't be recovered from $X_s(\omega)$. This proves the Sampling Theorem.

There are two basic conditions to be satisfied if $x(t)$ is to be recovered from its samples.

(1) $x(t)$ should be band limited to some frequency $\omega_m$.

(2) The Sampling Frequency $\omega_s$ should be atleast twice the band-limiting frequency $\omega_m$ (i.e., $\omega_s \geq 2\omega_m$).

From fig, we can observe that:

(1) $X_s(\omega)$ is a repetitive version of $X(\omega)$ with $X(\omega)$ repeating itself at regular intervals of $\omega_s$, the sampling frequency.

(2) When $\omega_s > 2\omega_m$, the spectral replicates have a larger separation between them, known as Guard band, which makes the process of filtering much easier & effective. Even a non-ideal filter which does not have a sharp cutoff can also be used.

(3) When $\omega_s = 2\omega_m$ (fig ©) there is no separation between the replicates, so no Guard band exists, & $X(\omega)$ can be obtained from $X_s(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.

(4) When $\omega_s < 2\omega_m$ the low frequency components in $X_s(\omega)$ overlap on the higher frequency components of $X(\omega)$, there is distortion & $X(\omega)$ can't be recovered from $X_s(\omega)$ by using any filter.

(1) Find the fourier transform of the following signals: (7)

(a) $e^{3t}u(t)$.

Given signal is not absolutely integrable.

$$\int_{-\infty}^{\infty} e^{3t}u(t) = \infty$$

Fourier transform of given signal does not exist.

(b) $x(t) = \cos\omega_0 t\, u(t)$

$$= \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) u(t)$$

$$X(\omega) = \frac{1}{2}\int_0^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt = \frac{1}{2}\left(\frac{e^{-j t(\omega-\omega_0)}}{-j(\omega-\omega_0)} + \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\right)_0^{\infty}$$

$$X(\omega) = \frac{1}{2}\left(-\left(\frac{1}{-j(\omega-\omega_0)} + \frac{1}{-j(\omega+\omega_0)}\right)\right) = \frac{1}{2}\left(\frac{1}{j(\omega-\omega_0)} + \frac{1}{j(\omega+\omega_0)}\right)$$

$$X(\omega) = \frac{1}{2j}\left(\frac{\omega+\omega_0+\omega-\omega_0}{\omega^2-\omega_0^2}\right) = \frac{2\omega}{2j(\omega^2-\omega_0^2)} = \frac{\omega}{j(\omega^2-\omega_0^2)}$$

$$X(\omega) = \frac{-j\omega}{\omega^2-\omega_0^2} = \frac{j\omega}{\omega_0^2-\omega^2}$$

with impulse strength $\pi$ at $\omega=\omega_0$ & $\omega=-\omega_0$

$$X(\omega) = \frac{j\omega}{\omega_0^2-\omega^2} + \frac{1}{2}\left(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right)$$

(c) ~~$x(t) = \sin\omega_0 t\, u(t)$~~

(d) $x(t) = e^{-t}\sin 5t\, u(t)$.

$$x(t) = e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right)u(t) = \frac{e^{-t(1-j5)} - e^{-t(1+j5)}}{2j}$$

$$F(x(t)) = X(\omega) = \int_0^\infty e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right)e^{-j\omega t}\,dt.$$

$$= \frac{1}{2j}\int_0^\infty e^{-t(1-j5+j\omega)} - e^{-t(1+j5+j\omega)}\,dt$$

$$= \frac{1}{2j}\left(\frac{e^{-t(1-j5+j\omega)}}{-(1-j5+j\omega)} - \frac{e^{-t(1+j5+j\omega)}}{-(1+j5+j\omega)}\right)_0^\infty$$

$$= \frac{1}{-2j}\left(\frac{-1}{1-j5+j\omega} - \left(\frac{-1}{1+j5+j\omega}\right)\right) = \frac{+1}{+2j}\left(\frac{1}{1-j5+j\omega} - \frac{1}{1+j5+j\omega}\right)$$

$$X(\omega) = \frac{+1}{2j}\left(\frac{1+j5+j\omega - (1-j5+j\omega)}{((1+j\omega)-j5)((1+j\omega)+j5)}\right) = \frac{+1}{2j}\left(\frac{j10}{(1+j\omega)^2 - (j5)^2}\right)$$

$$X(\omega) = \frac{5}{(1+j\omega)^2 + (5)^2}$$

(e) $x(t) = e^{at}u(-t)$.

$$F\{x(t)\} = \int_{-\infty}^{0} e^{at}u(-t)e^{-j\omega t}dt = \left(\frac{e^{at-j\omega t}}{a-j\omega}\right)_{-\infty}^{0}$$

$$F\{x(t)\} = \frac{1}{a-j\omega}(1-0) = \frac{1}{a-j\omega} = \frac{a+j\omega}{(a-j\omega)(a+j\omega)}$$

$$F\{x(t)\} = \frac{a+j\omega}{a^2+\omega^2}.$$

(f) $x(t) = te^{-at}u(t) = \frac{1}{(a+j\omega)^2}$

② Using properties of Fourier Transform, find the fourier transform of the following:

ⓐ $e^{-at}u(t)$ ⓑ $te^{-2t}u(t)$ ⓒ $e^{j2t}u(t)$ ⓓ $e^{j\omega_0 t}$ ⓔ $\cos\omega_0 t\, u(t)$

ⓕ $\sin\omega_0 t\, u(t)$ ⓖ $e^{3t}u(t-2)$ ⓗ $e^{-a|t-2|}$ ⓘ $\delta(t+2)+\delta(t+1)+\delta(t-1)+\delta(t-2)$.

ⓐ → $x(t) = e^{-at}u(t)$

$$F\{e^{at}u(-t)\} = \frac{1}{a-j\omega} \quad \text{(use Time Reversal)}$$

Replace $t$ by $-t$ implies $\omega$ by $-\omega$

$$F\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$$

ⓑ $x(t) = te^{-2t}u(t)$

$$F\{e^{-2t}u(t)\} = \frac{1}{2+j\omega}$$

(use Frequency differentiation)

$$tx(t) \leftrightarrow j\frac{d}{d\omega}X(\omega)$$

$$F\{te^{-2t}u(t)\} = j\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right) = j\left(\frac{0-1(j)}{(2+j\omega)^2}\right) = \frac{1}{(2+j\omega)^2}$$

ⓒ $x(t) = e^{j2t}u(t)$

$$F\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega} \quad \text{(use Frequency shifting)}$$

$$F\{e^{j2t}u(t)\} = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)_{\omega=\omega-2} = \pi\delta(\omega-2) + \frac{1}{j(\omega-2)}$$

ⓓ $x(t) = e^{j\omega_0 t}$

$$F\{1\} = 2\pi\delta(\omega) \quad \text{(use Frequency shift)}$$

$$F\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0)$$

ⓔ $$F\{\cos\omega_0 t\, u(t)\} = F\left\{\frac{e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)}{2}\right\}$$

$$= \frac{1}{2}F\{e^{j\omega_0 t}u(t)\} + \frac{1}{2}F\{e^{-j\omega_0 t}u(t)\}$$

$$= \frac{1}{2}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\}_{\omega=\omega-\omega_0} + \frac{1}{2}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\}_{\omega=\omega+\omega_0}$$

$$= \frac{1}{2}\left\{\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega+\omega_0) + \frac{1}{j(\omega+\omega_0)}\right\}$$

$$= \frac{1}{2}\left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) + \frac{1}{j(\omega-\omega_0)} + \frac{1}{j(\omega+\omega_0)}\right\}$$

$$= \frac{1}{2}\left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) + \frac{j\omega+j\omega_0+j\omega-j\omega_0}{j^2(\omega^2-\omega_0^2)}\right\}$$

$$F\{\cos\omega_0 t\, u(t)\} = \frac{1}{2}\left\{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) + \frac{2j\omega}{\omega_0^2-\omega^2}\right\}$$

(f) $x(t) = \sin\omega_0 t\, u(t)$.

By $F\{\sin\omega_0 t\, u(t)\} = \frac{1}{2}\left\{\frac{j\omega_0}{\omega_0^2-\omega^2} + \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)\right\}$

$$= \frac{\omega_0}{\omega_0^2-\omega^2} + \frac{\pi}{j2}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right).$$

(g) $x(t) = e^{-3t}u(t-2)$.

$\Rightarrow$ $F\{e^{-3t}u(t)\} = \frac{1}{3+j\omega}$

$F\{e^{-3(t-2)}u(t-2)\} = e\frac{e^{-j2\omega}}{(3+j\omega)}$

$F\{e^{-6}e^{-3(t-2)}u(t-2)\} = \frac{e^{-6}e^{-j2\omega}}{3+j\omega} = \frac{e^{-2(3+j\omega)}}{3+j\omega}$.

(h) $x(t) = e^{-a|t-2|}$

$\Rightarrow$ $F\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2}$ $\quad F\{e^{-a|t-2|}\} = \frac{e^{-j2\omega}\,2a}{a^2+\omega^2}$.

(i) $x(t) = \delta(t+2) + \delta(t+1) + \delta(t-1) + \delta(t-2)$

Ans $X(\omega) = e^{j2\omega} + e^{j\omega} + e^{-j\omega} + e^{-j2\omega} = 2\cos 2\omega + 2\cos\omega$.

Ex:- Find the Fourier transform of the signal $x(t) = e^{-a|t|}\,sgn(t)$.

$$x(t) = e^{-a|t|}\,sgn(t)$$

i.e., $$x(t) = e^{-a|t|}\,sgn(t) = \begin{cases} e^{-a(-t)}(-1) & \text{for } t<0 \\ e^{-a(t)}(1) & \text{for } t>0. \end{cases}$$

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} -e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

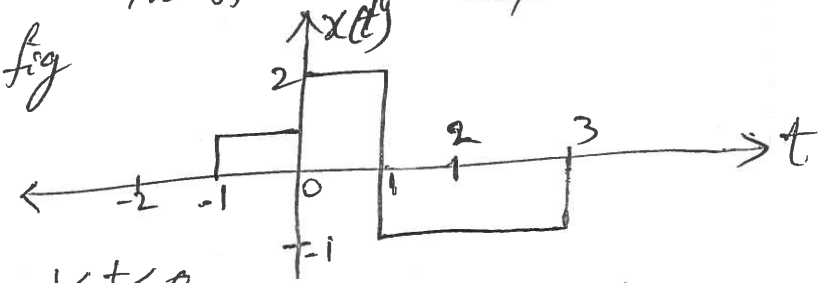
$$= -\int_{0}^{\infty} e^{-(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-at(a+j\omega)}\, dt$$

$$= \left( -\frac{e^{-(a-j\omega)t}}{-(a-j\omega)} + \frac{e^{-(a+j\omega)}}{-(a+j\omega)} \right)_{0}^{\infty}$$

$$= \left( -\frac{e^{-\infty} - e^{0}}{-(a-j\omega)} + \frac{e^{-\infty} - e^{0}}{-(a+j\omega)} \right) = \frac{-1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{-a-j\omega+a-j\omega}{a^2+\omega^2} = \frac{-2j\omega}{a^2+\omega^2}$$

Ex:- Find the Fourier Transform of the signal $x(t)$ shown in fig




$$x(t) = \begin{cases} 1 & -1<t<0 \\ 2 & 0<t<1 \\ -1 & 1<t<3 \\ 0 & \text{otherwise.} \end{cases}$$

$$X(\omega) = F(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-1}^{0} e^{-j\omega t}\, dt + \int_{0}^{1} 2\, e^{-j\omega t} + \int_{1}^{3} (-1) e^{-j\omega} dt$$

$$= \left( \frac{e^{-j\omega t}}{-j\omega} \right)_{-1}^{0} + \left( \frac{2\, e^{-j\omega t}}{-j\omega} \right) - \left( \frac{e^{-j\omega t}}{-j\omega} \right)_{1}^{3}$$

$$= \frac{-1}{j\omega}\left(1 - e^{j\omega} + 2e^{-j\omega} - 2e^{-j2\omega} + e^{-j3\omega}\right)$$

$$X(\omega) = \frac{1}{j\omega}\left(1 + e^{j\omega} - 3e^{-j\omega} + e^{-j3\omega}\right)$$

Ex:- By using partial fractions, find the inverse Fourier Transform of the following:

(a) $X(\omega) = \dfrac{4(j\omega)+6}{(j\omega)^2 + 6(j\omega) + 8}$ (b) $X(\omega) = \dfrac{1+3j\omega}{(j\omega+3)^2}$.

A (a) $X(\omega) = \dfrac{4(j\omega)+6}{(j\omega)^2 + 6j\omega + 8} = \dfrac{4j\omega + 6}{(j\omega+2)(j\omega+4)}$

$$X(\omega) = \frac{A}{j\omega+2} + \frac{B}{j\omega+4} \qquad A = -1 \quad B = 5$$

$$X(\omega) = \frac{-1}{j\omega+2} + \frac{5}{j\omega+4}$$

Taking Inverse Fourier Transform, we have

$$x(t) = -e^{-2t}u(t) + 5e^{-4t}u(t).$$

(b) $X(\omega) = \dfrac{1+3j\omega}{(j\omega+3)^2}$

Taking Partial fractions we have

$$X(\omega) = \frac{3(j\omega)+1}{(j\omega+3)^2} = \frac{A}{3+j\omega} + \frac{B}{(3+j\omega)^2}$$

$$= \frac{3}{3+j\omega} - \frac{8}{(j\omega+3)^2}$$

$$x(t) = 3e^{-3t}u(t) - 8te^{-3t}u(t).$$

Ex:- Using Fourier Transform, find the Convolution of the signals: $x_1(t) = e^{-2t}u(t)$ & $x_2(t) = e^{-3t}u(t)$

A $F\{x_1(t) * x_2(t)\} = X_1(\omega)\,X_2(\omega)$

$x_1(t) = e^{-2t}u(t)$ $\quad X_1(\omega) = \frac{1}{2+j\omega}$

$x_2(t) = e^{-3t}u(t)$ $\quad X_2(\omega) = \frac{1}{3+j\omega}$

~~$x_1(\omega) X_2(\omega) = F$~~

$x_1(t) * x_2(t) = F^{-1}\left(X_1(\omega) X_2(\omega)\right)$

$= F^{-1}\left(\frac{1}{(j\omega+2)(j\omega+3)}\right) = \frac{1}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$

$A=1 \quad B=-1$

$= F^{-1}\left(\frac{1}{j\omega+2} - \frac{1}{j\omega+3}\right) = F^{-1}\left(\frac{1}{2+j\omega}\right) - F^{-1}\left(\frac{1}{3+j\omega}\right)$

$x_1(t) * x_2(t) = e^{-2t}u(t) - e^{-3t}u(t).$

Ex:- Using the Fourier Transform Find the Convolution of Signals

$x_1(t) = te^{-t}u(t)$ & $x_2(t) = te^{-2t}u(t)$.

$X_1(\omega) = \frac{1}{(1+j\omega)^2}$ $\quad X_2(\omega) = \frac{1}{(2+j\omega)^2}$

$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$

$x_1(t) * x_2(t) = \frac{1}{(j\omega+1)^2(2+j\omega)^2} = \frac{A}{1+j\omega} + \frac{B}{(j\omega+1)^2} + \frac{C}{(j\omega+2)} + \frac{D}{(j\omega+2)^2}$

$= \frac{-2}{j\omega+1} + \frac{1}{(1+j\omega)^2} + \frac{2}{(j\omega+2)} + \frac{1}{(j\omega+2)^2}$

Taking Inverse F.T

$$x(t) = -2e^{-t}u(t) + te^{-t}u(t) + 2e^{-2t}u(t) + te^{-2t}u(t).$$

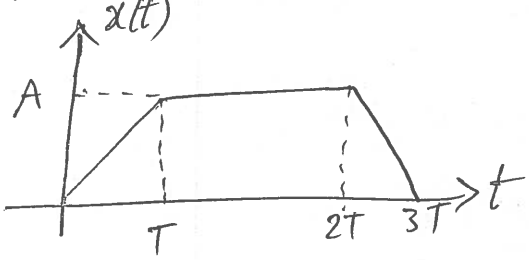
Ex:- Find the F.T of $x(t) = 5\sin^2 3t$.

$x(t) = 5\sin^2 3t = 5\left(\frac{1-\cos 6t}{2}\right) = \frac{5}{2} - \frac{5}{2}\cos 6t.$

$X(\omega) = F\left(\frac{5}{2}\right) - \frac{5}{2}F(\cos 6t) = \frac{5}{2}(2\pi\delta(\omega)) - \frac{5}{2}\left(\pi\delta(\omega-6) + \delta(\omega+6)\right)$

$X(\omega) = 5\pi\delta(\omega) - \frac{5\pi}{2}\left(\delta(\omega-6) + \delta(\omega+6)\right).$

Ex:- Find the F.T of Signal $x(t)$ shown in fig

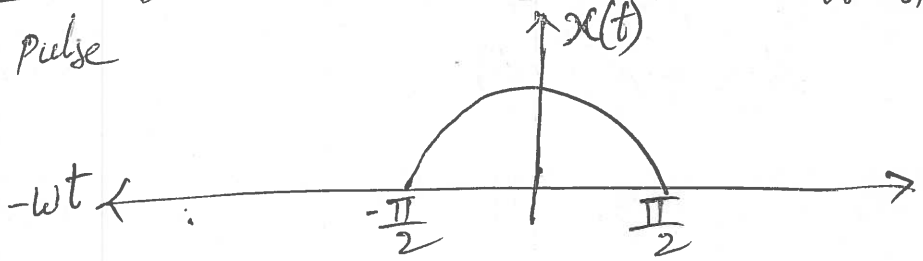



⇒

$$x(t) = \begin{cases} \frac{A}{T}t & 0 < t < T \\ A & T < t < 2T \\ \frac{-AT}{(t-3T)} & 2T < t < 3T. \end{cases}$$

$$X(\omega) = \frac{4A}{T(j\omega)^2} e^{-j(3/2)\omega T} \cos\left(\frac{\omega T}{2}\right) \cos\omega T.$$

Ex- Determine the ~~inverse~~ Fourier transform of the cosinusoidal pulse




⇒

$T = 2\pi$ $\quad x(t) = \cos\omega_0 t \quad -\frac{\pi}{2} < \omega t < \frac{\pi}{2}$

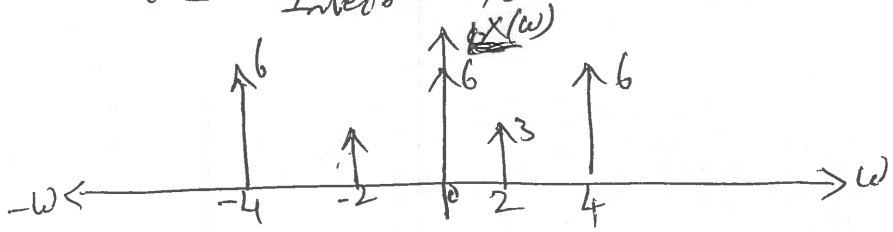
$\omega_0 = 1$ $\quad x(t) = \cos\omega t \quad -\frac{\pi}{2} < \omega t < \frac{\pi}{2}.$

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} d(\omega t) = \int_{-\pi/2}^{\pi/2} \cos\omega t \; e^{-j\omega t} d(\omega t)$$

$$= 2\int_{0}^{\pi/2} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) e^{-j\omega t} d(\omega t)$$

$$= \frac{\pi}{2} - j.$$

Ex- Find the Inverse Fourier Transform of spectrum given by




⇒

$F\{1\} = 2\pi\delta(\omega) \qquad e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega+\omega_0)$

$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \qquad F^{-1}\{\delta(\omega+2)\} = \frac{1}{2\pi} e^{-j2t}$

Using Time Reversal property $x(-t) \longleftrightarrow X(-\omega)$

$$F\left(\frac{1}{2-jt}\right) = 2\pi e^{-2\omega} u(\omega)$$

$$e^{-2\omega} u(\omega) = \frac{1}{2\pi} F\left(\frac{1}{2-jt}\right)$$

$$F^{-1}\left(e^{-2\omega} u(\omega)\right) = \frac{1}{2\pi(2-jt)}$$

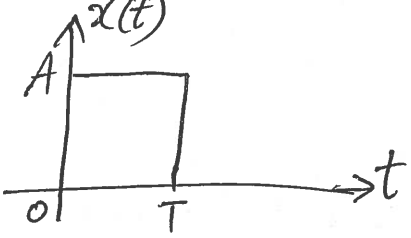
(or)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\omega} u(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{0}^{\infty} e^{-2\omega} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-(2-jt)\omega} d\omega = \frac{1}{2\pi} \left(\frac{e^{-(2-jt)\omega}}{-(2-jt)}\right)_0^{\infty} = \frac{1}{2\pi(2-jt)}.$$

Ex:- Find the F.T of the rectangular pulse shown in fig



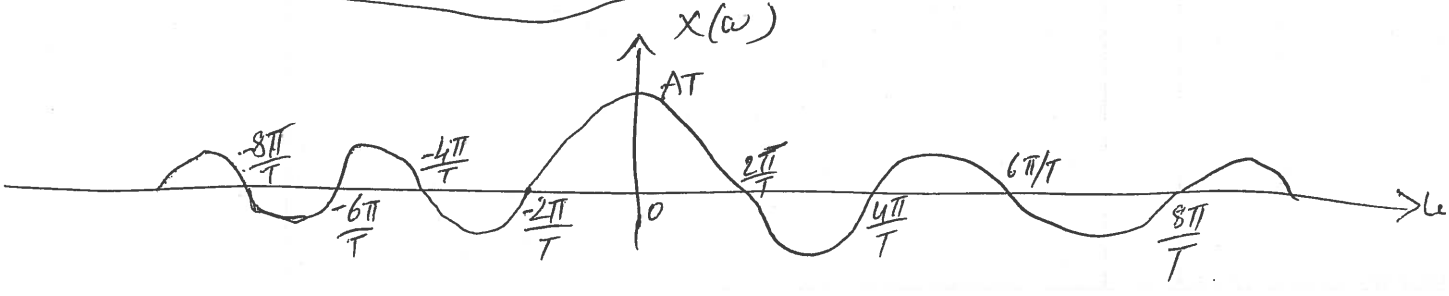

$$x(t) = \begin{cases} A & 0 < t < T \\ 0 & \text{elsewhere} \end{cases}$$

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{T} A e^{-j\omega t} dt = A\left(\frac{e^{-j\omega t}}{-j\omega}\right)_0^T$$

$$= A\left(\frac{e^{-j\omega T} - 1}{-j\omega}\right) = A\left(\frac{1 - e^{-j\omega T}}{j\omega}\right) = A e^{-\frac{j\omega T}{2}} \left(\frac{e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}}}{j\omega}\right)$$

$$= A e^{-\frac{j\omega T}{2}} \left(\frac{2 \sin\frac{\omega T}{2}}{\omega}\right) = \frac{2A}{\omega} e^{-\frac{j\omega T}{2}} \sin\frac{\omega T}{2}.$$

$$X(\omega) = AT e^{-j\omega T/2} \operatorname{sinc}\frac{\omega T}{2}$$



1) Find the Inverse Fourier transform of $X(\omega) = \frac{j\omega}{(2+j\omega)^2}$

Ans $F(t e^{-at} u(t)) = \frac{1}{(a+j\omega)^2}$

$F\{t e^{-2t} u(t)\} = \frac{1}{(2+j\omega)^2}$

$t e^{-2t} u(t) = x_1(t)$

$X_1(\omega) = \frac{1}{(2+j\omega)^2}$

using differentiation in time domain property $\left(\frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)\right)$ we have

$F\left(\frac{d}{dt} x_1(t)\right) = j\omega X_1(\omega)$

$F^{-1}\left(j\omega X_1(\omega)\right) = \frac{d}{dt}\left(x_1(t)\right) = \frac{d}{dt}\left(t e^{-2t} u(t)\right)$

$F^{-1}\left(j\omega X_1(\omega)\right) = t(-2)e^{-2t} + e^{-2t} = e^{-2t}(1-2t)$

Ex:- Find the inverse Fourier transform of $X(\omega) = e^{-|\omega|}$.

Δ we know that $F\{e^{-|t|}\} = \frac{2}{1+\omega^2}$

using Duality property $X(t) = 2\pi x(-\omega)$, we have

$F\left\{\frac{2}{1+t^2}\right\} = 2\pi\left(e^{-|-\omega|}\right) = 2\pi e^{-|\omega|}$

$e^{-|\omega|} = \frac{1}{2\pi} F\left\{\frac{2}{1+t^2}\right\}$

$F^{-1}\left(e^{-|\omega|}\right) = \frac{1}{2\pi}\left(\frac{2}{1+t^2}\right) = \frac{1}{\pi(1+t^2)}$.

Ex:- Find the inverse Fourier Transform of $X(\omega) = e^{-2\omega} u(\omega)$

Δ $F\{e^{-2t} u(t)\} = \frac{1}{2+j\omega}$.

using Duality property $X(t) \longleftrightarrow 2\pi x(-\omega)$, we have

$F\left(\frac{1}{2+jt}\right) = 2\pi e^{-2(-\omega)} u(-\omega) = 2\pi e^{2\omega} u(-\omega)$

# Fourier Transforms of Standard Signals:-

(1) $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$ $a>0$

(2) $e^{at}u(-t) \longleftrightarrow \frac{1}{a-j\omega}$ $a>0$

(3) $e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$

(4) $\frac{a^2}{a^2+t^2} \longleftrightarrow \pi a e^{-a|\omega|}$ $a>0$

(5) $te^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$ $a>0$

(6) $t^n e^{-at}u(t) \longleftrightarrow \frac{n!}{(a+j\omega)^{n+1}}$ $a>0$

(7) $\text{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau \,\text{Sinc}\left(\frac{\omega\tau}{2}\right)$

(8) $\text{Sinc}(Wt) \longleftrightarrow \frac{\pi}{W}\,\text{rect}\left(\frac{\omega}{2W}\right)$

(9) $\Delta\left(\frac{t}{\tau}\right) \longleftrightarrow \frac{\tau}{2}\,\text{Sinc}^2\left(\frac{\omega\tau}{4}\right)$

(10) $\text{Sinc}^2\left(\frac{Wt}{2}\right) \longleftrightarrow \frac{2\pi}{W}\Delta\left(\frac{\omega}{2W}\right)$

(11) $e^{-at}\sin\omega_0 t\, u(t) \longleftrightarrow \frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$, $a>0$

(12) $e^{-at}\cos\omega_0 t\, u(t) \longleftrightarrow \frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$, $a>0$

(13) $\delta(t) \longleftrightarrow 1$

(14) $1 \longleftrightarrow 2\pi\delta(\omega)$

(15) $\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$

(16) $e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

(17) $\cos\omega_0 t \longleftrightarrow \pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$

(18) $\sin\omega_0 t \longleftrightarrow j\pi\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)$

(19) $\cos\omega_0 t\, u(t) \longleftrightarrow \frac{\pi}{2}\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)+\frac{j\omega}{\omega_0^2-\omega^2}$

(20) $\sin\omega_0 t\, u(t) \longleftrightarrow j\frac{\pi}{2}\left(\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right)+\frac{\omega_0}{\omega_0^2-\omega^2}$

(21) $\text{Sgn}(t) \longleftrightarrow \frac{2}{j\omega}$

(22) $u(t) \longleftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$

(23) $\sum_{n=-\infty}^{\infty}\delta(t-nT) \longleftrightarrow \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-n\frac{2\pi}{T}\right)$

(24) $\sum_{n=-\infty}^{\infty}f(t)\delta(t-nT) \longleftrightarrow \sum_{n=-\infty}^{\infty}\frac{1}{T}F\left(\omega-n\frac{2\pi}{T}\right)$

$$F^{-1}\{\delta(\omega-2)\} = \frac{1}{2\pi} e^{j2t} \qquad F^{-1}\{\delta(\omega+4)\} = \frac{1}{2\pi} e^{-j4t}$$

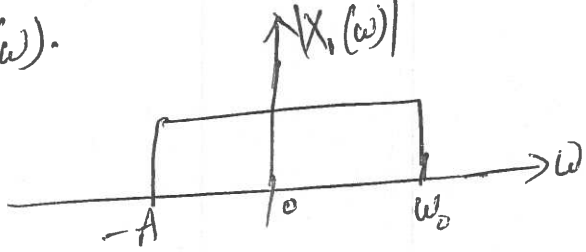
$$F^{-1}\{\delta(\omega-4)\} = \frac{1}{2\pi} e^{j4t}.$$

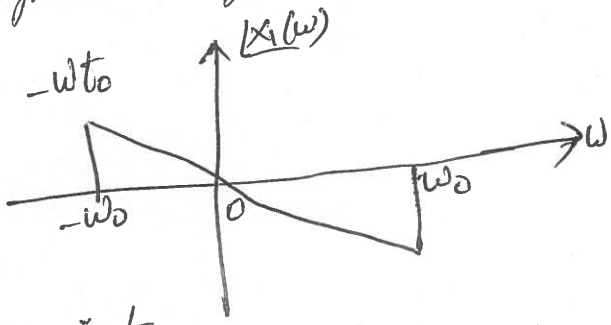
Inverse Fourier transform of $X(\omega)$ is:

$$x(t) = \frac{6}{2\pi} + 3\left(\frac{e^{-j2t}+e^{j2t}}{4}\right) + 6\left(\frac{e^{-j4t}+e^{j4t}}{2\pi}\right)$$

$$= \frac{3}{\pi}\left(1+\cos 2t + 2\cos 4t\right).$$

Ex:- Find the time domain signal for given spectra of $X(\omega)$.



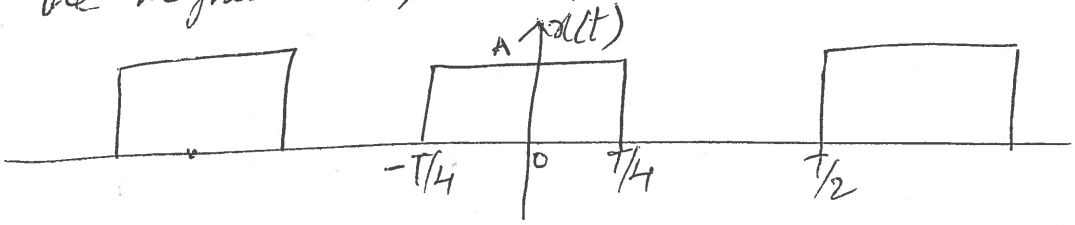

$$X_1(\omega) = |X_1(\omega)|\, e^{-j\angle X_1(\omega)} = A e^{-j\omega t_0} \quad \text{for } -\omega_0 < \omega < \omega_0$$

$$x_1(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} A\, e^{-j\omega t_0} e^{j\omega t}\, d\omega$$

$$= \frac{A}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\, d\omega = \frac{A}{\pi(t-t_0)} \sin\omega_0(t-t_0)$$

$$x_1(t) = \frac{A\omega_0}{\pi}\, \text{Sin c}\left(\omega_0(t-t_0)\right)$$

(1) obtain the Exponential Fourier Series for the following waveform. plot the magnitude & phase spectra.




$$x(t) = A \quad \text{for } -\frac{T}{4} < t < T/4$$
$$= 0 \quad \frac{T}{4} < t < \frac{T}{2}$$
$$= 0 \quad -\frac{T}{2} < t < \frac{-T}{4}$$

$$C_0 = \frac{A}{2} \qquad C_n = \frac{A}{n\pi} \sin\left(\frac{\pi}{2} n\right)$$

$-\frac{T}{2}$ to $\frac{T}{2}$ but Signal Exist only $-\frac{T}{4}$ to $T/4$.

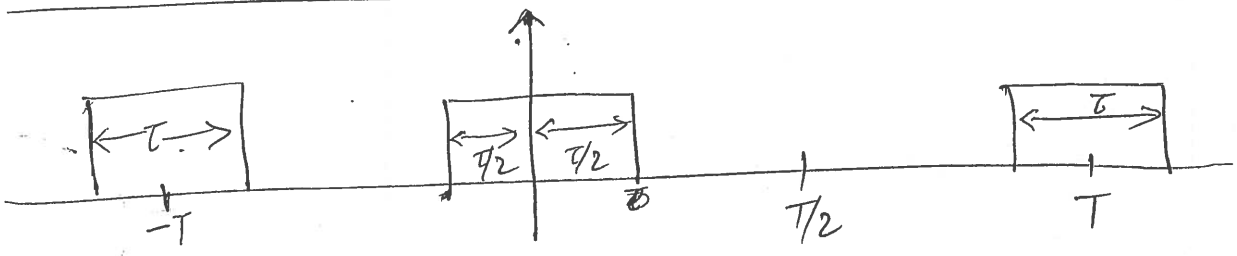
$$x(t) = \left(\sum_{n=-\infty}^{\infty} \frac{A}{n\pi} \sin\left(\frac{\pi}{2} n\right) e^{jn\omega_0 t}\right) + \underbrace{(C_0)}_{A/2}$$

~~$C_n = \frac{A}{n\pi} S$~~

$$C_1 = C_{-1} = \frac{A}{\pi}$$
$$C_2 = C_{-2} = 0$$
$$C_3 = C_{-3} = \frac{A}{-3\pi}$$
$$\vdots$$

Fullwave Rectifier

$$C_0 = \frac{2A}{\pi} \qquad C_n = \frac{2A}{\pi(1-n^2 4)}.$$




$$C_0 = \frac{a\tau}{T} \qquad C_n = \frac{A\tau}{T} + \frac{A}{T}\tau \sum_{n=-\infty}^{\infty} \text{Sinc}\left(n\omega_0 \frac{\tau}{2}\right) e^{jn\omega_0 t}$$

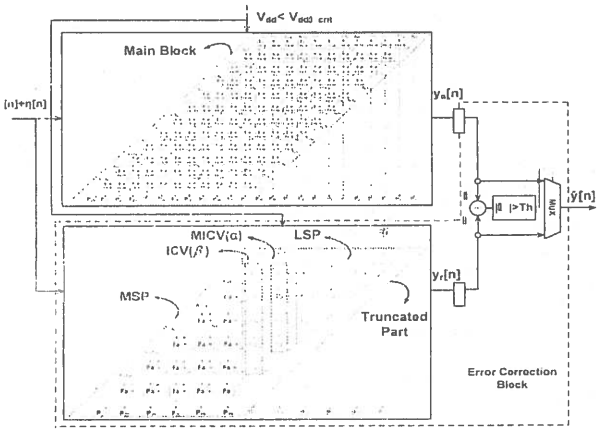



Fig. 3. 12 × 12 bit ANT multiplier is implemented with the six-bit fixed-width replica redundancy block.

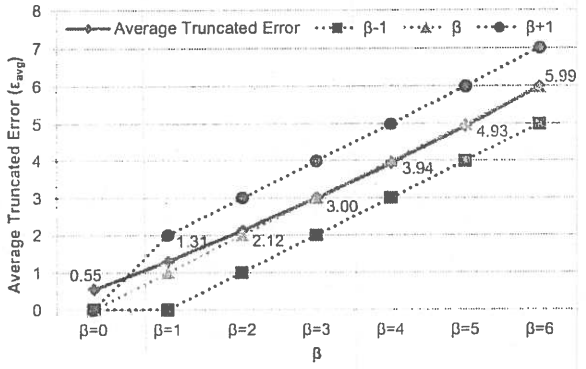



Fig. 4. Statistical curves of average truncation error in the LSP block and the curves of compensation function with $\beta - 1$, $\beta$, and $\beta + 1$ in the 12-bit fixed-width RPR-based ANT multiplier.

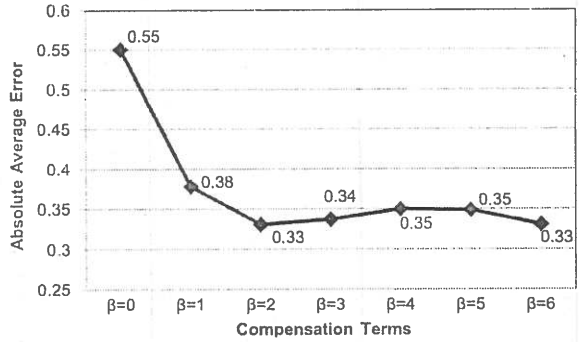



Fig. 5. Analysis of absolute average compensation error under various $\beta$ values in the 12-bit fixed-width RPR-based ANT multiplier.

ICV [MICV($\alpha$)], and LSP, as shown in Fig. 3. In the fixed-width RPR, only MSP part is kept and the other parts are removed. Therefore, the other three parts of ICV($\beta$), MICV($\alpha$), and LSP are called as truncated part. The truncated ICV($\beta$) and MICV($\alpha$) are the most important parts because of their highest weighting. Therefore, they can be applied to construct the truncation error compensation algorithm.

To evaluate the accuracy of a fixed-width RPR, we can exploit the difference between the $(n/2)$-bit fixed-width RPR output and the $2n$-bit full-length MDSP output, which is expressed as

$$\varepsilon = P - P_t \tag{5}$$

where $P$ is the output of the complete multiplier in MDSP and $P_t$ is the output of the fixed-width multiplier in RPR. $P_t$ can be expressed as

$$\begin{aligned} P_t &= \sum_{j=\frac{n}{2}+1}^{n-1} y_j 2^j \sum_{i=\frac{3n}{2}-j}^{n-1} x_i 2^i + f\left(x_{n-1}y_{\frac{n}{2}}, x_{n-2}y_{\frac{n}{2}+1}, x_{n-3}y_{\frac{n}{2}+2}, \ldots, x_{\frac{n}{2}}y_{\frac{n}{2}+2}\right) \\ &\quad + f\left(x_{n-2}y_{\frac{n}{2}}, x_{n-3}y_{\frac{n}{2}+1}, x_{n-4}y_{\frac{n}{2}+2}, \ldots, x_{\frac{n}{2}}y_{n-2}\right) \\ &= \sum_{j=\frac{n}{2}+1}^{n-1} y_j 2^j \sum_{i=\frac{3n}{2}-j}^{n-1} x_i 2^i + f\,(\mathrm{ICV}) + f(\mathrm{MICV}) \\ &= \sum_{j=\frac{n}{2}+1}^{n-1} y_j 2^j \sum_{i=\frac{3n}{2}-j}^{n-1} x_i 2^i + f\,(\mathrm{EC}) \end{aligned} \tag{6}$$

where $f$(EC) is the error compensation function, $f$(ICV) is the error compensation function contributed by the input correction vector ICV($\beta$), and $f$(MICV) is the error compensation function contributed by minor input correction vector MICV($\alpha$).

The source of errors generated in the fixed-width RPR is dominated by the bit products of ICV since they have the largest weight. In [8], it is reported that a low-cost EC circuit can be designed easily if a simple relationship between $f$(EC) and $\beta$ is found. It is noted that $\beta$ is the summation of all partial products of ICV. By statistically analyzing the truncated difference between MDSP and fixed-width RPR with uniform input distribution, we can find the relationship between $f$(EC) and $\beta$. As shown in Fig. 4, the statistical results show that the average truncation error in the fixed-width RPR multiplier is approximately distributed between $\beta$ and $\beta+1$. More precisely, as $\beta = 0$, the average truncation error is close to $\beta + 1$. As $\beta > 0$, the average truncation error is very close to $\beta$. If we can select $\beta$ as the compensation vector, the compensation vector can directly inject into the fixed-width RPR as compensation, which does not need extra compensation logic gates [17].

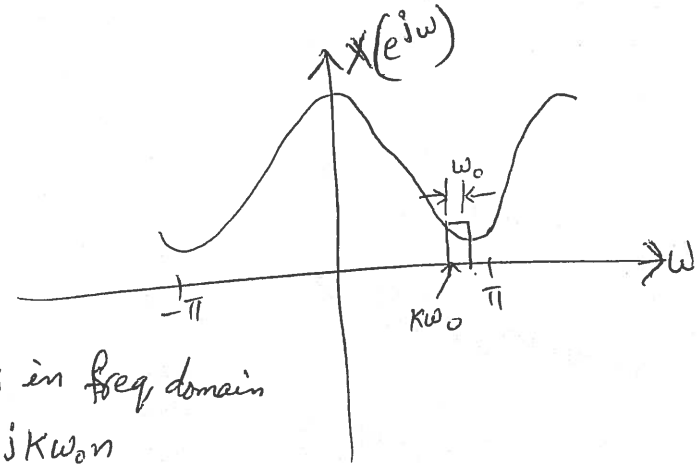
We go further to analyze the compensation precision by selecting $\beta$ as the compensation vector. We can find that the absolute average error in $\beta = 0$ is much larger than that in other $\beta$ cases, as shown in Fig. 5. Moreover, the absolute average error in $\beta = 0$ is larger than $0.5*2^{(3n/2)}$, while the absolute average error in other $\beta$ situations is smaller than $0.5*2^{(3n/2)}$. Therefore, we can apply multiple input error compensation vectors to further enhance the error compensation precision. For the $\beta > 0$ case, we can still select $\beta$ as the compensation vector. For the $\beta = 0$ case, we select $\beta + 1$ combining with MICV as the compensation vector.

Before directly injecting the compensation vector $\beta$ into the fixed-width RPR, we go further to double check the weight for the partial product terms in ICV with the same partial product summation value $\beta$ but with different locations. As shown in

$$c_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-jK\frac{2\pi}{N}n}$$

$$c_K = \frac{1}{N} \sum_{n=-(\frac{N}{2}-1)}^{\frac{N}{2}} x(n)\, e^{-\frac{j2\pi Kn}{N}} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x(n)\, e^{-\frac{j2\pi Kn}{N}}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$




$$c_K = \frac{1}{N} X(e^{jK\omega_0})$$

$\omega_0 = \frac{2\pi}{N}$ is the spacing of samples in freq domain

$$\tilde{x}(n) = \sum_K \frac{1}{N} X(e^{jK\omega_0})\, e^{jK\omega_0 n}$$

$$\tilde{x}(n) = \sum_K \frac{2\pi}{N} X(e^{jK\omega_0})\, e^{jK\omega_0 n} = \sum_K X(e^{jK\omega_0})\, e^{jK\omega_0 n}\, \omega_0$$

$N \uparrow \; \omega_0 \downarrow$ and As $N \to \infty$

$$K\omega_0 = \omega$$

$\omega_0 \to 0$ Summation becomes integral.

$N \to \infty \quad \tilde{x}(n) = x(n) \quad (\text{or}) \quad x_N(n) = x(n)$

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

---

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) \Rightarrow X_s(f) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j2\pi fnT}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j\omega nT} \qquad T = 1\,\text{sec} \quad X(e^{j\omega}) = \sum x(n)\, e^{-j\omega n}$$

$$\sum_{n=-\infty}^{\infty} x(nT)\, e^{-j(\omega+\omega_s)nT} = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j\omega nT} \qquad f_s = \frac{1}{T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi nf_0 t}\, dt$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

**VAAGDEVI INSTITUTE OF TECHNOLOGY AND SCIENCE**

**Peddasettipally, Proddatur.**

**Gate Model Exam -3 Key**

**Subjects: EDC , NT and Aptitude** **Max Marks: 100** **Date: 06/10/12**

**Questions 1 to 30 carry one mark:**

1) C
2) B
3) D
4) C
5) A
6) D
7) B
8) C
9) C
10) D
11) B
12) C
13) B
14) A
15) D
16) C
17) B
18) A
19) B
20) C
21) A
22) A
23) D
24) B
25) C
26) A
27) C
28) D
29) C
30) C

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2) N.NAGA TEJASWINI-09L21A0234
   D. PRATHAP REDDY-09L21A0422
3) D.DIVYA-09L21A0211
   S.AHMED HUSSAIN-09L21A0243
4) O.ROHITH-09L21A0235
   M.MOUNIKA-09L21A0227
   P.HARINATH REDDY-09L21A0238
5) P.NARASAIAH-09L21A0239

**Questions 31 to 65 carry two marks:**

31) B
32) C
33) B
34) C
35) A
36) D
37) C
38) A
39) A
40) D
41) B
42) B
43) B
44) B
45) A
46) C
47) D
48) C
49) C
50) B
51) A
52) B
53) B
54) C
55) D
56) B
57) B
58) C
59) D
60) C
61) A
62) A
63) D
64) C
65) D

$$\sum_{n=-\infty}^{\infty} x(n) \left(\frac{e^{j\omega(m-n)}}{j(m-n)}\right)_{-\pi}^{\pi}$$

$$\sum_{n=-\infty}^{\infty} \frac{x(n)}{j(m-n)} \left( e^{j\pi(m-n)} - e^{-j\pi(m-n)} \right)$$

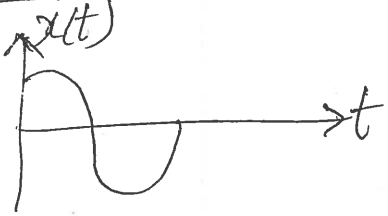
$$\sum \frac{x(n)}{(m-n)} 2\sin\pi(m-n)$$

$$\frac{2\pi\, x(m)}{2\pi} \quad \begin{matrix} m=n \\ m\neq n \end{matrix}$$

12/10/12

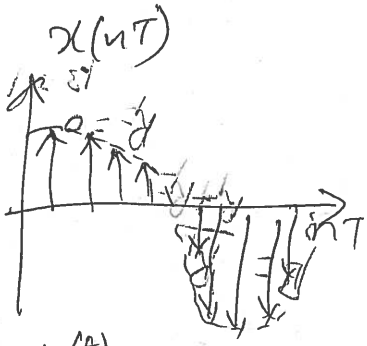
**Signature of the HOD.**

# Sampling theorem:-

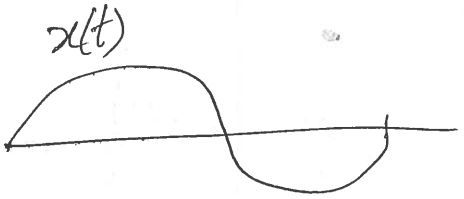


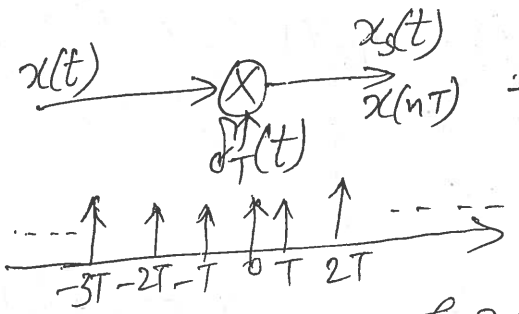

$x(t) \xrightarrow{\text{Sampling}} x_s(t)$

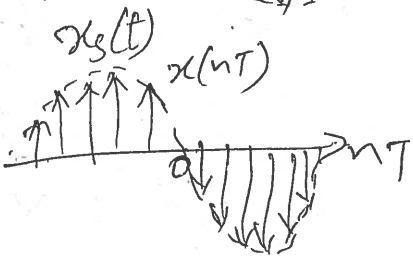
$x(n) = x(nT)$




Sampling operation



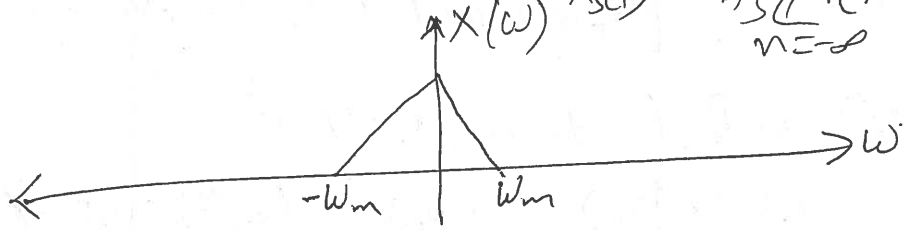

$x_s(t) = x(t)\,\delta_T(t)$ where $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

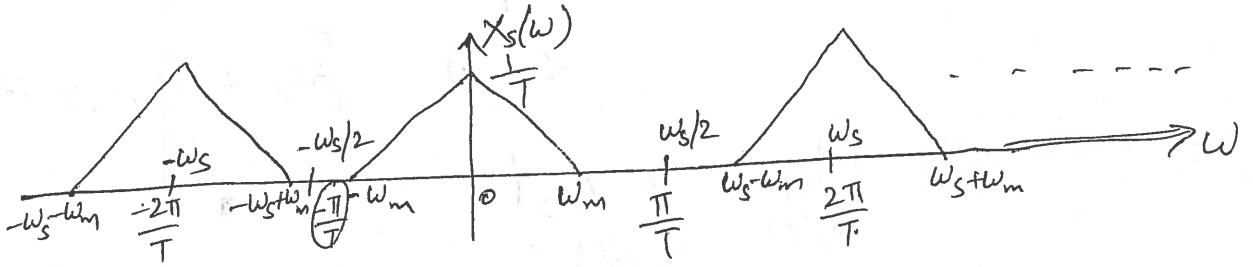
$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t}$ where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_s t} = \frac{1}{T}$

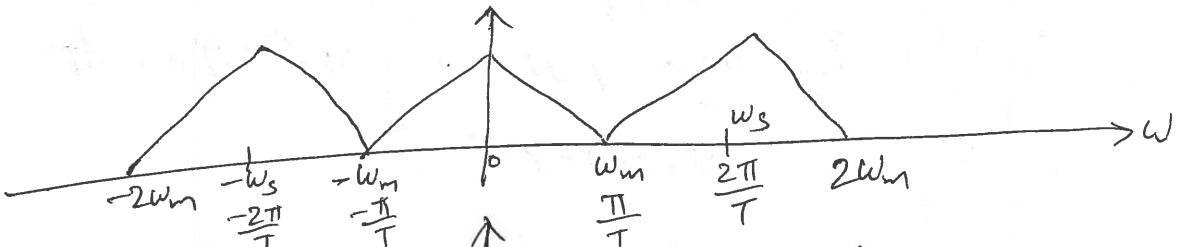
$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{jn\omega_s t}$

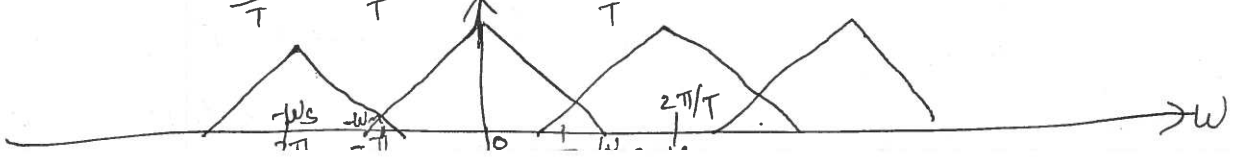
$F\{x_s(t)\} = F\{x(t)\,\delta_T(t)\} = F\left\{x(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right\}$

$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$



$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$F\{x(t)\} = X(\omega) = F\left\{\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right\} = \sum_{n=-\infty}^{\infty} C_n F\{e^{jn\omega_0 t}\}$$

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$

even × even = even
odd × odd = even
odd × even = odd

## FT of Complex & Real functions

$$x(t) = x_R(t) + j x_I(t)$$

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} = \int (x_R(t) + j x_I(t))(\cos\omega t - j\sin\omega t)\, dt$$

$$= \underbrace{\int (x_R(t)\cos\omega t + x_I(t)\sin\omega t)\, dt}_{X_R(\omega)} + j \underbrace{\int (x_I(t)\cos\omega t - x_R(t)\sin\omega t)\, dt}_{X_I(\omega)}$$

Ily $\quad x(t) = \frac{1}{2\pi}\int X(\omega) e^{j\omega t} d\omega$

$$x_R(t) = \frac{1}{2\pi}\int X_R(\omega)\cos\omega t - X_I(\omega)\sin\omega t\, d\omega$$

$$x_I(t) = \frac{1}{2\pi}\int X_R(\omega)\sin\omega t + X_I(\omega)\cos\omega t\, d\omega.$$

**Case 1:-** If $x(t)$ is Real then $x_I(t) = 0$

$$X(-\omega) = X^*(\omega) \qquad X_R(\omega) = \int_{-\infty}^{\infty} x(t)\cos\omega t\, dt$$

$$X_I(\omega) = -\int_{-\infty}^{\infty} x(t)\sin\omega t\, dt$$

**Case 2:-** $x(t)$ is even & Real

$$x(t) = x_e(t)$$

$$X_R(\omega) = \int_{-\infty}^{\infty} x_e(t)\cos\omega t\, dt = 2\int_{0}^{\infty} x_e(t)\cos\omega t\, dt$$

$$X_I(\omega) = 0.$$

$$X(\omega) = X_R(\omega)$$

**Case 3:-** $x(t) = x_o(t) \rightarrow$ is odd $\qquad X_R(\omega) = 0.$

$$X_I(\omega) = jX(\omega) = -j\int x_o(t)\sin\omega t = -j2\int x_o(t)\sin\omega t\, dt$$

(1) Find the complex exponential Fourier series representation of the following signals:

(a) $x(t) = 4\cos 2\omega_0 t$.

$$x(t) = 4\cos 2\omega_0 t = 4\left(\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2}\right)$$

$$x(t) = 2e^{j2\omega_0 t} + 2e^{-j2\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_2 = c_{-2} = 2 \qquad c_n = 0 \text{ for } n \neq 2.$$

(b) $x(t) = 3\sin 4\omega_0 t$

$$x(t) = 3\left(\frac{e^{j4\omega_0 t} - e^{-j4\omega_0 t}}{2j}\right) = \frac{1.5}{j}e^{j4\omega_0 t} - \frac{1.5}{j}e^{-j4\omega_0 t}$$

$$c_4 = \frac{1.5}{j} \qquad c_{-4} = \frac{-1.5}{j}$$

(c) $x(t) = \sin\left(2t + \frac{\pi}{4}\right)$

$$x(t) = \frac{e^{j\left(2t+\frac{\pi}{4}\right)} - e^{-j\left(2t+\frac{\pi}{4}\right)}}{2j} = \left(\frac{1}{j2}e^{j\pi/4}\right)e^{j2t} + \left(\frac{-1}{j2}e^{-j\pi/4}\right)e^{-j2t}$$

$$c_2 = \frac{1}{2j}e^{j\pi/4} = -0.5j\,e^{j\pi/4}$$

$$c_{-2} = \frac{-1}{j2}e^{j(\pi/4)} = 0.5j\,e^{-j\pi/4}$$

$$c_n = 0 \quad \text{for } |n| \neq 2$$

(d) $x(t) = \cos^2 t$

$$x(t) = \left(\frac{e^{jt} + e^{-jt}}{2}\right)^2 = \frac{1}{4}\left(e^{j2t} + e^{-j2t} + 2e^{jt}e^{-jt}\right) = \frac{1}{4}\left(e^{j2t} + e^{-j2t} + 2\right)$$

$$x(t) = \frac{1}{4}e^{j2t} + \frac{1}{2} + \frac{1}{4}e^{-j2t}$$

$$c_{-2} = \frac{1}{4} \qquad c_2 = \frac{1}{4} \qquad c_0 = \frac{1}{2}$$

# UNIT-3

## Laplace Transforms & Z-Transforms

### Introduction:-

A Linear Time Invariant (LTI) System is described by Differential Equations. The response of a system for a given i/p is obtained by solving the D.Es relating its output & i/p Signals. Solution of higher order D.Es is quite complex & tedious and time Consuming Compared to the Solution of Algebraic Equations. So the laplace transform is used to solve the Differential Equations. we Convert the D.Es in time domain in to algebraic equations in frequency domain by inserting the initial Conditions, solve the resultant algebraic equations and take inverse Laplace transforms of the Solution

The Laplace transform of a time domain function $x(t)$ is defined a

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t) e^{-st} dt. \longrightarrow (1)$$

S is a Complex Variable and is equal to $S = \sigma \pm j\omega$.

L is a Operator called the Laplace transform operator.

The above eq(1) is called Bilateral laplace transform (or) two Sided Laplace transform.

If the lower limit is changed to '0', we get the one sided La transform (or) unilateral Laplace transform.

$$L\{x(t)\} = X(S) = \int_{0}^{\infty} x(t) e^{-st} dt \longrightarrow (2)$$

Unilateral & Bilateral laplace transforms are equivalent only if $x(t$ for $t<0$ ie., $x(t)$ must be a Causal Signal. The laplace transfo pair is denoted by $x(t) \longleftrightarrow X(S)$.

## Region of Convergence :- (ROC)

For a given value of $x(t)$, the laplace transform $X(s)$ may not converge for all values of the complex variable $s$. Since every value of $s$ corresponds to a particular point in the s-plane ($\sigma$ along the x-axis and $j\omega$ along y-axis). We say that the set of points in the s-plane for which the integral on the right hand side of equation for $X(s)$ converges is the Region of Convergence (ROC) of the Laplace transform of $x(t)$. If there is no value of $s$, i.e., no point on the s-plane for which the integral converges, we say that $x(t)$ does not have a ROC and hence does not have a laplace transform, or that it is not Laplace transformable. Most of the functions which are of some use in engineering practice are Laplace transformable. For an $x(t)$ to be Laplace transformable, the usual condition is that it should be piece wise continuous & must be of exponential order.

If the ROC is not specified the inverse Laplace transform is not unique. Also in the one-sided Laplace transform, all the time functions are assumed to be positive and hence there is a one-to-one correspondence between the Laplace transform & its inverse laplace transform. ∴ No Ambiguity will arise even if the ROC is not specified in the one sided laplace Laplace transform. However in two-sided Laplace transform the specification of ROC is essential.

The bilateral Laplace transform of a signal $x(t)$ exists if

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \text{ is finite.}$$

$s = \sigma + j\omega$ in the above equation, we get

$$L(x(t)) = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt = \int_{-\infty}^{\infty} \left(x(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt$$

$$L\{x(t)\} = F\{x(t)\, e^{-\sigma t}\} = F\{x(t) e^{-\sigma t}\}.$$

This equation indicates that $X(s)$ is basically the Continuous time Fourier transform of $x(t)e^{-\sigma t}$. So we can say that

The Laplace transform of $x(t)$ is the Fourier transform of $x(t)e^{-\sigma t}$.

## Existence of Laplace Transform :-

The necessary & sufficient conditions for the existence of the Laplace transform are:

① $x(t)$ should be continuous (or) piece-wise continuous in the given closed interval.

② $x(t)\, e^{-\sigma t}$ must be absolutely integrable.

That is $X(s)$ exists only if $\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}|\, dt < \infty$.

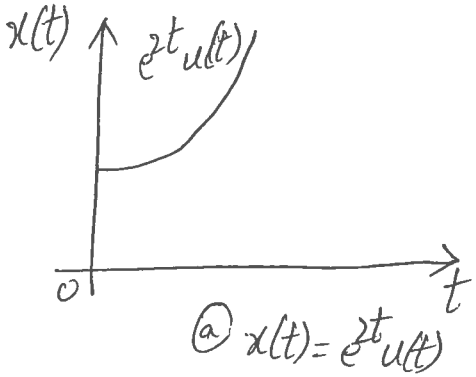
(or) only if $\lim_{t \to \infty} e^{-st} x(t) = 0$.

The range of $\sigma$ for which the laplace transform converges is known as the Region of Convergence (ROC). So the functions which are not Fourier transformable may be laplace transformable.

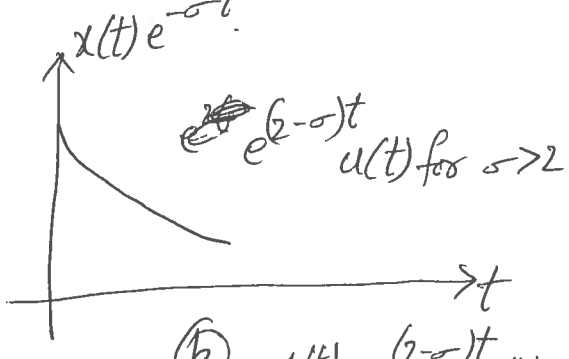
The range of $\sigma$ for which the laplace transform converges is known as the Region of Convergence (ROC). So the functions which are not Fourier transformable may be laplace transformable.

$x(t) = e^{2t} u(t)$ → Fourier transform does not exist because $\int_{-\infty}^{\infty} e^{2t} u(t)\, e^{-j\omega t}\, dt$ doesn't exist because it is not absolutely integrable.

Now multiply $x(t)$ with $e^{-\sigma t}$. Then the resultant signal $x(t) e^{-\sigma t}$ $u(t) = e^{(2-\sigma)t} u(t)$ is absolutely integrable for $\sigma > 2$. ∴ the laplace transform of $x(t)$ which is the F.T of $x(t) e^{-\sigma t}$ exist for some values of $\sigma$. we conclude that the L.T exists for signals for which the Fourier transform does not exist.




(a) $x(t) = e^{2t} u(t)$




(b) $x(t) = e^{(2-\sigma)t} u(t)$.

### Advantages:-

① Signals which are not convergent in Fourier transform are convergent in Laplace transform.

(2) Convolution in time domain can be obtained by multiplication in S-domain.

(3) Integro differential equations of a system can be converted into simple algebraic equations. So LTI systems can be analysed easily using Laplace transforms.

Limitations:-

(1) Frequency response of the system can't be drawn (or) estimated. Instead only the pole-zero plot can be drawn.

(2) $S=j\omega$ is used only for sinusoidal steady-state analysis.

Relationship between Fourier Transform & Laplace Transform:-

The Fourier transform of a function $x(t)$ can be represented by a continuous sum of exponential functions of the form $e^{j\omega t}$ i.e., it is a summation of waves of positive & negative frequencies. Moreover Fourier transform does not exist for many signals as $|x(t)|dt$ is not absolutely integrable. Moreover the Fourier transform can't be used to Analyse unstable system.

Laplace transform can be used where Fourier transform can't be used.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

where

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Including the convergence factor by replacing $\omega$ by $\sigma + j\omega$, we have

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

and $x(t)$ can be expressed as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega$$

$$S = \sigma + j\omega \qquad \frac{ds}{d\omega} = j \qquad d\omega = \frac{ds}{j}$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - \infty}^{\sigma + \infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} ds = \frac{1}{2\pi j} \int_{\sigma - \infty}^{\sigma + \infty} X(s) e^{st} ds.$$

Laplace transform is nothing but the complex fourier transform i.e., the laplace transform of $x(t)$ is the fourier transform of $x(t)e^{-\sigma t}$. (3-21)

If $x(t)$ is absolutely integrable then the fourier transform of $x(t)$ can be obtained from its Laplace transform using the relation

$$X(\omega) = X(s)\big|_{s=j\omega}. \qquad \boxed{X(\omega) = X(s)\big|_{s=j\omega}}$$

This means that the F.T is the laplace transform along the imaginary axis of the s-plane.

One-Sided Laplace Transform (unilateral) of Some Commonly used Signals:-

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt = \int_0^\infty \left(x(t)e^{-\sigma t}\right)e^{-j\omega t}\,dt$$

Because $|e^{j\omega t}| = 1$ the integral converges if $\int_0^\infty |x(t)e^{-\sigma t}\,dt| < \infty$

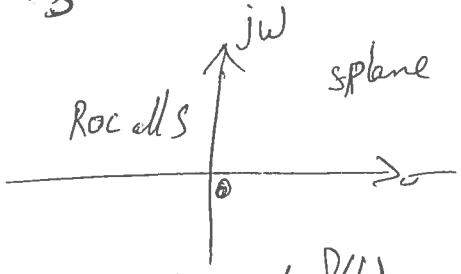
Impulse function:- $x(t) = \delta(t)$.

$$\delta(t) = 1 \quad t = 0$$
$$= 0 \quad t \neq 0$$

$$L\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st}\,dt = \left(e^{-st}\right)_{t=0} = e^0 = 1.$$

$$\delta(t) \xleftrightarrow{LT} 1 \quad \text{for all } s$$

ROC is entire ~~s-plane~~ s-plane.

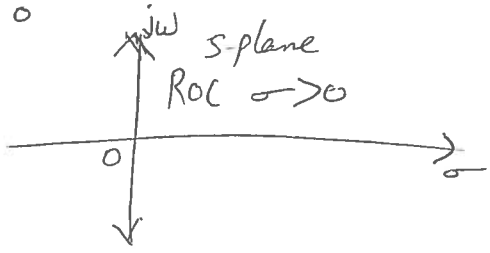



Step function:- $x(t) = u(t)$

$$u(t) = 1 \quad t \geq 0$$
$$= 0 \quad t < 0$$

$$L\{x(t)\} = L\{u(t)\} = X(s) = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt$$

$$X(s) = \left(\frac{e^{-st}}{-s}\right)_0^\infty = \left(\frac{0-1}{-s}\right) = \frac{1}{s}$$




ROC is ~~Re(s)~~ $\mathrm{Re}(s) > 0$

ROC is the entire right half of s-plane as shown in fig.

$L\{x(t)\} = L\{K\} = \frac{K}{s}$ if $K$ is a constant.

Since the one-sided Laplace transform doesnot take into account the values of $x(t)$ for $t<0$, the one-sided Laplace transform of $x(t)=1$ & $x(t)=u(t)$ will be same

Ramp function $x(t) = t\,u(t)$ :-

$$L\{t\,u(t)\} = X(s) = \int_0^\infty t\,u(t)\,e^{-st}\,dt = \int_0^\infty t\,e^{-st}\,dt$$

$$= \left(t\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt = 0 - \left(\frac{e^{-st}}{s^2}\right)_0^\infty$$

$$L\{t\,u(t)\} = \frac{1}{s^2}$$

ROC:- $Re\{s\} > 0$

$$t\,u(t) \longleftrightarrow \frac{1}{s^2}.$$

Parabolic function $x(t) = t^2 u(t)$ :-

$$X(s) = L\{x(t)\} = \int_0^\infty t^2 u(t) e^{-st}\,dt = \int_0^\infty t^2 e^{-st}\,dt$$

$$X(s) = \left(t^2\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(2t)\,dt$$

$$X(s) = 0 + \frac{2}{s}\int_0^\infty t\,e^{-st}\,dt = \frac{2}{s}\left\{\left(t\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt\right\}$$

$$X(s) = \frac{2}{s}\left(0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt\right) = \frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{2}{s^3}$$

ROC is $Re(s) > 0$.

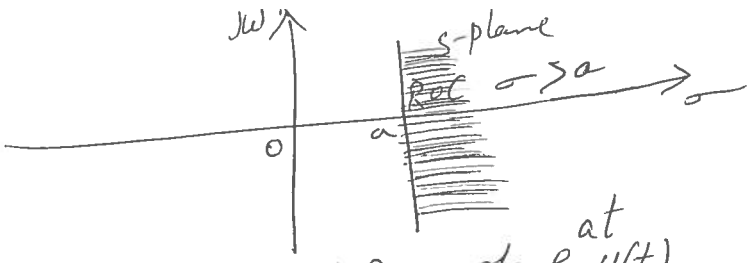
Real Exponential function $x(t) = e^{at}u(t)$.

$$X(s) = L\{e^{at}u(t)\} = \int_0^\infty e^{at}u(t)\,e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$X(s) = \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty = -\left(\frac{0-1}{s-a}\right) = \frac{1}{s-a}.$$

The above integral converges for $Re(s-a) > 0$ ie, $Re(s) > a$.

Hy $e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$ ROC is $Re(s) > -a$.




Roc of $e^{at}u(t)$

Complex Exponential function $x(t) = e^{jwt}u(t)$ :-

$$X(s) = L\{e^{jwt}u(t)\} = \int_0^\infty e^{jwt}u(t)\,e^{-st}\,dt$$

$$X(s) = \int_0^\infty e^{-t(s-jw)}\,dt = \left(\frac{e^{-t(s-jw)}}{-(s-jw)}\right)_0^\infty = \left(\frac{0-1}{-(s-jw)}\right)$$

$$X(s) = \frac{1}{s-jw}$$

$$e^{jwt}u(t) \xleftrightarrow{LT} \frac{1}{s-jw}$$

Ily $e^{-jwt}u(t) \longleftrightarrow \frac{1}{s+jw}$

Sine & Cosine functions :- $x(t) = \cos wt\, u(t)$.

$$X(s) = L\{\cos wt\, u(t)\} = L\left\{\frac{e^{jwt}+e^{-jwt}}{2}u(t)\right\}$$

$$X(s) = \int_0^\infty \frac{e^{jwt}}{2}u(t)\,e^{-st}\,dt + \int_0^\infty \frac{e^{-jwt}}{2}u(t)\,e^{-st}\,dt$$

$$X(s) = \frac{1}{2}\left\{\frac{1}{s-jw}\right\} + \frac{1}{2}\left\{\frac{1}{s+jw}\right\} = \frac{1}{2}\left\{\frac{2s}{(s-jw)(s+jw)}\right\}$$

$$X(s) = \frac{s}{s^2+w^2} \quad \{\text{Roc for } Re(s)>0\}$$

$\therefore \cos wt\, u(t) \xleftrightarrow{LT} \frac{s}{s^2+w^2}$ Roc $Re(s)>0$

$\sin wt\, u(t) \xleftrightarrow{LT} \frac{w}{s^2+w^2}$

Hyperbolic Sine & Cosine functions :- $x(t) = \sinh wt\, u(t)$.

$$L\{x(t)\} = X(s) = L\{\sinh wt\, u(t)\} = L\left\{\frac{e^{wt}-e^{-wt}}{2}u(t)\right\}$$

$$= \frac{1}{2}L\{e^{wt}u(t)\} - \frac{1}{2}L\{e^{-wt}u(t)\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-\omega} - \frac{1}{s+\omega}\right\} = \frac{\omega}{s^2-\omega^2}.$$

$$\sinh\omega t\, u(t) \xleftrightarrow{LT} \frac{\omega}{s^2-\omega^2} \quad \text{ROC is } Re(s) > 0.$$

By $\cosh\omega t\, u(t) \xleftrightarrow{LT} \frac{s}{s^2-\omega^2}$ ROC is $Re(s) > 0$.

Damped Sine & Cosine functions $x(t) = e^{-at}\sin\omega t\, u(t)$ :-

$$L\{x(t)\} = X(s) = L\left\{e^{-at}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)u(t)\right\}$$

$$X(s) = \frac{1}{2j} L\left\{\left(e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right)u(t)\right\}$$

$$= \frac{1}{2j}\left(\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right) = \frac{1}{2j}\left(\frac{2j\omega}{s^2+(a+j\omega)^2}\right)$$

$$= \frac{1}{2j}\left(\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right) = \frac{1}{2j}\left(\frac{2j\omega}{(s+a)^2+\omega^2}\right)$$

$$X(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

i.e., $e^{-at}\sin\omega t\, u(t) \xleftrightarrow{LT} \frac{\omega}{(s+a)^2+\omega^2}$ ROC is $Re(s) > -a$

By $e^{-at}\cos\omega t\, u(t) \xleftrightarrow{LT} \frac{s}{(s+a)^2+\omega^2}$ ROC is $Re(s) > -a$.

Damped Hyperbolic Sine & Cosine functions $x(t) = e^{-at}\sinh\omega t\, u(t)$ :-

$$L\{x(t)\} = X(s) = L\{e^{-at}\sinh\omega t\, u(t)\}$$

$$= L\left\{\frac{1}{2}\left[e^{-at}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right)\right]u(t)\right\}$$

$$= \frac{1}{2}\left\{L\left(e^{-(a-\omega)t}u(t)\right) - L\left(e^{-(a+\omega)t}u(t)\right)\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s+a-\omega} - \frac{1}{s+a+\omega}\right) = \frac{\omega}{(s+a)^2-\omega^2}$$

$e^{-at}\sinh\omega t\, u(t) \xleftrightarrow{LT} \frac{\omega}{(s+a)^2-\omega^2}$ ROC is $Re(s) > -a$ (5)

$e^{-at}\cosh\omega t\, u(t) \xleftrightarrow{LT} \frac{s}{(s+a)^2-\omega^2}$ ROC is $Re(s) > -a$.

## Properties & Theorems of Laplace Transform:-

(1) **Linearity:-** Laplace transform of a weighted sum of 2 signals is equal to the weighted sum of individual laplace transforms

$$x_1(t) \xleftrightarrow{LT} X_1(s)$$

$$x_2(t) \xleftrightarrow{LT} X_2(s)$$

$$a x_1(t) + b x_2(t) \xleftrightarrow{LT} a X_1(s) + b X_2(s)$$

**Proof:-**

$$L\{a x_1(t) + b x_2(t)\} = \int_{-\infty}^{\infty} (a x_1(t) + b x_2(t)) e^{-st} dt$$

$$= a \int x_1(t) e^{-st} dt + b \int x_2(t) e^{-st} dt$$

$$= a X_1(s) + b X_2(s)$$

(2) **Time shift property:-**

If $x(t) \xleftrightarrow{} X(s)$

$$x(t-t_0) \xleftrightarrow{} e^{-st_0} X(s)$$

**Proof:-**

$$L\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$$

$t - t_0 = p \quad dt = dp \;\&\; t = p + t_0$.

$$L\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(p) e^{-s(p+t_0)} dp = e^{-st_0} \int_{-\infty}^{\infty} x(p) e^{-ps} dp = e^{-st_0} X(s)$$

$$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$$

By $x(t+t_0) \xleftrightarrow{LT} e^{st_0} X(s)$.

(3) **Time Scaling property:-**

$$x(t) \xleftrightarrow{LT} X(s)$$

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$L\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt \qquad at = P$$

$$dt = \frac{dP}{a}$$

$$= \int_{-\infty}^{\infty} x(P)\, e^{-s(P/a)} \frac{dP}{a} = \frac{1}{a} \int_{-\infty}^{\infty} x(P)\, e^{-P(\frac{s}{a})} dP$$

$$X(s) = \frac{1}{a} X\left(\frac{s}{a}\right)$$

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

4) Time Reversal :-

$$x(t) \longleftrightarrow X(s)$$

$$x(-t) \longleftrightarrow X(-s)$$

Proof:-

$$L\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)\, e^{-st}\, dt$$

$$-t = P \qquad dt = -dP$$

$$L\{x(-t)\} = \int_{\infty}^{-\infty} x(P)\, e^{sP}(-dP) = \int_{-\infty}^{\infty} x(P)\, e^{-(-s)P} dP = X(-s)$$

$$L\{x(-t)\} = X(-s).$$

(5) Transform of Derivatives property (or) Differentiation in time property

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\frac{d}{dt}(x(t)) \xleftrightarrow{LT} sX(s) - x(0^-) \quad \text{(For unilateral Laplace Transform)}$$

Proof:-

$$L\left\{\frac{d}{dt}x(t)\right\} = \int_{0}^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt = \left(e^{-st}x(t)\right)_0^{\infty} - \int_0^{\infty} x(t)\left(-s e^{-st}\right) dt$$

$$L\left\{\frac{d}{dt}x(t)\right\} = \left(0 - x(0^-)\right) + s\int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$= -x(0^-) + sX(s)$$

$$L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0^-)$$

$$\text{By } L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx(0)}{dt} \cdots\cdots$$

$$L\left\{\frac{d^n x(t)}{dt}\right\} = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx(0^-)}{dt} - \cdots - \frac{d^{n-1}x(0^-)}{dt^{n-1}}$$

(6) Transform of Integrals Property:-

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s} + \int_{-\infty}^{0} \frac{x(\tau)}{s}\,d\tau$$

Proof:-

$$f(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$\frac{d}{dt}f(t) = x(t)$$

$$f(0^-) = \int_{-\infty}^{0} x(\tau)\,d\tau$$

$$L\left\{\frac{d}{dt}f(t)\right\} = L\{x(t)\}$$

$$sF(s) - f(0^-) = X(s)$$

$$F(s) = \frac{X(s)}{s} + \frac{f(0^-)}{s} = \frac{X(s)}{s} + \int_{-\infty}^{0} \frac{x(\tau)}{s}\,d\tau$$

$$L\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) = \frac{X(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} x(\tau)\,d\tau$$

For Bilateral L.T

$$L\left\{\frac{d}{dt}x(t)\right\} = sX(s)$$

$$x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)\,e^{st}\,dt$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)\,e^{st}(s)\,dt$$

$$\frac{d}{dt}x(t) = \left(L^{-1}(sX(s))\right)$$

$$L\left(\frac{d}{dt}x(t)\right) = sX(s)$$

Hy $L\left[\int x(t)dt\right] = \frac{X(s)}{s}$

(7) Differentiation in s-domain:-

$$t\,x(t) \xleftrightarrow{LT} -\frac{d}{ds}X(s)$$

proof:-

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left(\int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt\right)$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{\infty} x(t)\left(e^{-st}\right)(-t)\,dt$$

$$-\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}(t)\,dt = \int_{-\infty}^{\infty} t\,x(t)\,e^{-st}\,dt = L\{t\,x(t)\}$$

$$\therefore t x(t) \longleftrightarrow -\frac{d}{ds} X(s)$$

By $L\{t^n x(t)\} = (-1)^n \frac{d^n X(s)}{ds^n}$

If $t^n \longleftrightarrow \frac{n!}{s^{n+1}}$.

(8) Frequency shifting property:-

$$e^{-at} x(t) \longleftrightarrow X(s+a)$$

Proof:-

$$L\{e^{-at} x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-at} e^{-st} dt$$

$$L\{e^{-at} x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-t(s+a)} dt = X(s+a).$$

By $L\{e^{at} x(t)\} = X(s-a)$.

9) Time Convolution property:-

$$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s)\, X_2(s).$$

Proof:-

$$L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1(\tau)\, x_2(t-\tau)\, d\tau) e^{-st} dt$$

$t = \tau + p$, $t - \tau = p$, $dt = dp$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} (x_1(\tau)\, x_2(p)\, d\tau) e^{-s(\tau+p)} dp$$

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(p) e^{-sp} dp$$

$$L\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s).$$

(10) Multiplication (or) Modulation (or) Convolution in S-domain:-

$$x_1(t)\, x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j} (X_1(s) * X_2(s))$$

Proof:-

$$L\{x_1(t)\, x_2(t)\} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \left( \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(u)\, e^{ut}\, du \right) x_2(t)\, e^{-st}\, dt$$

$$= \frac{1}{2\pi j} \left( \int_{-\infty}^{\infty} X_1(u)\, du \int_{-\infty}^{\infty} x_2(t)\, e^{-t(s-u)}\, dt \right)$$

$$= \frac{1}{2\pi j} \left( \int_{-\infty}^{\infty} X_1(u)\, X_2(s-u)\, du \right) = \frac{1}{2\pi j} \left( X_1(s) * X_2(s) \right).$$

(11) Conjugation & Conjugate Symmetry:-

Conjugate:- $x^*(t) \longleftrightarrow X^*(s^*)$

Proof:- $L\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)\, e^{-st}\, dt = \left( \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt \right)^*$

$$L\{x^*(t)\} = \left( \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt \right)^* = \left( X(s^*) \right)^* = X^*(s^*).$$

Conjugate Symmetry:-

$$X(s^*) = \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt$$

$$X^*(s^*) = \left( \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)t}\, dt \right)^* = \int_{-\infty}^{\infty} x(t)\, e^{-(s^*)^* t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = X(s)$$

$$X(s) = X^*(s^*).$$

(12) Parseval's Relation (or) Theorem (or) property:-

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt \xleftrightarrow{LT} \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s)\, X_2^*(-s^*)\, ds$$

Proof:- $x_1(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s)\, e^{st}\, ds$

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s)\, e^{st}\, ds \right) x_2^*(t)\, dt = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s) \left( \int_{-\infty}^{\infty} x_2(t) e^{-(s^*)^* t}\, dt \right)$$

$$= \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s) \left( \int_{-\infty}^{\infty} x_2(t)\, e^{-(s^*)t}\, dt \right)^* ds$$

$$= \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s) \left(X_2(-s^*)\right)^* ds$$

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s) \left(X_2(-s^*)\right)^* ds = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X_1(s) \cdot X_2^*(-s^*) ds$$

## Initial Value Theorem:-

If $L\{x(t)\} = X(s)$

then $\lim_{t \to 0} x(t) = x(0) = \lim_{s \to \infty} X(s)$

$$L\left\{\frac{d}{dt} x(t)\right\} = sX(s) - x(0^-) = \int_0^{\infty} \frac{d}{dt} x(t)\, e^{-st} dt$$

$s \to \infty$ on both sides

$$\lim_{s \to \infty} L\left\{\frac{d}{dt} x(t)\right\} = \lim_{s \to \infty} \left( \int_0^{\infty} \frac{d}{dt} x(t) e^{-st} dt \right) = \lim_{s \to \infty} \left( sX(s) - x(0) \right)$$

$$0 = \lim_{s \to \infty} sX(s) - x(0)$$

$$x(0) = \lim_{s \to \infty} sX(s)$$

i.e. $\lim_{t \to 0} x(t) = x(0) = \lim_{s \to \infty} X(s)$

## Final Value Theorem:-

$$\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} sX(s)$$

$$L\left\{\frac{d}{dt} x(t)\right\} = \int_0^{\infty} \frac{d}{dt} x(t) e^{-st} dt = sX(s) - x(0^-)$$

Taking $\lim_{s \to 0}$ on both sides

$$\lim_{s \to 0} \left( \int_0^{\infty} \frac{d}{dt} x(t) e^{-st} dt \right) = \lim_{s \to 0} \left( sX(s) - x(0^-) \right)$$

$$\int_0^{\infty} \frac{d}{dt} x(t) dt = \lim_{s \to 0} \left( sX(s) - x(0^-) \right)$$

$$\left(x(t)\right)_0^{\infty} = \lim_{s \to 0} \left( sX(s) - x(0^-) \right)$$

$$x(\infty) - x(0^-) = \lim_{s \to 0} \left( sX(s) \right) - x(0^-)$$

$$x(\infty) = \lim_{s \to 0} sX(s)$$

$\therefore \lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} sX(s)$

## Existence of Laplace Transform:-

$$X(s) = L(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

The condition for the Laplace transform to exist is

$$\int_{-\infty}^{\infty} |x(t)\, e^{-st}|\, dt < \infty \quad \text{for some finite value of } '\sigma'.$$

The integral will converge if the signal $x(t)$ is sectionally continuous in every finite interval of '$t$' and if it is of exponential order as '$t$' approaches to $\infty$.

A causal signal $x(t)$ is said to be exponential order if a real, positive constant $\sigma$ (where $\sigma$ is a real part of $s$) exist such that the function $e^{-\sigma t}|x(t)|$ approaches to zero as '$t$' approaches $\infty$.

For a causal signal, if $\lim_{t\to\infty} e^{-\sigma t}|x(t)| = 0$ for $\sigma > \sigma_c$ & if $\lim_{t\to\infty} e^{-\sigma t}|x(t)| = \infty$ for $\sigma < \sigma_c$ then $\sigma_c$ is called abscissa of convergence (where $\sigma_c$ is a point on real axis in s-plane).

The integral $\int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ converges only if the real part of $s$ is greater than the abscissa of convergence $\sigma_c$. The values of '$s$' for which the integral $\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ converges is called Region of Convergence (ROC). ∴ The for a causal signal, the ROC includes all points on the s-plane to the right of abscissa of convergence

## Region of Convergence:-

### Case 1:- Right sided (causal) signal:-

Let $x(t) = e^{-at} u(t)$ where $a > 0$

$= e^{-at}$ for $t \geq 0$

Laplace transform of $x(t)$ is given by

$$L(x(t)) = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-st} = \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$X(s) = \left(\frac{e^{-(s+a)t}}{-(s+a)}\right)_0^{\infty} = \frac{1}{-(s+a)}\left(0 - 1\right) = \frac{1}{s+a}$$

Let $K = \sigma + a = \sigma - (-a)$

When $\sigma > -a$, $K = \sigma - (-a) =$ positive $\quad \therefore e^{-K\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $K = \sigma - (-a) =$ -ve $\quad \therefore e^{-K\infty} = e^{\infty} = \infty$.

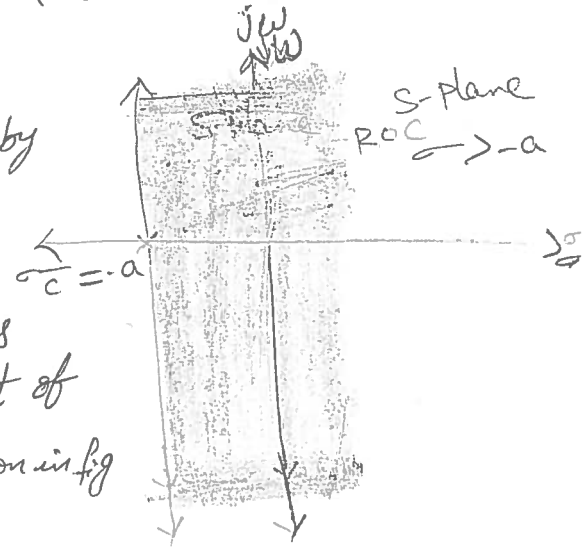
$$X(s) = \left(\frac{e^{-(\sigma + j\omega + a)t}}{-(s+a)}\right)_0^{\infty} = \left(\frac{e^{-(\sigma+a)t}\, e^{-j\omega t}}{-(s+a)}\right)_0^{\infty} = \frac{e^{-K\infty}\, e^{-(j\omega)\infty}}{-(s+a)} + \frac{1}{s+a}$$

Hence $X(s)$ converges when $\sigma > -a$ & does not converge for $\sigma < -a$.

Abscissa of convergence $\sigma_c = -a$

When $\sigma > -a$, the $X(s)$ is given by

$$X(s) = \frac{1}{s+a}$$




∴ For a causal signal the ROC includes all points on the S-plane to the right of abscissa of convergence $\sigma_c = -a$ as shown in fig

Case 2:- Left Sided (anticausal) Signal:-

Let $x(t) = e^{-bt}\, u(-t) = e^{-bt}$ for $t \le 0$ where $b > 0$

$$X(s) = L(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-bt}\, u(-t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt = \int_{-\infty}^{0} e^{-(s+b)t}\, dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$X(s) = \frac{-1}{s+b} + \frac{e^{K\infty}\, e^{j\omega\infty}}{(s+b)}$$

where $K = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $K = \sigma - (-b) =$ positive $\therefore e^{K\infty} = e^{\infty} = \infty$

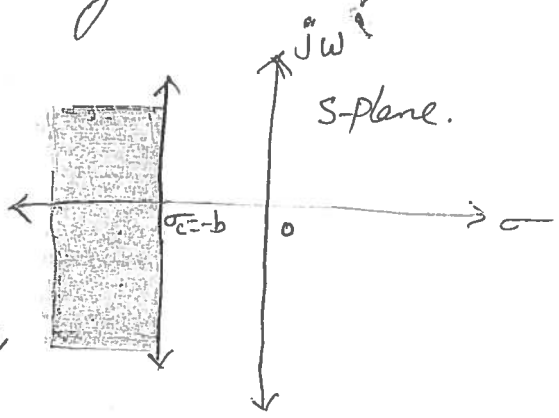
when $\sigma < -b$, $K = \sigma - (-b) =$ Negative $\therefore e^{K\infty} = e^{-\infty} = 0$

$\therefore X(s)$ converges when $\sigma < -b$, and does not converge for $\sigma > -b$.

Abscissa of Convergence $\sigma_c = -b$.

When $\sigma < -b$, then $X(s)$ is given by

$$L(x(t)) = X(s) = \frac{-1}{s+b}.$$




$\therefore$ for an Anticausal signal, the ROC includes all points on the s-plane to the left of abscicca of Convergence, $\sigma_c = -b$, as shown in fig.

Case 3:- Two Sided Signal:-

$x(t) = e^{-at}u(t) + e^{-bt}u(-t)$ where $a>0$, $b>0$ & $a>b$ (or) $(-a<-b)$

$$L(x(t)) = \int_{-\infty}^{\infty} e^{-at}u(t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-bt}u(-t)\,e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-t(a+s)}\,dt + \int_{-\infty}^{0} e^{-t(b+s)}\,dt$$

$$= \left[\frac{e^{-t(a+s)}}{-(a+s)}\right]_0^{\infty} + \left[\frac{e^{-t(b+s)}}{-(s+b)}\right]_{-\infty}^{0} = \frac{e^{-(m)\infty}\,e^{-(jw)\infty}}{-(s+a)} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{n(\infty)}\,e^{(jw)\infty}}{s+b}$$

$$\boxed{X(s) = \frac{1}{s+a} - \frac{1}{s+b}}$$

Let $m = \sigma + a = \sigma - (-a)$

$n = \sigma + b = \sigma - (-b)$

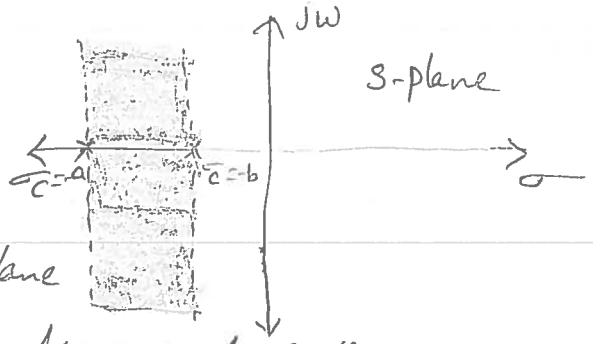
When $\sigma > -a$, $m$ is +ve, $e^{-m\infty} = e^{-\infty} = 0$

$\sigma < -a$, $m$ is -ve, $e^{-m\infty} = e^{\infty} = \infty$

$\sigma > -b$, $n$ is +ve, $e^{n\infty} = e^{\infty} = \infty$

$\sigma < -b$, $n$ is -ve, $e^{n\infty} = e^{-\infty} = 0$

$X(s)$ converges when $\sigma$ lies between $-a$ & $-b$ and does not converges outside.




∴ ROC includes all points on the s-plane in the region in between the two abscicca of convergence.

$$-a < \sigma < -b$$

Ex:- Find the Laplace transform of the signal

$$x(t) = e^{-b|t|}$$

$$x(t) = e^{-b|t|}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{+bt} e^{-st}\, dt + \int_{0}^{\infty} e^{-bt} e^{-st}\, dt$$

$$X(s) = \underbrace{\int_{-\infty}^{0} e^{-(s-b)t}\, dt}_{\text{Converges if } Re(s) < b} + \underbrace{\int_{0}^{\infty} e^{-(s+b)t}\, dt}_{\text{Converges if } Re(s) > -b}$$

$$X(s) = \frac{1}{-(s-b)} e^{-(s-b)t}\Big|_{-\infty}^{0} + \frac{1}{-(s+b)} e^{-(s+b)t}\Big|_{0}^{\infty}$$

$$X(s) = -\frac{1}{s-b} + \frac{1}{s+b} \qquad ROC: -b < Re(s) < b.$$

# Properties of ROC:-

The Properties of R.O.C are as follows;

① The ROC of $X(s)$, the laplace transform of $x(t)$ is bounded by poles (or) extends up to infinity.

② The ROC does not Contain any poles.

③ If $x(t)$ is a Right hand sided signal, the ROC of $X(s)$ extends to the right of the right most pole and no pole is located inside the ROC.

④ If $x(t)$ is a left sided signal, the ROC of $X(s)$ extends to the Left of the left most pole & no pole is located inside the ROC.

⑤ If $x(t)$ is a 2-sided signal, the ROC of $X(s)$ is a strip in the S-plane bounded by poles & no pole is located inside the ROC.

⑥ Impulse function is the only function for which the ROC is the Entire S-plane.

⑦ The ROC must be a Connected region.

⑧ The ROC of an LTI System Contains the imaginary axis of S-plane.

⑨ The ROC of the sum of two (or) more signals is equal to the intersection of the ROCs of those signals.

## Inversion of Unilateral Laplace Transform :-

The direct method of finding the inverse Laplace transform of $X(s)$ is using the equation

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)\, e^{st}\, ds.$$

is quite tedious & time consuming. So the inverse Laplace transform of $X(s)$ is usually obtained by using partial Fraction Expansion. It uses the one-to-one relationship between a signal & its unilateral laplace transform. The partial fraction expansion can b used only for proper rational functions, i.e., for functions in which th order of the numerator is smaller than that of the denomitor.

Let

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + b_{M-2} s^{M-2} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

If $M = N$, Degree of $N(s)$ = Degree of $D(s)$, then divide the numerator by the denominator

$$X(s) = C + \frac{N'(s)}{D(s)}$$

where '$C$' is a Constant & $\text{Deg } N'(s) < \text{Deg } D(s)$

Since a Constant $C$ has an inverse laplace transform $C\delta(t)$, $\text{Deg } N(s) = \text{Deg } D(s)$ implies that $x(t)$, the inverse Laplace transform of $X(s)$ has an impulse.

If $\text{Deg } N(s) > \text{Deg } D(s)$, divide the numerator by the denominator & write $X(s)$ as

$$X(s) = \text{Terms with powers of } s \text{ greater than (or) equal to zero} + \frac{N'(s)}{D(s)}$$

where $\text{Deg } N'(s) < \text{Deg } D(s)$

In this case $x(t)$ will include impulses & their derivatives.

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{\prod_{K=1}^{N} (s - P_K)}$$

$P_K \rightarrow$ Pole

The roots of the characteristics equation $D(S)=0$ are called the poles of $X(S)$.



We shall now consider 2 possible cases that arise.

① Distinct poles ② Multiple poles

① Distinct poles:-

If all poles $P_K$ are distinct, then we may write $X(S)$ as a sum of single pole terms given by

$$X(S) = \frac{K_1}{S-P_1} + \frac{K_2}{S-P_2} + \cdots\cdots + \frac{K_i}{S-P_i} + \cdots\cdots + \frac{K_n}{S-P_n}$$

To determine the value of $K_i$, multiply both sides by $(S-P_i)$ and put $S=P_i$, we then have

$$(S-P_i)X(S)\Big|_{S=P_i} = \left( (S-P_i)\frac{K_1}{S-P_1} + (S-P_i)\frac{K_2}{S-P_2} + \cdots\cdots + (S-P_i)\frac{K_i}{S-P_i} + \cdots\cdots + (S-P_i)\frac{K_n}{S-P_n} \right)_{S=P_i}$$

In the RHS of the above equation all the terms except $K_i$ vanish. Hence, we get

$$K_i = (S-P_i)X(S)\Big|_{S=P_i}.$$

② Multiple poles:-

In this case all the poles are not distinct, some of the poles may repeat. Let the pole $P_i$ repeat $l$ times. Then if all other poles are distinct, we write

$$X(S) = \frac{K_1}{S-P_1} + \frac{K_2}{S-P_2} + \cdots\cdots + \frac{K_{i1}}{(S-P_i)} + \frac{K_{i2}}{(S-P_i)^2} + \frac{K_{il}}{(S-P_i)^l} + \cdots\cdots + \frac{K_n}{(S-P_n)}.$$

The coefficient $K_{il}$ is obtained by multiplying both sides of the above equation by $(S-P_i)^l$ and evaluating it at $S=P_i$.

$$K_{il} = (S-P_i)^l X(S)\Big|_{S=P_i}$$

The coefficients $K_{i(l-b)}$ are evaluated by multiplying both sides of the preceding equation by $(S-P_i)^l$ differentiating $(l-b)$ times & then evalu

the resultant equation at $S=P_i$. Thus

$$K_{i(l-1)} = \frac{d}{ds}\left((S-P_i)^l X(S)\right)\Big|_{S=P_i}$$

$$\vdots$$

$$K_{i(l-b)} = \frac{1}{b!}\left\{\frac{d^b}{ds^b}\left((S-P_i)^l X(S)\right)\right\}\Big|_{S=P_i}$$

(3) Complex roots :-

If $X(S)$ has complex poles then the partial fraction Expansion of the same can be Expressed as:

$$X(S) = \frac{K_1}{S-P_1} + \frac{K_1^*}{S-P_1^*}$$

Where $K_1^*$ is Complex Conjugate of $K_1$ & $P_1^*$ is complex Conjugate of $P_1$. Inotherwords Complex Conjugate poles result in Complex Conjugate Coefficients.

Alternate method

The above Procedure of finding partial fractions involves manipulation of Complex numbers. When the Denominator has Complex roots, the Complex pair Can be Expressed as a single quadratic fac instead of having 1st order partial fractions. That is if $S=-a\pm jb$, we Can retain the quadratic factor.

$$D(S) = (s+a+jb)(s+a-jb) = (S+a)^2+b^2.$$

(1) Find the Inverse Laplace Transforms of the following:

(a) $\frac{1}{(s+1)^2}$ (b) $\frac{1}{(s+1)^2+1}$ (c) $\frac{s}{(s+2)^2}$ (d) $\frac{s}{(s+2)^2+1}$ (e) $\frac{s}{(s-b)^2+a^2}$

(f) $\frac{s}{2s^2-8}$ (g) $\frac{s}{s^2a^2+b^2}$ (h) $\frac{s}{(s^2+a^2)^2}$ (i) $\frac{s^2}{(s-1)^4}$ (j) $\frac{1}{s(s+2)^3}$

(K) $X(S) = \frac{2s+1}{s+2}$ $Re(S) > -2$.

(a) $X(s) = \frac{1}{(s+1)^2}$

$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2}\right) = te^{-t}u(t)$.

$L[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}$

(b) $X(s) = \frac{1}{(s+1)^2+1}$

$x(t) = L^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right) = e^{-t}(\sin t\, u(t))$

(c) $X(s) = \frac{s}{(s+2)^2}$

$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$

$x(t) = L^{-1}\left(\frac{s+2}{(s+2)^2}\right) - 2L^{-1}\left(\frac{2}{(s+2)^2}\right) = L^{-1}\left(\frac{1}{s+2}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2}\right)$

$x(t) = e^{-2t}u(t) - 2te^{-2t}u(t) = e^{-2t}(1-2t)u(t)$.

(d) $X(s) = \frac{s}{(s+2)^2+1}$

$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{s}{(s+2)^2+1}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2+1}\right)$

$= L^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}L^{-1}\left(\frac{s}{s^2+1}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$

$= e^{-2t}\cos t\, u(t) - 2e^{-2t}\sin t\, u(t)$

$x(t) = e^{-2t}(\cos t - 2\sin t)u(t)$.

(e) $X(s) = \frac{s}{(s-b)^2+a^2}$

$x(t) = L^{-1}\left\{\frac{s+b-b}{(s-b)^2+a^2}\right\} = L^{-1}\left\{\frac{s-b}{(s-b)^2+a^2} + \frac{b}{(s-b)^2+a^2}\right\}$

$= e^{bt}\cos at\, u(t) + \frac{a}{a}L^{-1}\left\{\frac{b}{(s-b)^2+a^2}\right\} = e^{bt}\cos at\, u(t) + \frac{b}{a}L^{-1}\left\{\frac{a}{(s-b)^2+a^2}\right\}$

$x(t) = e^{bt}\cos at\, u(t) + \frac{b}{a}e^{bt}\sin at\, u(t) = \left(e^{bt}\cos at + \frac{b}{a}\sin at\, e^{bt}\right)u(t)$

(f) $X(s) = \frac{s}{2s^2-8}$

$$X(s) = \frac{1}{2}\frac{s}{(s^2-4)} = \frac{1}{2}\left(\frac{s}{s^2-2^2}\right)$$

$$\cosh \omega t = \frac{s}{s^2-\omega^2}$$

$$x(t) = \frac{1}{2}L^{-1}\left(\frac{s}{s^2-4}\right) = \frac{1}{2}L^{-1}\left(\frac{s}{s^2-2^2}\right) = \frac{1}{2}\cosh 2t.$$

(g) $X(s) = \frac{s}{s^2a^2+b^2} = \frac{1}{a^2}\left(\frac{s}{s^2+\left(\frac{b}{a}\right)^2}\right)$

$$x(t) = \frac{1}{a^2}\cos\left(\frac{b}{a}\right)t$$

(h) $X(s) = \frac{s^2}{(s-1)^4}$

$$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{s^2}{(s-1)^4}\right)$$

$$L^{-1}\left(\frac{s^2}{(s-1)^4}\right) = \frac{d}{dt}\left\{L^{-1}\left(\frac{s}{(s-1)^4}\right)\right\} = \frac{d}{dt}\left\{\frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s-1)^4}\right)\right]\right\}$$

$$= \frac{d}{dt}\left\{\frac{d}{dt}\left(e^t L^{-1}\left(\frac{1}{s^4}\right)\right)\right\} = \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{1}{6}t^3e^t\right)\right)$$

$$= \frac{1}{6}\frac{d}{dt}\left(t^3e^t + e^t(3t^2)\right) = \frac{1}{6}\frac{d}{dt}\left(t^3e^t + 3t^2e^t\right)$$

$$x(t) = \frac{1}{6}\left(t^3e^t + 3t^2e^t + 3t^2e^t + 6te^t\right) = te^t\left(1 + t + \frac{t^2}{6}\right)$$

(j) $X(s) = \frac{1}{s(s+2)^3}$

$$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{1}{s(s+2)^3}\right) = \int_0^t L^{-1}\left(\frac{1}{(s+2)^3}\right)dt$$

$$x(t) = \int_0^t e^{-2t}\,L^{-1}\left(\frac{1}{s^3}\right)dt = \int_0^t e^{-2t}\frac{t^2}{2}dt = \frac{1}{2}\left\{\left(\frac{t^2e^{-2t}}{-2}\right)_0^t - \int_0^t \frac{e^{-2t}}{-2}(2t)dt\right\}$$

$$= \frac{1}{2}\left(\frac{t^2e^{-2t}}{-2} + \left(\frac{te^{-2t}}{-2}\right)_0^t - \int_0^t \frac{e^{-2t}}{-2}\,dt\right)$$

$$= \frac{1}{4}\left(\left(-t^2e^{-2t} - te^{-2t}\right) + \left(\frac{e^{-2t}}{-2}\right)_0^t\right) = \frac{1}{4}\left(-t^2e^{-2t} - te^{-2t} - \frac{1}{2}e^{-2t} + \frac{1}{2}\right)$$

$$x(t) = \frac{1}{8}\left(-2t^2e^{-2t} - 2te^{-2t} - e^{-2t} + 1\right)$$

(K) $X(s) = \dfrac{2s+1}{s+2}$

$$X(s) = \frac{2s+4-3}{s+2} = \frac{2s+4}{s+2} - \frac{3}{s+2} = \frac{2(s+2)}{(s+2)} - \frac{3}{(s+2)}$$

$$X(s) = 2 - \frac{3}{s+2}$$

$$x(t) = L^{-1}(X(s)) = 2\delta(t) - 3e^{-2t}u(t).$$

Ex:- Find the Inverse Laplace transform of the following:

(a) $X(s) = \dfrac{s^3+1}{s(s+1)(s+2)}$ (b) $X(s) = \dfrac{s-1}{(s+1)(s^2+2s+5)}$

(a) $X(s) = \dfrac{s^3+1}{s(s+1)(s+2)} = \dfrac{s^3+1}{s^3+3s^2+2s}$

Nr Degree = Dr Degree

$$s^3+3s^2+2s \;\overline{)\; s^3+1 \;(\;1}$$
$$\underline{-s^3 \mp 3s^2 \mp 2s}$$
$$-3s^2-2s+1$$

$$X(s) = 1 + \frac{1-2s-3s^2}{s^3+3s^2+2s}$$

$$\frac{1-2s-3s^2}{s^3+3s^2+2s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$A(s+1)(s+2) = 1-2s-3s^2$ $\quad s=0 \rightarrow A(1)(2) = 1 \Rightarrow A = \frac{1}{2}$

$B(s)(s+2) = 1-2s-3s^2$ $\quad s=-1 \rightarrow B(-1)(1) = 1+2-3 = 0 \Rightarrow B = 0$

$C(s)(s+1) = 1-2s-3s^2$ $\quad s=-2 \rightarrow C(-2)(-1) = 1+4-12 = -7 \Rightarrow C = -7/2$

$$X(s) = 1 + \frac{1}{2s} + \frac{-\frac{7}{2}}{s+2} = 1 + \frac{1}{2s} - \frac{7}{2(s+2)}$$

$$\boxed{x(t) = \delta(t) + \frac{1}{2}u(t) - \frac{7}{2}e^{-2t}u(t)}$$

(b) $$X(s) = \frac{s-1}{(s+1)(s^2+2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

$$A(s^2+2s+5) + (Bs+C)(s+1) = s-1$$

Put $s = -1$.

$$A(1-2+5) = -2$$

$$A(4) = -2$$

$$\boxed{A = -\frac{1}{2}}$$

s-Coefficients

$$2A + B + C = 1$$

$$2\left(\frac{-1}{2}\right) + B + C = 1$$

$$\boxed{B + C = 2}$$

Constants

$$5A + C = -1$$

$$-\frac{5}{2} + C = -1$$

$$C = -1 + \frac{5}{2} = \frac{3}{2} \checkmark$$

$$B + \frac{3}{2} = 2$$

$$B = 2 - \frac{3}{2} = \frac{1}{2}$$

$$X(s) = \frac{\left(\frac{-1}{2}\right)}{s+1} + \frac{\left(\frac{1}{2}\right)s + \frac{3}{2}}{s^2+2s+5}$$

$$= \frac{-1}{2(s+1)} + \frac{\left(\frac{1}{2}\right)(s+1+2)}{(s+1)^2+2^2}$$

$$X(s) = \frac{-1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{s+1}{(s+1)^2+2^2}\right) + \frac{1}{2}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$x(t) = \frac{-1}{2}e^{-t}u(t) + \frac{1}{2}e^{-t}\cos 2t\,u(t) + \frac{1}{2}e^{-t}\sin 2t\,u(t)$$

## Inversion of Bilateral Laplace Transform :-

So far we discussed about the inversion of unilateral laplace Transform which can be used only for causal signals. To determine the inverse of unilateral laplace transform, the ROC need not be specified because a given pole always corresponds to a +ve time portion of $x(t)$, i.e., ROC always lies to the right of the pole, i.e, ROC is always $Re(s) >$ some constant. The $L^{-1}\left(\frac{1}{s+a}\right)$ is always equal to $e^{-at}u(t)$ for unilateral Laplace transform & the ROC is $Re(s) > -a$. on the other hand if $x(t)$ is a non-causal signal, then to find $x(t)$ from $X(s)$ we must know the ROC of $X(s)$. The location of pole along with ROC determines whether a given pole corresponds to a +ve (or) -ve time portion of $x(t)$. If the ROC ~~is given pole to~~ is $Re(s) > a$ then the signal is a causal signal. on the other hand if the ROC is $Re(s) < a$, then the signal $x(t)$ is noncausal signal. on the other hand if the ROC is $Re(s) < a$, then the signal $x(t)$ is a Non causal signal. In otherwoods if a pole of $X(s)$ lies to the left of the ROC, them this pole gives rise to a causal signal, & if a pole of $X(s)$ lies to the right of the ROC, then this pole gives rise to a Non-causal signals i.e, $L^{-1}\left(\frac{1}{s+a}\right)$ may be equal to $e^{-at}u(t)$ if ROC is $Re(s) > -a$ & may be equal to $-e^{-at}u(-t)$ if ROC is $Re(s) < -a$.

① Find the inverse Laplace transform of $X(s) = \frac{2s+1}{s+2}$

ⓐ For ROC; $Re(s) > -2$ ⓑ For ROC; $Re(s) < -2$.

↳
$$X(s) = \frac{2s+1}{s+2} = 2\left(\frac{s+\frac{1}{2}}{s+2}\right) = 2\left(\frac{s+2-3/2}{s+2}\right) = 2\left(1 - \frac{3/2}{s+2}\right)$$

$$X(s) = 2 - \frac{3}{s+2}$$

ⓐ ROC is $Re(s) > -2$ the pole at $s = -2$ gives rise to a causal signal.

$$x(t) = 2\delta(t) - 3e^{-2t}u(t)$$

(b) ROC is $Re(s) < -2$, the pole at $s=-2$ gives rise to a Non Causal Signal

$$x(t) = L^{-1}\left(2 - \frac{3}{s+2}\right) = 2\delta(t) + 3e^{-2t}u(-t)$$

Ex:- Find the inverse Laplace transform of $X(s) = \frac{3}{(s+1)(s+4)}$ ; ROC; $-4 < Re(s)$

$$X(s) = \frac{3}{(s+1)(s+4)}$$ Two poles at $s=-1$ & $s=-4$

$$L^{-1}\left(\frac{1}{s+1}\right) = -e^{-t}u(-t)$$ since ROC is $Re(s) < -1$ → Non Causal Signal

$$L^{-1}\left(\frac{1}{s+4}\right) = e^{-4t}u(t)$$ since ROC is $Re(s) > -4$ → Causal Signal.

$$L^{-1}(X(s)) = x(t) = e^{-t}u(-t) - e^{-4t}u(t).$$

Ex:- Find the Signal whose Laplace transform (Bilateral) is:

$$X(s) = \frac{1}{(s+2)(s+5)}$$ ROC; $-5 < Re(s) < -2$.

$s=-2$ & $s=-5$ are the poles.

$$X(s) = \frac{A}{s+2} + \frac{B}{s+5} = \frac{1}{3}\left(\frac{1}{s+2}\right) - \frac{1}{3}\left(\frac{1}{s+5}\right)$$

$$\boxed{x(t) = \frac{-1}{3}e^{-2t}u(t) - \frac{1}{3}e^{-5t}u(t)}$$

Ex:- Find the laplace transform & ROC of the Right sided Signal

$x(t) = 4e^{-2t}u(t) + 3e^{-3t}u(t)$

$$X(s) = \frac{4}{s+2} + \frac{3}{s+3}$$ $Re(s) > -2$ & $Re(s) > -3$

i.e., $Re(s) > -2$ & $Re(s) > -3 = Re(s) > -2$.



## Solution of Differential Equations Using Laplace Transform:-

An LTI system is described by constant coefficient D.Es relating the i/p & o/p. The response of the system is obtained by solving the differential equations. The D.Es can be solved either by using classical method (or) by using the Laplace transform.

The total response of the system can be expressed as the sum of 2 components, that is

Total response = Natural response + Forced response.

The Natural response is the response due to the initial conditions alone of the system & the forced response of the system is the response due to i/p alone.

The total response of the system is equal to the forced response when the initial conditions are zero.

Let us consider a linear time invariant system

$$\sum_{K=0}^{N} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{M} b_K \frac{d^K x(t)}{dt^K}$$

When the initial conditions are neglected, we have

$$y(0^-) = \frac{dy(0^-)}{dt} = \frac{d^2y(0^-)}{dt^2} = \cdots\cdots = \frac{d^{N-1}y(0^-)}{dt^{N-1}} = 0$$

&

$$x(0^-) = \frac{dx(0^-)}{dt} = \frac{d^2x(0^-)}{dt^2} = \cdots\cdots = \frac{d^{M-1}x(0^-)}{dt^{M-1}} = 0$$

Applying L.T on both sides of the D.E & assuming zero initial conditions, we have

$$a_N S^N Y(S) + a_{N-1} S^{N-1} Y(S) + \cdots\cdots + a_0 Y(S) = b_M S^M X(S) + b_{M-1} S^{M-1} X(S) + \cdots\cdots + b_0 X(S)$$

$$\left(a_N S^N + a_{N-1} S^{N-1} + \cdots\cdots + a_0\right) Y(S) = \left(b_M S^M + b_{M-1} S^{M-1} + \cdots\cdots + b_0\right) X(S)$$

$$\frac{Y(S)}{X(S)} = \frac{b_M S^M + b_{M-1} S^{M-1} + \cdots\cdots + b_0}{\qquad\qquad}$$

The ratio of the Laplace transform of the output to the Laplace transform of the i/p with the initial conditions neglected is called the transfer function of the system.

The transfer function of a system is also defined as the L.T of the impulse response of the system.

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{L(\text{Zero state response})}{L(\text{Input})}$$

The function $H(s)$ depends upon the coefficients of the D.E. It does depend on the i/p signal (or) the initial Energy storage.

Response $Y(s) = X(s)\,H(s)$.

Inverse Laplace transform of $Y(s)$, i.e.,

$$y(t) = L^{-1}\left(X(s)H(s)\right) \rightarrow \text{is known as zero state response}$$

If i/p $x(t) = \delta(t)$ then $X(s) = 1$

$$y(t) = L^{-1}(H(s)) = h(t) \text{ is known as the impulse response of the system.}$$

Ex:- obtain the transfer function of the system if $y(t) = u(t) + e^{-t}u(t) - e^{-3t}u(t) - e^{-4t}u(t)$ & $x(t) = e^{-2t}u(t)$.

$\Rightarrow$ $x(t) = e^{-2t}u(t)$

$$X(s) = \frac{1}{s+2}$$

$$y(t) = u(t) + e^{-t}u(t) - e^{-3t}u(t) - e^{-4t}u(t)$$

$$Y(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{1}{s+3} - \frac{1}{s+4}$$

$$Y(s) = \frac{(s+4)(s+3)(s+1) + s(s+2)(s+4) - s(s+1)(s+4) - s(s+1)(s+3)}{s(s+1)(s+3)(s+4)}.$$

$$Y(s) = \frac{6(s^2+4s+2)}{s(s+1)(s+3)(s+4)}$$

$$H(s) = \frac{6(s+2)(s^2+4s+2)}{s(s+1)(s+3)(s+4)} = \frac{6(s^3+6s^2+10s+4)}{s^4+8s^3+19s^2+12s}.$$

Ex:- Find the system transfer function for each of the systems described by the following D.Es.

(a) $\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 14y(t) = 4\frac{dx(t)}{dt} + 7x(t)$

(b) $\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 6y(t) = 2\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 7x(t)$

⇝ (a) $\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 14y(t) = 4\frac{dx(t)}{dt} + 7x(t)$

$s^2Y(s) + 8sY(s) + 14Y(s) = 4sX(s) + 7X(s)$ $\left[\text{L.T of } \frac{d^2y(t)}{dt^2} = s^2Y(s) - sy(0) - y'(0)\right]$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{4s+7}{s^2+8s+14}$$

(b) $\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 6y(t) = 2\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 7x(t)$

$s^3Y(s) + 5s^2Y(s) + 12sY(s) + 6Y(s) = 2s^2X(s) + 5sX(s) + 7X(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s^2+5s+7}{s^3+5s^2+12s+6}$$

Ex:- Solve the integro D.E $\frac{dy(t)}{dt} + 2y(t) + 5\int_0^t y(t)dt = e^{-2t}$ when $y(0) = 0$.

⇝ $\frac{dy(t)}{dt} + 2y(t) + 5\int_0^t y(t)dt = e^{-2t}$

$sY(s) - y(0) + 2Y(s) + \frac{5Y(s)}{s} = \frac{1}{s+2}$

$y(0) = 0$ $\quad s^2Y(s) - sy(0) + 2sY(s) + 5Y(s) = \frac{s}{s+2}$

$s^2Y(s) + 2sY(s) + 5Y(s) = \frac{s}{s+2}$

$$Y(s) = \frac{s}{(s+2)(s^2+2s+5)} = \frac{s}{(s+2)((s+1)^2+2^2)}$$

$$Y(s) = \frac{1}{s+2} + \frac{Bs+C}{s^2+2s+5} = \frac{-2/5}{s+2} + \frac{(2/5)s+1}{s^2+2s+5}$$

$$Y(s) = \frac{-2/5}{s+2} + \frac{(2/5)(s+1)+(3/5)}{(s+1)^2+2^2} = \frac{-2/5}{s+2} + \frac{(2/5)(s+1)}{(s+1)^2+2^2} + \frac{3}{10}\left(\frac{2}{(s+1)^2+2^2}\right)$$

Take Inverse L.T. we have

$$y(t) = \frac{-2}{5}e^{-2t}u(t) + \frac{2}{5}e^{-t}\cos 2t\,u(t) + \frac{3}{10}e^{-t}\sin 2t\,u(t).$$

Ex:- A System is described by the Differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + x(t),\ \frac{dy(0)}{dt} = 3,\ y(0) = 1,\ x(t) = u(t)$$

Find the transfer function & the o/p signal $y(t)$.

▷ $x(t) = u(t)$ $\quad \frac{dy(0)}{dt} = 3$ $\quad y(0) = 1$

$$s^2Y(s) + 6sY(s) + 8Y(s) = sX(s) + X(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{s+1}{s^2+6s+8}$$

Taking Laplace transform on both sides for D.E, we have

$$\left(s^2Y(s) - sY(0) - \frac{dy(0)}{dt}\right) + 6\left(sY(s) - y(0)\right) + 8Y(s) = \left(sX(s) - x(0)\right) + X(s)$$

$$\left(s^2Y(s) - s(1) - 3\right) + 6\left(sY(s) - 1\right) + 8Y(s) = sX(s) + X(s)$$

$$Y(s)\left(s^2+6s+8\right) = \left(s+3+6\right) + (s+1)X(s) = \left(s+9\right) + (s+1)\frac{1}{s}$$

$$Y(s)\left(s^2+6s+8\right) = (s+9) + \left(1+\frac{1}{s}\right) = s+10+\frac{1}{s} = \frac{s^2+10s+1}{s}$$

$$Y(s) = \frac{s^2+10s+1}{s(s^2+6s+8)} = \frac{s^2+10s+1}{s(s+4)(s+2)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$Y(s) = \frac{1/8}{s} + \frac{15/4}{s+2} + \frac{-23/8}{s+4} = \frac{1}{8}u(t) + \frac{15}{4}e^{-2t}u(t) - \frac{23}{8}e^{-4t}u(t).$$

Ex:- A System is described by the Differential Equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

(a) Find the impulse response of the system.

(b) For the initial conditions, $dy(0^-)/dt = 2$ & $y(0^-) = 1$ & the i/p $x(t) = u(t)$, find the free & forced response of the system.

Sol (a) $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Taking LT on both sides & neglecting initial conditions.

$$s^2Y(s) + 5sY(s) + 6Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)}$$

$$H(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$$h(t) = e^{-2t}u(t) - e^{-3t}u(t).$$

(b) $$s^2Y(s) - sy(0^-) - \frac{dy(0^-)}{dt} + 5(sY(s) - y(0^-)) + 6Y(s) = X(s)$$

$$s^2Y(s) - (s+5)y(0^-) - \frac{dy(0^-)}{dt} + 5sY(s) + 6Y(s) = X(s)$$

$$(s^2+5s+6)\,Y(s) = X(s) + (s+5)y(0^-) + \frac{dy(0^-)}{dt}$$

$$Y(s) = \left(\frac{(s+5)y(0^-) + (dy(0^-)/dt)}{s^2+5s+6}\right) + \left(\frac{X(s)}{s^2+5s+6}\right)$$

The 1st term on the Right side of the response depends on the initial conditions, hence it is the natural response. It is called free response. The 2nd term represents the response due to i/p & is the forced response. The sum of these 2 ~~terms~~ responses gives the total response of the system.

Substituting the initial conditions, $\frac{dy(0^-)}{dt} = 2$, $y(0^-) = 1$ & the i/p $X(s) = 1/s$, we get

$$X(s) = \frac{1}{s}$$

$$Y(s) = \frac{(s+5)(1)+2}{s^2+5s+6} + \frac{1/s}{s^2+5s+6} = \frac{s+7}{s^2+5s+6} + \frac{1}{s(s^2+5s+6)}$$

$$Y(s) = \frac{s+7}{(s+2)(s+3)} + \frac{1}{s(s+2)(s+3)} = \frac{5}{s+2} - \frac{4}{s+3} + \frac{1}{6}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s+3}\right)$$

$$y(t) = 5e^{-2t}u(t) - 4e^{-3t}u(t) + \frac{1}{6}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)$$

Natural Response — Forced Response

$$y(t) = \frac{1}{6}u(t) + \frac{9}{2}e^{-2t}u(t) - \frac{11}{3}e^{-3t}u(t)$$

Ex:- Find the forced response of the system with transfer function

$$H(s) = \frac{1}{(s+2)(s+3)}$$

for an i/p $x(t) = e^{-t}u(t)$.

$$X(s) = \frac{1}{s+1}$$

$$Y(s) = X(s)H(s) = \frac{1}{(s+2)(s+3)} \times \frac{1}{(s+1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$Y(s) = \frac{1/2}{s+1} - \frac{1}{s+2} + \frac{1/2}{s+3}$$

$$Y(s) = \frac{1/2}{s+1} - \frac{1}{s+2} + \frac{1/2}{s+3} = \frac{1/2}{s+1} + \frac{1/2}{s+3} - \frac{1}{s+2}$$

$$y(t) = \underbrace{\frac{1}{2}e^{-t}}_{\text{Steady state response}} + \underbrace{\frac{1}{2}e^{-3t} - e^{-2t}}_{\text{Transient Response.}}$$

Due to i/p pole

Due to poles of transfer function

Ex: Using Laplace transform Solve the Differential equations:

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

if $\frac{dy(0^-)}{dt} = 2$, $y(0^-) = 1$ for input $x(t) = \cos 2t$.

→

$$s^2Y(s) - sy(0^-) - \frac{dy(0^-)}{dt} + Y(s) = X(s)$$

$$X(s) = L(x(t)) = L(\cos 2t) = \frac{s}{s^2+4}$$

Substituting the initial conditions, we have

$$s^2Y(s) - s - 2 + Y(s) = \frac{s}{s^2+4}$$

$$(s^2+1)Y(s) = (s+2) + \frac{s}{s^2+4}$$

$$Y(s) = \frac{s+2}{s^2+1} + \frac{s}{(s^2+1)(s^2+4)} = \frac{(s+2)(s^2+4)+s}{(s^2+1)(s^2+4)}$$

$$Y(s) = \frac{s^3+2s^2+5s+8}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} = \frac{\left(\frac{4}{3}s+2\right)}{s^2+1} + \frac{\left(\frac{-1}{3}\right)s}{s^2+4}$$

$$Y(s) = \frac{\left(\frac{4}{3}\right)s+2}{s^2+1} + \frac{\frac{-1}{3}s}{s^2+4} = \frac{4}{3}\left(\frac{s}{s^2+1}\right) + 2\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{s}{s^2+4}\right).$$

$$y(t) = \frac{4}{3}\cos t\,u(t) + 2\sin t\,u(t) - \frac{1}{3}\cos 2t\,u(t)$$

Ex:- For a system with transfer function $H(s) = \left(\frac{s+2}{s^2+4s+3}\right)$

Find the Zero state response, if the i/p $x(t)$ is $e^{-t}u(t)$.

→

$$H(s) = \frac{s+2}{s^2+4s+3} \qquad x(t) = e^{-t}u(t) \quad X(s) = \frac{1}{s+1}$$

Response $Y(s) = H(s)X(s) = \left(\frac{s+2}{s^2+4s+3}\right)\left(\frac{1}{s+1}\right) = \frac{s+2}{(s+1)^2(s+3)}$

$$Y(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$A = \frac{d}{ds}\left((s+1)^2 Y(s)\right)\Big|_{s=-1} = \frac{d}{ds}\left(\frac{s+2}{s+3}\right)_{s=-1} = \frac{(s+3)1-(s+2)1}{(s+3)^2}\Big|_{s=-1} = \frac{1}{4}$$

$$B = (s+1)^2 \frac{s+2}{(s+1)^2(s+3)}\Big|_{s=-1} = \frac{-1+2}{-1+3} = \frac{1}{2}$$

$$C = (s+3)\frac{(s+2)}{(s+1)^2(s+3)}\Big|_{s=-3} = \frac{-3+2}{(-3+1)^2} = \frac{-1}{4}$$

$$Y(s) = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{2}}{(s+1)^2} + \frac{\left(\frac{-1}{4}\right)}{s+3}$$

$$\boxed{y(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)}$$

Ex:- A system has transfer function $H(s) = \frac{3s+2}{(s+4)(s-1)}$. Find the impulse response assuming the system is causal & stable.

⇒

$$H(s) = \frac{3s+2}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1} = \frac{2}{s+4} + \frac{1}{s-1}$$

Poles are at $s=-4$ & $s=1$.

For an LTI system to be stable, its ROC must contain the imaginary axis of S-plane. So in this case there would not be any combination which can be stable as well as causal.

The impulse response which is stable but not causal is;

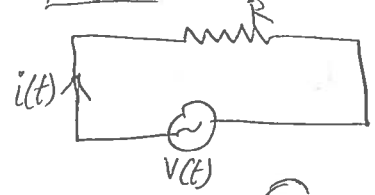
$$h(t) = 2e^{-4t}u(t) - e^{t}u(-t); \quad ROC; \quad -4 < Re(s) < 1$$

By the impulse response which is causal but not stable is:

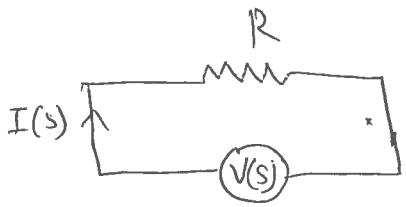
$$h(t) = 2e^{-4t}u(t) + e^{t}u(t); \quad ROC; \quad Re(s) > 1.$$

# Circuit Analysis using Laplace Transforms:-

## Pure Resistive element:-




(a) Pure Resistive Circuit




(b) Laplace transformed version.

For the fig (a)

$$V(t) = R\, i(t)$$

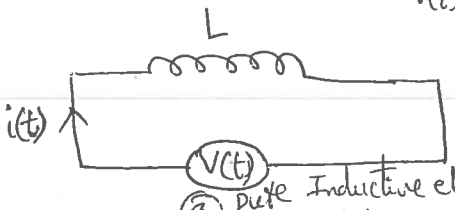
$$V(s) = R\, I(s)$$

Resistance R in time domain remains as R in S domain also.

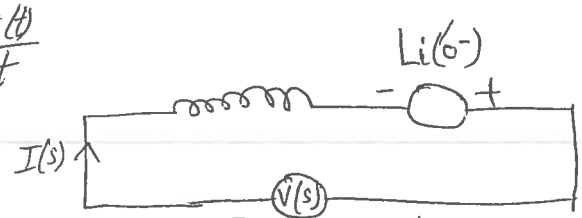
It has no initial conditions.

## Pure Inductive element:-

$$V(t) = L\frac{di(t)}{dt}$$




(a) Pure Inductive element



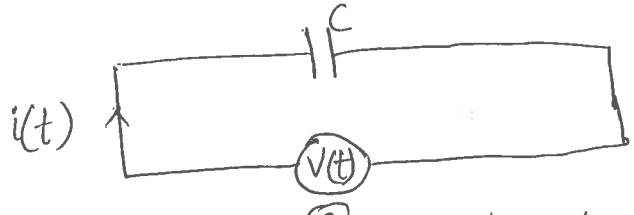

(b) Its Laplace transformed in s-domain

$$V(s) = L\left(sI(s) - i(0^-)\right)$$

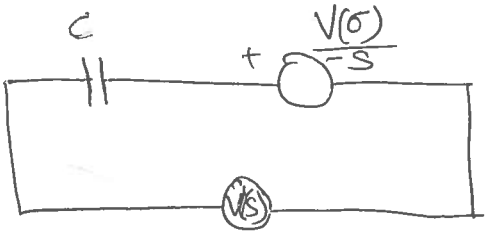
$$V(s) = LsI(s) - Li(0^-) = LsI(s) - Li(0^-)$$

$i(0^-)$ is the initial current through the inductor. Inductor L in time domain becomes Ls in S-domain.
Also the current I(s) is opposed by a voltage source of $Li(0^-)$.

## Pure Capacitive element:-




(a) Pure Capacitive element




(b) Laplaced Transformed version

For the circuit shown in fig (a)

$$i(t) = C\frac{dV(t)}{dt}$$

$$I(s) = C\left(sV(s) - V(0^-)\right) = CsV(s) - CV(0^-) = CsV(s) - q(0^-)$$

$$V(s) = \frac{I(s)}{Cs} + \frac{q(0^-)}{Cs} = \frac{I(s)}{Cs} + \frac{V(0^-)}{s}.$$

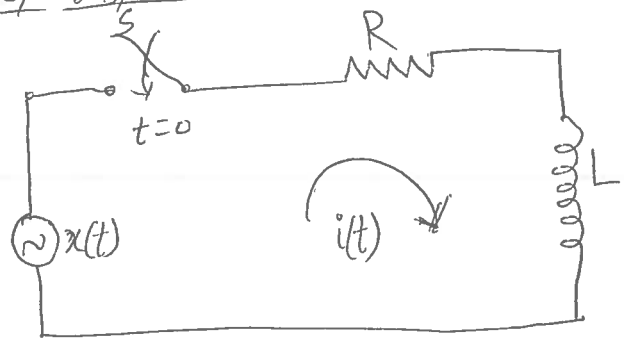
$v(0^-)$ -- initial voltage across capacitor.

$\frac{V(0^-)}{s}$ is introduced which opposes the flow of current.

Capacitor 'C' in time domain becomes $\frac{1}{cs}$ in s-domains.

## Step and Impulse responses of Series R-L Circuit :-

### Evaluation of step response :-




R-L Circuit

In the above circuit consider that the switch is closed at time t=0 For the step response i/p excitation is $x(t) = Vu(t)$. Apply KVL to the ckt

$$x(t) = Ri(t) + L\frac{di(t)}{dt}$$

$$Vu(t) = Ri(t) + L\frac{di(t)}{dt}$$

Apply L.T on both sides

$$\frac{V}{S} = RI(s) + L\left(sI(s) - i(0^+)\right)$$

Since the current through an inductor can't change instantaneously & since the current before t=0 is zero, $i(0^+) = 0$.

$$\therefore \frac{V}{S} = (Ls + R)I(s)$$

$$I(s) = \frac{V}{s}\left(\frac{1}{Ls+R}\right) = \frac{V}{L}\left(\frac{1}{s(s+R/L)}\right) = \frac{V}{L}\frac{L}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)$$

$$\boxed{I(s) = \frac{V}{R}\left(\frac{1}{s} - \frac{L}{sL+R}\right)} \qquad I(s) = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right)$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{V}{R}\left(1 - e^{-(R/L)t}\right).$$

### Evaluation of Impulse Response :-

To Evaluate impulse response the i/p Excitation is $x(t) = \delta(t)$.

$$\therefore \quad x(t) = Ri(t) + L\frac{di(t)}{dt}$$

$$\delta(t) = Ri(t) + L\frac{di(t)}{dt}$$

Taking L.T. on both sides, we have

$$RI(s) + L\left(sI(s) - i(0^-)\right) = 1$$

Now $i(0^-) = 0$.

$$(R + LS)\, I(s) = 1$$

$$I(s) = \frac{1}{R+LS} = \frac{1}{L}\left(\frac{1}{S+\frac{R}{L}}\right)$$

Taking Inverse Laplace transform, we obtain

$$i(t) = \frac{1}{L} e^{-t(R/L)}\, u(t)$$

## Step Response & Impulse responses of Series R-C Circuit:-

### Step Response:-

In series RC circuit as shown in fig let us consider that the switch is closed at $t=0$.

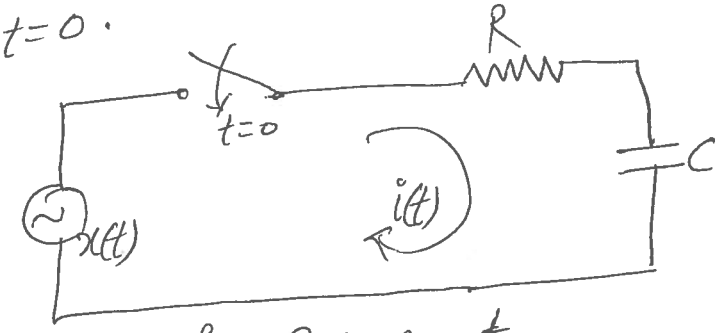



fig:- R-C Circuit.

writing KVL

$$x(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

$$Vu(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

$$Vu(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

Apply L.T on both sides

$$\frac{V}{s} = RI(s) + \frac{1}{C}\left(\frac{I(s)}{s}\right) + \frac{1}{C} L\left(q(0^+)\right) = \frac{V}{s}$$

(or)

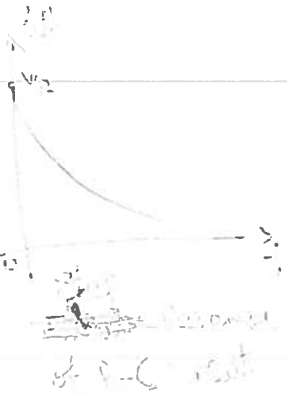
$$RI(s) + \frac{1}{C}\left(\frac{I(s)}{s} + \frac{q(0^+)}{s}\right) = \frac{V}{s}$$

where $q(0^+)$ is the charge on the capacitor at $t=0^+$. If the capacitor is initially relaxed, $q(0^+)=0$.

$$I(s)\left(R+\frac{1}{Cs}\right)=\frac{V}{s}$$

$$I(s)=\frac{V}{s}\left(\frac{1}{R+\frac{1}{Cs}}\right)=\frac{V}{s}\left(\frac{Cs}{RCs+1}\right)=\frac{VC}{RC\left(s+\frac{1}{RC}\right)}$$

$$I(s)=\frac{V/R}{s+\left(\frac{1}{RC}\right)}$$



Taking Inverse Laplace transform on both sides

$$i(t)=\frac{V}{R}e^{-t/RC} \longrightarrow \text{This is the step response}$$

Impulse response:-

$$x(t)=\delta(t)$$

$$R\,i(t)+\frac{1}{C}\int_{0}^{t} i(t)\,dt+\frac{1}{C}\int_{-\infty}^{0} i(t)\,dt=\delta(t)$$

Taking L.T on both sides

$$RI(s)+\frac{1}{C}\left(\frac{I(s)}{s}\right)+\frac{1}{C}L\left(q(0^+)\right)=1$$

$$RI(s)+\frac{1}{C}\left(\frac{I(s)}{s}+\frac{q(0^+)}{s}\right)=1$$

Now, since $q(0^+)=0$, we have

$$I(s)\left(R+\frac{1}{Cs}\right)=1$$

$$I(s)=\frac{1}{R+\frac{1}{Cs}}=\frac{1}{R\left(1+\frac{1}{RCs}\right)}=\frac{RCs}{R(RCs+1)}=\frac{\cancel{C}s}{R\cancel{C}\left(s+\frac{1}{RC}\right)}$$

$$I(s)=\frac{s}{R\left(s+\frac{1}{RC}\right)}=\frac{1}{R}\left(\frac{s+\frac{1}{RC}-\frac{1}{RC}}{s+\frac{1}{RC}}\right)$$

$$I(s)=\frac{1}{R}\left\{1-\frac{1}{RC}\left(\frac{1}{s+\frac{1}{RC}}\right)\right\}$$

Taking Inverse L.T on both sides

$$i(t)=\frac{1}{R}\left\{\delta(t)-\frac{1}{RC}e^{-\frac{1}{RC}t}u(t)\right\}$$

## Step Response of series RLC Circuit

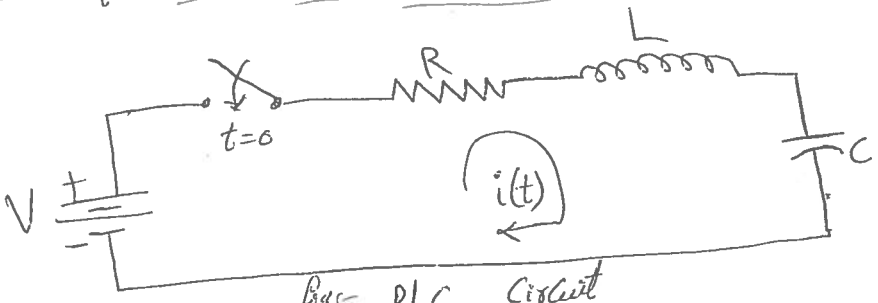



Fig:- RLC Circuit

The KVL equation for the circuit is

$$x(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

$$V u(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

Apply L.T on both sides

$$\frac{V}{S} = R\,I(S) + L\left(S\,I(S) - i(0^+)\right) + \frac{1}{C}L\left(q(0^+)\right) + \frac{1}{C}\frac{I(S)}{S}.$$

$$\frac{V}{S} = R\,I(S) + L\left(S\,I(S) - i(0^+)\right) + \frac{1}{C}\frac{q(0^+)}{S} + \frac{1}{C}\frac{I(S)}{S}$$

Where $i(0^+)$ is the initial current through the inductor & $q(0^+)$ is the initial charge on the capacitor. Neglecting the initial conditions we get

$$R\,I(S) + LS\,I(S) + \frac{1}{CS}I(S) = \frac{V}{S}$$

$$I(S)\left(LS + R + \frac{1}{CS}\right) = \frac{V}{S}$$

$$I(S) = \frac{V}{LS^2 + RS + \frac{1}{C}} = \frac{V}{L\left(S^2 + \frac{R}{L}S + \frac{1}{LC}\right)} = \frac{V}{L(S-P_1)(S-P_2)}$$

$$P_1, P_2 = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{R^2 - \frac{4L}{C}}$$

$$\boxed{i(t) = \frac{V}{L(P_1 - P_2)}\left(e^{P_1 t} - e^{P_2 t}\right)}$$

Ex: Find the Laplace transform of the signals: (a) $x(t)=u(-t)$ (b) $x(t)=e^{-t}u(t)+e^{-4t}u(t)$ & find their ROCs.

(a) $L\{x(t)\} = L\{u(-t)\} = \int_{-\infty}^{\infty} u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{-st}dt$

$$X(s) = \left(\frac{e^{-st}}{-s}\right)_{-\infty}^{0} = \frac{e^{0}-e^{-s(-\infty)}}{-s} = \frac{0-1}{s} \quad (\text{if } Re(s)<0)$$

$$X(s) = \frac{-1}{s} \qquad ROC\ Re(s)<0$$

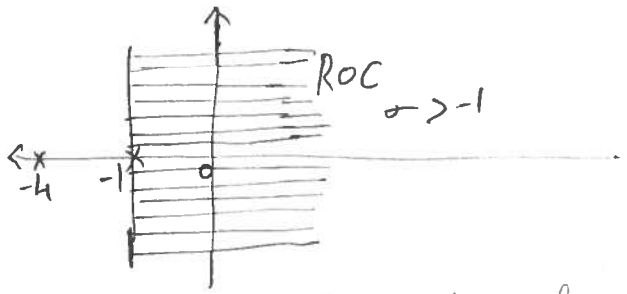
(b) $x(t)=e^{-t}u(t)+e^{-4t}u(t)$

$$L\{x(t)\} = L\{e^{-t}u(t)\} + L\{e^{-4t}u(t)\} = \frac{1}{s+1} + \frac{1}{s+4} = \frac{2s+5}{s^2+5s+4}$$

ROC for $L\{e^{-t}u(t)\}$ is $Re(s)>-1$

$L\{e^{-4t}u(t)\}$ is $Re(s)>-4$

∴ ROC for sum of 2 signals is equal to the Intersection of the ROCs of two signals.

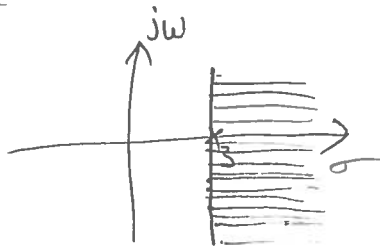



Ex:- Find the Laplace transform of $x(t)=e^{-2t}u(t)+e^{3t}u(t)$

$$X(s) = \frac{1}{s+2} + \frac{1}{s-3} = \frac{2s-1}{s^2-s-6}$$

ROC is for 1st function is $Re\{s\}>-2$

2nd function is $Re\{s\}>3$




ROC of 2 functions is $Re\{s\}>3$ $(\infty) \rightarrow 3$

Ex:- Find the Laplace transform of the signal $x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$ & find its ROC.

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{e^{-2t}u(-t)\} + \mathcal{L}\{e^{-3t}u(-t)\}$$

$$= \int_{-\infty}^{\infty} e^{-2t}u(-t)\,e^{-st}\,dt + \int_{-\infty}^{\infty} e^{-3t}u(-t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-2t}e^{-st}\,dt + \int_{-\infty}^{0} e^{-3t}e^{-st}\,dt = \int_{-\infty}^{0} e^{-(s+2)t}\,dt + \int_{-\infty}^{0} e^{-(s+3)t}\,dt$$

$$= \left(\int_{0}^{\infty} e^{(s+2)t}\,dt + \int_{0}^{\infty} e^{(s+3)t}\,dt\right)$$

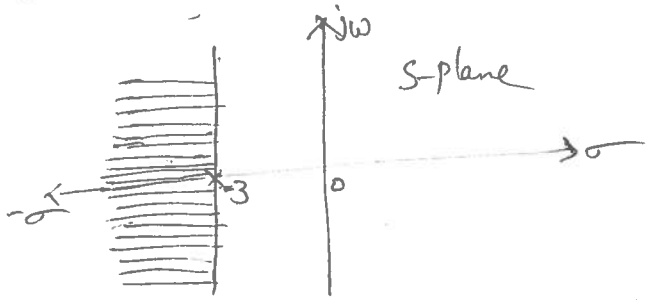
$$X(s) = \left(\left(\frac{e^{(s+2)t}}{s+2}\right)_0^{\infty} + \left(\frac{e^{(s+3)t}}{s+3}\right)_0^{\infty}\right)$$

The above transform converges if $\sigma+2<0$ & $\sigma+3<0$

ROC of 2 functions is $\sigma<-2$ & $\sigma<-3$

i.e., $(\sigma<-2) \cap (\sigma<-3) = \sigma<-3$

ROC of $x(t)$ is $Re(s)<-3$ (or) $\sigma<-3$




Ex:- Find the Laplace transform of $x(t) = e^{2t}u(t) + e^{-2t}u(-t)$

$$X(s) = \int_{0}^{\infty} e^{-t(s-2)}\,dt + \int_{-\infty}^{0} e^{-t(s+2)}\,dt \quad \sigma>2 \text{ & } \sigma<-2$$

ROC is $\sigma-2>0$ & $\sigma+2<0$ No Intersection → i.e., NO ROC

No Laplace Transform

Ex:- Find the Laplace transform & ROC for the following 20

3.23

ⓐ $x(t) = e^{-5t}u(t-1)$ ⓑ $x(t) = e^{t}\sin 2t$ for $t \le 0$

ⓒ $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$ ⓓ $x(t) = te^{-2|t|}$

⇒ ⓐ $x(t) = e^{-5t}u(t-1)$

$$L(x(t)) = X(s) = \int_{1}^{\infty} e^{-5t}u(t-1)e^{-st}dt = \int_{1}^{\infty} e^{-t(s+5)}dt$$

$$X(s) = \left\{\frac{e^{-t(s+5)}}{-(s+5)}\right\}_{1}^{\infty} = \frac{0 - e^{-(s+5)}}{-(s+5)} = \frac{e^{-(s+5)}}{s+5}$$

$$X(s) = \frac{e^{-(s+5)}}{s+5}$$

The above integral converges if $\sigma + 5 > 0$ i.e., $\sigma > -5$

ROC is $Re(s) > -5$, $(\sigma > -5)$.

ⓑ $x(t) = e^{t}\sin 2t = e^{t}\left(\frac{e^{j2t} - e^{-j2t}}{2j}\right) = \frac{1}{2j}\left\{e^{t(1+j2)} - e^{t(1-j2)}\right\}$

for $t \le 0$

$$X(s) = \frac{1}{2j}\left\{\int_{-\infty}^{\infty}\left(e^{t(1+j2)} - e^{t(1-j2)}\right)e^{-st}dt\right\}$$

$$X(s) = \frac{1}{2j}\left\{\int_{-\infty}^{0} e^{t(1+j2-s)}dt - \int_{-\infty}^{0} e^{t(1-j2-s)}dt\right\}$$

$$= \frac{1}{2j}\left\{\left(\frac{e^{(1-s+j2)t}}{(1-s+j2)}\right)_{-\infty}^{0} - \left(\frac{e^{(1-s-j2)t}}{(1-s-j2)}\right)_{-\infty}^{0}\right\}$$

$$= \frac{1}{2j}\left[\frac{1}{1-s+j2} - \frac{1}{1-s-j2}\right] = \frac{1}{2j}\left[\frac{1-s-j2-(1-s+j2)}{(1-s+j2)(1-s-j2)}\right]$$

$$= \frac{1}{2j}\left\{\frac{-j4}{(s-1)^2+2^2}\right\} = \frac{1}{2j}\left\{\frac{-2}{(s-1)^2+2^2}\right\}$$

$$X(s) = \frac{-2}{(s-1)^2+2^2}$$ ©

(c) $x(t) = e^{2t}u(-t) + e^{3t}u(-t)$

$$X(s) = \int_{-\infty}^{0} e^{2t}e^{-st}\,dt + \int_{-\infty}^{0} e^{3t}e^{-st}\,dt$$

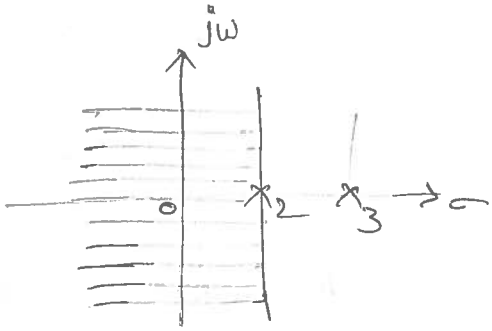
$$= \int_{-\infty}^{0} e^{-(s-2)t}\,dt + \int_{-\infty}^{0} e^{-(s-3)t}\,dt$$

$$= \int_{\infty}^{0} e^{(s-2)t}\,dt + \int_{\infty}^{0} e^{(s-3)t}\,dt$$

$\sigma - 2 < 0$ ; $\sigma < 3$

$\sigma < 2$

$Roc = \{\sigma < 2 \cap \sigma < 3\}$




Roc of $x(t) = \sigma < 2$ i.e., $Re(s) < 2$.

(d) $x(t) = te^{-2|t|}$

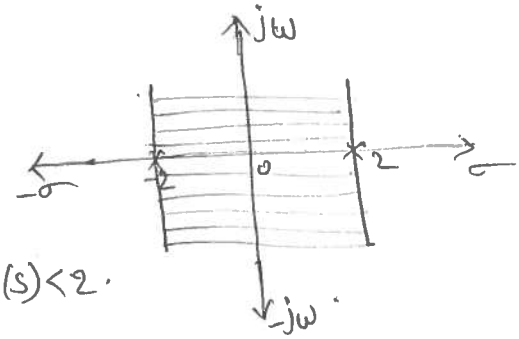
$$X(s) = \int_{-\infty}^{\infty} te^{-2|t|}e^{-st}\,dt = \int_{-\infty}^{0} te^{2t}e^{-st}\,dt + \int_{0}^{\infty} te^{-2t}e^{-st}\,dt$$

$$X(s) = \int_{-\infty}^{0} t\left(e^{-t(s-2)}\right)dt + \int_{0}^{\infty} t\left(e^{-t(s+2)}\right)dt$$

Converges if $\sigma - 2 \leq 0$, $\sigma \leq 2$, $\sigma < 2$

Converges if $\sigma + 2 > 0$, $\sigma > -2$

Roc is $\{\sigma < 2 \cap \sigma > -2\}$




$$X(s) = -\frac{1}{(s-2)^2} + \frac{1}{(s+2)^2} \quad Roc;\ -2 < Re(s) < 2.$$

★ Find the Inverse Laplace transform for the following:

ⓐ $\log\left(\frac{1+s}{s^2}\right)$ ⓑ $\log\left(\frac{s(s+1)}{s^2+1}\right)$

ⓐ $X(s) = \log\left(\frac{1+s}{s^2}\right)$

$$x(t) = L^{-1}(X(s)) = L^{-1}\left(\log\left(\frac{1+s}{s^2}\right)\right)$$

$$L(x(t)) = \log\left(\frac{1+s}{s^2}\right)$$

$$L(t\,x(t)) = -\frac{d}{ds}\left(\log\left(\frac{1+s}{s^2}\right)\right) = -\frac{d}{ds}\left(\log(1+s) - \log s^2\right)$$

$$L(t\,x(t)) = \frac{-1}{1+s} - 2s\left(\frac{1}{s^2}\right) = \frac{2}{s} - \frac{1}{s+1}$$

$$t\,x(t) = L^{-1}\left(\frac{2}{s} - \frac{1}{s+1}\right) = 2L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$t\,x(t) = 2(1) - e^{-t}$$

$$\boxed{x(t) = \frac{2 - e^{-t}}{t}}$$

ⓑ $X(s) = \log\left(\frac{s(s+1)}{s^2+1}\right)$

$$x(t) = L^{-1}(X(s))$$

$$x(t) = L^{-1}\left(\log\frac{s(s+1)}{s^2+1}\right) = L^{-1}\left(\log(s(s+1)) - \log(s^2+1)\right)$$

$$x(t) = L^{-1}\left(\log s + \log(s+1) - \log(s^2+1)\right)$$

$$L(x(t)) = \log s + \log(s+1) - \log(s^2+1)$$

$$L(t\,x(t)) = -\frac{d}{ds}\left(\log s + \log(s+1) - \log(s^2+1)\right) = -\left(\frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2+1}(2s)\right)$$

$$L[t\,x(t)] = -\frac{1}{s} - \frac{1}{s+1} + \frac{2s}{s^2+1}$$

$$t\,x(t) = L^{-1}\left(\frac{-1}{s} - \frac{1}{s+1} + \frac{2s}{s^2+1}\right) = -L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{2s}{s^2+1}\right)$$

$$t\,x(t) = -1 - e^{-t} + 2\cos t$$

$$\boxed{x(t) = \frac{2\cos t - e^{-t} - 1}{t}}$$

Ex:- Find the inverse Laplace Transform of the following functions:

(a) $X(s) = \dfrac{s+4}{s^2+5s+6}$ (b) $X(s) = \dfrac{s^2+3s+4}{s^3+5s^2+7s+3}$

(a) $$X(s) = \frac{s+4}{s^2+5s+6} = \frac{s+4}{(s+2)(s+3)} = \frac{2}{s+2} - \frac{1}{s+3}$$

$$\boxed{x(t) = 2e^{-2t}u(t) - e^{-3t}u(t)}$$

(b) $$X(s) = \frac{s^2+3s+4}{s^3+5s^2+7s+3} = \frac{s^2+3s+4}{(s+3)(s+1)^2} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = 1, \quad B = \frac{1}{1!}\frac{d}{ds}\left[(s+1)^2 X(s)\right]_{s=-1} = \frac{d}{ds}\left(\frac{s^2+3s+4}{s+3}\right)\Bigg|_{s=-1}$$

$$B = 0$$

$$\text{By } C = (s+1)^2 X(s)\Big|_{s=-1} = 1$$

$$X(s) = e^{-3t}u(t) + te^{-t}u(t) \quad \frac{1}{(s+3)} + \frac{1}{(s+1)^2} = e^{-3t}u(t) + te^{-t}u(t).$$

Ex:- Given $X(s) = \dfrac{s^2+5s+1}{s^2+3s+2}$ find $x(t)$ $t \geq 0$

$$s^2+3s+2 \;\big)\; s^2+5s+1 \;\big(\;1$$
$$\underline{s^2+3s+2}$$
$$2s-1$$

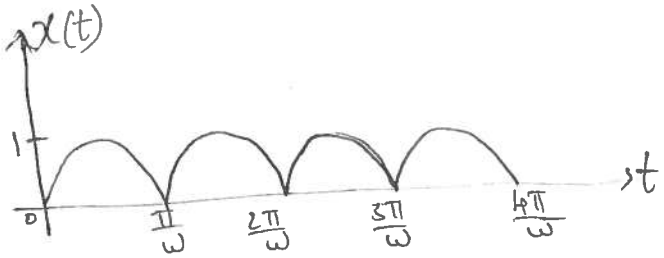
$$X(s) = 1 + \frac{2s-1}{s^2+3s+2} = 1 + \frac{2s-1}{(s+1)(s+2)}$$

$$X(s) = 1 - 3\left(\frac{1}{s+1}\right) + 5\left(\frac{1}{s+2}\right)$$

$$x(t) = \delta(t) - 3e^{-t}u(t) + 5e^{-2t}u(t).$$

**5:-** Find the Inverse Laplace Transform of the wave form shown below

(3-2-4)




$x(t) = \sin\omega t \quad 0 < t < \frac{\pi}{\omega}$.

$$X(s) = L(x(t)) = \frac{1}{1-e^{-sT}} X_1(s)$$

$$X_1(s) = \int_0^T x(t)\, e^{-st}\, dt$$

$$X(s) = \frac{1}{1-e^{-s\pi/\omega}} \int_0^{\pi/\omega} \sin\omega t\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-s\pi/\omega}} \left( \int_0^{\pi/\omega} \left( \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt \right)$$

$$= \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)} \left( \int_0^{\pi/\omega} e^{-t(s-j\omega)}\, dt - \int_0^{\pi/\omega} e^{-t(s+j\omega)}\, dt \right)$$

$$X(s) = \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)} \left( \left( \frac{e^{-t(s-j\omega)}}{-(s-j\omega)} \right)_0^{\pi/\omega} - \left[ \frac{e^{-t(s+j\omega)}}{-(s+j\omega)} \right]_0^{\pi/\omega} \right)$$

$$X(s) = \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)} \left( \left( \frac{e^{-t(s+j\omega)}}{s+j\omega} \right)_0^{\pi/\omega} - \left( \frac{e^{-t(s-j\omega)}}{(s-j\omega)} \right)_0^{\pi/\omega} \right)$$

$$= \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)} \left( \frac{e^{-\pi/\omega\,(s+j\omega)} - 1}{s+j\omega} - \frac{e^{-\pi/\omega\,(s-j\omega)} - 1}{s-j\omega} \right)$$

$$= \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)} \left( \frac{\left(e^{-\pi/\omega(s+j\omega)} - 1\right)(s-j\omega) - \left(e^{-\pi/\omega(s-j\omega)} - 1\right)(s+j\omega)}{s^2+\omega^2} \right)$$

$$= \frac{1}{2j\left(1-e^{-s\pi/\omega}\right)(s^2+\omega^2)} \left( \omega e^{-s\pi/\omega} + \omega \right)(2j)$$

$$X(s) = \frac{1}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}\left(\omega e^{-s\pi/\omega}+\omega\right) = \omega\left(\frac{1+e^{-s\pi/\omega}}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}\right)$$

$$X(s) = \frac{\omega}{s^2+\omega^2}\left(\frac{e^{s\pi/2\omega}+e^{-s\pi/2\omega}}{e^{s\pi/2\omega}-e^{-s\pi/2\omega}}\right) = \frac{\omega}{s^2+\omega^2}\coth\left(\frac{s\pi}{2\omega}\right)$$

Ex:- Find the Inverse Laplace transform of $X(s) = \frac{1}{s(s+3)}$ using Integral property.

↳ Let $X_1(s) = \frac{1}{s+3}$

$$X(s) = \frac{X_1(s)}{s}$$

$$x_1(t) = L^{-1}(X_1(s)) = L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}u(t)$$

$$L\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) = \frac{X(s)}{s}$$

$$L^{-1}\left(\frac{X(s)}{s}\right) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$L^{-1}\left(\frac{1}{s(s+3)}\right) = L^{-1}\left(\frac{X_1(s)}{s}\right) = \int_{-\infty}^{t} x_1(\tau)\,d\tau = \int_{0}^{t} e^{-3\tau}u(\tau)\,d\tau$$

$$L^{-1}\left(\frac{1}{s(s+3)}\right) = \int_{0}^{t} e^{-3\tau}\,d\tau = \left(\frac{e^{-3\tau}}{-3}\right)_0^t = \frac{e^{-3t}-1}{-3} = \frac{1-e^{-3t}}{3}\ //.$$

Ex:- Find the Laplace transform of the following signals using properties of Laplace transform:

(a) $x(t) = \delta(t)$ (b) $x(t) = \frac{d}{dt}\delta(t)$ (c) $x(t) = tu(t)$ (d) $x(t) = u(t-3)$

(e) $x(t) = te^{-t}u(t)$ (f) $x(t) = t^2e^{-3t}u(t)$ (g) $x(t) = e^{-2t}\cos(2\pi 100t)u(t)$

(h) ~~$x(t) = te^{-5t}$~~ $x(t) = e^{-5t}\sin 10at\,u(t)$ (i) $x(t) = te^{-2t}\sin 2t\,u(t)$

(j) $x(t) = 2e^{-6t}u(t) - 10e^{4t}u(-t)$ (k) $x(t) = \sin at\cos bt\,u(t)$ (l) $x(t) = \frac{\sin at}{t}$

(m) $x(t) = \frac{\cos at}{t}$ (n) $x(t) = \frac{\sin^2 at}{t}$ (o) $x(t) = \frac{e^{at}-\cos 3t}{t}$ (p) $x(t) = 3e^{-2t}\cos 4t - 5e^{-3t}\sin 2t + 2$

a) $x(t) = \delta(t)$

$\delta(t) = \frac{d}{dt} u(t)$

$L\left(\frac{d}{dt} u(t)\right) = sX(s)$

$L(u(t)) = \frac{1}{s}$ "Use Differentiation in time"

$L(\delta(t)) = L\left\{\frac{d}{dt} u(t)\right\} = s \times \frac{1}{s} = 1$

$\therefore L(\delta(t)) = 1$

b) $x(t) = \frac{d}{dt}\delta(t)$

$L(\delta(t)) = 1$

$L\left(\frac{d}{dt}\delta(t)\right) = sX(s)$ Use Differentiation in time.

$= s \times 1 = \underline{s}$.

c) $x(t) = t\,u(t)$

$L\{u(t)\} = \frac{1}{s}$

$L\{t\,u(t)\} = -\frac{d}{ds}\left(L\{u(t)\}\right)$

$L\{t\,u(t)\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = -\left(\frac{-1}{s^2}\right) = \frac{1}{s^2}$

d) $x(t) = u(t-3)$

$L(u(t)) = \frac{1}{s}$

$L(u(t-3)) = e^{-3s} L(u(t))$

$= \frac{e^{-3s}}{s}$

use Time shift

$$\boxed{L(u(t-3)) = \frac{e^{-3s}}{s}}$$

e) $x(t) = t\,e^{-t}u(t)$

we know that

$L\{t\,u(t)\} = \frac{1}{s^2}$

Apply Frequency shift

$L\{e^{-t}t\,u(t)\} = L\{t\,u(t)\}\Big|_{s=s+1} = \frac{1}{(s+1)^2}$

(or)

$L(x(t)) = X(s)$

$L(e^{-at}x(t)) = X(s+a)$

$L(u(t)) = \frac{1}{s}$

$L(e^{-t}u(t)) = \frac{1}{s+1}$

$L(t\,x(t)) = -\frac{d}{ds}X(s)$

use Differentiation in 's' Domain

$L(t\,e^{-t}u(t)) = -\frac{d}{ds}\left(\frac{1}{s+1}\right)$

$= -\left(\frac{-1}{(s+1)^2}\right)$

$L(te^{-t}u(t)) = \frac{1}{(s+1)^2}$ //

(f) $x(t) = t^2 e^{-3t} u(t)$

$\mathcal{L}\{e^{-3t}u(t)\} = \frac{1}{s+3}$

use Differentiation in s-Domain property.

$\mathcal{L}\{t^2 e^{-3t}u(t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$

$\mathcal{L}\{t^2 e^{-3t}u(t)\} = \frac{d}{ds}\left(\frac{-1}{(s+3)^2}\right) = \frac{2}{(s+3)^3}$

$\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{d}{ds}\left(\frac{d}{ds}(s+3)^{-1}\right) = \frac{d}{ds}\left(-1(s+3)^{-1-1}\right) = \frac{d}{ds}\left(\frac{-1}{(s+3)^2}\right)$

$= \frac{d}{ds}\left(-1(s+3)^{-2}\right) = -\frac{d}{ds}\left((s+3)^{-2}\right) = 2(s+3)^{-2-1} = \frac{2}{(s+3)^3}$.

$\mathcal{L}\{t^2 u(t)\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

use time shift

(oo) $\mathcal{L}\{e^{-t}t^2u(t)\} = \frac{2}{(s+1)^3}$

(g) $x(t) = e^{-2t}\cos(2\pi 100t)\,u(t)$

$\mathcal{L}\{\cos 2\pi 100t\, u(t)\} = \frac{s}{s^2 + (2\pi\times 100)^2} = \frac{s}{s^2+(200\pi)^2}$

use time shift

$\mathcal{L}\{e^{-2t}\cos(2\pi 100t)u(t)\} = \frac{(s+2)}{(s+2)^2+(200\pi)^2}$

(h) $x(t) = e^{-5t}\sin 10at\, u(t)$

$\mathcal{L}(\sin 10at\, u(t)) = \frac{10a}{s^2+(10a)^2} = \frac{10a}{s^2+(10a)^2}$

$\mathcal{L}(e^{-5t}\sin 10at\, u(t)) = \frac{10a}{(s+5)^2+(10a)^2} = \frac{10a}{(s+5)^2+(10a)^2}$

(i) ~~$\mathcal{L}\{e^{-2}\sin 2t\, u(t)\}$~~ $x(t) = te^{-2t}\sin 2t\, u(t)$

$\mathcal{L}\{\sin 2t\, u(t)\} = \frac{2}{s^2+4}$

$\mathcal{L}\{e^{-2t}\sin 2t\, u(t)\} = \frac{2}{(s+2)^2+4}$

(j) $x(t) = x_1(t) + x_2(t)$

$x_1(t) = 2e^{-6t}u(t) \qquad x_2(t) = -10e^{4t}u(-t)$

$$L\{x_1(t)\} = L\{2e^{-6t}u(t)\} = 2L\{e^{-6t}u(t)\} = 2L\{u(t)\}\Big|_{s=s+6}$$

$$= \left(\frac{2}{s}\right)\Big|_{s=s+6} = \frac{2}{s+6}; \quad ROC;\ Re(s) > -6.$$

$$L\{x_2(t)\} = L\{-10e^{4t}u(-t)\} = -10L\{e^{4t}u(-t)\} = -10L(u(-t))\Big|_{s=s-4}$$

$$= -10\left(\frac{-1}{s}\right)\Big|_{s=s-4} = \frac{10}{s-4}; \quad ROC;\ Re(s) < 4$$

$$L(x(t)) = X(s) = L(x_1(t)) + L(x_2(t))$$

$$L(x(t)) = \frac{2}{s+6} + \frac{10}{s-4} = \frac{2s-8+10s+60}{(s-4)(s+6)}; \quad ROC\ -6 < Re(s) < 4.$$

(k) $x(t) = \sin at \cos bt\, u(t) = \frac{1}{2}\left(\sin(a+b)t + \sin(a-b)t\right)u(t)$

$$X(s) = L\{x(t)\} = L\left\{\frac{1}{2}\left(\sin(a+b)t + \sin(a-b)t\right)u(t)\right\}$$

$$= \frac{1}{2}\left\{L(\sin(a+b)t\,u(t)) + L(\sin(a-b)t\,u(t))\right\}$$

$$X(s) = \frac{1}{2}\left(\frac{a+b}{s^2+(a+b)^2} + \frac{a-b}{s^2+(a-b)^2}\right)$$

(l) Given ~~$X(s) =$~~

$$x(t) = \frac{\sin at}{t}$$

$$L\{x(t)\} = X(s) = L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at)\,ds = \int_s^\infty \left(\frac{a}{s^2+a^2}\right)ds$$

$$= a\int_s^\infty \frac{1}{s^2+a^2}\,ds = a\left(\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)_s^\infty = \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right) =$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

$$L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right).$$

(m) $x(t) = \frac{\cos at}{t}$

$$\lim_{t\to 0} x(t) = \lim_{t\to 0} \frac{\cos at}{t} = \frac{1}{0} = \infty.$$

Laplace transform doesnot Exist

(n) $x(t) = \frac{\sin^2 at}{t}$

$$L\{\sin^2 at\} = L\left\{\frac{1-\cos 2at}{2}\right\} = L\left\{\frac{1}{2}\right\} - L\left\{\frac{\cos 2at}{2}\right\}$$

$$L\{\sin^2 at\} = \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right)$$

$$L\left\{\frac{\sin^2 at}{t}\right\} = \int_s^\infty L(\sin^2 at)\, ds = \int_s^\infty \left(\frac{1}{2s} - \frac{s}{2(s^2+4a^2)}\right) ds$$

$$= \frac{1}{2}\left(\log s - \frac{1}{2}\log(s^2+4a^2)\right)_s^\infty = \frac{1}{2}\left(\log\frac{s}{\sqrt{s^2+4a^2}}\right)_s^\infty$$

$$= \frac{1}{2}\left(\log\left(\frac{1}{\sqrt{1+\frac{4a^2}{s^2}}}\right)\right)_s^\infty = \frac{1}{2}\left(\log(1) - \log\left(\frac{1}{\sqrt{1+\left(\frac{4a^2}{s^2}\right)}}\right)\right)$$

$$= \frac{1}{2}\left(0 - \log\frac{s}{\sqrt{s^2+4a^2}}\right) = -\frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4a^2}}\right)$$

$$\boxed{L\left\{\frac{\sin^2 at}{t}\right\} = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4a^2}}{s}\right)}$$

(o) $x(t) = \frac{e^{at} - \cos 3t}{t}$

$$L\left\{\frac{e^{at}-\cos 3t}{t}\right\} = \int_s^\infty L(e^{at} - \cos 3t)\, ds$$

$$L\{e^{at} - \cos 3t\} = \frac{1}{s-a} - \frac{s}{s^2+9}$$

$$L\left\{\frac{e^{at}-\cos 3t}{t}\right\} = \int_s^\infty \left(\frac{1}{s-a} - \frac{s}{s^2+9}\right) ds = \left(\log(s-a) - \frac{1}{2}\log(s^2+9)\right)_s^\infty$$

$$= \log\frac{\sqrt{s^2+9}}{s-a}$$

(P) $x(t) = \left(3e^{-2t}\cos 4t - 5e^{-3t}\sin 2t + 2\right)u(t)$.

$$L\{x(t)\} = 3\left(\frac{4}{(s+2)^2+16}\right) - 5\left(\frac{2}{(s+3)^2+4}\right) + \frac{2}{s}.$$

Ex:- By 1st Determining $x(t)$, verify the initial & final value theorems for each of the following:

(a) $X(s) = \frac{1}{s+3}$ (b) $X(s) = \frac{5}{(s+1)(s+2)}$

Sol:- (a) Given $X(s) = \frac{1}{s+3}$

$$x(t) = L^{-1}(X(s)) = L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

Initial value $x(0) = (x(t))_{t=0} = (e^{-3t})_{t=0} = e^0 = 1$

Final value $x(\infty) = (x(t))_{t=\infty} = e^{-\infty} = 0$.

By initial value theorem, we have

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left(\frac{1}{s+3}\right) = \lim_{s\to\infty}\left(\frac{1}{1+\frac{3}{s}}\right) = 1$$

By final value theorem, we have

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left(\frac{1}{s+3}\right) = 0$$

(b) Given $X(s) = \frac{5}{(s+1)(s+2)}$

Taking Partial fractions, we get.

$$X(s) = \frac{5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$x(t) = -e^{-t} + 2e^{-2t}$$

$x(0) = -1+2 = 1$

$x(\infty) = 0$.

By initial value Theorem

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left(\frac{5}{(s+1)(s+2)}\right) = \lim_{s\to\infty}\left(\frac{5s^2}{s^2\left(1+\frac{1}{s}\right)\left(1+\frac{2}{s}\right)}\right) = 1$$

By Final Value Theorem

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left(\frac{s}{(s+1)(s+2)}\right) = 0.$$

Ex:- $X(s) = \left(\frac{s^3+5s^2+13s+8}{s^2+4s+8}\right)$ find $x(t)$ for $t \geq 0$..

⇒

$X(s)$ is an improper function as the degree of the Numerator polynomial is greater than the degree of the Denominator polynomial Hence Divide the Nor by the Dor to obtain a proper function.

$$\begin{array}{r} s^2+4s+8 \;\big)\; s^3+5s^2+13s+8 \;\big(\; s+1 \\ \underline{s^3+4s^2+8s} \\ s^2+5s+8 \\ \underline{s^2+4s+8} \\ (s) \end{array}$$

$$X(s) = (s+1) + \frac{s}{s^2+4s+8} = (s+1) + \frac{s}{(s+2)^2+2^2}$$

$$= (s+1) + \frac{(s+2-2)}{(s+2)^2+2^2} = (s+1) + \frac{(s+2)}{(s+2)^2+4} - \frac{2}{(s+2)^2+4}$$

$$x(t) = \delta'(t) + \delta(t) + \left(e^{-2t}\cos 2t - 2e^{-2t}\sin 2t\right)u(t).$$

$\delta'(t)$ is the 1st Derivative of $\delta(t)$ w.r.t time.

Ex:- Determine the function of time $x(t)$ for each of the following Laplace transforms with their associated region of Convergence.

(a) $X(s) = \frac{s^2-s+1}{(s+1)^2}$ $Re(s) > -1$ (b) $X(s) = \frac{(s+1)^2}{s^2-s+1}$ $Re(s) > \frac{1}{2}$.

⇒ (a) $X(s) = \frac{s^2-s+1}{(s+1)^2} = \frac{s^2-s+1}{s^2+2s+1}$ $Re(s) > -1$

$$\begin{array}{r} s^2+2s+1 \;\big)\; s^2-s+1 \;\big(\; 1 \\ \underline{s^2+2s+1} \\ -3s \end{array}$$

$$X(s) = 1 - \frac{3s}{s^2+2s+1}$$

$$X(s) = 1 - \frac{3s}{(s+1)^2} = 1 - \left(\frac{3s+3-3}{(s+1)^2}\right) = 1 - \left(\frac{3s+3}{(s+1)^2}\right) + \frac{3}{(s+1)^2}$$

ROC is $Re(s) > -1$, the signals must be causal signals.

$$x(t) = \delta(t) - 3te^{-t}u(t) + \frac{3}{(s+1)^2}$$

$$X(s) = 1 - \frac{3(s+1)}{(s+1)^2} + \frac{3}{(s+1)^2} = 1 - \frac{3}{s+1} + \frac{3}{(s+1)^2}$$

$$x(t) = \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t).$$

(b) $$X(s) = \frac{(s+1)^2}{s^2 - s + 1} = \frac{s^2+2s+1}{s^2-s+1} \quad ; Re(s) > \frac{1}{2}$$

$$s^2 - s + 1 \;\overline{)\; s^2 + 2s + 1 \;(\; 1}$$
$$\underline{s^2 - s + 1}$$
$$3s$$

$$X(s) = 1 + \frac{3s}{s^2 - s + 1} = 1 + \frac{3s}{\left(s - \left(\frac{1}{2}\right)^2\right) + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$X(s) = 1 + \left(\frac{3\left(s - \frac{1}{2}\right) + \frac{3}{2}}{\left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right) = 1 + \frac{3\left(s - \frac{1}{2}\right)}{\left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \sqrt{3}\left(\frac{\frac{\sqrt{3}}{2}}{\left(s - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$x(t)$ must be causal

$$x(t) = \delta(t) + 3e^{t/2}\cos\frac{\sqrt{3}}{2}t\,u(t) + \sqrt{3}e^{t/2}\sin\frac{\sqrt{3}}{2}t\,u(t).$$

Ex:- Find the ILT of the following.

(a) $X(s) = \frac{s^2+6s+7}{s^2+3s+2}$ $Re(s) > -1$ (b) $X(s) = \frac{s^3+2s^2+6}{s^2+3s}$ $Re(s) > 0$.

Sol:- (a) $$X(s) = \frac{s^2+6s+7}{s^2+3s+2} = 1 + \frac{3s+5}{s^2+3s+2} = 1 + \frac{2}{s+1} + \frac{1}{s+2}$$

$$x(t) = \delta(t) + 2e^{-t}u(t) + e^{-2t}u(t).$$

(b) $$X(s) = \frac{s^3+2s^2+6}{s^2+3s} = (s-1) + \frac{(3s+6)}{s(s+3)} = (s-1) + \frac{2}{s} + \frac{1}{s+3}$$

$$x(t) = \delta'(t) - \delta(t) + 2u(t) + e^{-3t}u(t).$$

Ex:- Find the signal $x(t)$ that corresponds to the Laplace transform.

$$X(s) = \frac{3s^2+22s+27}{(s^2+3s+2)(s^2+2s+5)}$$

⇒

$$X(s) = \frac{3s^2+22s+27}{(s+1)(s+2)(s^2+2s+5)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{(s^2+2s+5)}$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s+2} + \left(\frac{-3s+1}{s^2+2s+5}\right)$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s+2} + \left(\frac{-3s+4-3}{(s+1)^2+2^2}\right)$$

$$= \frac{2}{s+1} + \frac{1}{s+2} - \frac{3(s+1)}{(s+1)^2+2^2} + \frac{4}{(s+1)^2+2^2}$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s+2} - 3\left(\frac{s+1}{(s+1)^2+2^2}\right) + 2\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$X(s) = \left(2e^{-t} + e^{-2t} - 3e^{-t}\cos 2t + 2e^{-t}\sin 2t\right)u(t).$$

Ex:- Given $x(t) = e^{-t}u(t)$. Find the Inverse Laplace transform of $e^{-3s}X(2s)$.

⇒ $X(s) = L(x(t)) = \frac{1}{s+1}$

$$X(2s) = \frac{1}{2s+1} = \frac{\frac{1}{2}}{s+\frac{1}{2}}$$

$$L^{-1}(X(2s)) = \frac{1}{2}e^{-t/2}u(t)$$

$$L^{-1}\left(e^{-3s}X(2s)\right) = L^{-1}\left(X(2s)\right)\Big|_{t=t-3} = \frac{1}{2}e^{-t/2}u(t)\Big|_{t=t-3} = \frac{1}{2}e^{-(t-3)/2}u(t-3)$$

Ex:- using Convolution Theorem, find $y(t) = x_1(t) * x_2(t)$ where:

ⓐ $x_1(t) = e^{-2t}u(t)$ $x_2(t) = u(t-3)$ ⓑ $x_1(t) = u(t)$ $x_2(t) = u(t)$

⇒ ⓐ $X_1(s) = \frac{1}{s+2}$ $X_2(t) = \frac{e^{-3s}}{s}$ ⓒ $x_1(t) = tu(t)$ $x_2(t) = u(t-2)$

$$L(x_1(t) * x_2(t)) = \left(\frac{1}{s+2} \times \frac{e^{-3s}}{s}\right) = X_1(s)X_2(s) = \frac{e^{-3s}}{s(s+2)}$$

$$x_1(t) * x_2(t) = L^{-1}\left(\frac{e^{-3s}}{s(s+2)}\right) = \text{@}\,L^{-1}\left(\frac{1}{s(s+2)}\right)\Big|_{t=t-3} \Rightarrow$$

$$Y_1(s) = \frac{1}{s(s+2)} = \frac{\frac{1}{2}}{s} + \frac{\frac{-1}{2}}{s+2}$$

$$y_1(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$$

$$y(t) = y_1(t)\Big|_{t=t-3} = x_1(t) * x_2(t) = \frac{1}{2}u(t-3) - \frac{1}{2}e^{-2(t-3)}u(t-3).$$

(b) $x_1(t) = u(t) \quad x_2(t) = u(t)$

$$L\left(x_1(t) * x_2(t)\right) = X_1(s)X_2(s) = \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^2}$$

$$x_1(t) * x_2(t) = L^{-1}\left(\frac{1}{s^2}\right) = t\,u(t) = t\,u(t).$$

(c) $x_1(t) = t\,u(t) \qquad x_2(t) = u(t-2)$

$$X_1(s) = \frac{1}{s^2} \qquad X_2(s) = \frac{e^{-2s}}{s}$$

$$L\left(x_1(t) * x_2(t)\right) = \frac{1}{s^2(s+2)}$$

$$= X_1(s)X_2(s) = \left(\frac{1}{s^2}\right)\left(\frac{e^{-2s}}{s}\right) = \frac{e^{-2s}}{s^3}$$

$$L^{-1}\left(\frac{e^{-2s}}{s^3}\right) = \left(\frac{t^2}{2}\right)_{t=t-2} = \frac{(t-2)^2}{2}$$

Ex:- Find the Inverse Laplace transform of the functions

(a) $X(s) = \frac{1}{s(s+1)}$ (b) $X(s) = \frac{1}{s^2(s+2)}$

(a) $X(s) = \frac{1}{s(s+1)} = \frac{1}{s}\left(\frac{1}{s+1}\right) = X_1(s) \cdot X_2(s)$

$$x_1(t) = L^{-1}\left(X_1(s)\right) = L^{-1}\left(\frac{1}{s}\right) = u(t)$$

$$x_2(t) = L^{-1}\left(X_2(s)\right) = L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}u(t)$$

$$L^{-1}\left(X_1(s)X_2(s)\right) = x_1(t) * x_2(t)$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} u(\tau)\,e^{-(t-\tau)}u(t-\tau)\,d\tau = \int_0^t e^{-t}e^{\tau}\,d\tau = e^{-t}\left(e^{\tau}\right)_0^t = e^{-t}(e^t - 1)$$

$$x_1(t) * x_2(t) = 1 - e^{-t}$$

(b) $Y(s) = \frac{1}{s^2(s+2)}$

$X_1(s) = \frac{1}{s^2} \qquad X_2(s) = \frac{1}{s+2}$

$x_1(t) = t\,u(t) \qquad x_2(t) = e^{-2t}u(t)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau\, u(\tau)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= \int_0^t \tau\, e^{-2t} e^{2\tau}\, d\tau = e^{-2t}\left(\int_0^t \tau e^{2\tau} d\tau\right)$$

$$= e^{-2t}\left(\left(\tau\frac{e^{2\tau}}{2}\right)_0^t - \left(\frac{e^{2\tau}}{4}\right)_0^t\right) = e^{-2t}\left(\frac{te^{2t}}{2} - \left(\frac{e^{2t}}{4} - \frac{1}{4}\right)\right)u(t)$$

$$x_1(t) * x_2(t) = \left(\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2t}\right)u(t)$$

# Unit - IV

## Signal Transmission through Linear Systems

**Introduction:-**

A system is defined as an entity that acts on an i/p signal & transforms it into an output signal. The two most important attributes of a system are linearity & time invariance. A system which is both linear & time invariant is called a linear time invariant system (LTI). A system which is linear but time variant is called a linear time variant (LTV) system. The change of shape of a signal when it is transmitted through a system is called Distortion.

**Systems:-**

**Linear System:-**

A system for which the principle of superposition & the principle of homogeneity are valid is called a linear system.

Superposition principle states that the response to a weighted sum of i/p signals be equal to the corresponding weighted sum of the o/ps of the system to each of the i/p signals.

If an Arbitary signal $x_1(t)$ produces an o/p $y_1(t)$ & an arbitary i/p $x_2(t)$ produces an o/p $y_2(t)$ then the system is linear if the weighted sum of i/ps

$a.x_1(t) + b x_2(t)$ produces an o/p $a y_1(t) + b y_2(t)$

where $a$ & $b$ are constants

$$T[a x_1(t) + b x_2(t)] = a y_1(t) + b y_2(t)$$

**Impulse Response:-**

Impulse response is the o/p of the system for a unit impulse input. For understanding the system behaviour, unit impulse response is very important.

If i/p $x(t) = \delta(t)$

then o/p $y(t) = h(t)$

$$L\{\delta(t)\} = 1 \text{ & } F\{\delta(t)\} = 1$$

once the transfer function $H(w)$ of an LTI system is known in frequency domain, the impulse response of the system can be fo

by finding the inverse Fourier transform of $H(\omega)$.

$$h(t) = F^{-1}(H(\omega))$$

Ily once the transfer function $H(s)$ of an LTI system is known in s-domain, the impulse response of the system can be found by finding the inverse Laplace transform of $H(s)$.

ie, $h(t) = L^{-1}(H(s))$.

Response of a Linear System :-

Once the impulse response $h(t)$ is known, the response of the linear system $y(t)$ for any given i/p $x(t)$ can be obtained by convolving the i/p with the impulse response of the system.

$$y(t) = x(t) * h(t) = h(t) * x(t).$$

If a non-causal signal is applied to a non causal system, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

If a causal signal is applied to a Non-causal system, then

$$y(t) = \int_{-\infty}^{t} h(\tau)\, x(t-\tau)\, d\tau = \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

If a Non causal signal is applied to a causal system, then

$$y(t) = \int_{0}^{\infty} h(\tau)\, x(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

If a causal signal is applied to a causal system, then

$$y(t) = \int_{0}^{t} h(\tau)\, x(t-\tau) d\tau = \int_{0}^{t} x(\tau)\, h(t-\tau) d\tau$$

Step response :-

The step response of an LTI continuous-time system can be obtained by using convolution integral. If $u(t)$ is an i/p to a system with impulse response $h(t)$, then the step response is given by

$$s(t) = h(t) * u(t)$$

If the system is non-causal, then the step response

$$s(t) = \int_{-\infty}^{t} h(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{t} h(\tau)\, d\tau.$$

If the system is Causal then

$$S(t) = \int_0^t h(\tau)\, d\tau.$$

That is the unit step response of a Continuous time LTI system is the running integral of its impulse response.

Linear time Invariant (LTI) System:-

A System which is linear as well as time Invariant is Called a linear time Invariant System. That is a system for which the Principle of Superposition and the principle of homogeneity are valid and in addition, the I/O characteristics do not change with time is called an LTI system.

For an LTI system, the o/p due to delayed i/p $x(t-T)$, i.e, $y(t,T)$ must be equal to the delayed o/p $y(t-T)$.

$$y(t,T) = y(t-T)$$

That is if the i/p is delayed by T units the corresponding output will also be delayed by T units. For a linear time Invariant system all the coefficients of the Differential Equation describing the System are Constants.

Linear Time Variant (LTV) System:-

A System which is linear but time variant is called a linear Time Variant System. That is a system for which the principle of Superposition & the principle of Homogeneity are valid but the I/o characteristics change with time is called an LTV System.

For an LTV System the o/p due to the delayed i/p $x(t-T)$ i.e., $y(t,T)$ is not equal to the delayed output $y(t-T)$.

i.e., $y(t,T) \neq y(t-T)$.

That is for an LTV system if the i/p is delayed by T units th

corresponding o/p will not be delayed by Exactly T units.

For a linear time variant system atleast one of the coefficients of the Differential Equation describing the system vary with time.

Continuous-time systems are described by Differential Equations and they can be analysed using either Fourier transform (or) laplace transform. Analysis using laplace transform is easy & more convenient compared to analysis using Fourier transform. Also laplace transform is more powerful than Fourier transform because there are many functions for which Fourier transform doesnot exist, but laplace transform exist. Moreover Fourier transform of a function can be easily obtained from its laplace transform by just replacing S by jw.

Properties of Linear Time Invariant Systems:-

An Extremely important representation of continuous time LTI Systems is in terms of its unit impulse response. It takes the form of Convolution integral. The characteristics of an LTI System are completely determined by its impulse response. The unit impulse response of a non-linear System doesnot completely characterize the behavior of the system.

Commutative Property:-

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau.$$

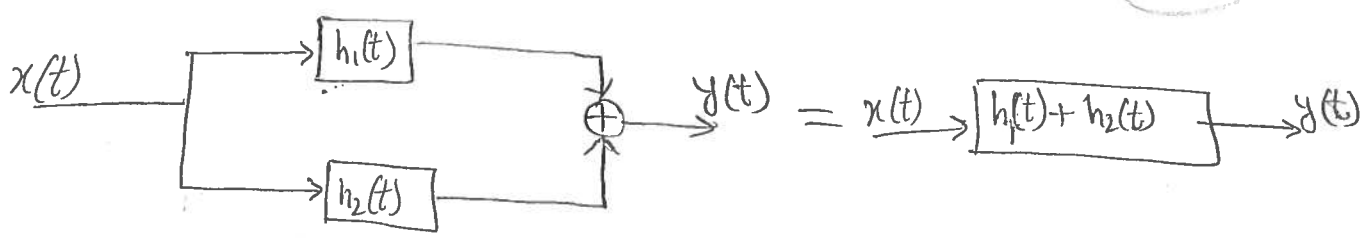
i.e., the o/p of an LTI system with i/p $x(t)$ & impulse response $h(t)$ is identical to the o/p of an LTI system with i/p $h(t)$ & impulse response $x(t)$.

Distributive property:- Specifically Convolution distributes over addition Sothat

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$

The distribution property has a useful interpretation in terms of system interconnection. Two systems with impulse responses $h_1(t)$ & $h_2(t)$ connected in parallel can be replaced by a single system with impulse response $h_1(t) + h_2(t)$.

The distributive property of convolution can be used to break a complicated convolution into several simpler ones. (4-3)



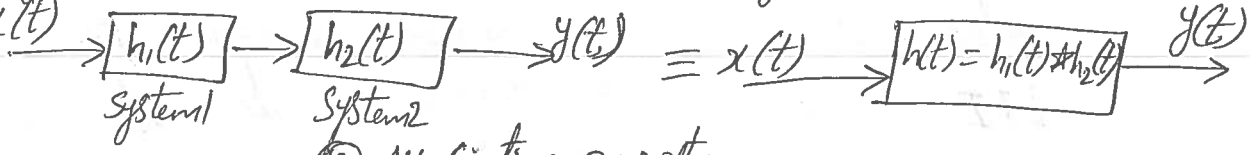

Associative property:-

Another important & useful property of convolution is that it is associative. That is

$$x(t) * \left(h_1(t) * h_2(t)\right) = \left(x(t) * h_1(t)\right) * h_2(t)$$

i.e., it is immaterial in which order the signals are convolved.

According to the associative property, the series interconnection of 2 systems is equivalent to a single system.




(or) Associative property

Systems with & without Memory:-

A system is said to be static (or) memoryless if its o/p at any time depends only on the value of the i/p at that time. A continuous time LTI system is memoryless if $h(t)=0$ for $t \neq 0$ & such a memoryless system has the form

$$y(t) = Kx(t)$$

for some constant K & has the impulse response

$$h(t) = K\delta(t)$$

$\therefore$ if $h(t) \neq 0$ for $t \neq 0$.

the continuous time system has memory. A memory system is also called a Dynamic system. If $K=1$ $h(t)=\delta(t)$ the system becomes an identity system.

Causality:- A causal system is non anticipatory & does not produce an o/p before an i/p is applied. Its o/p depends only on the pres

∴ for a Causal LTI System we have $h(t)=0$ for $t<0$

The o/p of a Causal & LTI System for a non-Causal i/p is

$$y(t)=\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau=\int_{-\infty}^{t} x(\tau)\cdot h(t-\tau)\,d\tau.$$

The o/p of a Causal LTI System for a Causal Signal is

$$y(t)=\int_{0}^{t} h(\tau)\,x(t-\tau)\,d\tau=\int_{0}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

A Non Causal System is anticipatory if

$$h(t)\neq 0 \text{ for } t<0.$$

Stability:-

A System is stable if every bounded i/p produces a bounded output. The BIBO Stability of an LTI System can be easily determined from its impulse response. For a Continuous time LTI System to be BIBO Stable, its impulse respons $h(t)$ must be absolutely integrable i.e.,

$$\int_{-\infty}^{\infty}|h(\tau)|\,d\tau<\infty.$$

Invertibility:-

A Continuous time LTI System with impulse response $h(t)$ is Said to be invertible if an inverse System with impulse response $h_1(t)$ which when Connected in series with the original System produces an o/p equal to the i/p of the 1st System.

$$h(t)*h_1(t)=\delta(t).$$

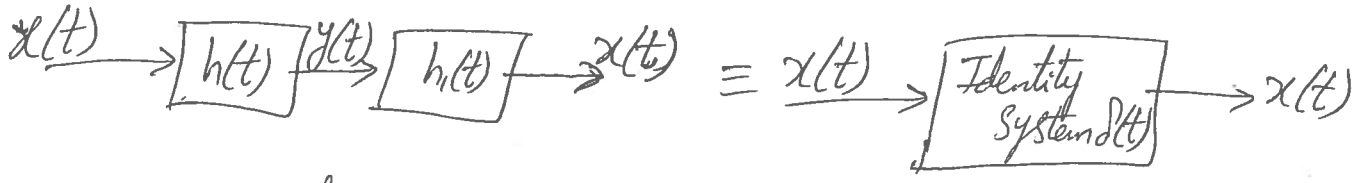



fig:- Invertibility of System.

## Transfer function of an LTI System:-

A Continuous time System is shown in fig(1).

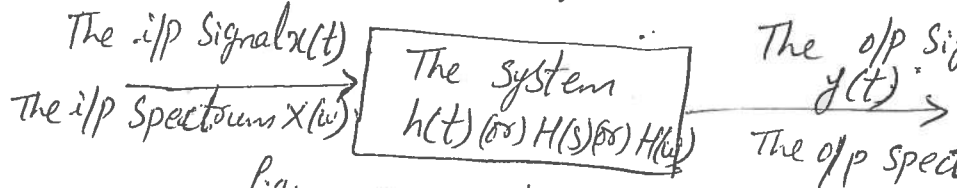



fig:- A System.

The transfer function of a Continuous time LTI System may be defined using fourier transform (or) Laplace transform. The transfer function is defined only under zero initial Condition

The transfer function of a LTI System $H(\omega)$ is define as the ratio of the F.T of the o/p Signal to the Fourie transform of the i/p Signal when the initial Conditions are Zero. (or) Simply we Can say that it is the ratio of o/p to i/p in frequency domain when the initial Conditions are neglected. (or) Simply we Can say that the Transfer function $H(\omega)$ of a LTI System is the Fourier transform of its impulse response.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$H(\omega)$ is a Complex quantity having magnitude & phase.

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$$

The transfer function in frequency domain $H(\omega)$ is also Called the frequency response of the System. The frequency response is the amplitude response plus phase response.

$|H(\omega)|$ = Amplitude response of the System.

$\theta(\omega) = \angle H(\omega)$ = phase response of the System.

We Can say that $H(\omega)$ is a frequency domain representation of a System.

$$Y(\omega) = H(\omega) X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

and $H(\omega)$ has Conjugate Symmetry property.

$$H(-\omega) = H^*(\omega)$$

$$|H(-\omega)| = |H(\omega)|$$

& $$\angle H(-\omega) = -\angle H(\omega)$$

The impulse response $h(t)$ of a system is the inverse Fourier Transform of its transfer function (frequency response) $H(\omega)$.

$$H(\omega) = F(h(t))$$

(or) $$h(t) = F^{-1}(H(\omega))$$

The Transfer function of a system in S-domain (Laplace domain) is defined as the ratio of the Laplace transform of the o/p of System to the laplace transform of the i/p of the system when the initial Conditions are neglected. (or) Simply we can say that it is the ratio of the o/p to i/p in S-domain when the initial Conditions are zero. The transfer function of a system can also be defined as the Laplace Transform of the impulse respo of the system.

$$H(S) = \frac{Y(S)}{X(S)}$$

$$H(S) = L(h(t))$$

The impulse response is nothing but the inverse Laplace Trans form of the transfer function $H(S)$.

$$h(t) = L^{-1}(H(S))$$

once the transfer function in S-domain $H(S)$ is known, the transfer function in frequency domain $H(\omega)$ can be found by replacing $S$ in $H(S)$ by $j\omega$. $H(\omega) = H(S)\big|_{S=j\omega}$.

## Filter characteristics of linear System:-

For a given system an i/p signal $x(t)$ gives rise to a response signal $y(t)$. The system ∴ processes the signal $x(t)$ in a way that is characteristics of the system. The spectral density function of the i/p signal $x(t)$ is given by $X(s)$ (or) $X(\omega)$, and the spectral density function of the response signal $y(t)$ is given by $Y(s)$ (or) $Y(\omega)$.

$$Y(s) = H(s)X(s)$$

(or)

$$Y(\omega) = H(\omega)X(\omega)$$

Where $H(s)$ (or) $H(\omega)$ is the transfer function (or) system function of the system.

The system ∴ modifies the spectral density function of the input. The system acts as a kind of filter for various frequency components. Some frequency components are boosted in strength i.e., they are amplified. Some frequency components are weakend in strength i.e., they are attenuated and some may remain unaffected. Ily each frequency component suffers a different amount of phase shift in the process of transmission. The system ∴ modifies the spectral density function of the i/p according to its filter characteristics. The modification is carried out according to the transfer function $H(s)$ (or) $H(\omega)$, which represents the response of the system to various frequency components. $H(\omega)$ acts as a weighing function (or) spectral shaping function to the different frequency components in the i/p signal. An LTI signal system ∴ acts as a filter. A filter is basically a frequency selective network.

Some LTI systems allow the transmission of low frequency components & stop all high frequency components. They are called Low pass filters (LPFs).

Some LTI systems allow the transmission of only high frequency components and stop all low frequency components. They are called high pass filters. (HPF)

Some LTI systems allow transmission of only a particular band of frequencies & stop all other frequency components. They are called Band pass filters (BPFs).

Some LTI systems ~~allow transmission of~~ reject only a particular band of frequencies and allow all other frequency components. They are called Band-rejection filters (BRFs).

The band of frequency that is allowed by the filter is called Pass-Band.

The band of frequency that is severely attenuated and not allowed to pass through the filter is called StopBand (or) rejection Band.

An LTI System may be characterised by its pass-Band, Stop Band and Half power Band width.

Distortionless Transmission through a System:-

The change of shape of the signal when it is transmitted through a system is called Distortion.

Transmission of a signal through a system is said to be distortio-less if the o/p is an Exact replica of the i/p signal. This replica may have different magnitude and also it may have different time delay. A Constant change in magnitude & a Constant time delay are not considered as Distortion. only the shape of the signal is important. Mathematically we can say that a signal $x(t)$ is Transmitted without distortion if the o/p

$$y(t) = Kx(t-t_d)$$

Where K is a constant representing the change in amplitude (amplification or attenuation) and $t_d$ is delay time. A distortionless system & typical i/p & o/p waveforms are shown in fig

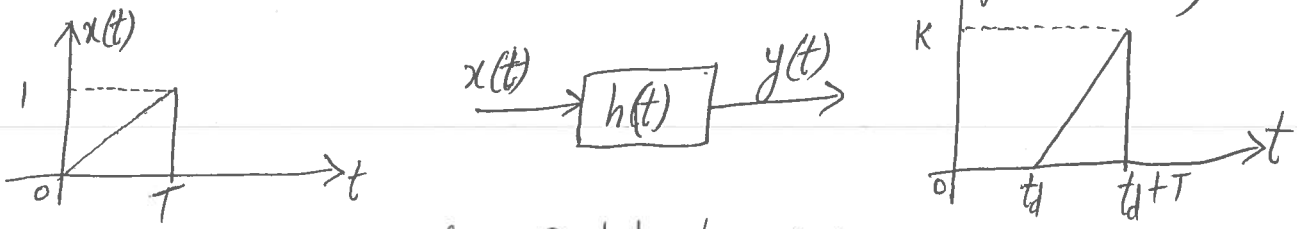



fig:- Distortionless System.

Taking Fourier Transform on both sides of the equation for $y(t)$ and using the shifting property, we have

(4-6)

$$Y(\omega) = K e^{-j\omega t_d} X(\omega).$$

$\therefore$ For Distortionless transmission, the transfer function of the system must be of the form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = K e^{-j\omega t_d}$$

Taking inverse Fourier transform, the corresponding impulse response must be

$$h(t) = K\delta(t - t_d)$$

It is evident that the magnitude of the transfer function

$$|H(\omega)| = K.$$

and that it is constant for all values of $\omega$.

The phase shift

$$\theta(\omega) = \angle H(\omega) = -\omega t_d.$$

and it varies linearly with frequency.

In general $\theta(\omega) = n\pi - \omega t_d$ ($n$ integral).

So for distortionless transmission of a signal through a system the magnitude $|H(\omega)|$ should be a constant, i.e, all the frequency components of the ~~frequency components~~ i/p signal must undergo the same amount of amplification (or) attenuation, i.e, the system bandwidth is $\infty$ and the phase spectrum should be proportional to frequency. But in practice no system can have $\infty$ Bandwidth and hence distortionless conditions are never met exactly.

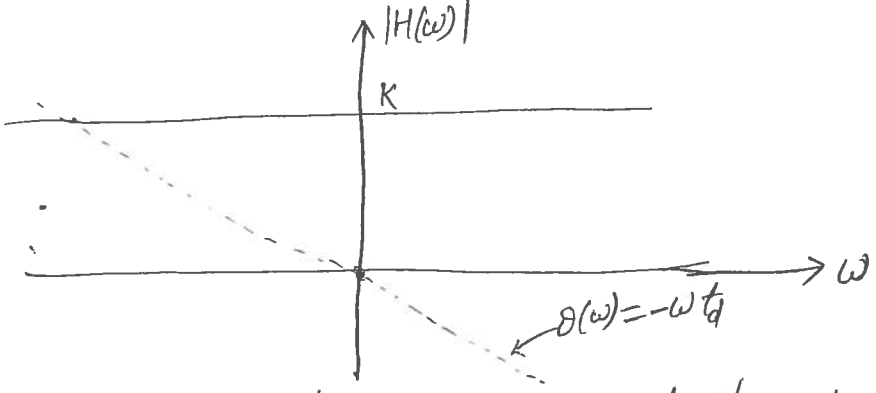



Fig:- The magnitude & phase characteristics of a Distortionless Txn System.

## Linear phase systems:

For a Distortionless transmission, there should not be any phase distortion. No phase distortion means the phase should be linear.
∴ for distortionless transmission, the system must be of linear phase type. For linear phase system the impulse response is symmetrical about $t=t_d$. This can be proved as follows:

For linear phase system,

$$H(\omega) = |H(\omega)|\, e^{-j\omega t_d}$$

The impulse response of such a system is obtained by finding the invetse Fourier transform.

$$h(t) = F^{-1}\{H(\omega)\} = F^{-1}\{|H(\omega)|\, e^{-j\omega t_d}\}$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|\, e^{-j\omega t_d} e^{j\omega t}\, d\omega$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} |H(\omega)|\, e^{+j\omega(t-t_d)}\, d\omega + \int_{0}^{\infty} |H(\omega)|\, e^{j\omega(t-t_d)}\, d\omega\right\}.$$

$$= \frac{1}{2\pi}\left\{\int_{0}^{\infty} |H(\omega)|\, e^{-j\omega(t-t_d)}\, d\omega + \int_{0}^{\infty} |H(\omega)| e^{j\omega(t-t_d)}\, d\omega\right\}.$$

$$= \frac{1}{2\pi}\left\{\int_{0}^{\infty} |H(\omega)|\left[e^{-j\omega(t-t_d)} + e^{j\omega(t-t_d)}\right] d\omega\right\}$$

$$h(t) = \frac{1}{2\pi}\int_{0}^{\infty} |H(\omega)|\,(2\cos\omega(t-t_d))\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|\cos\omega(t-t_d)\, d\omega$$

$$h(t) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|\cos\omega(t-t_d)\, d\omega$$

$$h(t_d+t) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|\cos\omega t\, d\omega.$$

$$h(t_d-t) = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|\cos\omega t\, d\omega.$$

$\therefore h(t_d+t) = h(t_d-t)$. This shows that for a linear phase system, the impulse response $h(t)$ is symmetrical about $t_d$.

and it is non causal (non zero for $t<0$).



The maximum value of $h(t)$ is at $t=t_d$ and is given by

$$h_{max} = h(t_d) = \frac{1}{\pi}\int_0^{\infty} |H(\omega)|\, d\omega.$$

## Signal Bandwidth :-

The Spectral Components of a signal Extends from $\omega = -\infty$ to $\infty$. Any typical practical Signal have finite Energy. As a result the Spectral Components approaches to Zero as $\omega \to \infty$ ($\omega \to \infty$). ∴ we can neglect the spectral components that has negligible Energy & Select a band of frequency components that contains most of the signal Energy. This band of frequencies that contains most of the Signal Energy is known as Bandwidth of the signal and it is denoted by $f_m$. Normally the band is selected such that it contains atleast 95% (lower (or) higher than 95%) of the total Energy depending on the precision.

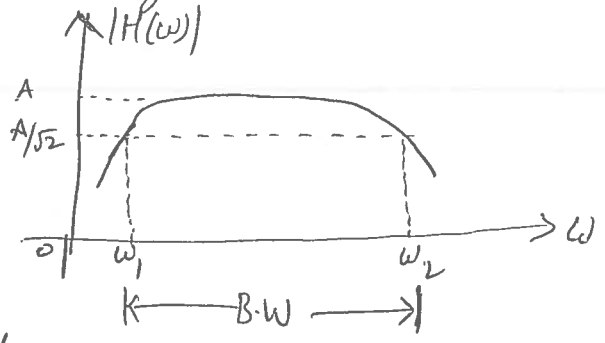
## System Bandwidth :-

In the previous section we have studied about the requirement of a distortionless system. But in practices it is impossible to design an ideal distortionless system, because the response of all practical systems diminishes at higher frequencies. But it is possible to design a system that is distortionless over a certain range of frequencies. This range of frequencies is known as Bandwidth of the System. For Example if $|H(j\omega)|$ (or) $H(j\omega)$ is the transfer function of a System, the Bandwidth of the System can be defined as the interval of frequencies for which $|H(j\omega)|$ remain within $\frac{1}{\sqrt{2}}$ times of its maximum value (-3dB down). Fig below shows the frequency response of a practical Bandpass filter from which one can find the Bandwidth as $(\omega_2 - \omega_1)$ (or) $(\omega_2 - \omega_1)$ rad/sec.

For Distortionless Transmission we need a System with $\infty$ Bandwidth. Due to physical limitations it is impossible to Construct a System with

of Bandwidth. Actually a satisfactory distortionless transmission can be achieved by systems with finite, but fairly large Bandwidths, if the magnitude $|H(\omega)|$ is constant over this band.

$\omega_2$ is the upper cutoff frequency (or) upper 3dB frequency (or) upper Half power frequency and $\omega_1$ is called lower cutoff frequency (or) lower 3dB frequency (or) lower half power frequency.



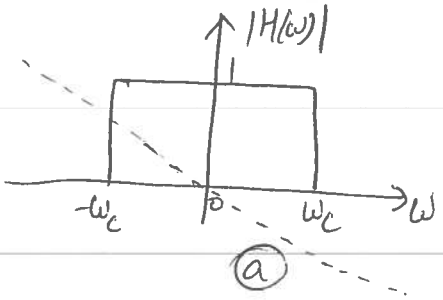

The Band limited Signals can be transmitted without distortion, if the System Bandwidth is atleast equal to the Signal Bandwidth.

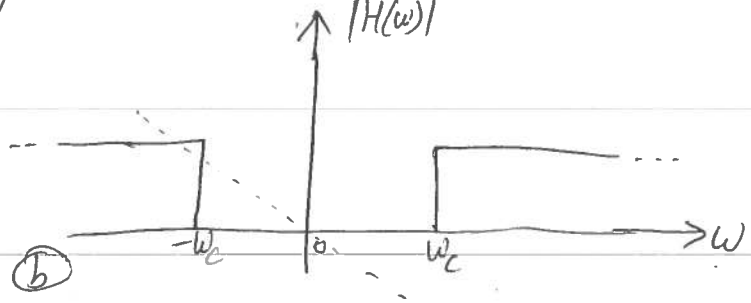
## Ideal Filter characteristics:-

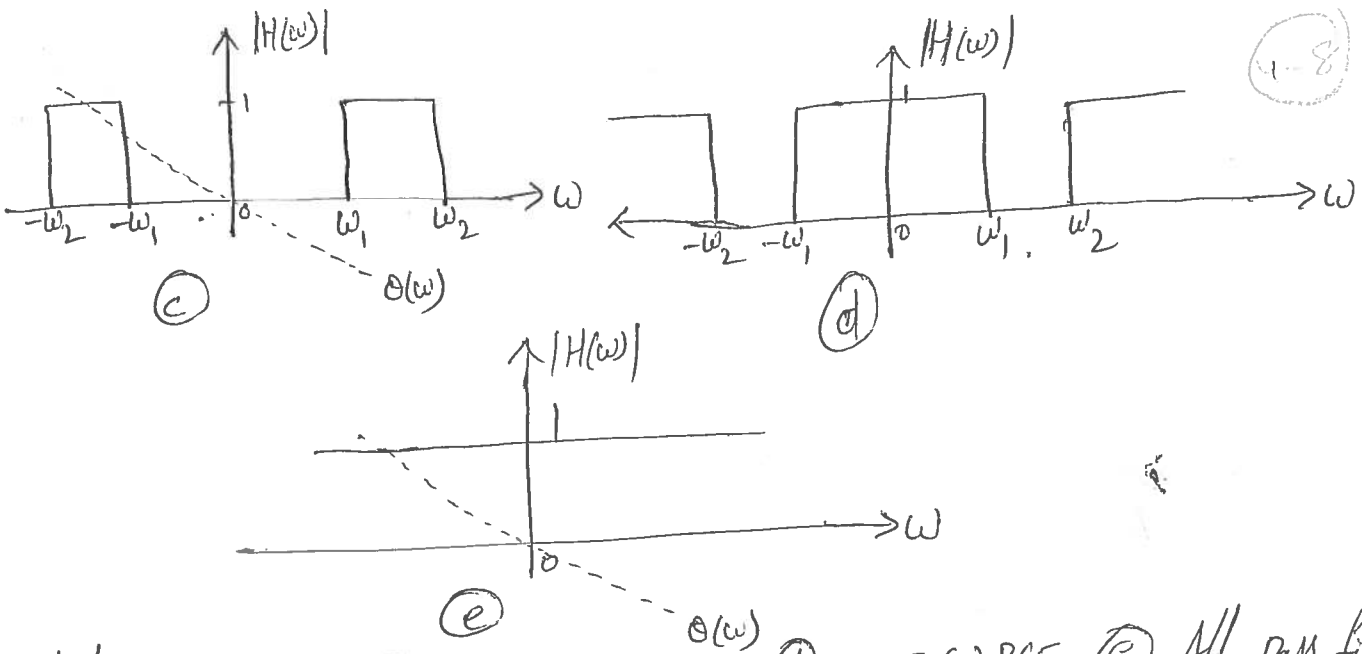
A filter is a frequency selective Network. It allows transmission of Signals of certain frequencies with no attenuation (or) with very little attenuation, and it rejects (or) heavily attenuates Signals of all other frequencies.

An ideal filter has very sharp cutoff characteristics and it Passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band. Its phase Spectrum is linear.

Filters are classified according to their frequency response characteristics as Low pass Filter (LPF), High pass Filter (HPF), Band pass Filter (BPF) and Band-Elimination (or) Band stop (or) Band Reject filter (BSF). Ideal versions of these filters are described below and their magnitude responses are shown in fig.@






Ideal (a) LPF (b) HPF (c) BPF (d) BRF (or) BSF (e) All pass filter.

Ideal LPF:-

An ideal low pass filter transmits, with out any distortion all of the signals of frequencies below a certain frequency $\omega_c$ radians per second. The signals of frequencies above $\omega_c$ rad/sec are completely attenuated. $\omega_c$ is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

The transfer function of an ideal LPF is given by

$$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal LPF are shown in fig (a). It is a gate function.

Ideal HPF:-

An ideal HPF transmits with out any distortion, all of the signals of frequencies above a certain frequency $\omega_c$ rad/sec & attenuates completely the signals of frequencies below $\omega_c$ rad/sec, where $\omega_c$ is called the cutoff frequency. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal HPF is specified by

$$|H(\omega)| = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

The frequency response characteristics of an ideal HPF are shown in fig (b).

## Ideal BPF:-

An ideal Bandpass filter transmits, without any distortion, all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ rad/second & attenuates completely the signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the bandwidth of the Bandpass filter. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BPF is specified by

$$|H(\omega)| = \begin{cases} 1, & |\omega_1| < \omega < \omega_2 \\ 0, & \omega < |\omega_1| \text{ \& } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BPF are shown in (c)

## Ideal BRF:-

An ideal Band Rejection filter rejects totally all of the signals of frequencies within a certain frequency band $(\omega_2 - \omega_1)$ rad/second & transmits without any distortion all signals of frequencies outside this band. $(\omega_2 - \omega_1)$ is the rejection band. The corresponding phase function for distortionless transmission is $-\omega t_d$.

An ideal BRF is specified by

$$|H(\omega)| = \begin{cases} 0 & |\omega_1| < \omega < |\omega_2| \\ 1 & \omega < |\omega_1| \text{ and } \omega > |\omega_2| \end{cases}$$

The frequency response characteristics of an ideal BRF are shown in fig (d).

An all pass filter transmits signals of all frequencies without any distortion, that is, its Bandwidth is $\infty$ as shown in fig (e).

An ideal all pass filter is specified by

$$|H(\omega)| = 1 \quad \text{(for all frequencies)}$$

The corresponding phase function for distortionless transmission is $-\omega t_d$.

All ideal filters are non causal systems. Hence None of them is physically realizable.

## Causality & Paley-Wiener Criterion for Physical Realization:

A System is said to be causal if it does not produce an output before the i/p is applied. For an LTI system to be causal, the condition to be satisfied is its impulse response must be Zero for t less than Zero i.e.,

$$h(t) = 0 \quad \text{for } t < 0.$$

Physically realizability implies that it is physically possible to construct that system in real time. A physically realizable system cannot have a response before the i/p is applied. This is known as causality condition. It means the unit impulse response $h(t)$ of a physically realizable system must be causal. This is the time domain criterion of physical realizability. In the frequency domain the criterion implies that a necessary & sufficient condition for a magnitude function $H(\omega)$ to be physically realizable is:

$$\int_{-\infty}^{\infty} \frac{\ln|H(\omega)|}{1+\omega^2} d\omega < \infty$$

The magnitude function $|H(\omega)|$ must, however be square-integrable before the paley-wiener criterion is valid, that is

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty.$$

A System whose magnitude function violates the paley-wiener criterion has a non-causal impulse response, that is the response exists prior to the application of the driving function.

The following conclusions can be drawn from paley-wiener criterion:

(1) The magnitude function $|H(\omega)|$ may be Zero at some discrete frequencies, but it can't be Zero over a finite band of frequencies since this will cause the integral in the equation of paley-wiener criterion to become infinity. That means ideal filters are not physically realizable.

(5) The magnitude function $|H(\omega)|$ can't fall off to zero faster than function of Exponential order. It implies a realizable magnitude characteristic can't have too great a total attenuation.

## Relationship Between Bandwidth & Rise time:-

We know that the transfer function of an ideal LPF is given by

$$H(\omega) = |H(\omega)|\, e^{-j\omega t_d}$$

where

$$|H(\omega)| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

$\omega_c$ is called the cut off frequency.

$$\therefore H(\omega) = e^{-j\omega t_d}, \quad -\omega_c \le \omega \le \omega_c, \text{ i.e., } |\omega| \le \omega_c.$$

$$= 0 \qquad \omega > |\omega_c|$$

The impulse response of the LPF $h(t)$ is obtained by taking the inverse fourier transfer function $H(\omega)$.

$$h(t) = F^{-1}(H(\omega)) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t - t_d)}\, d\omega = \frac{1}{2\pi} \left( \frac{e^{j\omega(t-t_d)}}{j(t-t_d)} \right)_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \left( \frac{e^{j\omega_c(t-t_d)} - e^{-j\omega_c(t-t_d)}}{j(t-t_d)} \right)$$

$$h(t) = \frac{1}{\pi(t-t_d)} \sin \omega_c (t-t_d) = \frac{\omega_c}{\pi} \left( \frac{\sin \omega_c (t-t_d)}{\omega_c (t-t_d)} \right)$$

The impulse Response of the ideal LPF is shown in fig. The impulse response has a peak value at $t = t_d$. This value $\omega_c/\pi$ is proportional to cutoff frequency $\omega_c$. The width of the main lobe is $2\pi/\omega_c$. As $\omega_c \to \infty$ then the LPF becomes an all Pass filter. As $t_d \to 0$, the o/p response peak $\to \infty$, that is the

o/p response approaches i/p:

Further the impulse response $h(t)$ is non zero for $t<0$, even though the i/p $\delta(t)$ is applied at $t=0$. That is the impulse response begins before the i/p is applied. In real life no system exhibits such type of characteristics. Hence we can conclude that ideal LPF is physically not realizable.

If the impulse response is known, the step response can be obtained by convolution.

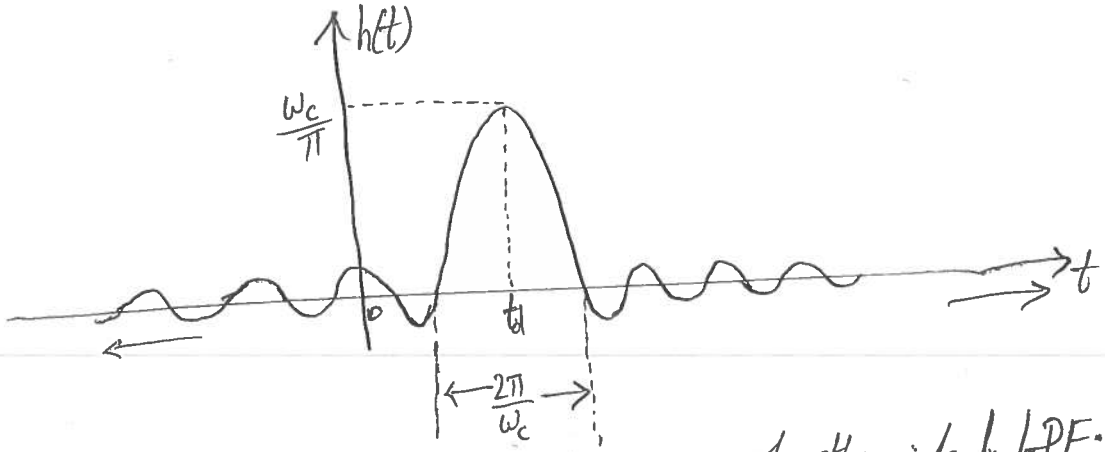



Fig:- Impulse Response of the ideal LPF.

The step response

$$y(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

we have $h(t) = \frac{\omega_c}{\pi} \frac{\sin(\omega_c(t-t_d))}{\omega_c(t-t_d)}$

$$y(t) = \int_{-\infty}^{t} \frac{\omega_c}{\pi} \frac{\sin \omega_c(\tau - t_d)}{\omega_c(\tau - t_d)}\, d\tau$$

$$x = \omega_c(\tau - t_d)$$

$$dx = \omega_c\, d\tau \quad (\text{or}) \quad d\tau = \frac{dx}{\omega_c}$$

$$\therefore y(t) = \int_{-\infty}^{\omega_c(t-t_d)} \frac{\omega_c}{\pi} \frac{\sin x}{x} \frac{dx}{\omega_c} = \frac{1}{\pi} \int_{-\infty}^{\omega_c(t-t_d)} \frac{\sin x}{x}\, dx.$$

$$y(t) = \frac{1}{\pi} \left( Si(x) \right)_{-\infty}^{\omega_c(t-t_d)}$$

where Si is the sine Integral function.

① $Si(x)$ is an odd function, that is $Si(-x) = -Si(x)$

② $Si(0) = 0$

③ $Si(\infty) = \pi/2$ & $Si(-\infty) = -\left(\frac{\pi}{2}\right)$

A sketch of $Si(x)$ is shown in fig ⓐ

The step response can be expressed as;

$$y(t) = \frac{1}{\pi}\left[Si(\omega_c(t-t_d)) - Si(-\infty)\right]$$

$$= \frac{1}{\pi}\left\{Si(\omega_c(t-t_d) + \frac{\pi}{2}\right\} = \frac{1}{2} + \frac{1}{\pi} Si(\omega_c(t-t_d))$$

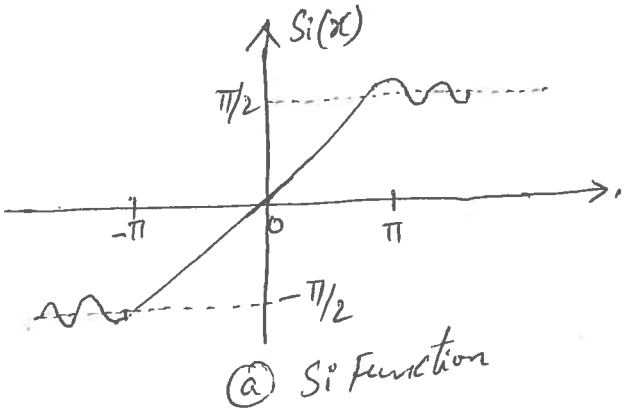
If $\omega_c \longrightarrow \infty$ then the response is:

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(\infty) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{\pi}{2}\right) = 1$$

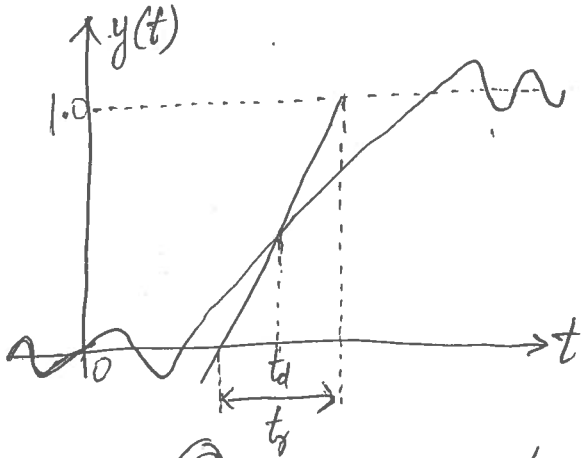
If $\omega_c \longrightarrow -\infty$ then the response is

$$y(t) = \frac{1}{2} + \frac{1}{\pi} Si(-\infty) = \frac{1}{2} + \frac{1}{\pi}\left(-\frac{\pi}{2}\right) = 0.$$

The step response of LPF is shown in fig ⓑ.




ⓐ Si Function




ⓑ Step response of an ideal LPF.

From fig ⓑ we can observe that $y(t)$ approaches a delayed unit step $u(t-t_d)$. But the abrupt rise of i/p corresponds to more gradual rise of the o/p.

The rise time $t_r$ is defined as the time required for the response to reach from 10% to 90% of the final value. To find it draw a tangent at $t=t_d$ with the line $y(t)=0$ &

$y(t)=1$. From fig (b) we have

$$\left.\frac{dy(t)}{dt}\right|_{t=t_d} = \frac{1}{t_r} = \frac{\omega_c}{\pi}\left.\frac{\sin\omega_c(t-t_d)}{\omega_c(t-t_d)}\right|_{t=t_d} = \frac{\omega_c}{\pi}.$$

$$t_r = \frac{\pi}{\omega_c}.$$

For a low pass filter

Cut off frequency = Bandwidth

So the rise time is inversely proportional to the B.W

Bandwidth x Rise time = Constant.

Energy Density Spectrum :-

Spectral Density :- It is the distribution of energy (or) power of a signal per unit Bandwidth as a function of frequency.

Energy Signals :-

Signals with finite energy & Zero average power i.e., $0<E<\infty$ & $P=0$ are called energy signals. ex:- aperiodic signals like pulse.

Normalized Energy :-

The Normalized energy (or) Simply energy of a signal $x(t)$ is defined as the energy dissipated by a voltage signal applied across 1Ω resistor (or by a current signal flowing through 1Ω resistor).

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

The energy of a signal Exists only if E is finite i.e, only if $0<E<\infty$.

Parseval's Theorem for Energy Signals (Rayleigh's Energy theorem)

Parsevals theorem defines the energy of a signal in terms of its fourier transform. Using Parseval's Theorem, the energy of a signal $x(t)$ can be evaluated directly from its frequency spectrum $X(\omega)$ with out the knowledge of its time domain version.

$$E = \int |x(t)|^2 dt = \frac{1}{2\pi}\int |X(\omega)|^2 d\omega.$$

(or) $E = \int_{-\infty}^{\infty} |X(f)|^2 df$.

Proof:- $x(t) \longleftrightarrow X(\omega)$

$x^*(t) \longleftrightarrow X^*(-\omega)$.

Energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt$$

Replace $x(t)$ by its Inverse F.T, we have

$$E = \int_{-\infty}^{\infty} x^*(t) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \right) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left( \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt \right) d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, X^*(-\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

$\omega = 2\pi f \qquad d\omega = 2\pi df$

Normally $X(2\pi f)$ can be written as $X(f)$, then we have

$$E = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

This is called Parsevals Theorem for Energy Signals.

Energy Spectral Density:-

The ESD function gives the distribution of Energy of a signal in the frequency domain.

$$\psi(\omega) = |X(\omega)|^2 .$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$\psi_y(\omega) = |Y(\omega)|^2 = |H(\omega)\, X(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2 = |H(\omega)|^2 \psi_x(\omega)$$

Energy $E_y = \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \psi_y(f)\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_y(\omega) d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega = \frac{1}{2\pi} 2 \int_{0}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$$

$$E_y = \frac{1}{\pi} \int_{0}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega .$$

If the LTI system is an ideal LPF with $f_L$ & $f_H$ as cutoff frequencies of $|H(\omega)| = 1$ for $f_L < f < f_H$.

$$E_y = \frac{1}{\pi} \int_{f_L}^{f_H} \psi_x(\omega) d\omega = \frac{1}{\pi} \int_{f_L}^{f_H} \psi_x(f) 2\pi df = 2 \int_{f_L}^{f_H} \psi_x(f)\, df,$$

The ESD function gives the distribution of Energy of a Signal in the frequency domain. For an Energy Signal, the total area under the spectral density curve plotted as a function of frequency is equal to the total Energy of the Signal. It is also called Energy density Spectrum (ESD or ED). It is designated by $\psi(\omega)$ & given by

$$\psi(\omega) = |X(\omega)|^2$$

Let $x(t)$ & $y(t)$ be the i/p & o/p respectively of a linear system.

Let $x(t) \longleftrightarrow X(\omega)$ & $y(t) \longleftrightarrow Y(\omega)$ & $H(\omega)$ be the system transfer function.

Then we have

$$Y(\omega) = H(\omega) X(\omega)$$

The ESD of the i/p is:

$$\psi_x(\omega) = |X(\omega)|^2$$

The ESD of the o/p $y(t)$ is:

$$\psi_y(\omega) = |Y(\omega)|^2$$

$$\psi_y(\omega) = |Y(\omega)|^2 = |X(\omega) H(\omega)|^2$$

$$= |X(\omega)|^2 |H(\omega)|^2 = |H(\omega)|^2 \psi_x(\omega)$$

$$\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega).$$

Thus the Energy Spectrum Density of the o/p (response) of a linear system is the product of ESD of i/p (excitation) & square of the magnitude of the transfer function.

Energy of the output signal $E_y = \int_{-\infty}^{\infty} \psi_y(f)\, df$

$$E_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_y(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega$$

$$E = \frac{1}{2\pi}\left(2\int_0^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega\right) = \frac{1}{\pi}\int_0^{\infty} |H(\omega)|^2 \psi_x(\omega)\, d\omega.$$

If the LTI system is an ideal LPF with lower & upper cutoff frequencies $f_L$ & $f_H$ respectively respectively, then $|H(\omega)| = 1$ for $f_L \le f \le f_H$

$$\therefore \quad E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(\omega)\, d\omega$$

$$(\text{or}) \quad E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(2\pi f)\, 2\pi\, df = 2\int_{f_L}^{f_H} \psi_x(f)\, df.$$

Properties of ESD :- The following are the properties of ESD.

(1) The total area under the Energy density spectrum is equal to the total Energy of the signal.

i.e., $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df.$$

(2) If $x(t)$ is the i/p to an LTI system with impulse response $h(t)$, then the i/p & o/p ESD functions are related as:

$$\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega)$$

(or)

$$\psi_y(f) = |H(f)|^2 \psi_x(f)$$

(3) The Auto correlation function $R(\tau)$ and ESD $\psi(\omega)$ form a Fourier transform pair i.e.,

$$R(\tau) \longleftrightarrow \psi(\omega)$$

(or)

$$R(\tau) \longleftrightarrow \psi(f)$$

Power Density Spectrum :-

Power Signals :- Signals with finite average power & Energy i.e., $0 < P < \infty$ & $E = \infty$ are called power signals.

Ex :- Periodic Signals.

**Average Power:-** It is defined as the average power dissipated by a voltage $x(t)$ applied across $1\Omega$ resistor (or by a current signal flowing through $1\Omega$ resistor). Mathematically,

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

The power $P$ defined above is actually the mean square value (or) the time average of the squared signal. Thus we may write

$$P = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

**Parseval's Power Theorem:-**

Parseval's power theorem defines the power of a signal in terms of its Fourier series coefficients, i.e., in terms of the Harmonic components present in the signal. Mathematically it is given by

$$P = \sum_{n=-\infty}^{\infty} |C(n)|^2.$$

**Proof:-** Consider a function $x(t)$. we know that

$$|x(t)|^2 = x(t)\, x^*(t)$$

where $x^*(t)$ is the conjugate of $x(t)$.

The average power of $x(t)$ for one cycle is:

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t) dt.$$

But we have the Exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$P = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}\, x^*(t) dt.$$

Interchanging the order of Summation & integration, we get

$$P = \sum_{n=-\infty}^{\infty} C_n \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) e^{jn\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \left( \frac{1}{T} \int x(t) e^{-jn\omega t} dt \right)^*$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^*$$

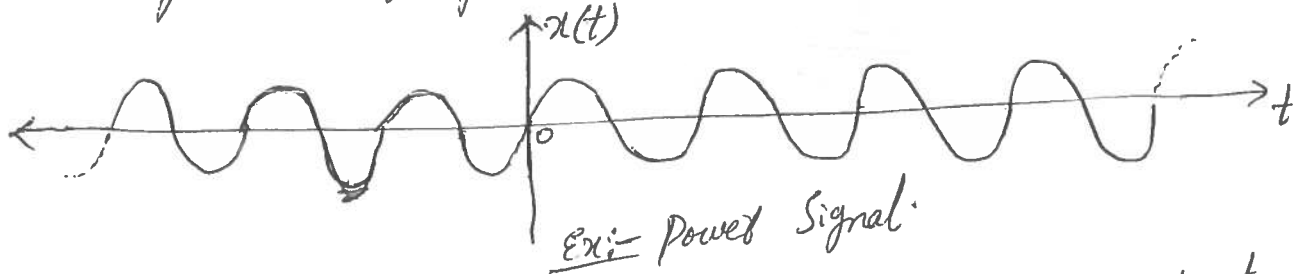
$$\boxed{P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

This is called Parseval's Power Theorem. It states that the Power of a Signal is equal to the Sum of Square of the magnitudes of various harmonics present in the discrete spectrum

Power Spectral Density:-

The distribution of average power of the Signal in the frequency domain is called power Spectral density (or) Power density spectrum (PSD or PD).

To derive the PSD, Assume the Power Signal as a limiting Case of the Energy Signal. Consider a Power Signal $x(t)$, Extending to infinity as shown in fig.




Ex:- Power Signal.

Let us truncate this Signal $x(t)$ so that it is zero outside the interval $|T/2|$ as shown in fig below. Let this truncated signal be $x_T(t)$

$$x_T(t) = \begin{cases} x(t), & |t| < T/2 \\ 0 & \text{elsewhere.} \end{cases}$$

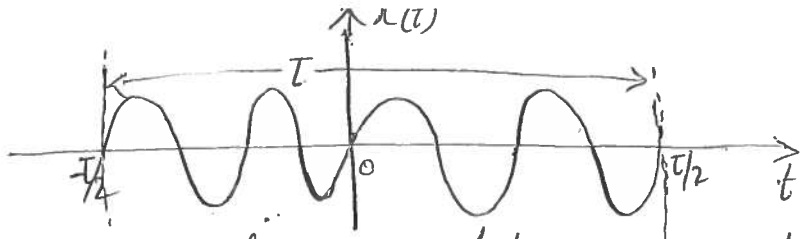



fig:- Truncated power signal.

The signal $x_T(t)$ is of finite duration T and hence it is an Energy signal with Energy E given by

$$E = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

$$x_T(t) \longleftrightarrow X_T(\omega).$$

As $x(t)$ over the interval $(-T/2 \text{ to } T/2)$ is same as $x_T(t)$ over the interval $-\infty$ to $\infty$, we have

$$\int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

$$\frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{2\pi} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega.$$

If $T \to \infty$, the left hand side of above equation represents the average power P of the function $x(t)$.

$$\therefore P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} d\omega$$

If $T \to \infty$, $|X_T(\omega)|^2/T$ approaches a finite value.

Let this finite value is denoted by $S(\omega)$ i.e.,

$$S(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T}.$$

The average power P of the function $x(t)$ is given by

$$P = \overline{x^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = \int_{-\infty}^{\infty} S(f) df.$$

Where $\overline{x^2(t)}$ is the mean square value of $x(t)$.

The average power is therefore given by

$$P = 2\left(\frac{1}{2\pi}\int_0^{\infty} S(\omega)\,d\omega\right) = \frac{1}{\pi}\int_0^{\infty} S(\omega)\,d\omega = 2\int_0^{\infty} S(f)\,df.$$

The PSD of a periodic function is given by

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0)$$

(or)

$$S(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0).$$

The i/p & o/p relation of a linear system in terms of PSD is given by

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega).$$

$$S_y(f) = |H(f)|^2 S_x(f).$$

Properties of PSD:-

The following are the properties of PSD:

(1) The area under the PSD function is equal to the average power of the signal i.e.,

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = \int_{-\infty}^{\infty} S(f)\,df$$

(2) The i/p & o/p PSDs of an LTI system are related as:

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega).$$

(3) The Auto correlation function $R(\tau)$ & PSD $S(\omega)$ form a Fourier transform pair; i.e.,

$$R(\tau) \longleftrightarrow S(\omega)$$

$\blacksquare$

## Relation Between Auto Correlation function & Energy/Power Spectral Density Function :-

### Relation between ESD & Auto correlation $R(\tau)$ :-

The Auto Correlation function $R(\tau)$ & Energy Spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e,

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Proof :- The Auto correlation of a function $x(t)$ is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t-\tau)$ by its inverse fourier transform we have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-\tau)}\, d\omega \right)^*$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left( \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t-\tau)}\, d\omega \right) dt$$

Interchanging the order of integration, we have

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left( \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right) e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, e^{j\omega\tau}\, d\omega$$

~~Interchanging the~~

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\omega)\, e^{j\omega\tau}\, d\omega \quad \left( \text{since } |X(\omega)|^2 = \psi(\omega) \right)$$

$$= F^{-1}\{\psi(\omega)\}$$

$$\therefore \psi(\omega) = F\{R(\tau)\}.$$

This proves that $R(\tau)$ & $\psi(\omega)$ form a fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega).$$

Relation between Auto Correlation Function $R(\tau)$ & Power Spectral Density (PSD):-

The Auto Correlation function $R(\tau)$ & the Power Spectral Density (PSD), $S(\omega)$ of a power signal form a fourier transform pair ie,

$$R(\tau) \longleftrightarrow S(\omega).$$

Proof:- The Auto Correlation function of a Power (Periodic) Signal $x(t)$ in terms of Fourier Series Coefficients is given as:

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0\tau}$$

Where $C_n$ & $C_{-n}$ are the Exponential fourier series coefficie

$$R(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau}$$

Taking F.T on both sides, we have

$$F\{R(\tau)\} = \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau} \right) e^{-j\omega\tau} d\tau$$

Interchanging the order of integration & Summation, we get

$$F\{R(\tau)\} = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)} d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0).$$

The RHS is the PSD $S(\omega)$ (or) $S(f)$ of the periodic function $x(t)$.

$$F\{R(\tau)\} = S(\omega) \quad (\text{or } S(f))$$

$$F^{-1}\{S(\omega)\} = R(\tau) \quad (\text{or } F^{-1}(S(f)) = R(\tau))$$

$$R(\tau) \longleftrightarrow S(\omega) \quad (\text{or } S(f)).$$

## Relationship between Convolution & Correlation

There is a striking resemblance between the operation of Convolution and Correlation. Indeed the 2 integrals are closely related. To obtain the Cross Correlation of $x_1(t)$ & $x_2(t)$ according to the Equation $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$,

we multiply $x_1(t)$ with function $x_2(t)$ displaced by $\tau$ sec

The area under the product curve is the Cross Correlation b/w $x_1(t)$ & $x_2(t)$ at $\tau$. on the other hand, the Convolution of $x_1(t)$ & $x_2(t)$ at $t=\tau$ is obtained by folding $x(t)$ backwards about vertical axis at the origin & taking the area under the product curve of $x_1(t)$ & the folded function $x_2(-t)$ displaced by $\tau$. It therefore follows that the Cross Correlation of $x_1(t)$ & $x_2(t)$ is the same as Convolution of $x_1(t)$ & $x_2(-t)$.

The same conclusion can be arrived at analytically as follows:

The Convolution of $x_1(t)$ & $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau.$$

Replacing the dummy variable $\tau$ in the above integral by another variable $n$, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n-t)\, dn.$$

changing the variable from $t$ to $\tau$ we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n-\tau)\, dn = R_{12}(\tau)$$

$$R_{12}(\tau) = x_1(t) * x_2(-t)\Big|_{t=\tau}$$

Thy

$$R_{21}(\tau) = x_2(t) * x_1(-t)\Big|_{t=\tau}.$$

All of the techniques used to evaluate the Convolution of 2 functions can be directly applied in order to find the Correlation of

two functions. By all of the results derived for Convolution also apply to correlation.

If one of the function is an even function of t, let us say $x_2(t)$ is an even function of t; i.e.,

$$x_2(t) = x_2(-t)$$

then the Cross correlation & Convolution are equivalent.

Ex:- Let the system function of an LTI system be $\left(\frac{1}{jw+2}\right)$. What is the output of the system for an i/p of $(0.8)^t u(t)$? (17)

$$H(\omega) = \frac{1}{j\omega+2}$$

$$x(t) = (0.8)^t u(t)$$

$$h(t) = e^{-2t}u(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, (0.8)^{t-\tau} u(t-\tau)\, d\tau$$

$$y(t) = (0.8)^t \int_0^{\infty} (0.8\, e^2)^{-\tau}\, d\tau$$

$$(0.8e^2)^{-1} = a$$

$$y(t) = (0.8) \int_0^{\infty} a^{\tau}\, d\tau = (0.8)^t \left(\frac{a^{\tau}}{\log a}\right)_0^{\infty} = \frac{(0.8)^t}{\left(\log (0.8e^2)^{-1}\right)} \left((0.8e^2)^{-\tau}\right)_0^{\infty}$$

$$y(t) = \frac{(0.8)^t \left((0.8e^2)^{-\infty} - 1\right)}{\log (0.8e^2)^{-1}} = \frac{(0.8)^t (0-1)}{-\log (0.8e^2)}$$

$$y(t) = \frac{(0.8)^t}{\log (0.8\, e^2)} = \frac{(0.8)^t}{\log 0.8 + \log e^2} = \frac{(0.8)^t}{\log 0.8 + 2}.$$

Ex:- Consider a causal LTI system with frequency response $H(\omega) = \frac{1}{4+j\omega}$ For a particular i/p $x(t)$ the system is observed to produce the o/p $y(t) = e^{-2t}u(t) - e^{-4t}u(t)$.

$$H(\omega) = \frac{1}{4+j\omega}$$

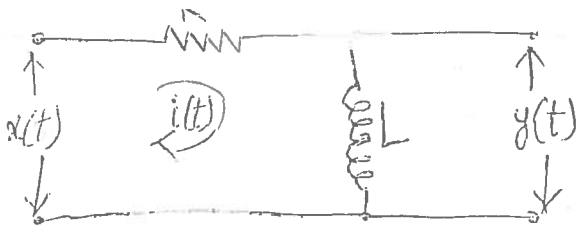
$$Y(\omega) = \frac{1}{2+j\omega} - \frac{1}{4+j\omega}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \qquad X(\omega) = \frac{Y(\omega)}{H(\omega)} = \frac{\left(\frac{1}{2+j\omega} - \frac{1}{4+j\omega}\right)}{\frac{1}{4+j\omega}}$$

$$X(\omega) = \frac{2}{2+j\omega}$$

$$x(t) = 2\, e^{-2t} u(t).$$

Ex:- Find the impulse response for the RL filter shown in fig. Also find the Transfer function. What would be its frequency response.




$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega}{j\omega + (R/L)}$$

$$h(t) = F^{-1}(H(\omega)) = F^{-1}\left(\frac{j\omega}{j\omega + (R/L)}\right)$$

$$h(t) = F^{-1}\left(1 - \frac{R}{L}\,\frac{1}{j\omega + (R/L)}\right) = \delta(t) - \frac{R}{L} e^{-(R/L)t} u(t)$$

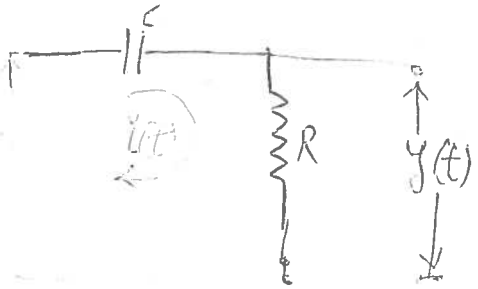
The frequency response of the system is:

$$H(\omega) = \frac{j\omega}{j\omega + (R/L)}$$

Magnitude response $|H(\omega)| = \dfrac{\omega}{\sqrt{\omega^2 + (R/L)^2}}$.

Phase response $\angle H(\omega) = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{L}{R}\right)\omega$.

Ex:- Find the impulse response of the system shown in fig. Also find the Transfer function. What would be its frequency response




$$H(\omega) = \frac{R}{R + \left(\frac{1}{j\omega C}\right)} = \frac{j\omega CR}{1 + j\omega CR} = \frac{j\omega}{j\omega + \left(\frac{1}{RC}\right)}$$

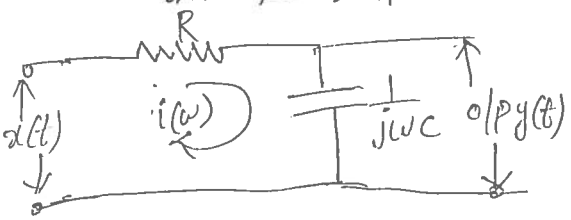
Impulse response $h(t) = F^{-1}(H(\omega)) = F^{-1}\left(\dfrac{j\omega}{j\omega + \left(\frac{1}{RC}\right)}\right)$

$$h(t) = F^{-1}\left(1 - \frac{1/RC}{j\omega + \frac{1}{RC}}\right) = \delta(t) - \frac{1}{RC} e^{-\left(\frac{1}{RC}\right)t} u(t)$$

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} \qquad |H(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \qquad \angle H(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

Ex:- Find the impulse response of the system shown in fig (a) Find the transfer function. what would be its frequency response? Sketch the response





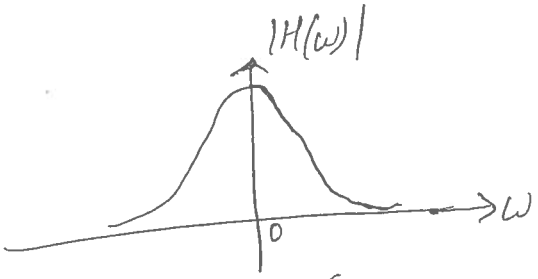

$$\Rightarrow \quad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{RC}\left(\frac{1}{j\omega + \left(\frac{1}{RC}\right)}\right)$$

$$h(t) = F^{-1}\{H(\omega)\} = F^{-1}\left\{\frac{1}{RC}\,\frac{1}{j\omega + \left(\frac{1}{RC}\right)}\right\} = \frac{1}{RC}\, e^{-(1/RC)t}\, u(t)$$

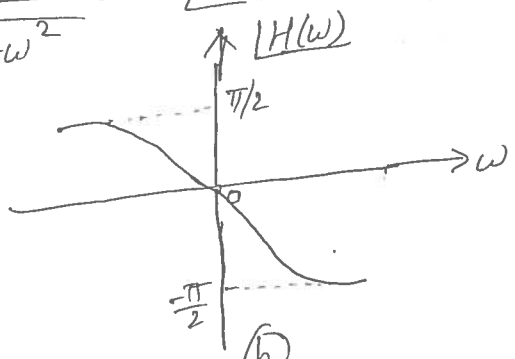
$$H(\omega) = \frac{1}{1 + j\omega RC} \qquad |H(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

let $RC = 1$ sec.

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}} \qquad \angle H(\omega) = -\tan^{-1}(\omega).$$




(a)




(b)

Ex:- For the system shown in fig below.

$$f(t) \longrightarrow \boxed{\text{Filter}} \longrightarrow y(t)$$

$$f(t) \begin{cases} e^{-at} & t \geq 0,\ |a| \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y(\omega) = \frac{1}{a + j\omega}, \quad -a < \omega < a$$

$$= 0 \qquad \text{otherwise}$$

Find the transfer function & impulse response of the system.

$$\Rightarrow \quad f(t) = e^{-at} \quad t \geq 0$$

$$f(t) = e^{-at}\, u(t)$$

$$F(\omega) = F(e^{-at}u(t)) = \frac{1}{j\omega+a}$$

$$Y(\omega) = \frac{1}{a+j\omega}.$$

The transfer function of the system

$$H(\omega) = \frac{Y(\omega)}{F(\omega)} = \frac{\frac{1}{j\omega+a}}{\frac{1}{j\omega+a}} = 1$$

$$h(t) = F^{-1}\{H(\omega)\} = F^{-1}\{1\} = \delta(t)$$

Ex:- Find the convolution of the following signals:

① $x_1(t) = e^{-2t}u(t)$ $\quad x_2(t) = e^{-4t}u(t)$.

→

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\, e^{-4(t-\tau)}u(t-\tau)\, d\tau.$$

$u(\tau) = 1$ for $\tau > 0$ & $u(t-\tau) = 1$ for $(t-\tau) \geq 0$ (or) for $\tau < t$.

Hence $u(\tau)\, u(t-\tau) = 1$ only for $0 < \tau < t$. For all other values of $\tau$, $u(\tau) \times u(t-\tau) = 0$

$$x_1(t) * x_2(t) = \int_0^t e^{-2\tau} e^{-4(t-\tau)}\, d\tau$$

$$x_1(t) * x_2(t) = e^{-4t}\left(\int_0^t e^{+2\tau}\, d\tau\right) = e^{-4t}\left(\frac{e^{2t}}{2}\right)_0^t = e^{-4t}\left(\frac{e^{2t}-1}{2}\right)$$

$$= \frac{e^{-2t} - e^{-4t}}{2} \quad (t \geq 0)$$

$$= \left(\frac{e^{-2t} - e^{-4t}}{2}\right)u(t)$$

② $x_1(t) = t\,u(t)$ $\quad x_2(t) = t\,u(t)$.

$u(\tau) = 1$ for $\tau > 0$ & $u(t-\tau) = 1$ for $(t-\tau) \geq 0$ (or) for $\tau < t$.

$$x_1(t) * x_2(t) = \int_0^t \tau\,(t-\tau)\, d\tau = \left(\frac{t^2}{2}\right) - \left(\frac{t^3}{3}\right).$$

$$x_1(t) * x_2(t) = \frac{t^3}{6}u(t).$$

(3) $x_1(t) = \cos t\, u(t)$ $\quad x_2(t) = u(t)$.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \cos\tau\, u(\tau)\, u(t-\tau)\, d\tau$$

$$x_1(t) * x_2(t) = \int_{0}^{t} \cos\tau\, d\tau = (\sin\tau)_0^t = \sin t \quad \text{for } t \geq 0$$

(4) $x_1(t) = e^{-3t} u(t)$ $\quad x_2(t) = u(t+3)$.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-3\tau} u(\tau)\, u(t+3-\tau)\, d\tau$$

$$= \int_{0}^{t+3} e^{-3\tau}\, d\tau = \left(\frac{e^{-3\tau}}{-3}\right)_0^{t+3}$$

$$= \frac{e^{-3(t+3)} - 1}{-3}$$

$y(t) = 0$ for $(t < -3)$

$$= \frac{1 - e^{-3(t+3)}}{3} \quad \text{for } (t > -3)$$

(5) A Signal $x(t) = e^{-2t} u(t)$ is passed through an ideal Low pass filter with cut off frequency of 1 rad/sec.

(a) Test whether the i/p is an Energy Signal

(b) Find the i/p & o/p Energy.

$x(t) = e^{-2t} u(t)$

$$E_i = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} e^{-4t} u(t)\, dt = \int_{0}^{\infty} e^{-4t} dt = \left(\frac{e^{-4t}}{-4}\right)_0^{\infty}$$

$$E_i = \left(\frac{e^{-4t}}{-4}\right)_0^{\infty} = \frac{1}{4} = 0.25 \text{ Joule}.$$

I/P Power $P_i = \lim_{T \to \infty} \frac{1}{2T}(E_i) = \left(\frac{1}{4}\right)\left(\frac{1}{2T}\right) = 0$.

Power = 0 Energy = finite. So it is an Energy Signal

(ii) $\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega)$

$\psi_x(\omega) = |X(\omega)|^2$

$\therefore X(\omega) = F\{x(t)\} = \frac{1}{2+j\omega}$

$\psi_x(\omega) = |X(\omega)|^2 = \frac{1}{2^2+\omega^2} = \frac{1}{4+\omega^2}$

$\boxed{|H(\omega)| = 1 \quad |\omega| < 1}$

$\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega) = \frac{1}{4+\omega^2} \quad |\omega| < 1.$

$= 0$ otherwise.

o/p Energy $E_o = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_y(\omega) d\omega = \frac{1}{2\pi} \times 2\int_0^{\infty} \psi_y(\omega) d\omega = \frac{1}{\pi} \int_0^{\infty} \psi_y(\omega) d\omega$

$E_o = \frac{1}{\pi} \int_0^{1} \frac{1}{4+\omega^2} d\omega = \frac{1}{\pi} \int_0^{1} \frac{1}{4+\omega^2} d\omega$

$E_o = \frac{1}{2\pi} \tan^{-1}\left(\frac{1}{2}\right) = 0.049$ Joule.

(6) Ex:- A function $x(t)$ has a PSD of $S_x(\omega)$. Find the PSD of (a) Integral of $x(t)$ (b) Time derivative of $x(t)$.

(i) $\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega)$

$Y(\omega) = \frac{1}{j\omega} X(\omega)$

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\left(\frac{1}{j\omega}\right) X(\omega)}{X(\omega)} = \frac{1}{j\omega}$

$|H(\omega)|^2 = \frac{1}{\omega^2}$

Now the output PSD is given by

$S_y(\omega) = |H(\omega)|^2 S_x(\omega) = \frac{1}{\omega^2} S_x(\omega)$

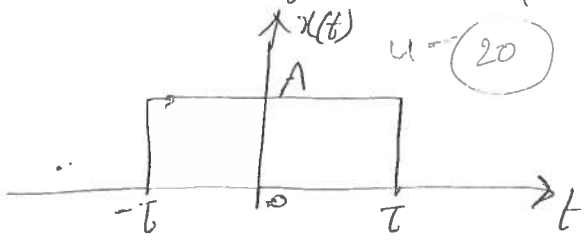
(ii) $x(t) \longleftrightarrow X(\omega)$

$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega) \qquad \therefore Y(\omega) = j\omega X(\omega)$

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = j\omega \qquad |H(\omega)|^2 = \omega^2$

$S_y(\omega) = |H(\omega)|^2 S_x(\omega) = \omega^2 S_x(\omega).$

Ex:- (7) Determine the ESD of a gate function of width $\tau$ & Amplitude $A$ as shown in fig.






$\Rightarrow$ $x(t) = A \quad -\tau \le t \le \tau$

$= 0 \quad$ else

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = 2A\tau\left(\frac{\sin\omega\tau}{\omega\tau}\right) = 2A\tau\,\mathrm{sinc}\,\omega\tau$$

$$\psi(\omega) = |X(\omega)|^2 = |2A\tau\,\mathrm{sinc}\,\omega\tau|^2 = 4A^2\tau^2\,\mathrm{sinc}^2\omega\tau.$$

Ex:- (8) The AutoCorrelation function of an aperiodic power signal is:

$$R(\tau) = e^{-\tau^2/2\sigma^2}$$

Determine the PSD & the normalized average power content of the signal.

$\Rightarrow$ Given $R(\tau) = e^{-\tau^2/2\sigma^2}$

$$R(\tau) \longleftrightarrow S(\omega).$$

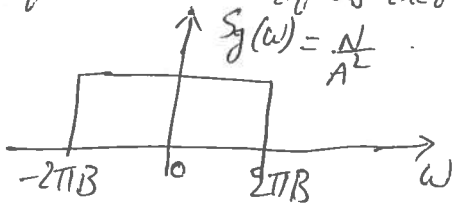
$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} e^{-(\tau^2/2\sigma^2)}\, e^{-j\omega\tau}\, d\tau$$

$$S(\omega) = \sqrt{2\pi\sigma^2}\; e^{-(\omega\sigma)^2/2}$$

Normalized average power $P = R(0) = e^0 = 1$ watt.

Ex:- (9) A power signal $g(t)$ has a PSD $S_g(\omega) = N/A^2$, $-2\pi B < \omega < 2\pi B$, shown in fig, where $A$ & $N$ are constants.

Determine the PSD & the mean square value of its derivative $dg(t)/dt$.




$\Rightarrow$ I/P Signal $= g(t)$

$$S_g(\omega) = \frac{N}{A^2}.$$

$$g(t) \longleftrightarrow G(\omega)$$

$$\frac{d}{dt}g(t) \longleftrightarrow j\omega\, G(\omega).$$

$\therefore$ Transfer function $H(\omega) = \dfrac{O/P}{I/P} = \dfrac{j\omega G(\omega)}{G(\omega)} = j\omega$

$H(\omega) = j\omega$

$|H(\omega)|^2 = \omega^2$

output PSD, $S_y(\omega) = |H(\omega)|^2 S_g(\omega)$

$$S_y(\omega) = \frac{\omega^2 N}{A^2}$$

Mean Square Value of the o/p = Area under o/p PSD.

$$= \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \frac{\omega^2 N}{A^2} d\omega$$

$$= \frac{N}{2\pi A^2} \left(\frac{\omega^3}{3}\right)_{-2\pi B}^{2\pi B} = \frac{N}{6\pi A^2}\left(8\pi^3 B^3 - (-8\pi^3 B^3)\right)$$

$$= \frac{N}{6\pi A^2} \left(16\pi^3 B^3\right) = \frac{8\pi^2 N B^3}{3A^2} \text{ ///}$$

ex:- Energies of Signal $x_1(t)$ & $x_2(t)$ are $E_1$ & $E_2$ respectively.

(10) (i) Show that, in general, the Energy of Signal $x_1(t)+x_2(t)$ is not $E_1+E_2$.

(ii) under what Condition is the Energy of $x_1(t)+x_2(t)$ is equal to $E_1+E_2$?

(iii) Can the Energy of the Signal $x_1(t)+x_2(t)$ be Zero? If so, under what conditions?

>

$$E_g = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad E_1 = \int_{-\infty}^{\infty} |x_1(t)|^2 dt$$

(i) Energy of signal $x_1(t)+x_2(t)$ is: $E_2 = \int_{-\infty}^{\infty} |x_2(t)|^2 dt$.

$$E = \int_{-\infty}^{\infty} |x_1(t) + x_2(t)|^2 dt$$

$$E = \int_{-\infty}^{\infty} |x_1(t)|^2 dt + \int_{-\infty}^{\infty} |x_2(t)|^2 dt + 2\int_{-\infty}^{\infty} |x_1(t)|\,|x_2(t)|\, dt$$

$$E = E_1 + E_2 + 2\int_{-\infty}^{\infty} |x_1(t)|\,|x_2(t)|\, dt$$

$$E \neq E_1 + E_2.$$

(ii) When $x_1(t)$ & $x_2(t)$ are orthogonal i.e,

$$\int_{-\infty}^{\infty} |x_1(t)|\,|x_2(t)|\,dt = 0$$

$$E = E_1 + E_2.$$

(iii) When $x_1(t) = x_2(t) = 0$ then the Energy $E = 0$.

Ex:- A Signal $x(t)$ has Energy $E$. Calculate the Energy of the (11) Signal $x(3t)$.

$$\Rightarrow \quad E = \int_{-\infty}^{\infty} |x(t)|^2\,dt.$$

Now for signal $x(3t)$, we have the Energy

$$E' = \int_{-\infty}^{\infty} |x(3t)|^2\,dt$$

$$3t = p \qquad dt = \frac{dp}{3}.$$

$$E' = \int_{-\infty}^{\infty} |x(p)|^2 \left(\frac{dp}{3}\right) = \frac{1}{3}\int_{-\infty}^{\infty} |x(p)|^2\,dp = \frac{1}{3}E.$$

Ex:- Verify Parseval's Theorem for the Energy Signal $x(t) = e^{-4t}u(t)$.

(12)

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |e^{-4t}u(t)|^2\,dt = \int_{0}^{\infty} |e^{-4t}|^2\,dt$$

$$E = \int_{0}^{\infty} e^{-8t}\,dt = \left(\frac{e^{-8t}}{-8}\right)_0^{\infty} = \frac{1}{8}\ \text{Joule}.$$

Now According to Parseval's Theorem, we have

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} e^{-4t}u(t)e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{0}^{\infty} e^{-(4+j\omega)t}\,dt = \left(\frac{e^{-(4+j\omega)t}}{-(4+j\omega)}\right)_0^{\infty} = \frac{1}{4+j\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{16+\omega^2}} \qquad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{16+\omega^2}\,d\omega = \frac{1}{8\pi}\tan^{-1}\left(\frac{\omega}{4}\right)_{-\infty}^{\infty}$$

$$E = \frac{1}{8\pi}\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) = \frac{1}{8}\ \text{Joule}.$$

Ex:- A filter has an i/p $x(t) = e^{-t}u(t)$ and its impulse response $h(t) = e^{-3t}u(t)$. Find the ESD of the o/p. (13)

$$x(t) = e^{-t}u(t)$$

$$X(\omega) = F\{e^{-t}u(t)\} = \frac{1}{1+j\omega}$$

ESD of i/p $\psi_x(\omega) = |X(\omega)|^2 = \frac{1}{1+\omega^2}$

$$h(t) = e^{-3t}u(t)$$

$$H(\omega) = F\{e^{-3t}u(t)\} = \frac{1}{j\omega+3}$$

$$|H(\omega)|^2 = \frac{1}{9+\omega^2}$$

ESD of o/p $\psi_y(\omega) = |H(\omega)|^2 \psi_x(\omega) = \left(\frac{1}{9+\omega^2}\right)\left(\frac{1}{1+\omega^2}\right)$.

Ex:- fig shows the PSD of the signal $x(t)$. Find out its average Power. (14)

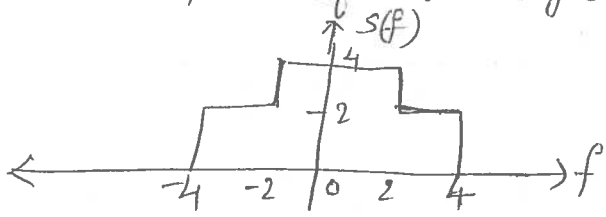



fig:- PSD for Example.

$$P = \int_{-\infty}^{\infty} S(f)\,df = \int_{-4}^{-2} 2\,df + \int_{-2}^{2} 4\,df + \int_{2}^{4} 2\,df = 2(f)_{-4}^{-2} + 4(f)_{-2}^{2} + 2(f)_{2}^{4}$$

$$P = 2(-2+4) + 4(2+2) + 2(4-2) = 24 \text{ watt}.$$

Ex:- For the signal $x(t) = e^{-at}u(t)$, find out the total Energy contained in the frequency band $|f| \le f_1$, where $f_1 = a/2\pi$. (15)

$$x(t) = e^{-at}u(t)$$

$$X(\omega) = \frac{1}{a+j\omega} = \frac{1}{a+j2\pi f}$$

$$ESD = |X(\omega)|^2 = \frac{1}{a^2+(2\pi f)^2} = \psi(f).$$

We know that the total Energy of $x(t)$ is given by area under the ESD curve, Energy contained in the band $|a| \le f_1$ is equal to area under the ESD curve from $-f_1$ to $f_1$ i.e.,

$$\text{Energy in } (-f_1, f_1) = \int_{-f_1}^{f_1} \psi(f)\,df = \int_{-f_1}^{f_1} \frac{1}{a^2+(2\pi f)^2}\,df$$

$$P = \frac{1}{2\pi a}\left(\tan^{-1}\left(\frac{2\pi f}{a}\right)\Big|_{-f_1}^{f_1}\right) = \frac{1}{4a}.$$

Ex:-(16) Determine the Autocorrelation function & Energy Spectral density of $x(t) = e^{-at}u(t)$.

≃ (1) To obtain Autocorrelation function:

$$x(t) = e^{-at}u(t)$$

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

If $x(t)$ is real, then the above equation becomes

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt.$$

$x(t) = e^{-at}u(t)$ is a real signal. Substituting this value of $x(t)$ in the equation for $R(\tau)$, we get.

$$R(\tau) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-a(t-\tau)}u(t-\tau)\, dt.$$

we know that $u(t) = 1$ for $t \geq 0$ & $u(t-\tau) = 1$ for $t \geq \tau$. Hence the limits of integrations will be from $\tau$ to $\infty$, i.e.,

$$R(\tau) = \int_{\tau}^{\infty} e^{-at}\, e^{-a(t-\tau)}\, dt = \int_{\tau}^{\infty} e^{-at}\, e^{-at}\, e^{a\tau}\, dt$$

$$R(\tau) = e^{a\tau}\int_{\tau}^{\infty} e^{-2at}\, dt = e^{a\tau}\left(\frac{e^{-2at}}{-2a}\right)_{\tau}^{\infty} = e^{a\tau}\left(\frac{e^{-\infty}}{-2a} + \frac{e^{-2a\tau}}{2a}\right)$$

$$R(\tau) = \frac{e^{-a\tau}}{2a}.$$

This is an Autocorrelation function for positive value of $\tau$. This is because we have considered $u(t-\tau) = 1$ for '$\tau$' positive. We know that autocorrelation has Conjugate Symmetry,

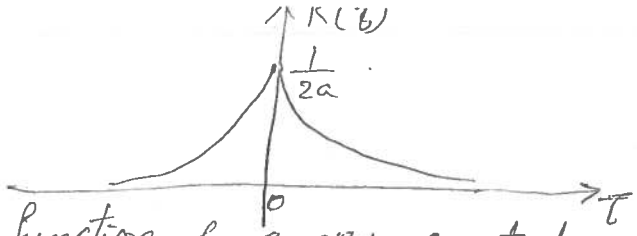
$$R(\tau) = R^*(-\tau)$$

In this example, we obtained $R(\tau)$, which is real. Hence the above equation becomes

$$R(\tau) = R(-\tau).$$

$R(\tau)$ is an even function

$\therefore R(\tau) = \dfrac{e^{-a|\tau|}}{2a}$




(i) We know that the auto correlation function & Energy spectral density form a fourier transform pair.

$$\psi(\omega) = F\{R(\tau)\} = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{e^{-a|\tau|}}{2a}\, e^{-j\omega\tau}\, d\tau$$

$$\psi(\omega) = \frac{1}{a^2+\omega^2}.$$

Ex:- (17) Consider a pulse function given by

$$x(t) = \begin{cases} \cos\left(\dfrac{\pi t}{T}\right) & \text{for } -\dfrac{T}{2} < t < \dfrac{T}{2}. \\ 0 & \text{otherwise.} \end{cases}$$

Find (i) Auto correlation function.

(ii) Energy spectral Density.

→ (i) To obtain Auto correlation function;

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt = \int_{-T/2}^{T/2} \cos\frac{\pi t}{T} \cos\frac{\pi}{T}(t-\tau)\, dt$$

$$R(\tau) = \int_{-T/2}^{T/2} \cos\frac{\pi}{T}t \cos\left(\frac{\pi t}{T} - \frac{\pi\tau}{T}\right) dt.$$

$$\cos x \cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right).$$

$$R(\tau) = \int_{-T/2}^{T/2} \frac{1}{2}\left\{\cos\left(\frac{\pi\tau}{T}\right) + \cos\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right)\right\} dt$$

$$R(\tau) = \frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{\pi\tau}{T}\right) dt + \frac{1}{2}\int_{-T/2}^{T/2} \cos\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right) dt$$

$$R(\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{T}\right)(t)_{-T/2}^{T/2} + \frac{1}{2}\left(\frac{1}{2\pi/T}\right)\left(\sin\left(\frac{2\pi t}{T} - \frac{\pi\tau}{T}\right)\right)_{-T/2}^{T/2}$$

$$R(\tau) = \frac{T}{2}\sin\left(\frac{\pi t}{T}\right).$$

(ii) To obtain Energy spectral density.

$$R(\tau) \longleftrightarrow \psi_x(\omega)$$

$$\psi(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{T}{2}\sin\left(\frac{\pi\tau}{T}\right) e^{-j\omega\tau}\, d\tau$$

We know that $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$.

$$\psi(\omega) = \frac{T}{2}\int_{-\infty}^{\infty} \frac{e^{j(\pi\tau/T)} - e^{-j(\pi\tau/T)}}{2j} e^{-j\omega\tau}\, d\tau$$

$$\psi(\omega) = \frac{T}{4j}\int_{-\infty}^{\infty} e^{-j(\omega-(\pi/T))\tau} - e^{-j(\omega+\pi/T)\tau}\, d\tau$$

$$= \frac{T}{4j}\left\{\int e^{-j(\omega-\pi/T)\tau}\, d\tau - \int_{-\infty}^{\infty} e^{-j((\omega+(\pi/T))\tau}\, d\tau\right\}$$

$$\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau = \delta(\omega)$$

$$\psi(\omega) = \frac{T}{4j}\left(\delta\left(\omega-\frac{\pi}{T}\right) - \delta\left(\omega+\frac{\pi}{T}\right)\right).$$

Ex:- If $x(t) = \sin\omega_0 t$, find

(i) $R(\tau)$ (ii) ESD.

$\Rightarrow$

$$x(t) = \sin\omega_0 t.$$

(i) $$R(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \sin\omega_0 t \sin(\omega_0(t-\tau))\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sin\omega_0 t\,(\sin\omega_0 t\cos\omega_0\tau - \cos\omega_0 t \sin\omega_0\tau)\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \cos\omega_0\tau\left(\frac{1-\cos 2\omega_0 t}{2}\right) - \sin\omega_0\tau\left(\frac{\sin 2\omega_0 t}{2}\right) dt$$

$$= \frac{1}{2T}\left\{\cos\omega_0\tau\,(t)_{-T/2}^{T/2}\right\} - 0 - 0 = \frac{1}{2}\cos\omega_0\tau.$$

(ii) ESD

$$\psi(\omega) = F\{R(\tau)\} = F\left\{\frac{1}{2}\cos\omega_0\tau\right\} = \frac{\pi}{2}\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right).$$

(1) Find out the Response of CT System to unit Step. Impulse response of the System is given by

$$h(t) = \frac{1}{RC} \cdot e^{-t/RC} u(t).$$

(4-14)

$$y(t) = x(t) * h(t)$$

$$x(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau) \left( \frac{1}{RC} e^{-(t-\tau)/RC} u(t-\tau) \right) d\tau$$

$$y(t) = \frac{1}{RC} \int_{-\infty}^{\infty} u(\tau)\, e^{-(t-\tau)/RC}\, u(t-\tau)\, d\tau$$

---

$u(\tau) = 1$, for $\tau \geq 0$

$= 0$ otherwise.

$u(t-\tau) = 1$ for $\tau \leq t$

$= 0$ otherwise.

$$y(t) = \frac{1}{RC} \int_0^t e^{-(t-\tau)/RC}\, d\tau = e^{-t/RC} \int_0^t e^{\tau/RC}\, d\tau$$

$$y(t) = \frac{e^{-t/RC}}{RC} \left( \frac{e^{\tau/RC}}{\frac{1}{RC}} \right)_0^t = \frac{e^{-t/RC}}{RC} (RC) \left( e^{t/RC} - e^{0} \right)$$

$$y(t) = e^{-t/RC} \left( e^{t/RC} - 1 \right) = 1 - e^{-t/RC} \quad \text{for } t \geq 0.$$

(2) An Impulse response of the RC. Circuit is $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$ Find the Step response of the circuit.

(3) Find the Response $y(t)$ of a LTI System with impulse response $h(t) = u(t-1)$ to the i/p $x(t) = e^{-2t} u(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau)\, u(t-\tau-1)\, d\tau$$

- $u(\tau) = 1$ for $\tau \geq 0$, $= 0$ otherwise
- $u(t-\tau-1) = 1$, for $\tau \leq t-1$, $= 0$ otherwise

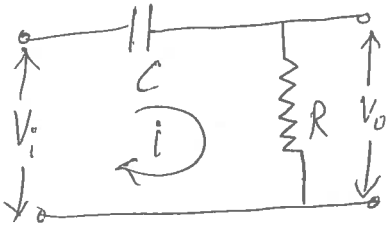
$u(t-\tau-1) = 1$, for $\tau \leq t-1$

$= 0$ otherwise

$$y(t) = \int_{0}^{t-1} e^{-2\tau}\, d\tau = \left(\frac{e^{-2\tau}}{-2}\right)_0^{t-1} = \left(\frac{e^{-2(t-1)}}{-2} - \frac{e^{0}}{-2}\right) = \frac{e^{-2(t-1)}}{-2} + \frac{1}{2}$$

$$y(t) = \frac{1}{2}\left(1 - e^{-2(t-1)}\right) \quad \text{for } t \geq 0$$

(3) Determine the step response of the following electrical system.




~~$V_o = V_i + \frac{1}{C}\int i\,dt$~~

~~$iR = V_i + \frac{1}{C}\int i\,dt$~~

~~$V_o(\omega) = V_i(\omega) + \frac{1}{C}\frac{i(\omega)}{j\omega}$~~

$$V_i = \frac{1}{C}\int i\,dt + iR$$

$$V_i(\omega) = \frac{1}{C}\frac{i(\omega)}{j\omega} + R\,i(\omega)$$

$$= \frac{i(\omega) + jRC\omega\, i(\omega)}{j\omega C}$$

$$V_i(\omega) = i(\omega)\left(\frac{1+j\omega RC}{j\omega C}\right)$$

~~$\frac{V_o(\omega)}{V_i(\omega)}$~~ ~~$V_o(\omega) = V_i(\omega) +$~~

$$V_o(\omega) = i(\omega)\, R$$

$$H(\omega) = \frac{j\omega RC}{1+j\omega RC} = 1 - \frac{1}{1+j\omega RC} = 1 - \frac{1}{RC\left(j\omega + \frac{1}{RC}\right)}$$

~~$h(t) = \delta(t) - e^{-\frac{1}{RC}t} u(t)$~~

$$h(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t)$$

$Y(\omega) = H(\omega)\, X(\omega)$ $\quad x(t) = u(t)$

$X(\omega) = \pi\delta(\omega) - \frac{1}{j\omega}$.

(3) Determine the unit step response of the system, whose $h(t) = e^{-5t}u(t)$

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} e^{-5\tau}u(\tau)\,d\tau$$

$$s(t) = \int_{0}^{t} e^{-5\tau}\,d\tau = \left(\frac{e^{-5\tau}}{-5}\right)_0^t = \frac{1}{-5}\left(e^{-5t} - e^{0}\right).$$

$$s(t) = \frac{1}{5}\left(1 - e^{-5t}\right) \quad \text{for } t \geq 0.$$

(4) Determine the unit step response of the following system whose impulse response is $h(t) = 4tu(t)$.

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau = \int_{-\infty}^{t} 4\tau\, u(\tau)\,d\tau$$

$u(\tau) = 1$ for $\tau \geq 0$

$= 0$ otherwise

$$s(t) = 4\int_{0}^{t} \tau\,d\tau = 4\left(\frac{\tau^2}{2}\right)_0^t = 2(\tau^2)_0^t = \underline{2t^2}.$$

(5) Find the step response of the system where

$h(t) = u(t+3) + u(t-3)$.

$h(t) = h_1(t) + h_2(t)$

unit step response $s(t) = h(t) * u(t)$

$$s(t) = h_1(t) * u(t) + h_2(t) * u(t) = s_1(t) + s_2(t)$$

$$s_1(t) = \int_{-\infty}^{t} h_1(\tau)\,d\tau = \int_{-\infty}^{t} u(\tau+3)\,d\tau$$

$u(t+3) = 1$ for $\tau \geq -3$

$= 0$ otherwise

$$s_1(t) = \int_{-3}^{t} 1\,d\tau = (\tau)_{-3}^{t} = t+3, \quad \text{for } t \geq -3$$

By $h_2(t) = u(t-3)$

$s_2(t) = t-3$ for $t \geq 3$.

$s(t) = s_1(t) + s_2(t) = t+3+t-3 = 2t$.

(6) Find the step response of the system whose impulse response is $h(t) = \delta(t) - \delta(t-2)$.

Ans $s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$

$s(t) = h(t) * u(t)$

$x(t) * \delta(t) = x(t)$.

$s(t) = u(t) - u(t-2)$.

(7) Find the step response of the system whose impulse response is $h(t) = e^{-4t}u(t) - e^{-2t}u(t)$.

Ans $s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$.

$s(t) = \int_{-\infty}^{t} e^{-4\tau} u(\tau)\, d\tau - \int_{-\infty}^{t} e^{-2\tau} u(\tau)\, d\tau$

$u(\tau) = 1$ for $\tau \geq 0$

$= 0$ otherwise

$s(t) = \int_{0}^{t} e^{-4\tau} d\tau - \int_{0}^{t} e^{-2\tau} d\tau$

$= \left(\frac{e^{-4\tau}}{-4}\right)_0^t \cdots \left(\frac{e^{-2\tau}}{-2}\right)_0^t = \frac{1}{4} - \frac{e^{-4t}}{4} + \frac{e^{-2t}}{2} - \frac{1}{2}$

$$\boxed{s(t) = \frac{e^{-2t}}{2} - \frac{e^{-4t}}{4} - \frac{1}{4}}$$

(8) Find the step response of the system whose impulse response is $h(t) = e^{-2t}u(t-2)$

Ans $s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \int_{-\infty}^{t} e^{-2\tau} u(\tau-2)\, d\tau$ $u(\tau-2) = 1$ for $\tau \geq 2$, $= 0$ otherwise

$s(t) = \int_{2}^{t} e^{-2\tau} d\tau = \left(\frac{e^{-2\tau}}{-2}\right)_2^t = \frac{-1}{2}\left(e^{-2\tau}\right)_2^t = \frac{-1}{2}\left(e^{-2t} - e^{-4}\right) = \frac{-e^{-2t}}{2} + \frac{e^{-4}}{2}$.

# UNIT-III

## (3) Discrete Time Fourier Transform

### Fourier Series Representation of Discrete Time periodic Signals:

A Discrete Time Signal $x(n)$ is periodic with period 'N' if

$$x(n+N) = x(n) \longrightarrow (1)$$

Fundamental period $N = \frac{2\pi}{\omega_0}$

The set of all discrete time complex exponential signals that are periodic with period 'N' is given by

$$\phi_k(n) = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n} \longrightarrow (2)$$

$k = 0, \pm 1, \pm 2, \ldots\ldots$

A linear combination of $\phi_k(n)$ has the form for the periodic sequences $x(n)$

$$x(n) = \sum_k c_k \phi_k(n) = \sum_k c_k e^{jk\omega_0 n}$$

$$x(n) = \sum_k c_k e^{jk\left(\frac{2\pi}{N}\right)n} \longrightarrow (3)$$

Eq(3) is referred to as the DT Fourier series & the coefficients $c_k$ are referred to as the Fourier Series coefficients.

Determination of Fourier Series representation of a periodic Signal:

$$x(n) = \sum c_k e^{j\left(\frac{2\pi}{N}\right)kn} \longrightarrow (1)$$

Multiply both sides by $e^{-jm\left(\frac{2\pi}{N}\right)n}$

$$x(n)\, e^{-j\left(\frac{2\pi}{N}\right)mn} = \sum_{\langle k \rangle} c_k e^{j\left(\frac{2\pi}{N}\right)kn} e^{-j\left(\frac{2\pi}{N}\right)mn}$$

$$\sum_{\langle n \rangle} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)mn} = \sum_{\langle n \rangle} \sum_{\langle k \rangle} c_k e^{j\frac{2\pi}{N}n(k-m)}$$

$$= \sum_k c_k \sum_{\langle n \rangle} e^{j\frac{2\pi}{N}(k-m)n}$$

$$\sum_{n=\langle N\rangle} e^{j(K-m)\frac{2\pi}{N}n} = \begin{cases} N & K=m \\ 0 & K\neq m \end{cases} \quad (K-m = \pm N, \pm 2N \dots\dots)$$

$$\sum_{n=\langle N\rangle} x(n)\, e^{-jm\left(\frac{2\pi}{N}\right)n} = \sum_{K=m} C_K N$$

$$\sum_{n=\langle N\rangle} x(n)\, e^{-j\frac{2\pi}{N}mn} = C_m N$$

$$C_m = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-j\frac{2\pi}{N}mn}$$

$$C_K = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-j\frac{2\pi}{N}Kn} \longrightarrow (2)$$

DTFS pair:-

$$x(n) = \sum_{K=\langle N\rangle} C_K\, e^{jk\omega_0 Kn} = \sum_{K=\langle N\rangle} C_K\, e^{jK\left(\frac{2\pi}{N}\right)n} \qquad K=0 \text{ to } N-1$$

$$C_K = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jK\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jK\left(\frac{2\pi}{N}\right)n} \qquad n=0 \text{ to } N-1$$

Periodicity of $C_K$:-

$$C_{K+N} = C_K$$

$$C_{K+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)n(k+N)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nK - j\frac{2\pi}{N}nN}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nK}\, e^{-j2\pi n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$\boxed{C_{K+N} = C_K}$$

## Discrete Frequency Spectrum & Frequency Range:-

Consider a periodic sequence $x(n)$ with period $N$. This sequence can be expressed in discrete Fourier series as follows:

$$x(n) = \sum_{K=0}^{N-1} C_K \, e^{j2\pi Kn/N}$$

The values of $C_K$, $K = 0, 1, \ldots, N-1$ are called the Discrete (or) line spectra of $x(n)$. Each $C_K$ appears at frequency $\omega_K = \frac{2\pi}{N}K$.

If $N=3$, the spectral $C_K$ appears at $\omega=0$, $\omega=\frac{2\pi}{3}$, & at $\omega=\frac{4\pi}{3}$. The spectral component at $\omega=2\pi$ is same as the spectral component at $\omega=0$ That is $C_K$ is periodic with period $2\pi$ radians. Thus on frequency scale $\omega$, $C_K$ repeats for every $2\pi$ intervals. Though $C_K$ appears at all integers $K$ from $(-\infty$ to $\infty)$, we need to plot them only in a frequency interval of $2\pi$. The interval can be choosen as $0 \le \omega < 2\pi$ (not including $2\pi$) (or) as $-\pi < \omega \le \pi$.

The Fourier coefficients $C_K$ are complex and they can be represented in the polar form as follows:

$$C_K = |C_K| e^{j \angle C_K}$$

The plot of $|C_K|$ (vs) $\omega$ (or) '$K$' is called magnitude spectrum & plot of $\angle C_K$ (vs) $\omega$ (or) $K$ is called the phase spectrum.

$|C_K| = |C_{-K}| \rightarrow$ Magnitude spectrum is an even fn of $\omega$.

$\angle C_K = -\angle C_{-K} \rightarrow$ phase spectrum is an odd fn of $\omega(K)$.

**Ex:-** Find the spectral components (coefficients) of

$$x(n) = \cos\frac{\pi}{4}n.$$

$$N = \frac{2\pi m}{\omega_0} = \frac{2\pi}{\frac{\pi}{4}} m = 8 \quad (\text{for } m=1) \qquad \boxed{N=8}$$

$$x(n) = \frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) = \frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n}$$

$$x(n) = \sum_{K=-2}^{4} C_K \, e^{j2\pi Kn/N} = \sum_{K=-3}^{4} C_K \, e^{j\frac{2\pi Kn}{8}}$$

$$x(n) = \sum_{K=-3} C_K e^{j\pi Kn/4}$$

$$C_1 = \frac{1}{2} \qquad C_{-1} = \frac{1}{2}$$

$$C_K = \frac{1}{2} \quad \text{for } K = -1 \text{ & } 1$$

$$= 0 \qquad -3 \le K \le 4 \quad \& \; K \neq \pm 1.$$

(2) Find the DTFS representation for

$$x(n) = 5 + \sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{4}\right).$$

and sketch amplitude & phase spectrum.

$$N_1 = \frac{2\pi}{\pi/2} m = 4 \quad (\text{for } m = 1)$$

$$N_2 = \frac{2\pi}{\pi/4} = 8 \quad (\text{for } m = 1)$$

$$\frac{N_1}{N_2} = \frac{4}{8} = \frac{1}{2}$$

$$2N_1 = N_2$$

The time period of $x(n)$ is $N_2 = 2N_1 = 8$

$N = 8$.

$$x(n) = 5 + \sin\frac{n\pi}{2} + \cos\frac{n\pi}{4}$$

$$= 5 + \left(\frac{e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}}{2}\right) + \left(\frac{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}}{2}\right)$$

$$x(n) = \sum_{K=-3}^{4} C_K e^{j2\pi Kn/N} = \sum_{K=-3}^{4} C_K e^{j\pi Kn/4}$$

$$x(n) = C_{-3} e^{-j3\pi n/4} + C_{-2} e^{-j\pi n/2} + C_{-1} e^{-j\pi n/4} + C_0 + C_1 e^{j\pi n/4} + C_2 e^{j\pi n/2} + C_3 e^{j3\pi n/4} + C_4 e^{j\pi n}$$

$$C_{-2} = \frac{-1}{2j} \qquad C_{-1} = \frac{1}{2} \qquad C_0 = 5 \qquad C_1 = \frac{1}{2} \qquad C_2 = \frac{1}{2j}$$

$$|C_K| = \frac{1}{2} \Rightarrow \text{for } K = -1, -2, 1, 2$$

$|C_K| = 5$ for $K=0$

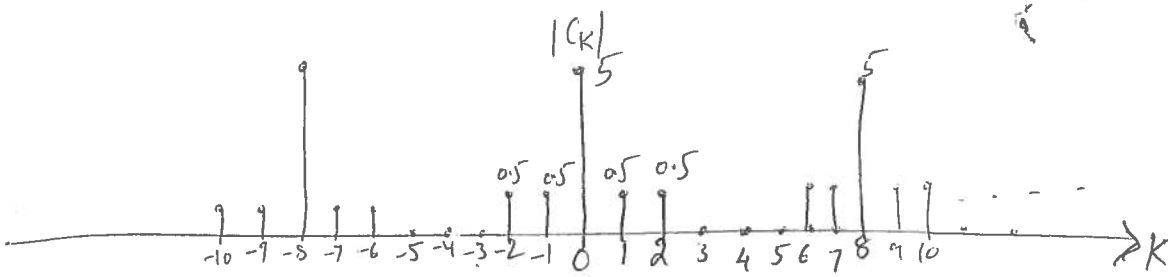
$= 0$ for $K = -3, 3, 4$

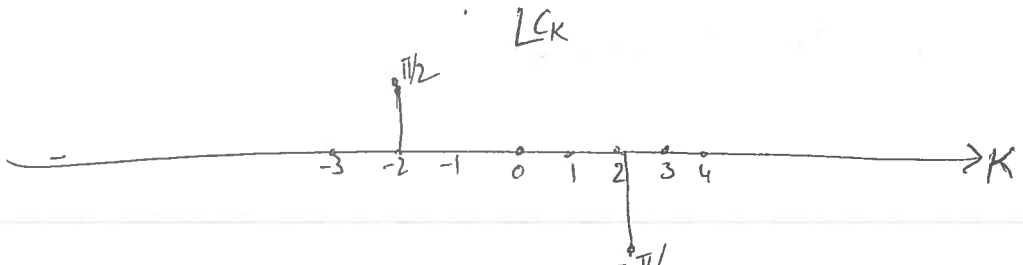


Hy $\angle C_K = \frac{\pi}{2}$ for $K = -2$

$= -\frac{\pi}{2}$ for $K = 2$

$= 0$ for $-3 \le K \le 4$ & $K \neq \pm 2$.




## Properties of Discrete time Fourier Series:-

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| Linearity | $a_1 x(n) + a_2 y(n)$ | $a_1 C_K + a_2 D_K$ |
| Time shifting | $x(n-n_0)$ | $C_K e^{-j2\pi K n_0/N}$ |
| Frequency shifting | $e^{j2\pi Mn/N} x(n)$ | $C_{K-M}$ |
| Time Reversal | $x(-n)$ | $C_{-K}$ |
| Conjugation | $x^*(n)$ | $C_{-K}^*$ |
| Convolution | $\sum_{r=\langle N \rangle} x(r) y(n-r)$ | $N C_K D_K$ |
| Multiplication | $x(n) y(n)$ | $\sum_{l=\langle N \rangle} C_l D_{K-l}$ |
| Conjugate Symmetry for real Signals | $x(n)$ real | $C_K = C_{-K}^*$ <br> $Re\{C_K\} = Re\{C_{-K}\}$ $Im\{C_K\} = -Im\{C_{-K}\}$ |

| | | |
|---|---|---|
| | | $\lvert C_K\rvert = \lvert C_{-K}\rvert$ |
| | | $\angle C_K = -\angle C_K$ |
| Real & Even Signals | $x(n)$ real & even | $C_K$ real & even |
| Real & odd Signals | $x(n)$ real & odd | $C_K$ imaginary & odd |
| | $x_e(n)$ | $Re\{C_K\}$ |
| | $x_o(n)$ | $jIm\{C_K\}$ |

Parsevals relation for periodic signals

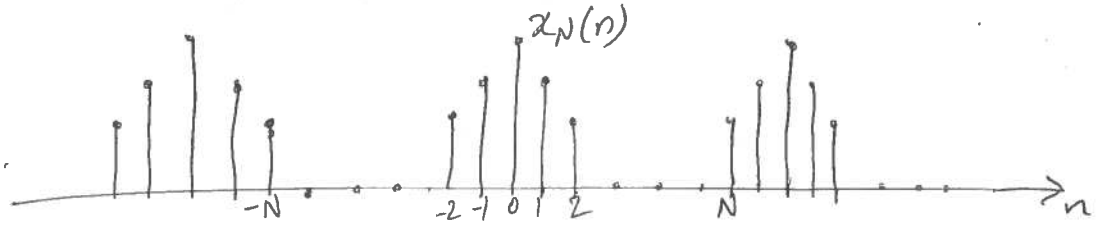
$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{K=\langle N\rangle}|C_K|^2$$

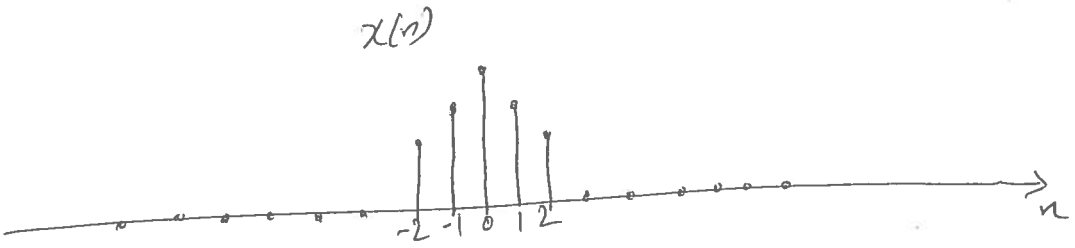
## Discrete Time Fourier Transform :-

We will develop the DTFT from DTFS concept.

Let us consider a periodic sequence $x_N(n)$ with period N as shown in fig. As $N\to\infty$ the periodic sequence $x_N(n)$ becomes aperiodic sequence $x(n)$.

$$x(n) = \lim_{N\to\infty} x_N(n) \quad \rightarrow (1)$$




$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_N(n)\, e^{-j2\pi Kn/N} \quad \rightarrow (2)$$

$$x_N(n) = \sum_{K=0}^{N-1} C_K\, e^{j2\pi Kn/N} \quad \rightarrow (3)$$

As $N\to\infty$ $C_K$ tend to zero. Hence let us define

$$NC_K = X\left(e^{j2\pi K/N}\right) \quad \rightarrow (4)$$

Let $\omega = \frac{2\pi}{N} K$ → (5)

5 → (4)

$$NC_K = X(e^{j\omega_K}) \rightarrow (6)$$

From (2)

$$C_K = \frac{1}{N} \sum_{n=-\left(\frac{N}{2}-1\right)}^{N/2} x_N(n)\, e^{-j2\pi Kn/N}$$

$$NC_K = \sum_{n=-\left(\frac{N}{2}-1\right)}^{\frac{N}{2}} x_N(n)\, e^{-j\omega_K n} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{N/2} x_N(n)\, e^{-j2\pi Kn/N}$$

As $N \to \infty$ periodic $x_N(n)$ becomes Aperiodic $x(n)$, $\omega_K = \frac{2\pi}{N} K$ becomes continuous variable & limits becomes $-\infty$ to $\infty$

$$NC_K = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\therefore X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \xrightarrow{(7)} DTFT.$$

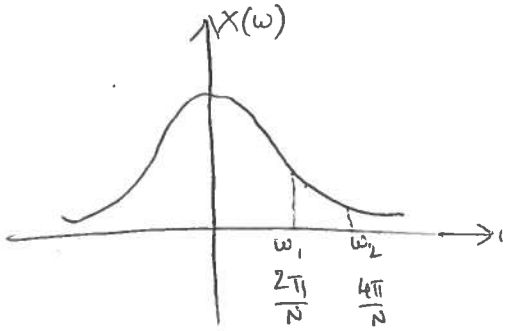
$N \to \infty$, $\frac{2\pi}{N} \to 0$.

$$x_N(n) = \sum_{n=0}^{N-1} C_K\, e^{j2\pi Kn/N}$$

$$= \frac{N}{N} \sum_{n=0}^{N-1} C_K\, e^{j2\pi Kn/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} NC_K\, e^{j2\pi Kn/N}$$

$$= \frac{2\pi}{N} \sum_{n=0}^{N-1} X(e^{j\omega_K})\, e^{j2\pi Kn/N} \left(\frac{1}{2\pi}\right)$$




$$x_N(n) = \frac{1}{2\pi} \sum_{n=0}^{N-1} X(e^{j\omega_K})\, e^{j2\pi Kn/N} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi} \sum_{n=0}^{N-1} X(e^{j\omega_K})\, e^{j2\pi Kn/N}\, \omega_K$$

As $N \to \infty$, $\frac{2\pi}{N} \to 0$ & $\omega_K = \frac{2\pi K}{N}$ becomes continuous & $\frac{2\pi}{N}$ can be written as $d\omega$. Further summation becomes an integral with limits 0 to $2\pi$.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \rightarrow (7)$$

we can take limits as $-\pi$ to $\pi$ also

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \longrightarrow (8)$$

$$F(x(n)) = X(e^{j\omega}) = X(\omega)$$

$$x(n) = F^{-1}(X(e^{j\omega})) = F^{-1}(X(\omega))$$

$$x(n) \Longleftrightarrow X(e^{j\omega})$$

## Existence of Discrete Time Fourier Transform :-

The discrete time fourier transform does not exists for all periodic sequences. A sufficient condition for the existence of DTFT for an aperiodic sequence $x(n)$ is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

That is if a sequence $x(n)$ is absolutely summable then DTFT exists for the sequence $x(n)$.

(1) Find the Fourier Transform of the following.

(a) $\delta(n)$ (b) $u(n)$ (c) $\delta(n-k)$ (d) $u(n-k)$ (e) $a^n u(n)$ (f) $\delta(n+2)-\delta(n-2)$

(a) $\delta(n)$

$x(n) = \delta(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n} = \delta(0) e^{j(0)} = 1$$

(b) $x(n) = u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{1}{1-e^{-j\omega}}$$

(c) $x(n) = \delta(n-k)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-k) e^{-j\omega n} = e^{-j\omega k}$$

(d) $x(n) = u(n-k)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u(n-k) e^{-j\omega n} = \sum_{n=k}^{\infty} 1 \times e^{-j\omega n} = e^{-j\omega k} + e^{-j\omega(k+1)} + e^{-j\omega(k+2)} + \cdots$$

$$X(e^{j\omega}) = e^{-j\omega k}\left(1 + e^{-j\omega} + e^{-j2\omega} + \cdots\right) = \frac{e^{-j\omega k}}{1-e^{-j\omega}}$$

(e) $x(n) = a^n u(n)$

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

(f) $x(n) = \delta(n+2) - \delta(n-2)$

$$X(\omega) = F\{\delta(n+2)\} - F\{\delta(n-2)\}$$

$$= e^{+j2\omega} - e^{-j2\omega}$$

$$= e^{j2\omega} - e^{-j2\omega}$$

$$\boxed{X(\omega) = 2j\sin\omega}$$

**Ex:-** Find the Discrete time Fourier transform of the following.

(a) $x(n) = \{1, -1, 2, 2\}$.

(b) $x(n) = 2^n u(n)$

(c) $x(n) = (0.5)^n u(n) + 2^{+n} u(-n-1)$

(a) ⇝ $x(n) = \{1, -1, 2, 2\}$

$x(0) = 1 \quad x(1) = -1 \quad x(2) = 2 \quad x(3) = 2$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{3} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = x(0)\, e^{0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$X(e^{j\omega}) = 1 + (-1) e^{-j\omega} + 2 e^{-j2\omega} + 2 e^{-j3\omega}$$

$$X(e^{j\omega}) = 1 - e^{-j\omega} + 2 e^{-j2\omega} + 2 e^{-j3\omega}.$$

(b) $x(n) = 2^n u(n)$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} 2^n e^{-j\omega n} = \sum_{n=0}^{\infty} (2e^{-j\omega})^n = \frac{1}{1 - 2e^{-j\omega}}$$

(c) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (0.5)^n u(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} 2^n u(-n-1)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} 2^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.5 e^{-j\omega})^n + \sum_{n=1}^{\infty} (2^{-1} e^{j\omega})^n$$

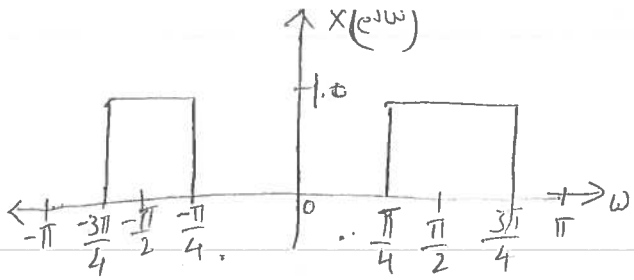
$$= \frac{1}{1 - 0.5 e^{-j\omega}} + \frac{0.5 e^{j\omega}}{1 - 0.5 e^{j\omega}} = \frac{1 - 0.5 e^{j\omega} + 0.5 e^{j\omega} - 0.25}{1 - 2\cos\omega + 0.25}$$

$$X(e^{j\omega}) = \frac{0.75}{1.25 - 2\cos\omega}$$

**Ex:-** Find the inverse fourier transform of the following

$$X(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \le |\omega| \le 3\pi/4 \\ 0 & 3\pi < |\omega| < \pi \quad 0 \le |\omega| \le \frac{\pi}{4} \end{cases}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$




$$= \frac{1}{2\pi}\left(\int_{-3\pi/4}^{-3\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}\, d\omega\right)$$

$$= \frac{1}{2\pi j n}\left(\left(e^{j\omega n}\right)_{-3\pi/4}^{-3\pi/4} + \left(e^{j\omega n}\right)_{\pi/4}^{3\pi/4}\right)$$

$$x(n) = \frac{1}{\pi n}\left(\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right)$$

**Ex:-** Find the Inverse Fourier transform of $X(e^{j\omega}) = 1 + 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$

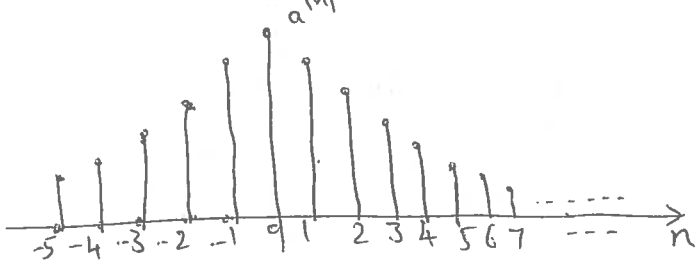
$0 \le n \le 3$

$$x(n) = \sum_{n=0}^{3} x(n)\, e^{-j\omega n} = x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega}$$

$x(0) = 1 \quad x(1) = 2 \quad x(2) = 2 \quad x(3) = 3$

$$x(n) = \{1, 2, 2, 3\}.$$

**Ex:-** Find the Fourier transform of the signal $x(n) = a^{|n|}$.




$$x(n) = a^{|n|}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n}\, e^{-j\omega n}$$

$$= \sum_{n=-1}^{\infty} \left(a\, e^{j\omega}\right)^{-n} + \sum_{n=0}^{\infty} \left(a\, e^{-j\omega}\right)^{n}$$

$$X(e^{j\omega}) = \sum_{n=1}^{\infty} \left(a\, e^{j\omega}\right)^{n} + \sum_{n=0}^{\infty} \left(a\, e^{-j\omega}\right)^{n} = \frac{a\, e^{j\omega}}{1 - a\, e^{j\omega}} + \frac{1}{1 - a\, e^{-j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

**Ex:-** Find the fourier transform of $x(n) = \sin\left(\frac{\pi n}{2}\right)u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j}\right) e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{1}{2j}\left(\sum_{n=0}^{\infty} e^{j\frac{\pi}{2}n}\, e^{-j\omega n} - \sum_{n=0}^{\infty} e^{-j\frac{\pi}{2}n}\, e^{-j\omega n}\right) = \frac{1}{2j}\left(\sum_{n=0}^{\infty} e^{\left(j\frac{\pi}{2} - j\omega\right)n} - \sum_{n=0}^{\infty} e^{\left(-j\frac{\pi}{2} - j\omega\right)n}\right)$$

$$X(e^{j\omega}) = \frac{e^{-j\omega}}{1 + e^{-j2\omega}}.$$

Ex: Determine the Signal $x(n)$ for the given fourier transform

(a) $X(e^{j\omega}) = e^{-j\omega} \quad -\pi \le \omega \le \pi$

(b) $X(e^{j\omega}) = e^{-j\omega}(1+\cos\omega)$

$\Rightarrow$ (a) $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega} e^{j\omega n} d\omega$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n-1)\omega} d\omega = \frac{1}{2\pi j(n-1)}\left(e^{j(n-1)\omega}\right)_{-\pi}^{\pi}$$

$$x(n) = \frac{e^{j(n-1)\pi} - e^{-j(n-1)\pi}}{2\pi j(n-1)} = \frac{\sin\pi(n-1)}{\pi(n-1)}$$

(b) $X(e^{j\omega}) = e^{-j\omega}(1+\cos\omega) = e^{-j\omega}\left(1+\frac{e^{j\omega}+e^{-j\omega}}{2}\right) = \frac{e^{-j\omega} + e^{0} + e^{-j2\omega}}{2}$

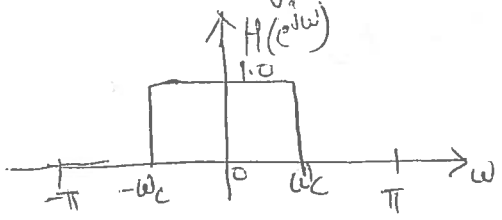
$$X(e^{j\omega}) = \frac{1}{2} + e^{-j\omega} + \frac{1}{2}e^{-j2\omega}$$

$$x(n) = \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}.$$

Ex: obtain the impulse response of the system described by

$H(e^{j\omega}) = 1 \quad$ for $|\omega| \le \omega_c$

$= 0 \quad$ for $\omega_c \le |\omega| \le \pi$




$\Rightarrow$ $h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n} d\omega = \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$h(n) = \frac{\sin\omega_c n}{\pi n}$$

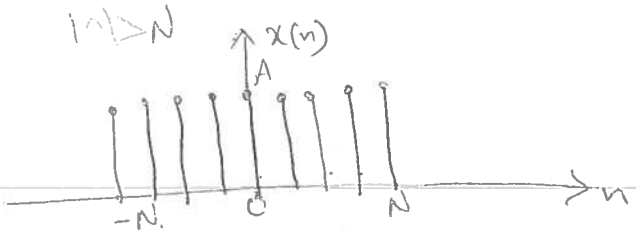
Ex: Find the Fourier transform of $x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$

$\Rightarrow$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} e^{-j\omega n} = e^{-j\omega} + \frac{1}{2}e^{-j2\omega} + \frac{1}{4}e^{-j3\omega} + \cdots$

$$X(e^{j\omega}) = e^{-j\omega}\left(1 + \frac{1}{2}e^{-j\omega} + \frac{1}{2^2}e^{-j2\omega} + \cdots\right) = e^{-j\omega}\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right)$$

$$X(e^{j\omega}) = \frac{e^{-j\omega}}{1-0.5e^{-j\omega}}$$

Ex:- Consider a Rectangular Pulse $x(n) = A \quad |n| \le N$
$= 0 \quad |n| > N$




→ Find $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-N}^{N} A\, e^{-j\omega n}$$

$$= \sum_{n=-N}^{-1} A e^{-j\omega n} + \sum_{n=0}^{N} A e^{-j\omega n} = \sum_{n=1}^{N} A\, e^{j\omega n} + \sum_{n=0}^{N} A e^{-j\omega n}$$

$$= \frac{A\left(1 - e^{j\omega N}\right)}{\left(1 - e^{j\omega}\right)} + A\left(\frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}}\right) = A\left(\frac{e^{j\omega} - e^{j\omega(N+1)}}{1 - e^{j\omega}} + \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}}\right)$$

$$X(e^{j\omega}) = A\left(\frac{2\cos\omega N - 2\cos\omega(N+1)}{2(1 - \cos\omega)}\right) = A\left(\frac{\sin\left(N + \frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}\right)$$

## Properties of DTFT :-

(1) Linearity :-

$$F(x_1(n)) = X_1(e^{j\omega}) \qquad F(x_2(n)) = X_2(e^{j\omega})$$

$$F(a_1 x_1(n) + a_2 x_2(n)) = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Proof :-

$$F(a_1 x_1(n) + a_2 x_2(n)) = a_1 F(x_1(n)) + a_2 F(x_2(n))$$
$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

(2) Periodicity :-

The DTFT is periodic in $\omega$ with period $2\pi$.

$$X(e^{j(\omega + 2K\pi)}) = X(e^{j\omega})$$

Proof :-

$$X(e^{j(\omega + 2K\pi)}) = \sum x(n)\, e^{-j(\omega + 2K\pi)n} = \sum x(n)\, e^{-j\omega n}\, e^{-j2\pi Kn}$$

$$= \sum x(n)\, e^{-j\omega n} = X(e^{j\omega}).$$

(3) Time shifting :- If $F(x(n)) = X(e^{j\omega})$

$$F(x(n-K)) = e^{-j\omega K} X(e^{j\omega})$$

Proof :-

$n - K = P \qquad n = K + P$

$$F(x(n-K)) = \sum_{n=-\infty}^{\infty} x(n-K)\, e^{-j\omega n} = \sum_{P=-\infty}^{\infty} x(P)\, e^{-j\omega(K+P)} = \sum x(P)\, e^{-j\omega P}\, e^{-j\omega K}$$

$$F(x(n-K)) = e^{-j\omega K} X(e^{j\omega}) \;//.$$

(4) Frequency Shifting Property:-

If $F(x(n)) = X(e^{j\omega})$

$$F\left(x(n)\, e^{j\omega_0 n}\right) = X\left(e^{j(\omega-\omega_0)}\right)$$

This property is the dual of the time shifting property

Proof:-

$$F\{x(n)\, e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

$$= X\left(e^{j(\omega-\omega_0)}\right)$$

(5) Time Reversal:-

If $F\{x(n)\} = X(e^{j\omega})$

then $F\{x(-n)\} = X(e^{-j\omega})$

Proof:-

$$F(x(-n)) = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}$$

$$F(x(-n)) = X\left(e^{-j\omega}\right).$$

That is folding in the time domain corresponds to the folding in the frequency domain.

(6) Differentiation in Frequency Domain:-

$$F(x(n)) = X(e^{j\omega})$$

$$F(n\,x(n)) = j\frac{d}{d\omega} X(e^{j\omega})$$

Proof:-

$$X(e^{j\omega}) = F(x(n)) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}(-jn)$$

$$j\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} n x(n)\, e^{-j\omega n}(-j)\, j = \sum_{n=-\infty}^{\infty} (n\,x(n))\, e^{-j\omega n} = F(n\,x(n))$$

$$F(n\,x(n)) = j\frac{d}{d\omega} X(e^{j\omega})$$

(7) Convolution in time domain.

$$F(x_1(n) * x_2(n)) = X_1(e^{j\omega}) X_2(e^{j\omega})$$

Proof:-

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$$

$$F(x_1(n) * x_2(n)) = \sum_{n=-\infty}^{\infty} \left[ \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K) \right] e^{-j\omega n}$$

$n-K=P \qquad n=K+P$

$$= \sum_{P=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(P)\, e^{-j\omega(K+P)}$$

$$= \sum_{P=-\infty}^{\infty} x_2(P)\, e^{-j\omega P} \sum_{K=-\infty}^{\infty} x_1(K)\, e^{-j\omega K}$$

$$F(x_1(n) * x_2(n)) = X_2(e^{j\omega})\, X_1(e^{j\omega})$$ ⇒ Convolution in time domain is equal to Multiplying their Spectrum in frequency Domain.

(8) Convolution in Frequency Domain:-

$$F(x_1(n)\, x_2(n)) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)})\, d\theta = X_1(e^{j\omega}) * X_2(e^{j\omega}).$$

Proof:-

$$F(x_1(n)\, x_2(n)) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_2(n) \left( \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) e^{j\theta n}\, d\theta \right) e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left( \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j(\omega-\theta)n} \right) d\theta$$

$$F(x_1(n)\, x_2(n)) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)})\, d\theta$$

(9) The Correlation Theorem:-

$F(x_1(n)) = X_1(e^{j\omega})$ & $F(x_2(n)) = X_2(e^{j\omega})$ then $F(\gamma_{x_1x_2}(l)) = X_1(e^{j\omega}) X_2(e^{-j\omega})$ ⇒

⇒ $F(\gamma_{x_1x_2}(l)) = \Gamma_{x_1x_2}(e^{j\omega}) = X_1(e^{j\omega}) X_2(e^{j\omega})$ → The function $\Gamma_{x_1x_2}(e^{j\omega})$ is the cross energy Spectrum density of signals $x_1(n)$ & $x_2(n)$.

(10) The Modulation Theorem:-

$$F(x(n) \cos\omega_0 n) = \frac{1}{2} \left( X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)}) \right)$$

Proof:- $F\{x(n)\cos\omega_0 n\} = F\left\{x(n)\left(\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\right)\right\}$

$F\{x(n)\cos\omega_0 n\} = \frac{1}{2}\left\{F(x(n)e^{j\omega_0 n}) + F(x(n)e^{-j\omega_0 n})\right\}$.

$F\{x(n)\cos\omega_0 n\} = \frac{1}{2}\left\{X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right\}$.

(IV) Symmetry properties :-

Fourier Transform $X(e^{j\omega})$ is a complex function of $\omega$ & can be Expressed as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \rightarrow (1)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \rightarrow (2)$$

$$= \sum_{n=-\infty}^{\infty} x(n)(\cos\omega n - j\sin\omega n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n - j\sum_{n=-\infty}^{\infty} x(n)\sin\omega n \rightarrow (3)$$

Comparing (1) & (3)

$$X_R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n \rightarrow (4a)$$

$$X_I(e^{j\omega}) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n \rightarrow (4b)$$

That is,

$X_R(e^{-j\omega}) = X_R(e^{j\omega})$ $(\because \cos(-\omega)n = \cos\omega n)$ → Even Symmetry.

$X_I(e^{-j\omega}) = -X_I(e^{j\omega})$ $(\because \sin(-\omega)n = -\sin\omega n)$ → odd Symmetry.

Therefore $X_R(e^{j\omega})$ is an even function of $\omega$ & $X_I(e^{j\omega})$ is an odd function of $\omega$.

we can write $X(e^{j\omega})$ in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)}$$

where $|X(e^{j\omega})|$ is the magnitude & $\theta(\omega)$ is the phase of $X(e^{j\omega})$.

$$\theta(\omega) = \angle H(e^{j\omega})$$

on Expanding

$$X(e^{j\omega}) = |X(e^{j\omega})| [\cos\theta(\omega) + j\sin\theta(\omega)]$$

$$X_R(e^{j\omega}) + jX_I(\omega) = |X(e^{j\omega})| (\cos\theta(\omega)) + j|X(e^{j\omega})| \sin\theta(\omega)$$

on comparing LHS & RHS we get

$$X_R(e^{j\omega}) = |X(e^{j\omega})| \cos\theta(\omega)$$

$$X_I(e^{j\omega}) = |X(e^{j\omega})| \sin\theta(\omega)$$

$$|X(e^{j\omega})|^2 = (X_R(e^{j\omega}))^2 + (X_I(e^{j\omega}))^2$$

$$|X(e^{j\omega})| = \sqrt{(X_R(e^{j\omega}))^2 + (X_I(e^{j\omega}))^2}$$

$$\tan\theta(\omega) = \frac{X_I(e^{j\omega})}{X_R(e^{j\omega})} \quad (\text{or}) \quad \theta(\omega) = \tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right)$$

$$|X(e^{-j\omega})| = \sqrt{(X_R(e^{-j\omega}))^2 + (X_I(e^{-j\omega}))^2} = \sqrt{(X_R(e^{j\omega}))^2 + (X_I(e^{j\omega}))^2}$$

$$|X(e^{-j\omega})| = |X(e^{j\omega})|$$

$\therefore |X(e^{j\omega})|$ is an even function of $\omega$.

$$\theta(-\omega) = \tan^{-1}\left(\frac{X_I(e^{-j\omega})}{X_R(e^{-j\omega})}\right) = \tan^{-1}\left(\frac{-X_I(e^{j\omega})}{X_R(e^{j\omega})}\right) = -\tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right)$$

$$\theta(-\omega) = -\theta(\omega).$$

That is

$$|H(e^{-j\omega})| = -|H(e^{j\omega})|$$

$\therefore$ $|H(e^{j\omega})|$ (or) $\theta(\omega)$ is an odd function of $\omega$.

Implication :

A sequence $x(n)$ can be expressed as

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n) = \frac{1}{2}\left(x(n) + x^*(-n)\right)$

$$x_o(n) = \frac{1}{2}\left(x(n) - x^*(-n)\right)$$

A Fourier transform $X(e^{j\omega})$ can be decomposed in to

$$X(e^{j\omega}) = \left(X_e(e^{j\omega}) + X_o(e^{j\omega})\right)$$

$$X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{-j\omega})\right)$$

$$X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right)$$

$F(x(n)) = X(e^{j\omega})$ then

$$F(x^*(n)) = X^*(e^{-j\omega})$$

and $F(x^*(-n)) = X^*(e^{j\omega})$

$$F(x_e(n)) = \frac{1}{2} F\left(x(n) + x^*(-n)\right) = \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right)$$

$$= Re\left(X(e^{j\omega})\right)$$

$$F(x_e(n)) = Re\left(X(e^{j\omega})\right)$$

Now Consider,

$$F\{x_o(n)\} = \frac{1}{2} F\{x(n) - x^*(-n)\}$$

$$= \frac{1}{2}\left(F(x(n)) - F(x^*(n))\right) = j\,Im\left(X(e^{jw})\right)$$

i.e.,

$$F(x_o(n)) = j\,Im(X(e^{jw}))$$

$$F(Re(x(n))) = F\left(\frac{1}{2}(x(n) + x^*(n))\right) = \frac{1}{2}\left(X(e^{jw}) + X^*(e^{-jw})\right)$$

$$F(Re(x(n))) = X_e(e^{jw}).$$

$$F(Re(x(n))) = X_e(e^{jw})$$

$$F(j\,Im(x(n))) = F\left(\frac{1}{2}(x(n) - x^*(n))\right)$$

$$= \frac{1}{2}\left(X(e^{jw}) - X^*(e^{-jw})\right) = X_o(e^{jw})$$

That is

$$F(j\,Im\,x(n)) = X_o(e^{jw})$$

(12) Parseval's Theorem:-

If $F(x(n)) = X(e^{jw})$

then

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} |X(e^{jw})|^2\, dw$$

Proof:-

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw\right)^*$$

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw}) \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}\right) dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw}) X(e^{jw})\, dw$$

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{jw})|^2\, dw.$$

Ex:- find the DTFT of $x(n)=\begin{cases}1 & n\le 5\\ 0 & \text{else.}\end{cases}$

$$X(\omega)=\sum_{n=0}^{5}x(n)\,e^{-j\omega n}=\sum_{n=0}^{5}e^{-j\omega n}$$

$$X(\omega)=e^{0}+e^{-j\omega}+e^{-j2\omega}+e^{-j3\omega}+e^{-j4\omega}+e^{-j5\omega}$$

(or)

$$X(\omega)=\frac{1-e^{-j\omega(6)}}{1-e^{-j\omega}}=\frac{e^{-j\omega 6/2}}{e^{-j\omega/2}}\left(\frac{e^{\frac{j\omega 6}{2}}-e^{\frac{-j\omega 6}{2}}}{e^{j\omega/2}-e^{-j\omega/2}}\right)$$

$$=e^{-j\omega\left(\frac{6}{2}-\frac{1}{2}\right)}\left(\frac{e^{\frac{j\omega 6}{2}}-e^{\frac{-j\omega 6}{2}}}{2j\sin\frac{\omega}{2}}\right)$$

$$=e^{-\frac{j\omega 5}{2}}\left(\frac{2\sin 3\omega}{2\sin\frac{\omega}{2}}\right)=e^{-\frac{j\omega 5}{2}}\left(\frac{\sin 3\omega}{\sin\frac{\omega}{2}}\right).$$

Ex:- Find the IDTFT of $X(\omega)=\frac{1}{1-e^{-j\omega}}$

$$\begin{array}{r|l}
1-e^{-j\omega} & 1+e^{-j\omega}+e^{-j2\omega}+e^{-j3\omega}+\cdots\\
\hline
1 & \\
1-e^{-j\omega} & \\
\hline
e^{-j\omega} & \\
e^{-j\omega}-e^{-j2\omega} & \\
\hline
e^{-j2\omega} & \\
e^{-j2\omega}-e^{-j3\omega} & \\
\hline
e^{-j3\omega} & \\
\vdots & \vdots
\end{array}$$

$$X(\omega)=1+e^{-j\omega}+e^{-2j\omega}+e^{-3j\omega}+\cdots$$

$$X(\omega)=\sum_{n=0}^{\infty}e^{-j\omega n}\Rightarrow x(n)=1\quad n\ge 0 \quad \text{i.e., } x(n)=u(n)$$

$$=0\quad \text{otherwise.}$$

* Find the Convolution of the Signals given below using Fourier Transform

$$x_1(n)=\left(\frac{1}{2}\right)^n u(n)\qquad x_2(n)=\left(\frac{1}{3}\right)^n u(n)$$

$$\text{DTFT}\{x_1(n)\}=X_1(e^{j\omega})=\frac{1}{1-0.5e^{-j\omega}}$$

$$X_2(e^{j\omega}) = \frac{1}{1-0.33e^{-j\omega}}$$

$$F(x_1(n) * x_2(n)) = X_1(e^{j\omega}) X_2(e^{j\omega})$$

$$= \left(\frac{1}{1-0.5e^{-j\omega}}\right)\left(\frac{1}{1-0.33e^{-j\omega}}\right)$$

$$x_1(n) * x_2(n) = F^{-1}\left\{\frac{1}{1-0.5e^{-j\omega}} \times \frac{1}{1-0.33e^{-j\omega}}\right\}$$

Let $Y(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} \times \frac{1}{1-0.33e^{-j\omega}} = \frac{e^{j\omega} \times e^{j\omega}}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}-\frac{1}{3}\right)}$

$$\frac{Y(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}-\frac{1}{3}\right)} = \frac{A}{\left(e^{j\omega}-\frac{1}{2}\right)} + \frac{B}{\left(e^{j\omega}-\frac{1}{3}\right)}$$

$$= \left(\frac{3}{e^{j\omega}-\frac{1}{2}}\right) - \left(\frac{2}{e^{j\omega}-\frac{1}{3}}\right)$$

$$Y(e^{j\omega}) = \frac{3e^{j\omega}}{e^{j\omega}-\frac{1}{2}} - \frac{2e^{j\omega}}{e^{j\omega}-\frac{1}{3}} = 3\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) - \frac{2}{1-\frac{1}{3}e^{-j\omega}}$$

$$y(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n).$$

Ex:- Find the DTFT of the signal $x(n) = \cos\omega_0 n\, u(n)$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \cos\omega_0 n\, u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \cos\omega_0 n\, e^{-j\omega n}$$

$$X(\omega) = \sum_{n=0}^{\infty}\left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) e^{-j\omega n} = \frac{1}{2}\left(\sum_{n=0}^{\infty} e^{-j(\omega-\omega_0)n} + \sum_{n=0}^{\infty} e^{-j(\omega+\omega_0)n}\right)$$

$$X(\omega) = \frac{1}{2}\left\{\frac{1}{1-e^{-j(\omega-\omega_0)}} + \frac{1}{1-e^{-j(\omega+\omega_0)}}\right\} = \frac{1}{2}\left(\frac{2-e^{-j\omega}(2\cos\omega_0)}{1-e^{-j\omega}2\omega(\omega_0)+e^{-j2\omega}}\right)$$

$$X(e^{j\omega}) = \frac{1 - e^{-j\omega}\cos\omega_0}{1 - 2e^{-j\omega}\cos\omega_0 + e^{-j2\omega}}.$$

## Properties of DTFT (Contd):-

### (13) Time Conjugation property:-

$$\text{DTFT}\{x^*(n)\} = X^*(-\omega)$$

$$\text{DTFT}\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(x(n)\, e^{j\omega n}\right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\omega)n}\right)^* = \left(X^*(e^{-j\omega})\right) \text{ (or) } X^*(-\omega).$$

### (14) Scaling Property:-

$$\text{DTFT}\{x(\alpha n)\} = X\left(\frac{\omega}{\alpha}\right)$$

$$\text{DTFT}\{x(\alpha n)\} = \sum_{n=-\infty}^{\infty} x(\alpha n)\, e^{-j\omega n} = \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\omega\frac{p}{\alpha}} \quad (n\alpha = p)$$

$$= \sum_{p=-\infty}^{\infty} x(p)\left(e^{-j\left(\frac{\omega}{\alpha}\right)p}\right) = X\left(\frac{\omega}{\alpha}\right).$$

Ex: Find the IDTFT of $X(\omega) = \dfrac{1 - e^{-6j\omega}}{1 - e^{-j\omega}}$

$\Rightarrow$

$$1 - e^{-j\omega} \;\Big)\; 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$$

$1 - e^{-6j\omega}$

$1 - e^{-6j\omega}$

$-e^{-6j\omega} + e^{-j\omega}$

$-e^{-2j\omega} + e^{-j\omega}$

$$1 - e^{-j\omega} \;\Big)\; 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega}$$

$1 - e^{-j6\omega}$

$(-)\quad (+)\quad 1 - e^{-j\omega}$

$e^{-j\omega} - e^{-j6\omega}$

$(-)e^{-j\omega} \quad (+) - e^{-j2\omega}$

$e^{-j2\omega} - e^{-j6\omega}$

$(-)e^{-j2\omega} \quad (+)e^{-j3\omega}$

$e^{-j3\omega} - e^{-j6\omega}$

$(-)e^{-j3\omega} \quad (+)e^{-j4\omega}$

$e^{-j4\omega} - e^{-j6\omega}$

$(-)e^{-j4\omega} \quad (+)e^{-j5\omega}$

$e^{-j5\omega} - e^{-j6\omega}$

$$X(\omega) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$$

$$x(n) = \{1, 1, 1, 1, 1, 1\} \quad \text{for } 0 \le n \le 5$$

$$x(n) = 0 \quad \text{otherwise.}$$

Ex:- Find the IDTFT of $X(\omega) = \frac{1}{(1 - a e^{-j\omega})^2}$, $|a| < 1$, use Convolution Property.

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}} \times \frac{1}{1 - a e^{-j\omega}} = X_1(\omega) X_2(\omega)$$

$$x_1(n) = a^n u(n)$$

$$x_2(n) = a^n u(n)$$

$$x(n) = x_1(n) \circledast x_2(n) = \sum_{K=-\infty}^{\infty} a^K u(K)\, a^{n-K} u(n-K)$$

$$x(n) = \sum_{K=0}^{\infty} a^K a^{n-K} u(n-K)$$

$$= \sum_{K=0}^{n} a^K a^n a^{-K} = \sum_{K=0}^{n} a^n = a^n \sum_{K=0}^{n} 1$$

$$\boxed{x(n) = a^n (n+1) u(n)}$$

Ex:- Find the DTFT of $x(n) = -a^n u(-n-1)$ if 'a' is real.

$$x(n) = -a^n u(-n-1)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) e^{-j\omega n} = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) e^{-j\omega n}$$

$$= -\sum_{n=-\infty}^{-1} a^n e^{-j\omega n} = -\sum_{n=1}^{\infty} a^{-n} e^{j\omega n} = -\left( \sum_{n=0}^{\infty} (a^{-1} e^{j\omega})^n - 1 \right)$$

$$= -\left( \frac{1}{1 - a^{-1} e^{j\omega}} - 1 \right) = 1 - \frac{1}{1 - a^{-1} e^{j\omega}} = 1 - \frac{a}{a - e^{j\omega}}$$

$$X(e^{j\omega}) = \frac{a - e^{j\omega} - a}{a - e^{j\omega}} = \frac{-e^{j\omega}}{a - e^{j\omega}} = \frac{-e^{j\omega} \times 1}{-e^{j\omega}(1 - a e^{-j\omega})}$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

Ex:- using the Appropriate property, find the DTFT of the signal

$x(n) = \left(\frac{1}{2}\right)^n u(n-2)$.



→

Consider $\left(\frac{1}{2}\right)^n u(n) = x_1(n)$

$x_1(n-2) = \left(\frac{1}{2}\right)^{n-2} u(n-2)$

$x(n) = \left(\frac{1}{2}\right)^2 x_1(n-2) = \left(\frac{1}{2}\right)^n u(n-2)$

$DTFT\{x_1(n)\} = DTFT\left(\left(\frac{1}{2}\right)^n u(n)\right) = \frac{1}{1-0.5e^{-j\omega}}$

$DTFT\{x_1(n-2)\} = \frac{e^{-j\omega(2)}}{1-0.5e^{-j\omega}}$

$DTFT\left\{\left(\frac{1}{2}\right)^2 x_1(n-2)\right\} = \left(\frac{1}{2}\right)^2 \left(\frac{e^{-j2\omega}}{1-0.5e^{-j\omega}}\right) = \frac{\frac{1}{4}\left(e^{-j2\omega}\right)}{1-0.5e^{-j\omega}}$

$DTFT\{x(n)\} = DTFT\left\{\left(\frac{1}{2}\right)^n u(n-2)\right\} = \frac{0.25e^{-j2\omega}}{1-0.5e^{-j\omega}}$

Ex:- Find the DTFT representation of the periodic Signal.

$x(n) = 2\cos\left(\frac{3\pi n}{8} + \frac{\pi}{3}\right) + 4\sin\left(\frac{\pi n}{2}\right)$

→

$2\cos\left(\frac{3\pi n}{8} + \frac{\pi}{3}\right) = 2\left(\frac{e^{j\left(\frac{3\pi n}{8}+\frac{\pi}{3}\right)} + e^{-j\left(\frac{3\pi n}{8}+\frac{\pi}{3}\right)}}{2}\right)$

$2\cos\left(\frac{3\pi n}{8} + \frac{\pi}{3}\right) = e^{j\frac{3\pi n}{8}} e^{j\pi/3} + e^{-j\frac{3\pi n}{8}} e^{-j\pi/3}$

$DTFT\left\{2\cos\left(\frac{3\pi n}{8} + \frac{\pi}{3}\right)\right\} = DTFT\left(e^{j\frac{3\pi n}{8}} e^{j\pi/3} + e^{-j\frac{3\pi n}{8}} e^{-j\pi/3}\right)$

$= e^{j\pi/3}\left(\sum_{n=-\infty}^{\infty} e^{j\frac{3\pi n}{8}} e^{-j\omega n}\right) + e^{-j\pi/3}\left(\sum_{n=-\infty}^{\infty} e^{-j\frac{3\pi n}{8}} e^{-j\omega n}\right)$

$$DTFT\left\{2\cos\left(\frac{3\pi n}{8}+\frac{\pi}{3}\right)\right\} = e^{j\pi/3}\sum_{n=-\infty}^{\infty} e^{-j\left(\frac{3\pi}{8}-\omega\right)n} + e^{-j\pi/3}\sum_{n=-\infty}^{\infty} e^{-j\left(\omega+\frac{3\pi}{8}\right)n}$$

$$\Rightarrow \quad 2\cos\left(\frac{3\pi n}{8}+\frac{\pi}{3}\right) \xleftrightarrow{DTFT} e^{\frac{j\pi}{3}}\left(\sum_{n=-\infty}^{\infty} e^{-j\left(\omega-\frac{3\pi}{8}\right)n}\right) + e^{\frac{-j\pi}{3}}\left(\sum_{n=-\infty}^{\infty} e^{-j\left(\omega+\frac{3\pi}{8}\right)n}\right)$$

$$\text{Ily} \quad 4\sin\left(\frac{\pi n}{2}\right) \xleftrightarrow{DTFT} \frac{2}{3}\sum_{n=-\infty}^{\infty} e^{-j(\omega-\pi/2)n} - \frac{2}{j}\sum_{n=-\infty}^{\infty} e^{-j\left(\omega+\frac{\pi}{2}\right)n}$$

Note:-

① $$\sum_{n=-\infty}^{\infty} e^{-j\left(\omega-\frac{3\pi}{8}\right)n} = 2\pi\delta\left(\omega-\frac{3\pi}{8}\right) = \begin{cases} 2\pi, & \omega=\frac{3\pi}{8} \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = 2\pi\left\{e^{j\pi/3}\delta\left(\omega-\frac{3\pi}{8}\right) + e^{-j\pi/3}\delta\left(\left(\omega+\frac{3\pi}{8}\right)+j2\left(\delta\left(\omega+\frac{\pi}{2}\right)-\delta\left(\omega-\frac{\pi}{2}\right)\right)\right\}$$

Note:-2 $$\sum_{n=-\infty}^{\infty} e^{-j(\omega-\pi/2)n} = 2\pi\,\delta\left(\omega-\frac{\pi}{2}\right) = \begin{cases} 2\pi & \omega=\frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$DTFT\left\{e^{j\omega_0 n}\right\} = 2\pi\delta(\omega-\omega_0)$$

Ex:- Find the Fourier Transform for

(a) $x(n) = \left(\frac{1}{3}\right)^{|n|} u(-n-2)$

(b) $x(n) = u(n-2) - u(n-6)$

Sol:- (a) $x(n) = \left(\frac{1}{3}\right)^{|n|} u(-n-2)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{|n|} u(-n-2)\,e^{-j\omega n}$$ put $n=-m$, $-n=m$; $n=-\infty \Rightarrow m=\infty$, $n=\infty \Rightarrow m=-\infty$

$$= \sum_{m=\infty}^{-\infty}\left(\frac{1}{3}\right)^{|m|} u(m-2)\,e^{j\omega m} = \sum_{m=2}^{\infty}\left(\frac{1}{3}\right)^{m} e^{j\omega m}$$

$$= \sum_{m=0}^{\infty}\left(\frac{1}{3}e^{j\omega}\right)^m - 1 - \frac{1}{3}e^{j\omega} = \left(\frac{1}{1-\frac{1}{3}e^{j\omega}} - 1 - \frac{1}{3}e^{j\omega}\right)$$

$$X(e^{j\omega}) = \frac{1-\left(1+\frac{1}{3}e^{j\omega}\right)\left(1-\frac{1}{3}e^{j\omega}\right)}{1-\frac{1}{3}e^{j\omega}} = \frac{\frac{1}{9}e^{j2\omega}}{1-\frac{1}{3}e^{j\omega}}.$$

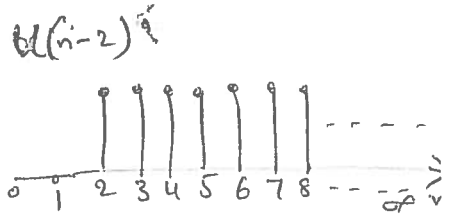
$$X(e^{j\omega}) = \frac{1}{9}\left(\frac{e^{j2\omega}}{1-\frac{1}{3}e^{j\omega}}\right).$$

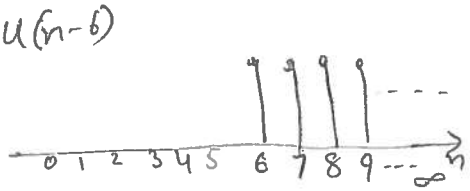
(b) $x(n) = u(n-2) - u(n-6)$

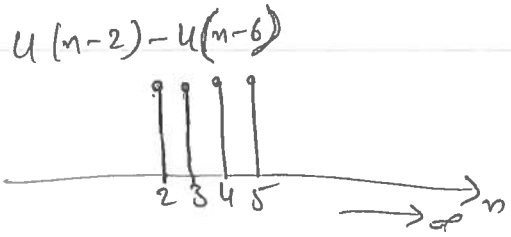
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(u(n-2) - u(n-6)\right) e^{-j\omega n}$$

$$= \sum_{n=2}^{5} 1 \times e^{-j\omega n}$$




Ex:- Verify parseval's Relation for the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n).$$

## Frequency Response of Discrete Time Systems :-

The o/p of an LTI discrete time system $y(n)$ can be obtained by convolution sum

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

where $x(n) = e^{j\omega n}$

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)\, e^{j\omega(n-K)}$$

$$= e^{j\omega n} \sum_{K=-\infty}^{\infty} h(K)\, e^{-j\omega K}$$

$$y(n) = \underbrace{e^{j\omega n}}_{\text{I/P}} \; \underbrace{H(e^{j\omega})}_{\text{Frequency Response.}}$$

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j \angle H(e^{j\omega})}$$

Thus if we force the system with a complex exponential $e^{j\omega n}$ then the o/p is of the form $H(e^{j\omega})\, e^{j\omega n}$ ∴ The o/p of the system is identical to the i/p but modified in amplitude & phase by $H(e^{j\omega})$. The quantity $H(e^{j\omega})$ is the frequency response of the system.

$$x(n) = e^{j\omega n} \longrightarrow \boxed{\text{LTI DT system}} \longrightarrow y(n) = H(e^{j\omega})\, e^{j\omega n}.$$

Ex:- Find the frequency response of a I$^{st}$ order system described by $y(n) = a\,y(n-1) + x(n)$

Plot the magnitude & phase response for $a = 0.5$.

⇒

$$y(n) = a\,y(n-1) + x(n) \Rightarrow Y(e^{j\omega}) = a e^{-j\omega} Y(e^{j\omega}) + X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1 - a e^{-j\omega}\right) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a e^{-j\omega}}$$

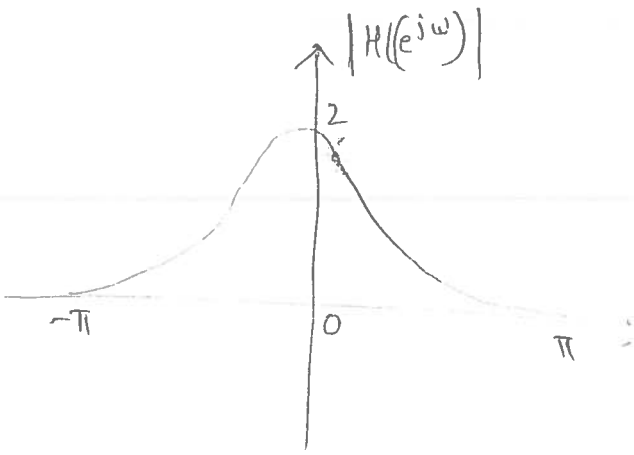
$$H(e^{j\omega}) = \frac{1}{1 - a\cos\omega + j a \sin\omega}$$

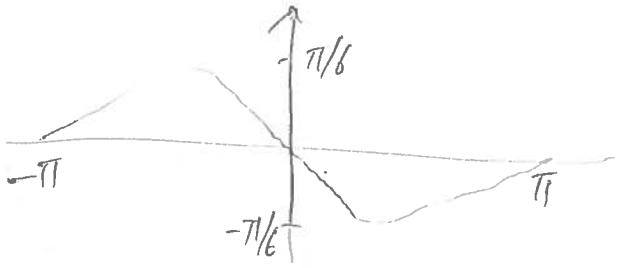
$$|H(e^{j\omega})| = \frac{1}{\left((1-a\cos\omega)^2 + a^2\sin^2\omega\right)^{\frac{1}{2}}}$$

$$\angle H(e^{j\omega}) = -\tan^{-1}\frac{a\sin\omega}{1-a\cos\omega}.$$

$a = 0.5$

| $\omega$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\lvert H(e^{j\omega})\rvert$ | 2 | 1.613 | 1.357 | 1.1547 | 0.8 | 0.756 | 0.7148 | 0.687 | 0.666 |
| $\angle H(e^{j\omega})$ | 0 | $-0.13\pi$ | $-0.16\pi$ | $-0.166\pi$ | $-0.147\pi$ | $-0.106\pi$ | $-0.081\pi$ | $-0.055\pi$ | 0 |




Transfer Function :-

The o/p $y(n)$ of an Discrete time LTI System is given by

$$y(n) = e^{j\omega n} H(e^{j\omega}) = x(n)\, H(e^{j\omega}) \quad (\because x(n) = e^{j\omega n})$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega\; H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

we know that $y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n}\, d\omega$

$$\therefore\; Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}.$$

Ex:- Consider a Causal and stable LTI System whose i/p & o/p are related through the 2nd order Equation $y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$. Determine its Freq.

2nd Dec 2

(a) Determine the Frequency Response $H(e^{j\omega})$ for the system (15)

(b) Determine the impulse response $h(n)$ for the system

a)

$$y(n) = \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

$$Y(e^{j\omega}) - \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}$$

$$h(n) = F^{-1}\left(H(e^{j\omega})\right)$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}} = \frac{e^{j2\omega}}{e^{j2\omega} - \frac{1}{6}e^{j\omega} - \frac{1}{6}}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{e^{j\omega}}{\left(e^{j\omega} - \frac{1}{2}\right)\left(e^{j\omega} + \frac{1}{3}\right)} = \frac{A}{\left(e^{j\omega} - \frac{1}{2}\right)} + \frac{B}{\left(e^{j\omega} - \frac{1}{3}\right)}$$

$$= \frac{3}{5}\left(\frac{1}{e^{j\omega} - \frac{1}{2}}\right) + \frac{2}{5}\left(\frac{1}{e^{j\omega} + \frac{1}{3}}\right)$$

$$H(e^{j\omega}) = \frac{3}{5}\left(\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right) + \frac{2}{5}\left(\frac{1}{1 + \frac{1}{3}e^{-j\omega}}\right)$$

$$h(n) = \frac{3}{5}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{5}\left(\frac{-1}{3}\right)^n u(n)$$

# Z - Transforms

Z-Transform is a powerful mathematical tool to convert the Difference equations into algebraic equations. Z-transform has the advantage that it is a simple and systematic method & the complete solution can be obtained in one step & the initial conditions can be introduced in the begining of the process itself. Z-transform is a very useful tool in the analysis of linear shift invariant systems (LSI). Like the L.T., the Z.T may be one-sided (or) 2 sided. unilateral (or) Bilateral.

The Bilateral Z-Transform of a discrete-time signal (or) a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow (1)$$

Where 'z' is a complex variable.

The one-sided (or) unilateral Z-Transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \longrightarrow (2)$$

## Region of Convergence (ROC):-

For any given sequence the Z-Transform may (or) may not converge. The set of values of Z (or) equivalently the set of points in Z-plane, for which $X(z)$ converges is called the Region of Convergence of $X(z)$. (ROC). If there is no value of z (i.e., no point in the Zplane) for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

## Advantages of Z-Transform:-

1. The Z-Transform converts the difference equations of a DT-System into linear Algebraic equations so that the analysis become easy & simple
2. Convolution in time domain is converted into multiplication in Z-domain
3. Z-Transform Exists for most of the signals for which D.T.F.T doesnot Exist.

## Limitation:-

1. Frequency domain response cannot be achieved & cant be plotted.

## Relationship between DTFT & Z Transform :-

The DTFT of a sequence $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge i.e., $x(n)$ must be absolutely summable.

$$Z(x(n)) = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Where $z$ is a complex variable given by

$$z = re^{j\omega}$$

where $r$ is the radius of a circle.

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x(n)\, r^{-n}\right) e^{-j\omega n}$$

For the existence of Z-Transform, the above summation should converge i.e., $x(n)r^{-n}$ must be absolutely summable i.e., $\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$.

The above equation represents the DTFT of $x(n)r^{-n}$. Hence we can say Z.T. of $x(n)$ is same as the DTFT of $x(n)r^{-n}$.

For the DTFT to exist the discrete sequence $x(n)$ must be absolutely summable i.e., $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$.

So for many sequences the DTFT may not exist but Z.T may exist. When $r=1$ DTFT is same as Z-Transform i.e., DTFT is nothing but the Z.T evaluated along the unit circle centred at the origin of the Z-plane.

## Z-Transforms of Some Common Sequences :-

### ① Unit Sample sequence (Unit Impulse sequence) :-

$x(n) = \delta(n)$

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$

$$X(z) = Z(x(n)) = Z(\delta(n)) = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = \delta(0)\, z^{0} = 1 \cdot (\text{for all } z).$$

R.O.C is Entire Z-plane.

$$\delta(n) \xleftrightarrow{ZT} 1 \quad \text{for all } z$$

(2) Unit step sequence :- $x(n) = u(n)$

$u(n) = 1 \quad n \geq 0$
$\quad\;\; = 0 \quad n < 0$

$$X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1-z^{-1}}$$

$$X(z) = \frac{z}{z-1} \qquad u(n) \xleftrightarrow{ZT} \frac{z}{z-1}$$

Above series converges if $|z^{-1}| < 1$ i.e, ROC is $|z| > 1$. So the ROC is exterior of the unit circle in the z-plane.

(3) Unit ramp sequence ($r(n)$) :-

$r(n) = n \quad n \geq 0$
$\quad\;\; = 0 \quad n < 0$

$$X(z) = Z(x(n)) = Z(r(n)) = \sum_{n=0}^{\infty} n z^{-n} = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$$

$$X(z) = z^{-1}\left(1 + 2z^{-1} + 3z^{-2} + \cdots\right) = z^{-1}(1-z^{-1})^{-2} = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$$

$$r(n) \longleftrightarrow \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}, \quad \text{ROC is } |z| > 1.$$

(4) Exponential sequence $x(n) = e^{-j\omega n} u(n)$

$x(n) = e^{-j\omega n} u(n) = e^{-j\omega n}$ for $n \geq 0$
$\qquad\qquad\;\; = 0$ for $n < 0$

$$X(z) = Z\left(e^{-j\omega n} u(n)\right) = \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} = \sum_{n=0}^{\infty} \left(e^{-j\omega} z^{-1}\right)^n = \frac{1}{1 - z^{-1} e^{-j\omega}} = \frac{z}{z - e^{-j\omega}}$$

$$e^{-j\omega n} u(n) \longleftrightarrow \frac{z}{z - e^{-j\omega}} \rightarrow \text{ROC is } |z| > 1$$

W.K.T $e^{j\omega n} u(n) \xleftrightarrow{ZT} \frac{1}{1 - z^{-1} e^{j\omega}} = \frac{z}{z - e^{j\omega}}$ (ROC is $|z| > 1$)

(5) Sinusoidal sequence $x(n) = \sin\omega n\, u(n)$

$$X(z) = Z\left(\sin\omega n\, u(n)\right) = \sum_{n=0}^{\infty} \sin\omega n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$X(z) = \frac{1}{2j} \sum_{n=0}^{\infty} \left(e^{j\omega n} z^{-n} - e^{-j\omega n} z^{-n}\right) = \sum_{n=0}^{\infty} \frac{\left(z^{-1} e^{j\omega}\right)^n - \left(z^{-1} e^{-j\omega}\right)^n}{2j}$$

$$X(z) = \frac{1}{2j}\left(\sum_{n=0}^{\infty} \left(z^{-1} e^{j\omega}\right)^n - \sum_{n=0}^{\infty} \left(z^{-1} e^{-j\omega}\right)^n\right)$$

The above series converges for $|z^{-1}| < 1$ i.e, $|z| > 1$.

$$X(z) = \frac{1}{2j}\left(\frac{1}{1-z^{-1}e^{j\omega}} - \frac{1}{1-z^{-1}e^{-j\omega}}\right) = \frac{1}{2j}\left(\frac{z}{z-e^{j\omega}} - \frac{z}{z-e^{-j\omega}}\right)$$

$$= \frac{1}{2j}\left(\frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right) = \frac{1}{2j}\left(\frac{z(e^{j\omega} - e^{-j\omega})}{z^2 - (e^{j\omega}+e^{-j\omega})z+1}\right)$$

$$X(z) = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$$

(6) Cosinusoidal Sequence $x(n) = \cos\omega n\, u(n)$ :-

$$Z(\cos\omega n\, u(n)) = Z\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2} u(n)\right) = \frac{1}{2}\left(Z(e^{j\omega n}u(n)) + Z(e^{-j\omega n}u(n))\right)$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}}\right) = \frac{1}{2}\left(\frac{z(z-e^{-j\omega}) + z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right); \quad |z|>1$$

$$= \frac{1}{2}\left\{\frac{z(2z - (e^{j\omega}+e^{-j\omega}))}{z^2 - z(e^{j\omega}+e^{-j\omega}) + 1}\right\} = \frac{z(z-\cos\omega)}{z^2 - 2z\cos\omega + 1} \quad \text{ROC is } |z|>1$$

$$\cos\omega n\, u(n) \xleftrightarrow{ZT} \frac{1 - z^{-1}\cos\omega}{1 - 2z^{-1}\cos\omega + z^{-2}} = \frac{z(z-\cos\omega)}{z^2 - 2z\cos\omega + 1}$$

Properties of ROC:-

1. The ROC is a ring (or) disk in the z-plane centred at origin
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z|>\alpha$, ie, it is the exterior of a circle of radius $\alpha$.
   If $x(n)$ is a finite duration causal sequence (Right sided sequence), the ROC is entire z-plane except at $z=0$
4. If $x(n)$ is an $\infty$ duration anticausal sequence, the ROC is $|z|<\beta$, ie, it is the interior of a circle of radius $\beta$.
   If $x(n)$ is a finite duration anticausal sequence, (left sided sequence) the ROC is entire z-plane except at $z=\infty$.
5. If $x(n)$ is a finite duration two sided sequence, the ROC is entire z-plane except at $z=0$ & $z=\infty$
6. If $x(n)$ is an $\infty$ duration, 2-sided sequence, the ROC consists of a ring in the z-plane (ROC: $\alpha<|z|<\beta$) bounded on the interior & exterior by a pole, not containing any poles
7. The ROC of an LTI stable system contains the unit circle.

(8) The ROC must be a connected region. If $X(z)$ is rational then its ROC is bounded by poles (or) extends upto infinity.

(9) $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

## Properties of Z-Transform:-

(1) **Linearity:-** It states that Z-Transform of a weighted sum of 2 signals is equal to the weighted sum of individual Z-Transforms.

$$x_1(n) \xleftrightarrow{ZT} X_1(z) \quad ROC = R_1$$

$$x_2(n) \xleftrightarrow{ZT} X_2(z) \quad ROC = R_2$$

Then $a x_1(n) + b x_2(n) \xleftrightarrow{ZT} a X_1(z) + b X_2(z)$ with $ROC = R_1 \cap R_2$.

**Proof:-**

$$Z\left(a x_1(n) + b x_2(n)\right) = \sum_{n=-\infty}^{\infty} \left(a x_1(n) + b x_2(n)\right) z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$Z\left(a x_1(n) + b x_2(n)\right) = a X_1(z) + b X_2(z) \quad ROC \text{ is } R_1 \cap R_2.$$

(2) **Time shifting property:-**

$$x(n) \xleftrightarrow{ZT} X(z) \text{ with zero initial conditions } ROC = R$$

$$x(n-m) \xleftrightarrow{ZT} z^{-m} X(z) \text{ with } ROC = R \text{ except for the possible addition (or) deletion of the origin (or) infinity.}$$

**Proof:-**

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

$$n-m = p \qquad n = m+p$$

$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(m+p)} = \sum_{p=-\infty}^{\infty} x(p) z^{-m} z^{-p} = z^{-m} \sum_{p=-\infty}^{\infty} x(p) z^{-p}$$

$$Z\{x(n-m)\} = z^{-m} X(z).$$

By $Z\{x(n+m)\} = z^{m} X(z)$

Similarly if the initial conditions are not neglected (unilateral Z.T)

$$Z\left(x(n-m)\right) = z^{-m} X(z) + z^{-m} \sum_{K=1}^{m} x(-K) z^{K}$$

$$Z\left(x(n+m)\right) = z^{m} X(z) - z^{m} \sum_{K=0}^{m-1} x(K) z^{-K}.$$

(3) Multiplication by an Exponential sequence property:-

$$x(n) \xleftrightarrow{ZT} X(z) \qquad ROC = R$$

$$a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right) \qquad ROC = |a|R$$

a is a Complex number.

Proof:- $Z(a^n x(n)) = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$.

By $e^{j\omega n} x(n) \Longleftrightarrow X\left(\frac{z}{e^{j\omega}}\right) = X\left(z e^{-j\omega}\right)$

$e^{-j\omega n} x(n) \Longleftrightarrow X\left(\frac{z}{e^{-j\omega}}\right) = X\left(z e^{j\omega}\right)$.

(4) Time Reversal Property:-

$$x(n) \xleftrightarrow{ZT} X(z) \qquad ROC = R$$

$$x(-n) \xleftrightarrow{ZT} X(z^{-1}) = X\left(\frac{1}{z}\right) \qquad ROC = \frac{1}{R}.$$

Proof:-

$$Z(x(n)) = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z(x(-n)) = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$ (put $-n = P$)

$$Z(x(-n)) = \sum_{P=-\infty}^{\infty} x(P) z^{P} = \sum_{P=-\infty}^{\infty} x(P) (z^{-1})^{-P} = X(z^{-1}) = X\left(\frac{1}{z}\right)$$

$$x(n) \xleftrightarrow{ZT} X\left(\frac{1}{z}\right).$$

(5) Time Expansion Property:-

$$x\left(\frac{n}{K}\right) \xleftrightarrow{ZT} X(z^K) \qquad ROC = R^{1/K}$$

Proof:-

$$Z\left(x\left(\frac{n}{K}\right)\right) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{K}\right) z^{-n}$$ (put $\frac{n}{K} = P$)

$$Z\left(x\left(\frac{n}{K}\right)\right) = \sum_{P=-\infty}^{\infty} x(P) z^{-PK} = \sum_{P=-\infty}^{\infty} x(P) (z^K)^{-P} = X(z^K)$$

(6) Multiplication by 'n' (or) differentiation in Z-domain property:-

$$n x(n) \xleftrightarrow{ZT} -z \frac{d}{dz} X(z) \qquad \text{with } ROC = R$$

Proof:- $X(z) = Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) (-n) z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} (-n) x(n) z^{-n}$$

$$\frac{d}{dz} X(z) = -z^{-1} \sum_{n=-\infty}^{\infty} n x(n) z^{-n} \qquad -z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$\boxed{Z(n x(n)) = -z \frac{d}{dz} X(z)} \qquad \boxed{\text{By } Z(n^K x(n)) = (-1)^K z^K \frac{d^K X(z)}{dz^K}}$$

$$n\,x(n) \xleftrightarrow{Z.T} -z\frac{d}{dz}X(z)$$

By $Z\{n^k x(n)\} = (-1)^k z^k \frac{d^k X(z)}{dz^k}$

(7) Conjugation property:-

$$x(n) \xleftrightarrow{ZT} X(z)$$

$$x^*(n) \longleftrightarrow X^*(z^*) \quad ROC = R$$

Proof:-

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} = \left(\sum x(n) (z^*)^{-n}\right)^* = (X(z^*))^*$$

$$Z\{x^*(n)\} = X^*(z^*)$$

(8) Convolution property:-

$$x_1(n) \longleftrightarrow X_1(z) \quad ROC = R_1$$

$$x_2(n) \longleftrightarrow X_2(z) \quad ROC = R_2$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z) X_2(z) \qquad ROC = R_1 \cap R_2$$

Proof:-

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$x(n) = x_1(n) * x_2(n)$$

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(x_1(n) * x_2(n)\right) z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-n}$$

$n-k = p$, $\quad n = k+p$

$$Z\left(x_1(n) * x_2(n)\right) = \sum_{p=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(p)\, z^{-(k+p)} = \sum_{p=-\infty}^{\infty} x_2(p) z^{-p} \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}$$

$$= X_1(z) X_2(z).$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z) X_2(z) \quad ROC \to R_1 \cap R_2.$$

(9) Multiplication property:-

$$x_1(n) \longleftrightarrow X_1(z)$$

$$x_2(n) \longleftrightarrow X_2(z)$$

$$Z\left(x_1(n) x_2(n)\right) = \frac{1}{2\pi j} \oint_C X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv$$

Proof:- $x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1} dz$

$$Z\left(x_1(n)\, x_2(n)\right) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi j} \oint X_1(v)\, v^{n-1} dv \right) x_2(n)\, z^{-n}$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left( \sum_{n=-\infty}^{\infty} x_2(n) \left(v^{-1} z\right)^{-n} \right) v^{-1} dv$$

$$Z\left(x_1(n)\, x_2(n)\right) = \frac{1}{2\pi j} \int X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1} dv$$

Accumulation

(10) ~~Correlation~~ property:-

$$x(n) \overset{ZT}{\longleftrightarrow} X(z)$$

$$\sum_{K=-\infty}^{n} x(K) \overset{ZT}{\longleftrightarrow} \frac{1}{1-z^{-1}} X(z)$$

Proof:-

$$Z\left\{ \sum_{K=-\infty}^{n} x(K) \right\} = \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{n} x(K)\, z^{-n}$$

$n-K = r \qquad n = K+r \qquad K = n-r$

$$Z\left\{ \sum_{K=-\infty}^{n} x(K) \right\} = \sum_{K=-\infty}^{\infty} \sum_{r=\infty} x(K) z^{-K} z^{-r}$$

$$= \sum_{r=0}^{\infty} z^{-r} \sum_{K=-\infty}^{\infty} x(K)\, z^{-K}$$

$$= \left( \sum_{r=0}^{\infty} z^{-r} \right) X(z) = \frac{1}{1-z^{-1}} X(z)$$

(11) Parseval's Theorem:-

$x_1(n) \longleftrightarrow X_1(z)$

$x_2(n) \longleftrightarrow X_2(z)$

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\left((z^*)^{-1}\right) z^{-1} dz \quad \text{for Complex } x_1(n) \& x_2(n)$$

Proof:-

$$LHS = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint \left( X_1(z)\, z^{n-1} dz \right) x_2^*(n)$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left( \sum_{n=-\infty}^{\infty} x_2^*(n) \left(z^{-1}\right)^{-n} \right) z^{-1} dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ \sum x_2(n) \left((z^*)^{-1}\right)^{-n} \right\}^* z^{-1} dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left( X_2(z^*)^{-1} \right)^* z^{-1} dz$$

$$= \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\left((z^*)^{-1}\right) z^{-1} dz = RHS$$

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*\left((z^*)^{-1}\right) z^{-1}\, dz$$

(13) Initial Value Theorem:-

$x(n) \xleftrightarrow{ZT} X(z)$ For a Causal signal $x(n)$

$$\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)$$

Proof:- we know that for a Causal signal

$$Z(x(n)) = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots\cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots\cdots$$

Taking limit $z\to\infty$ on both sides

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \left( x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots \right) = x(0) + 0 + 0 \cdots = x(0)$$

$$\boxed{\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)}$$

(14) Final Value Theorem:-

For a Causal signal $x(n)$

$$x(n) \longleftrightarrow X(z)$$

and if $X(z)$ has no poles outside the unit circle, and it has no double (or) higher order poles on the unit circle centered at the origin of the z-plane, then

$$\lim_{n\to\infty} x(n) = x(\infty) = \lim_{z\to 1} (z-1)X(z)$$

Proof:-

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z(x(n+1)) = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1) z^{-n}$$

$Z\{x(n+1)\} - Z\{x(n)\} = Z\{\quad\} \quad zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)z^{-n} - \sum_{n=0}^{\infty} x(n)z^{-n}$

$$(z-1)X(z) - zx(0) = \sum_{n=0}^{\infty}\left(x(n+1) - x(n)\right)z^{-n}$$

i.e., $$(z-1)X(z) - zx(0) = \left(x(1)-x(0)\right)z^{-0} + \left(x(2)-x(1)\right)z^{-1} + \left(x(3)-x(2)\right)z^{-2} + \cdots$$

Take $\lim_{z\to 1}$ on both sides, we have

$$\lim_{z\to 1}(z-1)X(z) - x(0) = \left(x(1)-x(0)+x(2)-x(1)+x(3)-x(2)+\cdots+x(\infty)-x(\infty-1)\right)$$

$$= x(\infty) - x(0)$$

$$x(\infty) = \lim_{z\to 1}(z-1)X(z)$$

(or)

$$\boxed{x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z)}$$

$$\boxed{x(\infty) = \lim_{z\to 1}(z-1)X(z)}$$

This Theorem enables us to find the steady state value of $x(n)$ i.e. $x(\infty)$ without taking the inverse Z-Transform of $X(z)$.

Ex:- Using the properties of Z-Transform, find the Z-Transform of the following signals.

(a) $x(n) = u(-n)$ (b) $x(n) = u(-n+1)$ (c) $x(n) = u(-n-2)$ (d) $x(n) = 2^n u(n-2)$

(a) $x(n) = u(-n)$

$$Z\{u(n)\} = \frac{z}{z-1}; \quad \text{ROC } |z|>1$$

use Time Reversal property

$$Z\{u(-n)\} = \left.\frac{z}{z-1}\right|_{z=\frac{1}{z}} = \frac{1/z}{\frac{1}{z}-1} = \frac{1}{1-z} = \frac{-1}{z-1} \quad \text{ROC } |z|<1$$

(b) $x(n) = u(-n+1) = u(-(n-1))$

$$Z\{u(-n+1)\} = Z\{u(-(n-1))\} = z^{-1}\left(Z\{u(-n)\}\right) = z^{-1}\left(\frac{1}{1-z}\right) = \frac{-1}{z(z-1)}.$$

(c) $x(n) = u(-n-2) = u(-(n+2))$

$$Z\{x(n)\} = z^2 Z\{u(-n)\} = \frac{z^2}{1-z} = \frac{-z^2}{z-1}$$

(d) $x(n) = 2^n u(n-2)$

$$Z\{u(n-2)\} = z^{-2}Z\{u(n)\} = z^{-2}\left(\frac{z}{z-1}\right) = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

$$Z\left\{2^n u(n-2)\right\} = Z\left\{u(n-2)\right\}_{z=(z/2)} = \frac{1}{(z/2)(z/2-1)} = \frac{4}{z(z-2)}$$

Ex: Prove that for Causal Sequences, the ROC is the Exterior of a Circle of Radius '$r$'. (6.21)

→ Causal sequences are the sequences defined for only positive integer values of $n$. & do not exist for negative times i.e.,

$$x(n) = 0 \quad \text{for } n < 0$$

Consider a Causal Sequence

$$x(n) = r^n u(n).$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} r^n u(n) z^{-n} = \sum_{n=0}^{\infty} r^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (r z^{-1})^n$$

The above Summation Converges for

$|r z^{-1}| < 1$, i.e., for $|z| > r$

$$\left|\frac{r}{z}\right| < 1$$
$$|r| < |z|$$
$$|z| > r$$

Hence for the Causal sequences, the ROC is the Exterior of a Circle of radius $r$.

Ex: Find the Z-Transform of

(a) $n\delta(n)$ (b) $\delta(n-4)$ (c) $\delta(n+3)$ (d) $n\delta(n-2)$ (e) $y(n) = x(n-1)u(n)$

(f) $y(n) = x(n+1)\delta u(n)$.

→ (a) $x(n) = n\delta(n)$ $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} n\delta(n) z^{-n} = 0 \times z^0 = 0$.

Since $\delta(n)$ exists only at $n = 0$

(b) $x(n) = \delta(n-4)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n-4) z^{-n} = \delta(4-4) z^{-4} = z^{-4}\delta(0) = z^{-4}.$$

ROC is entire plane except at $z = 0$

(c) $x(n) = \delta(n+3)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n+3) z^{-n} = \delta(-3+3) z^{-(-3)} = \delta(0) z^3 = z^3.$$

ROC is entire plane except at $z = \infty$

(d) $x(n) = n\delta(n-2)$

$$X(z) = \sum_{n=-\infty}^{\infty} n\delta(n-2) z^{-n} = 2\delta(2-2) z^{-2} = 2z^{-2}$$

ROC :- Except at $z = 0$

(e) $y(n) = x(n-1)\,u(n)$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-1)\,u(n)\,z^{-n} = \sum_{n=0}^{\infty} x(n-1)\,z^{-n}$$

$n-1 = K \qquad n = K+1$

$$Y(z) = \sum_{K=-1}^{\infty} x(K)\,z^{-(K+1)} = \sum_{K=-1}^{\infty} x(K)\,z^{-K}z^{-1}$$

$$Y(z) = z^{-1}\sum_{K=-1}^{\infty} x(K)\,z^{-K} \Rightarrow Y(z) = z^{-1}\left(x(-1)\,z^{1} + \sum_{K=0}^{\infty} x(K)\,z^{-K}\right)$$

$$Y(z) = z^{-1}\left(x(-1)\,z + X(z)\right) = z^{-1}X(z) + x(-1).$$

Ily $Z\left(x(n-2)u(n)\right) = z^{-2}X(z) + z^{-1}x(-1) + x(-2)$.

(f) $y(n) = x(n+1)\,u(n)$

$$Y(z) = \sum_{n=0}^{\infty} x(n+1)\,z^{-n} \Rightarrow \quad n+1 = K$$

$$n = K-1$$

$$Y(z) = \sum_{K=1}^{\infty} x(K)\,z^{-(K-1)} = \sum_{K=1}^{\infty} x(K)\,z^{-K}z^{1}$$

$$= z\,x(0) + \sum_{K=1}^{\infty} x(K)\,z^{-K}z^{1} \quad - z\,x(0)$$

$$= z\left(\sum_{K=1}^{\infty} x(K)\,z^{-K} + x(0)\right) - z\,x(0)$$

$$Y(z) = z\left(\sum_{K=0}^{\infty} x(K)\,z^{-K}\right) - z\,x(0) = z\,X(z) - z\,x(0)$$

$$Z\left(x(n+1)\,u(n)\right) = z\,X(z) - z\,x(0).$$

Ily $Z\left(x(n+2)u(n)\right) = z^{2}X(z) - z^{2}x(0) - z\,x(1)$.

Ex:- Prove that the sequences

(a) $x(n) = a^{n}u(n)$ (b) $x(n) = -a^{n}u(-n-1)$ have the same $X(z)$ & differ only in ROC. Also plot their ROCs

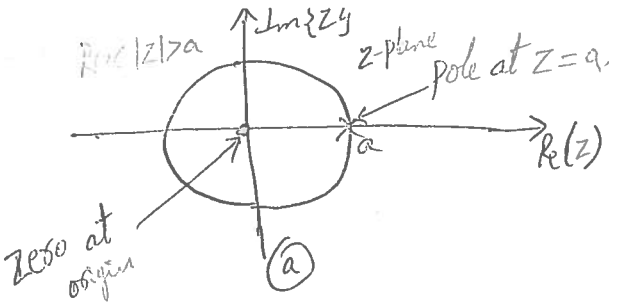
⇝ (a) $x(n) = a^{n}u(n)$ is a causal & duration sequence i.e.,

$$x(n) = \begin{cases} a^{n} & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad \text{because } u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

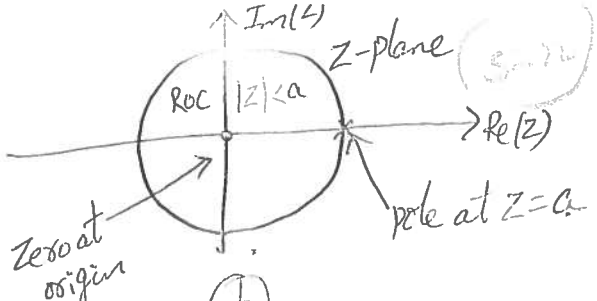
$$Z\left(x(n)\right) = Z\left(a^{n}u(n)\right) = \sum_{n=0}^{\infty} a^{n}u(n)\,z^{-n} = \sum_{n=0}^{\infty} a^{n}z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^{n}$$

$$X(z) = Z\left(x(n)\right) = 1 + az^{-1} + (az^{-1})^{2} + (az^{-1})^{3} + \cdots\cdots$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad \text{Converges if } |az^{-1}| < 1 \text{ (or) } |z| > |a|$$




ROC of $a^n u(n)$




ROC of $-a^n u(-n-1)$

(b)

$x(n) = -a^n u(-n-1)$ is a non causal & duration sequence

$x(n) = \begin{cases} -a^n, & n \le -1 \\ 0 & n \ge 0 \end{cases}$ because $u(-n-1) = \begin{cases} 1 & \text{for } n \le -1 \\ 0 & \text{for } n \ge 0. \end{cases}$

$$X(z) = Z\left(-a^n u(-n-1)\right) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} -a^{-n} z^{n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

This is a geometric series of & length & converges if $|a^{-1}z| < 1$

(or) $\left|\frac{z}{a}\right| < 1$ i.e., $|z| < |a|$.

$$X(z) = -\left(\sum_{n=1}^{\infty} (a^{-1}z)^n + 1 - 1\right) = -\left(\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right)$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z}$$

$$X(z) = \frac{-a^{-1}z}{-a^{-1}z\left(1-\frac{1}{a^{-1}z}\right)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \cdot \quad \text{ROC } |z| < |a|.$$

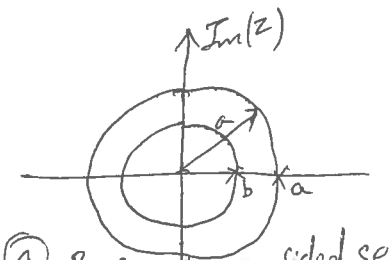
Ily $Z\{a^n u(-n)\} = \frac{a}{a-z} = \frac{1}{1-a^{-1}z}$ ROC $|z| < a$ → Non Causal & duration sequence

Ily $Z\{a^n u(n) - b^n u(-n-1)\} = \frac{z}{z-a} + \frac{z}{z-b}$ (a) If $|b| < |a|$ then $|z| > a$ & $|z| < b$

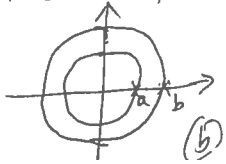
ROC is $|a| < |z| < |b|$ Z-Transform does not Exist

$= \frac{z}{z-a} + \frac{z}{z-b}$ (b) If $|b| > |a|$ then $|z| > |a|$ & $|z| < b$

Z-Transform Exist




(a) $|b| < |a|$




(b) $|b| > |a|$.

Ex:- Find the Z-Transform & ROC of $x(n)$ for

$$x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(\frac{-1}{3}\right)^n u(n).$$

Also find the pole zero location.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(3\left(\frac{5}{7}\right)^n u(n) + 2\left(\frac{-1}{3}\right)^n u(n)\right) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \left(3\left(\frac{5}{7}\right)^n z^{-n} + 2\left(\frac{-1}{3}\right)^n z^{-n}\right) = 3\sum_{n=0}^{\infty}\left(\frac{5}{7}\right)^n z^{-n} + 2\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n z^{-n}$$

$$X(z) = 3\sum_{n=0}^{\infty}\left(\frac{5}{7}z^{-1}\right)^n + 2\sum_{n=0}^{\infty}\left(\frac{-1}{3}z^{-1}\right)^n$$

The 1st Summation Converges for $\left|\frac{5}{7}z^{-1}\right| < 1$ (or) $|z| > \frac{5}{7}$ and the Second Summation Converges for $\left|\frac{-1}{3}z^{-1}\right| < 1$ (or) $|z| > \frac{1}{3}$

$X(z)$ Converges for $|z| > \frac{5}{7}$

$$X(z) = 3\left(\frac{1}{1-\frac{5}{7}z^{-1}}\right) + 2\left(\frac{1}{1+\frac{1}{3}z^{-1}}\right) = 3\left(\frac{z}{z-\frac{5}{7}}\right) + 2\left(\frac{z}{z+\frac{1}{3}}\right)$$

$$= \frac{3z\left(z+\frac{1}{3}\right) + 2z\left(z-\frac{5}{7}\right)}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)} = \frac{5z^2 - \frac{3}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

$$X(z) = \frac{5z^2 - \frac{3}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)} = \frac{z\left(5z-\frac{3}{7}\right)}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

Poles of $X(z)$ are at $z = \frac{5}{7}$ and $z = -\frac{1}{3}$. So the ROC is $|z| > \frac{5}{7}$ i.e., ROC is Exterior of the Circle of Radius $\frac{5}{7}$. The Zeros are at $z = 0$ & $z = \frac{3}{35}$.

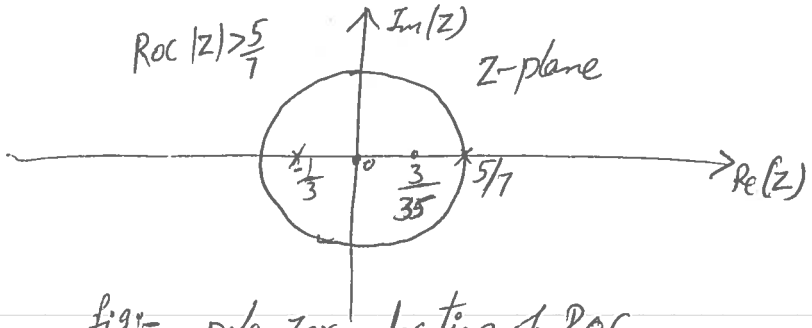



fig:- pole-zero location & ROC

Ex:- Find the Z-Transform & ROC of $x(n) = 2\left(\frac{5}{6}\right)^n u(-n-1) + 3\left(\frac{1}{2}\right)^n u(n)$

Ex:- Find the Z-Transform & ROC of $x(n) = a^{|n|}$ ; $|a|<1$

Sol:-

$$x(n) = a^{|n|} \; ; \; |a|<1$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^{n} z^{-n}$$

$$X(z) = \sum_{n=-1}^{-\infty} (az)^{-n} + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= \sum_{n=1}^{\infty} (az)^{n} + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$= 1 + \sum_{n=1}^{\infty} (az)^{n} - 1 + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

$$X(z) = \sum_{n=0}^{\infty} (az)^{n} - 1 + \sum_{n=0}^{\infty} (az^{-1})^{n}$$

1st series converges if $|az|<1$ i.e, $|z|<\left|\frac{1}{a}\right|$

2nd series converges if $|az^{-1}|<1$ i.e., $|z|>|a|$.

So $X(z)$ converges if $|a|<|z|<\frac{1}{|a|}$

$$X(z) = -1 + \frac{1}{1-az} + \frac{1}{1-az^{-1}} = -1 + \frac{1}{(1-az)} + \frac{z}{z-a}$$

$$X(z) = \frac{z\left(a - \frac{1}{a}\right)}{(z-a)\left(z-\frac{1}{a}\right)} \qquad \text{ROC } |a|<|z|<\left|\frac{1}{a}\right|$$

Ex:- Using properties of Z-Transform, find the Z-Transform of the sequence

(a) $x(n) = \alpha^{n-2} u(n-2)$ (b) $x(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$

⇒ (a) $x(n) = \alpha^{n} u(n) \longrightarrow X(z) = \frac{z}{z-\alpha} \quad |z|>|\alpha|$

$$Z(x(n-m)) = z^{-m} X(z)$$

$$Z(x(n-2)) = Z\left(\alpha^{n-2} u(n-2)\right) = z^{-2}\left(\frac{z}{z-\alpha}\right) = \frac{1}{z(z-\alpha)} \quad \text{ROC } |z|>|\alpha|$$

(b) $x(n) = u(n) - u(n-N)$

$$Z(u(n)) = \frac{z}{z-1} \qquad Z(u(n-N)) = z^{-N}\left(\frac{z}{z-1}\right)$$

$$Z\left(u(n)-u(n-N)\right) = \frac{z}{z-1} - z^{-N}\left(\frac{z}{z-1}\right) = \frac{z}{z-1}\left(1-z^{-N}\right)$$

Ex:- use appropriate properties, find the Z-Transform of the signal.

$$x(n) = 2(3)^n u(n)$$

$$x(n) = 2(3)^n u(-n)$$

$$Z\left(u(n)\right) = \frac{1}{1-z^{-1}} \quad ROC\ |z|>1$$

Time Reversal

$$Z\left(u(-n)\right) = \frac{1}{1-z} \quad ROC\ |z|<|\infty|$$

Using Scaling in Z-domain

$$Z\left(3^n u(-n)\right) = Z\left(u(n)\right)_{z\to z/3} = \frac{1}{1-z}\Big|_{z\to\frac{z}{3}} = \frac{1}{1-\frac{z}{3}} \quad ROC\ |z|<3$$

use linear property

$$Z\left(2(3)^n u(-n)\right) = \frac{2}{1-\frac{z}{3}} = \frac{2}{1-z\left(\frac{1}{3}\right)} \quad ROC\ |z|<3.$$

Ex:- Using appropriate properties, find the Z-Transform of

$$x(n) = n^2\left(\frac{1}{3}\right)^n u(n-2).$$

$$x(n) = n^2\left(\frac{1}{3}\right)^n u(n-2)$$

$$x(n) = n^2\left(\frac{1}{3}\right)^{n-2}\left(\frac{1}{3}\right)^2 u(n-2)$$

$$Z\left(u(n)\right) = \frac{z}{z-1} \quad ROC\ |z|>1$$

$$Z\left(\left(\frac{1}{3}\right)^n u(n)\right) = \frac{z}{z-\frac{1}{3}} \quad ROC\ |z|>\frac{1}{3}.$$

$$Z\left(\left(\frac{1}{3}\right)^{n-2} u(n-2)\right) = z^{-2}\left(\frac{z}{z-\frac{1}{3}}\right) = \frac{z^{-1}}{z-\frac{1}{3}} = \frac{1}{z\left(z-\frac{1}{3}\right)} = \frac{1}{z^2-\frac{1}{3}z}$$

$$Z\left(n\left(\frac{1}{3}\right)^{n-2} u(n-2)\right) = -z\frac{d}{dz}\left\{Z\left(\left(\frac{1}{3}\right)^{n-2} u(n-2)\right)\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{1}{z^2-\frac{1}{3}z}\right\} = -z\left\{\frac{-1\left(2z-\frac{1}{3}\right)}{\left(z^2-\frac{1}{3}z\right)^2}\right\} = \frac{2z^2-\frac{1}{3}z}{\left(z^2-\frac{1}{3}z\right)^2}.$$

Again using differentiation in Z-domain property, we have

$$Z\left(n^2\left(\frac{1}{3}\right)^{n-2}u(n-2)\right) = -Z\left(\frac{d}{dz}\left(n\left(\frac{1}{3}\right)^{n-2}u(n-2)\right)\right)$$

$$= -Z\frac{d}{dz}\left\{\frac{2z^2-\frac{1}{3}z}{\left(z^2-\frac{1}{3}z\right)^2}\right\}$$

$$= -Z\left\{\frac{\left(z^2-\frac{1}{3}z\right)^2\left(4z-\frac{1}{3}\right)-2\left(z^2-\frac{1}{3}z\right)\left(2z-\frac{1}{3}\right)\left(2z^2-\frac{1}{3}z\right)}{\left(z^2-\frac{1}{3}z\right)^4}\right\}$$

$$= \left\{\frac{4z^2-z+\frac{1}{9}}{z\left(z-\frac{1}{3}\right)^3}\right\}$$

Using the linearity property we get

$$Z\left(\frac{1}{9}n^2\left(\frac{1}{3}\right)^{n-2}u(n-2)\right) = \frac{1}{9}\left\{\frac{4z^2-z+\frac{1}{9}}{z\left(z-\frac{1}{3}\right)^3}\right\}.$$

Ex:- Determine the Z-Transform of the following signals & sketch the ROC

(a) $x_1(n) = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ \left(\frac{1}{4}\right)^{-n}, & n < 0 \end{cases}$

(b) Using Properties of Z-Transform, determine $x_2(n) = x_1(n+2)$.

→ (a) $$X_1(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{-1}\left(\frac{1}{4}\right)^{-n}z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}.$$

$$X_1(z) = \sum_{n=-1}^{-\infty}\left(\left(\frac{1}{4}\right)z\right)^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$X_1(z) = \sum_{n=1}^{\infty}\left(\frac{1}{4}z\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{4}z\right)^n - 1 + \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

Series 1 → $\left|\frac{1}{4}z\right| < 1$ i.e., $|z| < 4$

Series 2 → $\left|\frac{1}{2}z^{-1}\right| < 1$ $|z| > \frac{1}{2}$

ROC is $\frac{1}{2} < |z| < 4$

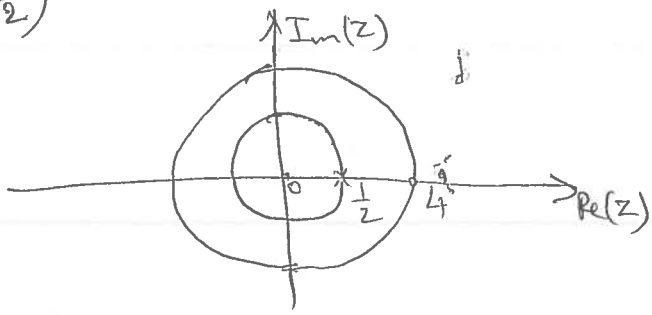
$$X_1(z) = -1 + \frac{1}{1-\left(\frac{1}{4}z\right)} + \frac{1}{1-\frac{1}{2}z^{-1}} = \left(\frac{\frac{1}{4}z}{1-\frac{1}{4}z}\right) + \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)$$

$$X_1(z) = \frac{z}{z-4} + \frac{z}{z-\frac{1}{2}} = \frac{\left(7/2\right)z}{(z-4)\left(z-\frac{1}{2}\right)}$$

(b) Using shifting property

$$X_2(z) = Z\left(x_1(n+2)\right) = z^2 Z\left(x_1(n)\right)$$

$$= z^2 X_1(z) = \frac{-(1/2)z^3}{(z-4)(z-\frac{1}{2})} \qquad \text{ROC; } \frac{1}{2} < |z| < 4.$$




Pole zero location

Ex:- Express Z-Transform of "$y(n) = \sum_{K=-\infty}^{n} x(k)$" in terms of $X(z)$.

$$y(n) = \sum_{K=-\infty}^{n} x(K)$$

$$y(n-1) = \sum_{K=-\infty}^{n-1} x(K)$$

$$y(n) - y(n-1) = \sum_{K=-\infty}^{n} x(K) - \sum_{K=-\infty}^{n-1} x(K)$$

$$y(n) - y(n-1) = x(n)$$

$$Y(z) - z^{-1}Y(z) = X(z)$$

$$Y(z)\left(1 - z^{-1}\right) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1-z^{-1}} \qquad \boxed{Y(z) = \frac{X(z)}{1-z^{-1}}}$$

Ex:- Find the Z-Transform of $x(n) = n^2 u(n)$

$$Z(u(n)) = \frac{z}{z-1}$$

~~$Z(nu(n)) = -z\frac{d}{dz}(\quad)$~~

$$Z(nu(n)) = -z\frac{d}{dz}\left(Z(u(n))\right) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left(\frac{(z-1)-z}{(z-1)^2}\right)$$

$$Z\left(nu(n)\right) = -z\left(\frac{-1}{(z-1)^2}\right) = \frac{z}{(z-1)^2}$$

Again using multiplication by 'n' property

$$Z\left(n^2 u(n)\right) = -z\frac{d}{dz}Z\left(nu(n)\right) = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = -z\left(\frac{(z-1)^2 - z2(z-1)}{(z-1)^4}\right) = \frac{z(z+1)}{(z-1)^3}$$

Ex: Find the Z-Transform of the signal $x(n) = n\left(\frac{1}{2}\right)^{n+2} u(n+2)$

Sol:- The given sequence $x(n) = n\left(\frac{1}{2}\right)^{n+2} u(n+2)$

Use Time shift

$$Z\left\{\left(\frac{1}{2}\right)^{n+2} u(n+2)\right\} = z^2\left(Z\left(\left(\frac{1}{2}\right)^n u(n)\right)\right) = z^2\left(\frac{z}{z-\frac{1}{2}}\right) = \frac{z^3}{z-\frac{1}{2}}$$

Use Multiplication by n

$$Z\left\{n\left(\frac{1}{2}\right)^n u(n+2)\right\} = -z\frac{d}{dz}\left\{\frac{z^3}{z-\frac{1}{2}}\right\} = -z\left\{\frac{\left(z-\frac{1}{2}\right)3z^2 - z^3}{\left(z-\frac{1}{2}\right)^2}\right\}$$

$$Z\left\{n\left(\frac{1}{2}\right)^n u(n+2)\right\} = \frac{\left(\frac{1}{2}\right)z^3(3-4z)}{\left(z-\frac{1}{2}\right)^2}$$

Ex:- Find the Z-Transform of the following Signal:

$$x(n) = n\left(\frac{1}{2}\right)^n u(n) * \left(\delta(n) - \frac{1}{2}\delta(n-1)\right).$$

>

$$x(n) = x_1(n) * x_2(n)$$

$$Z\left(x_1(n) * x_2(n)\right) = X_1(z)\,X_2(z)$$

$$Z(x_1(n)) = X_1(z) = Z\left(n\left(\frac{1}{2}\right)^n u(n)\right) = -z\frac{d}{dz}\left(\frac{z}{z-\frac{1}{2}}\right) = -z\left(\frac{\left(z-\frac{1}{2}\right)-z}{\left(z-\frac{1}{2}\right)^2}\right) = \frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}$$

$$Z\left(x_2(n)\right) = X_2(z) = 1 - \frac{1}{2}z^{-1} = \left(\frac{z-\frac{1}{2}}{z}\right)$$

$$X(z) = X_1(z)\,X_2(z) = \left(\frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2} \times \frac{\left(z-\frac{1}{2}\right)}{z}\right) = \frac{\left(\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)} \quad \text{ROC is } |z| > \frac{1}{2}$$

Ex:- Using Z-Transform find the convolution of the sequences

$x_1(n) = \{2, 1, 0, -1, 3\}$ $x_2(n) = \{1, -3, 2\}$

>

$$Z\left(x_1(n) * x_2(n)\right) = X_1(z)\,X_2(z)$$

$$X_1(z) = 2 + z^{-1} - z^{-3} + 3z^{-4}$$

$$X_2(z) = 1 - 3z^{-1} + 2z^{-2}$$

$$X_1(z)\,X_2(z) = \left(2 + z^{-1} - z^{-3} + 3z^{-4}\right)\left(1 - 3z^{-1} + 2z^{-2}\right) = 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

$$x(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

Ex: Using final value Theorem, find $x(\infty)$ if $X(z)$ is given by

(a) $\frac{z+1}{(z-0.6)^2}$ (b) $\frac{z+2}{4(z-1)(z+0.7)}$ (c) $\frac{2z+3}{(z+1)(z+3)(z-1)}$

(a) $X(z) = \frac{z+1}{(z-0.6)^2}$ ROC $|z| > 0.6$

$x(\infty) = \lim_{z\to 1} (z-1)X(z)$

$(z-1)X(z)$ has no poles on (or) outside the unit circle

$x(\infty) = \lim_{z\to 1} (z-1)\left(\frac{z+1}{(z-0.6)^2}\right) = 0$

(b) $X(z) = \frac{z+2}{4(z-1)(z+0.7)}$

$(z-1)X(z)$ has no poles on (or) outside unit ole

$x(\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}\left(\frac{z+2}{4(z-1)(z+0.7)}\right)(z-1)$

$= \frac{3}{4(1.7)} = \frac{3}{6.8} = 0.44$

(c) $X(z) = \frac{2z+3}{(z+1)(z+3)(z-1)}$

$(z-1)X(z) = \frac{2z+3}{(z+1)(z+3)}$

~~$x(\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}\frac{2z+3}{(z+1)(z+3)} = \frac{5}{8}$~~

$(z-1)X(z)$ has one pole on the unit circle & one pole outside the unit circle. So $x(\infty)$ tends to $\infty$ as $n\to\infty$.

Ex: Find $x(0)$ if $X(z)$ is given by

(a) $\frac{z^2+2z+2}{(z+1)(z+0.5)}$ (b) $\frac{z+3}{(z+1)(z+2)}$

(a) $x(0) = \lim_{z\to\infty} X(z)$ $X(z) = \frac{z^2+2z+2}{(z+1)(z+0.5)} = \frac{z^2(1+2z^{-1}+2z^{-2})}{z^2(1+z^{-1})(1+0.5z^{-1})}$

$X(z) = \frac{1+\frac{2}{z}+\frac{2}{z^2}}{(1+\frac{1}{z})(1+\frac{0.5}{z})}$ $\lim_{z\to\infty} X(z) = \frac{1}{1} = 1$

(b) $X(z) = \frac{z+3}{(z+1)(z+2)} = \frac{z(1+\frac{3}{z})}{z^2(1+\frac{1}{z})(1+\frac{2}{z})} = \frac{(1+\frac{3}{z})}{z(1+\frac{1}{z})(1+\frac{2}{z})}$ $x(0) = \lim_{z\to\infty} X(z) = 0$

## Inverse Z-Transform:-

The process of finding the time domain signal $x(n)$ from its Z-Transform $X(z)$ is called the Inverse Z-Transform which is denoted as:

$$x(n) = Z^{-1}(X(z)).$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x(n)\, r^{-n})\, e^{-j\omega n}$$ This is the DTFT of signal $x(n)r^{-n}$

Hence the inverse DTFT of $X(re^{j\omega})$ must be $x(n)r^{-n}$.

$$x(n)\, r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega}) (re^{j\omega})^n\, d\omega$$

$$z = re^{j\omega}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, (z)^n\, d\omega$$

$$\frac{dz}{d\omega} = r(j)e^{j\omega}$$

$$d\omega = \frac{dz}{jre^{j\omega}}$$

$$x(n) = \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(z)\, z^n \frac{dz}{re^{j\omega}}$$

$$= \frac{1}{2\pi j} \oint_C X(z)\, z^n \frac{dz}{z} = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} dz$$

where the symbol $\oint_C$ denotes integration around the circle of radius $|z| = r$ in a counter clockwise direction.

$$X(z) = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

The roots of the numerator polynomial are those values of z for which $X(z) = 0$ and are referred to as the zeros of $X(z)$. The roots of the denominator polynomial are those values of z for which $X(z) = \infty$ and are referred to as poles of $X(z)$. In z-plane, zero locations are denoted by $\circ$ (a small circle) and pole locations with $\times$ (cross) symbol. There are 4 methods that are often used to find the Inverse Z-Transform.

They are: (1) Power Series (or) long division method (2) partial fraction expansion method (3) Complex inversion integral (Residue) method (4) Convolution integral method.

Long Division method is simple, but does not give a closed form Expression of the time signal. Further it can be used only if the ROC of the given $X(z)$ is either of the form $|z|>\alpha$ (or) of the form $|z|<\alpha$ i.e., it is useful only if the sequence $x(n)$ is either purely Right sided (or) purely left sided. The partial fraction Expansion method enables us to determine the time signal $x(n)$ making use of knowledge of some basic Z-Transform pairs & Z-Transform theorems.

The Inversion integral method requires a knowledge of theory of Complex Variables but is quite powerful & useful. The Convolution integral method uses Convolution property of Z-Transforms & can be used when given $X(z)$ can be written as the product of 2-functions.

Long Division Method:-

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\,z^{-n} \rightarrow \text{Z-T of 2-Sided Sequence}$$

we can't obtain a 2-sided sequence by long division. If the sequence $x(n)$ is causal, then

$$X(z)=\sum_{n=0}^{\infty} x(n)\,z^{-n}=x(0)+x(1)z^{-1}+x(2)z^{-2}+\cdots\cdots$$

has only -ve powers of Z, with ROC $|z|>\alpha$.

If $x(n)$ is Anticausal, then

$$X(z)=\sum_{n=-\infty}^{0} x(n)\,z^{-n}=\cdots\cdots+x(-3)z^{3}+x(-2)z^{2}+x(-1)z^{1}+x(0)z^{0}.$$

has only positive powers of Z, with ROC; $|z|<\alpha$.

To determine ~~the~~ inverse Z-T of $X(z)$, if $X(z)$ is a ratio of 2 polynomials

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots\cdots+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\cdots\cdots+a_Nz^{-N}}$$

we can generate a series in Z by dividing the Nr of $X(z)$ by its denominator. If $X(z)$ converges for $|z|>\alpha$, we obtain the series

$$X(z)=x(0)+x(1)z^{-1}+x(2)z^{-2}+\cdots\cdots$$

We can identify the coefficients of $z^{-n}$ as $x(n)$ of a causal sequence.

If $X(z)$ converges for $|z|<\alpha$, we obtain the series

$$X(z)=x(0)+x(-1)z^{1}+x(-2)z^{2}+\cdots\cdots$$

we can identify the coefficients of $z^{-n}$ as $x(n)$ of a non-causal sequence.

~~If X(z) converges for |z|<α, we obtain the series~~

For getting a Causal sequence, 1st put N(z) & D(z) either in descending Powers of z (or) in ascending powers of $z^{-1}$ before long division.

For getting a Non-Causal sequence, 1st put N(z) & D(z) either in Ascending Powers of z (or) in descending powers of $z^{-1}$ before long division.

Ex: Find the inverse Z-Transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}.$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \cdots + x(-3)z^3 + x(-2)z^2 + x(1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + \cdots$$

$$x(n) = \{1, 2, 1, \underset{\uparrow}{1}, -2, -3, 4\}$$

(or) Alternatively taking Inverse Z-Transform of X(z), we have

$$x(n) = \delta(n+3) + 2\delta(n+2) + \delta(n+1) + \delta(n) - 2\delta(n-1) - 3\delta(n-2) + 4\delta(n-3)$$

Ex: Determine the Inverse Z.T of

(a) $X(z) = \frac{1}{z-a}$ $|z| > a$ (b) $X(z) = \frac{1}{1-az^{-1}}$ Roc $|z| > a$

(c) $X(z) = \frac{1}{1-z^{-4}}$ $|z| > 1$

(a) $X(z) = \frac{1}{z-a}$ Roc $|z| > a$

$$X(z) = \frac{1}{z(1-az^{-1})} = z^{-1}(1-az^{-1})^{-1} = z^{-1}(1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots)$$

$$X(z) = z^{-1} + az^{-2} + a^2z^{-3} + \cdots = \sum_{n=1}^{\infty} a^{n-1} z^{-n} = \sum_{n=1}^{\infty} a^{n-1} u(n-1) z^{-n}$$

$$\boxed{x(n) = a^{n-1} u(n-1)}$$

(b) $X(z) = \frac{1}{1-az^{-1}}$ Roc; $|z| > a$

$$X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$x(n) = a^n u(n).$$

(c) From ∞ sum formula, we have

$$\sum_{K=0}^{\infty} \alpha^K = \frac{1}{1-\alpha} \quad |\alpha| < 1$$

$$X(z) = \frac{1}{1-z^{-4}} = \sum_{K=0}^{\infty} (z^{-4})^K = \sum_{K=0}^{\infty} z^{-4K} \quad (|z^{-4}| < 1, \text{ i.e., } |z| > 1)$$

taking inverse Z-Transform on both sides

$$x(n) = \sum_{K=0}^{\infty} \delta(n-4K)$$

$x(n) = 1$, when $n = 4K$, when n is an integer multiple of 4.

$= 0$ otherwise.

Ex:- Find the inverse Z-Transform of $X(z) = \sum_{K=4}^{8} \left(\frac{1}{2K}\right) z^{-K}$; ROC; $|z| > 0$

$$X(z) = \sum_{K=4}^{8} \left(\frac{1}{2K}\right) z^{-K} = \frac{1}{8} z^{-4} + \frac{1}{10} z^{-5} + \frac{1}{12} z^{-6} + \frac{1}{14} z^{-7} + \frac{1}{16} z^{-8}$$

Taking inverse Z-Transform, we get

$$x(n) = \frac{1}{8}\delta(n-4) + \frac{1}{10}\delta(n-5) + \frac{1}{12}\delta(n-6) + \frac{1}{14}\delta(n-7) + \frac{1}{16}\delta(n-8)$$

$$x(n) = \sum_{K=4}^{8} \left(\frac{1}{2K}\right) \delta(n-K)$$

Ex:- use long division, determine the inverse Z-Transform of

$$X(z) = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1}$$

$$z^3 - 3z^2 + 4z + 1 \;\Big|\; z^2 + 2z \;\Big|\; z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} + 13z^{-5}$$

$z^2 - 3z + 4 + z^{-1}$

$5z - 4 - z^{-1}$

$5z - 15 + 20z^{-1} + 5z^{-2}$

$11 - 21z^{-1} - 5z^{-2}$

$11 - 33z^{-1} + 44z^{-2} + 11z^{-3}$

$12z^{-1} - 49z^{-2} - 11z^{-3}$

$12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4}$

$-13z^{-2} - 59z^{-3} - 12z^{-4}$

$$X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} + \cdots\cdots$$

$$x(n) = \{0, 1, 5, 11, 12, -13, \cdots\cdots\}.$$

## Partial Fraction Method:-

**Ex:-** Find the inverse Z-Transform of

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} \quad \text{ROC; } |z|>1.$$

⇝

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{1-4z+3z^2}$$

$$\frac{X(z)}{z} = \frac{1}{1-4z+3z^2} = \frac{1}{3\left(z^2-\frac{4}{3}z+\frac{1}{3}\right)} = \frac{1}{3(z-1)\left(z-\frac{1}{3}\right)}$$

$$\frac{X(z)}{z} = \frac{1}{3}\left(\frac{1}{(z-1)}\right)\left(\frac{1}{z-\frac{1}{3}}\right) = \frac{A}{z-1}+\frac{B}{z-\frac{1}{3}}$$

$$\frac{X(z)}{z} = \frac{3}{2}\left(\frac{1}{z-1}\right) - \frac{3}{2}\left(\frac{1}{z-\frac{1}{3}}\right)$$

ROC is $|z|>1$, both sequences must be causal.

$$x(n) = \frac{3}{2}\left(u(n) - \left(\frac{1}{3}\right)^n u(n)\right) \quad \text{ROC; } |z|>1$$

$$X(z) = \frac{3}{2}\left(\frac{z}{z-1} - \frac{z}{z-\frac{1}{3}}\right)$$

**Ex:-** Find the inverse Z-Transform of $X(z) = \frac{z(z-1)}{(z+1)^3(z+2)}$ ROC $|z|>2$

⇝

$$X(z) = \frac{z(z-1)}{(z+1)^3(z+2)} \quad \text{ROC; } |z|>2$$

$$\frac{X(z)}{z} = \frac{z-1}{(z+1)^3(z+2)} = \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2} + \frac{C_3}{(z+1)^3} + \frac{C_4}{z+2}$$

$$C_4 = (z+2)\left.\frac{X(z)}{z}\right|_{z=-2} = \left.\frac{z-1}{(z+1)^3}\right|_{z=-2} = \frac{-2-1}{(-2+1)^3} = 3$$

$$C_2 = \frac{1}{1!}\left.\frac{d}{dz}\left((z+1)^3\frac{X(z)}{z}\right)\right|_{z=-1} = \left.\frac{d}{dz}\left(\frac{z-1}{z+2}\right)\right|_{z=-1} = \left.\frac{(z+2)(1)-(z-1)(1)}{(z+2)^2}\right|_{z=-1} = 3$$

$$C_1 = \frac{1}{2!}\left.\frac{d^2}{dz^2}\left((z+1)^3\frac{X(z)}{z}\right)\right|_{z=-1} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z-1}{z+2}\right) = \frac{1}{2}\frac{d}{dz}\left(\frac{3}{(z+2)^2}\right)$$

$$C_1 = \frac{1}{2}\left(\frac{-3\times 2(z+2)}{(z+2)^4}\right)_{z=-1} = \frac{-3(-1+2)}{(-1+2)^3} = -3$$

$$s = (z+1)^3 \frac{X(z)}{z}\Big|_{z=-1} = \frac{z-1}{(z+2)}\Big|_{z=-1} = \frac{-1-1}{-1+2} = -2$$

$$\frac{X(z)}{z} = \frac{-3}{z+1} + \frac{3}{(z+1)^2} + \frac{2}{(z+1)^3} + \frac{3}{z+2}$$

$$X(z) = \frac{-3z}{z+1} + \frac{3z}{(z+1)^2} + \frac{2z}{(z+1)^3} + \frac{3z}{(z+2)} \quad \text{ROC } |z|>2$$

All the above sequences must be causal. Taking Inverse Z.T on both side we have

$$x(n) = -3(-1)^n u(n) + 3n(-1)^n u(n) - 2n(n-1)(-1)^n u(n) + 3(-2)^n u(n).$$

Convolution Method:-

Ex:- Find the inverse Z-Transform of $X(z) = \frac{z^2}{(z-2)(z-3)}$ using Convolution property of Z-Transforms.

$$X(z) = \frac{z^2}{(z-2)(z-3)} = \left(\frac{z}{z-2}\right)\left(\frac{z}{z-3}\right) = X_1(z)\,X_2(z)$$

$$x_1(n) = z^{-1}(X_1(z)) = z^{-1}\left(\frac{z}{z-2}\right) = 2^n u(n)$$

$$x_2(n) = z^{-1}(X_2(z)) = z^{-1}\left(\frac{z}{z-3}\right) = 3^n u(n)$$

$$x_1(n) * x_2(n) = \sum_{K=0}^{n} x_1(K)\,x_2(n-K) = \sum_{K=0}^{n} 2^K u(K)\,3^{n-K} u(n-K)$$

$$x_1(n)*x_2(n) = 3^n \sum_{K=0}^{n} \left(\frac{2}{3}\right)^K = 3^n \left(\frac{1-(2/3)^{n+1}}{1-(2/3)}\right) = 3^{n+1}\left(1-\left(\frac{2}{3}\right)^{n+1}\right) = 3^{n+1}u(n) - 2^{n+1}u(n)$$

Ex:- Find the inverse Z-Transform of $X(z) = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)}$ using Convolution property of Z-Transforms.

$$x_1(n) = z^{-1}\left(\frac{z}{z-1}\right) = u(n)$$

$$x_2(n) = z^{-1}(X_2(z)) = z^{-1}\left(\frac{1}{z-\frac{1}{2}}\right) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$x_1(n) * x_2(n) = \sum_{K=0}^{n} x_1(K)\,x_2(n-K) = \sum_{K=0}^{n-1} u(K)\left(\frac{1}{2}\right)^{n-1-K} u(n-1-K)$$

$$= \left(\frac{1}{2}\right)^{n-1}\left(\sum_{K=0}^{n-1}\left(\frac{1}{2}\right)^{-K}\right) = \left(\frac{1}{2}\right)^{n-1}\sum_{K=0}^{n-1}\left(\left(\frac{1}{2}\right)^{-1}\right)^K = \left(\frac{1}{2}\right)^{n-1}\left(\frac{1-\left(\left(\frac{1}{2}\right)^{-1}\right)^n}{1-\left(\frac{1}{2}\right)^{-1}}\right)$$

$$x_1(n)*x_2(n) = \left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{-1} = 2u(n) - 2\left(\frac{1}{2}\right)^n u(n).$$

## Transform Analysis of LTI Systems:-

The Z-Transform plays an important role in the analysis & design of discrete-time LTI Systems.

### System Function & Impulse Response:-

Consider a discrete time LTI System having an impulse response h(n) as shown in fig

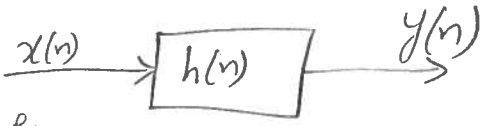



fig:- Discrete time LTI System.

Let us say it gives an o/p y(n) for an i/p x(n). Then we have

$$y(n) = x(n) * h(n)$$

Taking Z-Transform on both sides, we get

$$Y(z) = H(z)X(z)$$

Where $Y(z)$ = Z-Transform of the o/p $y(n)$

$X(z)$ = Z-Transform of the i/p $x(n)$.

$H(z)$ = Z-Transform of the impulse response $h(n)$

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is the ratio of Z-Transform of the o/p sequence $y(n)$ to the Z-Transform of the i/p sequence $x(n)$ when the initial conditions are neglected.

$H(z)$ is called as the System function (or) Transfer function of the LTI discrete System.

If the i/p $x(n)$ is an impulse sequence, then $X(z) = 1$. So $Y(z) = H(z)$

So the transfer function is also defined as the Z-Transform of the impulse response of the System.

The poles & Zeros of the System function offer an insight into the System characteristics.

## Relationship between Transfer function & Difference Equation:-

In terms of a difference equation, an $n^{th}$ order discrete time LTI System is defined as

$$\sum_{K=0}^{N} a_K y(n-K) = \sum_{K=0}^{M} b_K x(n-k).$$

Expanding it, we have

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

Taking Z-Transform on both sides & neglecting the initial conditions, we obtain

$$a_0 Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_M z^{-M} X(z)$$

$$Y(z)\left(a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}\right) = X(z)\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$H(z) = \frac{\sum_{K=0}^{M} b_K z^{-K}}{\sum_{K=0}^{N} a_K z^{-K}}$$

$H(z)$ is called the transfer function of the System (or) the System function. The frequency response of a System is obtained by Substituting $z = e^{j\omega}$ in $H(z)$

## Stability & Causality:-

We know that the necessary & sufficient Condition for a Causal linear time invariant discrete time system to be BIBO stable is

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

i.e, an LTI discrete time System is BIBO stable if its impulse response is absolutely Summable.

We also know that for a system to be causal, its impulse response must be equal to zero for $n<0$ ($h(n)=0$ for $n<0$). Alternatively if the system is causal, then the ROC for $H(z)$ will be outside the outermost pole.

For a causal LTI system to be stable, all the poles of $H(z)$ must lie inside the unit circle in the z-plane, i.e., for a causal LTI system to be stable, the ROC of the system function must include the unit circle. The pole locations of the system function & the corresponding impulse responses are shown in fig.

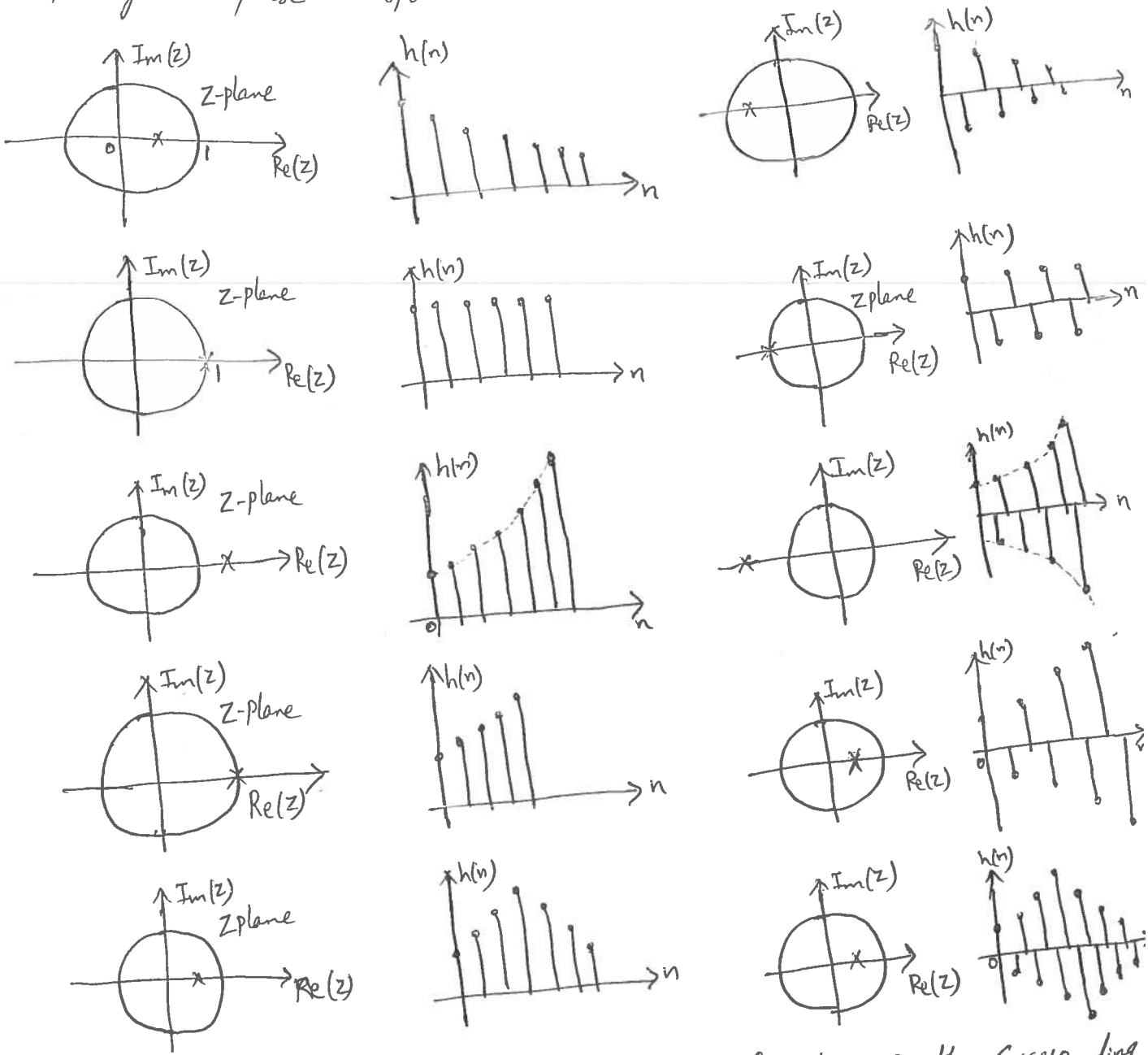



Fig:- pole locations of the system function & the corresponding Impulse Responses.

## Solution of Difference Equations using Z-Transforms :-

To solve the difference equation, 1st it is converted into algebraic equation by taking its Z-Transform. The solution is obtained in Z-domain and the time domain solution is obtained by taking its Inverse Z-Transform.

The response of the system due to i/p alone when the initial conditions are neglected is called the forced response of the system. The response of the system due to initial conditions alone when the i/p is neglected is called the free (or) natural response of the system. The response due to i/p & initial conditions considered simultaneously is called the total response of the system.

When i/p is a unit impulse input, the response is called the impulse response of the system & when the i/p is a unit step i/p, the response is called the step response of the system.

Ex:- A linear shift invariant system is described by the Difference Equation $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$. with $y(-1) = 0$ & $y(-2) = 1$

(a) Find the natural response of the system.
(b) The forced response of the system and
(c) The frequency response of the system for a step i/p.

(a) The Natural response is the response due to initial conditions only. So make $x(n) = 0$

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 0$$

$$Y(z) - \frac{3}{4}\left(z^{-1}Y(z) + y(-1)\right) + \frac{1}{8}\left(z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right) = 0$$

$$+Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) - \frac{1}{8} = 0$$

$$Y(z) = \frac{-1/8}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{-(1/8)z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$\frac{Y(z)}{z} = \frac{-\frac{1}{8}(z)}{(z - 1/2)(z - 1/4)} = \frac{A}{z - 1/2} + \frac{B}{z - \frac{1}{4}} = \frac{1/4}{z - \frac{1}{2}} - \frac{1/8}{z - \frac{1}{4}}$$

Ex:- (a) Determine the free response of the system described by the Difference equation $y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n)$ with $y(-1) = 1$ & $y(-2) = 0$

(b) Determine the forced response for an i/p $x(n) = \left(\frac{1}{4}\right)^n u(n)$.

→ (a) Make i/p terms = 0

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = 0$$

$$Y(z) - \frac{5}{6}\left(z^{-1}Y(z) + y(-1)\right) + \frac{1}{6}\left(z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right) = 0$$

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) - \frac{5}{6} + \frac{1}{6}z^{-1} = 0$$

$$Y(z) = \frac{\frac{5}{6} - \frac{1}{6}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} = \frac{5/6\left(z - \frac{1}{5}\right)z}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{\frac{5}{6}z\left(z - \frac{1}{5}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$\frac{Y(z)}{z} = \frac{\frac{5}{6}\left(z - \frac{1}{5}\right)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{A}{z - \frac{1}{2}} + \frac{B}{z - \frac{1}{3}} = \frac{3/2}{z - \frac{1}{2}} - \frac{2/3}{z - \frac{1}{3}}$$

$$Y(z) = \frac{3}{2}\left(\frac{z}{z - \frac{1}{2}}\right) - \frac{2}{3}\left(\frac{z}{z - \frac{1}{3}}\right)$$

$$y(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{3}\right)^n u(n)$$

(b)

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n)$$

Put $x(n) = \left(\frac{1}{4}\right)^n u(n)$

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = \left(\frac{1}{4}\right)^n u(n)$$

$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = \frac{z}{z - \frac{1}{4}}$$

$$Y(z) = \left(\frac{z}{z - \frac{1}{4}}\right)\left(\frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}\right) = \frac{z^3}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$\frac{Y(z)}{z} = \frac{z^2}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)} = \frac{A}{z - \frac{1}{4}} + \frac{B}{z - \frac{1}{3}} + \frac{C}{z - \frac{1}{2}}$$

$$Y(z) = \frac{1}{4}\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{1}{8}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{8}\right)\left(\frac{1}{4}\right)^n u(n)$$

(b) Forced response due to Step i/p, So put $x(n) = u(n)$

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

During Forced response initial Conditions are neglected

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = \frac{z+1}{z-1}$$

$$Y(z) = \frac{z+1}{(z-1)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)} = \frac{z^2(z+1)}{(z-1)\left(z^2-\frac{3}{4}z+\frac{1}{8}\right)}$$

$$Y(z) = \frac{z^2(z+1)}{(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{\frac{16}{3}}{z-1} - \frac{6}{z-\frac{1}{2}} + \frac{5/3}{z-\frac{1}{4}}$$ (using partial fractions)

$$Y(z) = \frac{16}{3}\left(\frac{z}{z-1}\right) - 6\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{5}{3}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$y(n) = \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

(c) Frequency Response is obtained by putting $z = e^{j\omega}$ in $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$H(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2 - \frac{3}{4}e^{j\omega} + \frac{1}{8}}$$

$$\frac{Y(z)}{z} = \frac{3}{z-\frac{1}{4}} - \frac{8}{z-\frac{1}{3}} + \frac{6}{z-\frac{1}{2}}$$

Multiply both sides by $z$

$$Y(z) = 3\left(\frac{z}{z-\frac{1}{4}}\right) - 8\left(\frac{z}{z-\frac{1}{3}}\right) + 6\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 8\left(\frac{1}{3}\right)^n u(n) + 6\left(\frac{1}{2}\right)^n u(n).$$

Ex:- Find the impulse response and step response of the system.

$$y(n) = 2x(n) - 3x(n-1) + x(n-2) - 4x(n-3).$$

⇒ For impulse response $x(n) = \delta(n)$

$$Y(z) = \left(2 - 3z^{-1} + z^{-2} - 4z^{-3}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 2 - 3z^{-1} + z^{-2} - 4z^{-3}$$

$$h(n) = 2\delta(n) - 3\delta(n-1) + \delta(n-2) - 4\delta(n-3).$$

(or) $x(n) = \delta(n)$

$$y(n) = 2\delta(n) - 3\delta(n-1) + \delta(n-2) - 4\delta(n-3).$$

For step Response $x(n) = u(n)$

$$y(n) = 2u(n) - 3u(n-1) + u(n-2) - 4u(n-3)$$

Ex:- Solve the following Difference equation

$$y(n) + 2y(n-1) = x(n)$$

with $x(n) = \left(\frac{1}{3}\right)^n u(n)$ and the initial conditions are $y(-1) = 1$

⇒ Solving the Difference equation gives solution, considering the initial condition & i/p simultaneously gives the total response of the system

The given Difference equation is

$$y(n) + 2y(n-1) = x(n) = \left(\frac{1}{3}\right)^n u(n) \text{ with } y(-1) = 1$$

$$Y(z) + 2\left(z^{-1}Y(z) + y(-1)\right) = X(z) = \frac{1}{1-\left(\frac{1}{3}\right)z^{-1}}$$

Substituting initial conditions, we have

$$Y(z)\left(1+2z^{-1}\right) = -2(1) + \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$Y(z) = \frac{-2}{1+2z^{-1}} + \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1+2z^{-1}\right)}$$

$$Y(z) = \frac{-2z}{z+2} + \frac{z^2}{\left(z-\frac{1}{3}\right)(z+2)}$$

$$Y(z) = -\frac{2z}{z+2} + \frac{z \times z}{\left(z-\frac{1}{3}\right)(z+2)}$$

$$\frac{Y(z)}{z} = \frac{-2}{z+2} + \frac{z}{\left(z-\frac{1}{3}\right)(z+2)}$$

$$\frac{Y(z)}{z} = \frac{-2}{z+2} + \frac{6}{7}\left(\frac{1}{z+2}\right) + \frac{1}{7}\left(\frac{1}{z-\frac{1}{3}}\right)$$

$$Y(z) = \frac{-2z}{z+2} + \frac{6}{7}\left(\frac{z}{z+2}\right) + \frac{1}{7}\left(\frac{z}{z-\frac{1}{3}}\right)$$

Taking Inverse Z.T.

$$y(n) = \frac{-8}{7}(-2)^n u(n) + \frac{1}{7}\left(\frac{1}{3}\right)^n u(n)$$

Ex:- Solve the following Difference equation using unilateral Z-Transform

$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n)$ for $n \geq 0$

with initial conditions $y(-1) = 2$, $y(-2) = 4$ & $x(n) = \left(\frac{1}{5}\right)^n u(n)$

$$\Rightarrow y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$Y(z) - \frac{7}{12}\left(z^{-1}Y(z) + y(-1)\right) + \frac{1}{12}\left(z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right) = \frac{1}{1-\frac{1}{5}z^{-1}}$$

$$Y(z)\left(1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}\right) = \frac{5}{6}\left(1-\frac{1}{5}z^{-1}\right) + \frac{1}{1-\frac{1}{5}z^{-1}}$$

$$Y(z) = \frac{z\left(\frac{11}{6}z^2 - \frac{1}{3}z + \frac{1}{30}\right)}{\left(z-\frac{1}{5}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-\frac{1}{5}} + \frac{B}{z-\frac{1}{4}} + \frac{C}{z-\frac{1}{3}} = \frac{6}{5}\left(\frac{1}{z-\frac{1}{5}}\right) + \frac{1}{8}\left(\frac{1}{z-\frac{1}{4}}\right) + \frac{102}{27}\left(\frac{1}{z-\frac{1}{3}}\right)$$

$$Y(z) = \frac{6}{5}\left(\frac{z}{z-\frac{1}{5}}\right) + \frac{1}{8}\left(\frac{z}{z-\frac{1}{4}}\right) + \frac{102}{27}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$y(n) = \frac{6}{5}\left(\frac{1}{5}\right)^n u(n) + \frac{1}{8}\left(\frac{1}{4}\right)^n u(n) + \frac{102}{27}\left(\frac{1}{3}\right)^n u(n).$$

Ex:- Consider an LTI system function $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$. Find the Difference Equation.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

$$y(n) - \frac{1}{2}y(n-1) = x(n).$$

Ex:- A Causal System is represented by $H(z) = \frac{z+2}{2z^2-3z+4}$. Find the Difference equation & the frequency response of the System.

$$H(z) = \frac{z+2}{2z^2-3z+4}$$

to get frequency response put $z = e^{j\omega}$

$$H(z) = H(e^{j\omega}) = \frac{e^{j\omega}+2}{2e^{2j\omega}-3e^{j\omega}+4} = \frac{2+\cos\omega+j\sin\omega}{2(\cos 2\omega+j\sin 2\omega)-3(\cos\omega+j\sin\omega)+4}$$

Freq Response $H(e^{j\omega}) = \frac{2+\cos\omega+j\sin\omega}{4+(2\cos 2\omega-3\cos\omega)+j(2\sin 2\omega-3\sin\omega)}$.

$$\frac{Y(z)}{X(z)} = \frac{z+2}{2z^2-3z+4} = \left(\frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}\right)$$

$$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2).$$

Ex:- plot the pole zero pattern & determine which of the following systems are stable:

(a) $y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

(b) $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

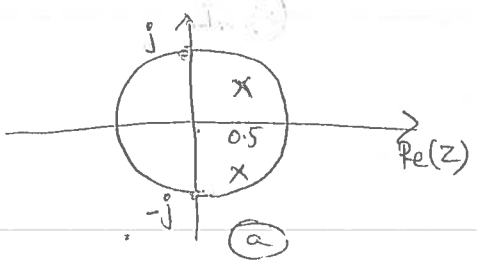
(a) $y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

$$Y(z)(1-z^{-1}+0.8z^{-2}) = X(z)(1+z^{-2})$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+z^{-2}}{1-z^{-1}+0.8z^{-2}} = \frac{z^2+1}{z^2-z+0.8}$$

$$H(z) = \frac{(z-j)(z-j)}{(z-0.5-j0.74)(z-0.5+j0.74)}$$



(a)

Zeroes of $H(z)$ are $z = +j1$ & $z = -j1$

Poles of $H(z)$ are $z = +0.5 - j0.74$, & $0.5 + j0.74$.

All the poles are inside the unit circle. Hence the system is stable

(b) $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

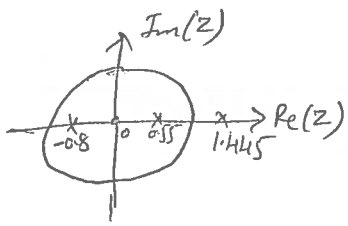
$$Y(z) = 2z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + 0.8z^{-1}X(z)$$

$$Y(z)\left(1 - 2z^{-1} + 0.8z^{-2}\right) = X(z)\left(1 + 0.8z^{-1}\right)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 0.8z^{-1}}{1 - 2z^{-1} + 0.8z^{-2}} = \frac{z(z+0.8)}{z^2 - 2z + 0.8} = \frac{z(z+0.8)}{(z-0.555)(z-1.445)}$$

Zeros of $H(z)$ are $z = 0$, & $z = -0.8$

Poles of $H(z)$ are $z = 1.445$ & $z = 0.555$




one pole is outside the unit circle.

The system is unstable.

Ex:- A Causal system has i/p $x(n)$ and output $y(n)$. Find the system funct frequency response & impulse response of the system if.

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and $y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$.

Ans

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2) \Rightarrow X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

$$y(n) = \delta(n) - \frac{2}{3}\delta(n-1) \Rightarrow Y(z) = 1 - \frac{2}{3}z^{-1}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{z\left(z - \frac{2}{3}\right)}{\left(z - \frac{1}{3}\right)\left(z + \frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{z - 2/3}{\left(z - \frac{1}{3}\right)\left(z + \frac{1}{2}\right)} = \frac{-2/5}{z - \frac{1}{3}} + \frac{7/5}{z + \frac{1}{2}}$$

Freq Response

$$H(e^{j\omega}) = \frac{e^{j\omega}\left(e^{j\omega} - \frac{2}{3}\right)}{\left(e^{j\omega} - \frac{1}{3}\right)\left(e^{j\omega} + \frac{1}{2}\right)}$$

Ily

$$H(z) = \frac{(-2/5)z}{z - \frac{1}{3}} + \frac{7}{5}\left(\frac{z}{z + \frac{1}{2}}\right)$$

$$h(n) = -\frac{2}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{5}\left(\frac{-1}{2}\right)^n u(n)$$

Ex:- Determine whether the following systems are both Causal & Stable.

(a) $H(z) = \dfrac{3+z^{-1}}{1+z^{-1}-\left(\frac{4}{9}\right)z^{-2}}$

$\Rightarrow$

$$H(z) = \frac{3+z^{-1}}{1+z^{-1}-\frac{4}{9}z^{-2}} = \frac{z(z+3)}{z^2+z-\frac{4}{9}} = \frac{z(z+3)}{\left(z-\frac{1}{3}\right)\left(z+\frac{4}{3}\right)}$$

The poles of $H(z)$ are: $z = \frac{1}{3}$ & $z = -4/3$

For a Causal system to be Stable, ROC must include the unit circle. Now for a Causal $H(z)$, ROC is $|z| > \frac{4}{3}$, Since one pole is lying outside the unit circle, the given system is not both Causal & Stable.

(b) $H(z) = \dfrac{1+2z^{-1}}{1+\frac{6}{5}z^{-1}+\left(\frac{9}{25}\right)z^{-2}} = \dfrac{z(z+2)}{z^2+\left(\frac{6}{5}\right)z+\frac{9}{25}} = \dfrac{z(z+2)}{\left(z+\frac{3}{5}\right)^2}$

location of poles is at $z = -3/5$

Since all the poles are lying inside the unit circle in the Z-plane, the system is both Causal & Stable.

Ex:- An LTI System is described by the equation.

$$y(n) = x(n) + 0.81x(n-1) - 0.81x(n-2) - 0.45y(n-2).$$

Determine the transfer function of the system. Sketch the pole & zeros on the Z-plane. Asses the stability.

$\Rightarrow$

$$y(n) = x(n) + 0.81x(n-1) - 0.81x(n-2) - 0.45y(n-2)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.81z^{-1}-0.81z^{-2}}{1+0.45z^{-2}}$$

$$H(z) = \frac{z^2+0.81z-0.81}{z^2+0.45}$$ poles are obtained by equating Dr to 0

$z^2+0.45 = 0$ $\quad z = \pm j\sqrt{0.45} = \pm j0.67.$

Poles are $z = j0.67$ & $z = -j0.67$.

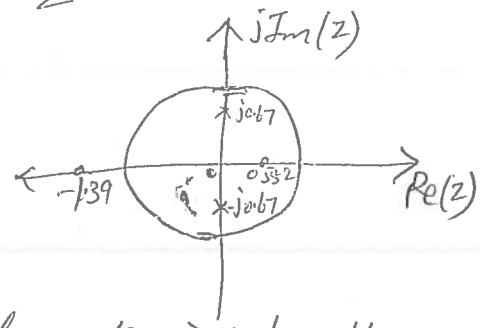
Equating the numerator to zero, we have

$$z^2 + 0.81z - 0.81 = 0$$

$$z = \frac{-0.81 \pm \sqrt{0.656 + 3.24}}{2} = \frac{-0.81 \pm \sqrt{3.896}}{2} = -1.39, 0.582$$

Zeros are at $z = -1.39$ & $z = 0.58$

$$H(z) = \frac{(z+1.39)(z-0.582)}{(z-j0.67)(z+j0.67)}$$




The System is stable as both the poles are inside the unit circle.

Using Residue Method find the Inverse Z-Transform of

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}}; \quad ROC\ |z|>3$$

Ans

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}} = \frac{z(z+2)}{z^2+4z+3} = \frac{z(z+2)}{(z+1)(z+3)}$$

$x(n) = \sum$ Residues of $X(z)z^{n-1}$ at poles of same within C.

$$x(n) = \sum \text{Residues of } \frac{z(z+2)}{(z+1)(z+3)} z^{n-1}$$

$$x(n) = \frac{z^n(z+2)}{(z+1)(z+3)} \text{ at the poles of same within C.}$$

$$x(n) = \sum \text{Residues of } \frac{z^n(z+2)}{(z+1)(z+3)} \text{ at poles } z=-1 \text{ \& } z=-3$$

$$= (z+1)\frac{z^n(z+2)}{(z+1)(z+3)}\bigg|_{z=-1} + \frac{(z+3)(z+2)z^n}{(z+1)(z+3)}\bigg|_{z=-3}$$

$$x(n) = (-1)^n \frac{1}{2}u(n) + (-3)^n \frac{1}{2}u(n).$$

Ex:- We want to design a Causal Discrete time LTI system with the property that if the i/p is $x(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{5}\left(\frac{1}{3}\right)^{n-1}u(n-1)$, then the o/p is $y(n) = \left(\frac{1}{2}\right)^n u(n)$. Determine the transfer function $H(z)$, the impulse response $h(n)$ & frequency response $H(e^{j\omega})$ of the system that satisfies this condition.

Δ $x(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{5}\left(\frac{1}{3}\right)^{n-1}u(n-1)$

$$X(z) = \frac{z}{z-\frac{1}{3}} - \frac{1}{5}\left(\frac{1}{z-\frac{1}{3}}\right) = \frac{z-\frac{1}{5}}{z-\frac{1}{3}}$$

$y(n) = \left(\frac{1}{2}\right)^n u(n) \qquad Y(z) = \frac{z}{z-\frac{1}{2}}$

$$\frac{Y(z)}{X(z)} = \left(\frac{z}{z-\frac{1}{2}}\right) \Big/ \frac{z-\frac{1}{5}}{z-\frac{1}{3}}$$

$$H(z) = \frac{z\left(z-\frac{1}{3}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{5}\right)}$$

$$\frac{H(z)}{z} = \frac{z-\frac{1}{3}}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{5}\right)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{5}} = \frac{5/9}{z-\frac{1}{2}} + \frac{4/9}{z-\frac{1}{5}}$$

$$H(z) = \frac{5}{9}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{4}{9}\left(\frac{z}{z-\frac{1}{5}}\right)$$

$$h(n) = \frac{5}{9}\left(\frac{1}{2}\right)^n u(n) + \frac{4}{9}\left(\frac{1}{5}\right)^n u(n)$$

$$H(e^{j\omega}) = \frac{e^{j\omega}\left(e^{j\omega}-\frac{1}{3}\right)}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}-\frac{1}{5}\right)}$$

Ex:- An LTI system is defined by the Difference equation

$$y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = x(n) - 3x(n-1)$$

Specify the ROC of $H(z)$, & determine $h(n)$ for the following conditions: (a) System is stable (b) System is causal.

$$y(n) - \frac{9}{4}y(n-1) + \frac{1}{2}y(n-2) = x(n) - 3x(n-1)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-3z^{-1}}{1-\frac{9}{4}z^{-1}+\frac{1}{2}z^{-2}} = \frac{z(z-3)}{\left(z-\frac{1}{4}\right)(z-2)}$$

$$\frac{H(z)}{z} = \frac{z-3}{\left(z-\frac{1}{4}\right)(z-2)} = \frac{11}{7}\left(\frac{z}{z-\frac{1}{4}}\right) - \frac{4}{7}\left(\frac{z}{z-2}\right)$$

2-poles at $z=\frac{1}{4}$ & $z=2$

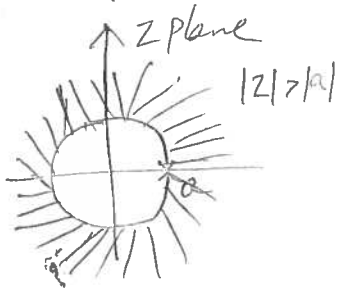
(a) For system to be stable, ROC must include inside the unit circle: $\frac{1}{4} < |z| < 2$. $\therefore h(n) = \frac{11}{7}\left(\frac{1}{4}\right)^n u(n) + \frac{4}{7}(2)^n u(-n-1)$ (system is stable, not causal)

(b) For the system to be causal, the ROC should be outside the outermost pole. ROC is $|z|>2$ $h(n) = \frac{11}{7}\left(\frac{1}{4}\right)^n u(n) - \frac{4}{7}(2)^n u(n)$ → System is causal, but not stable

Ex:- Prove that the sequences

(a) $x(n) = a^n u(n)$ & (b) $x(n) = -a^n u(-n-1)$

have the same $X(z)$ and differ only in their ROC.




(a) $x(n) = a^n u(n)$

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

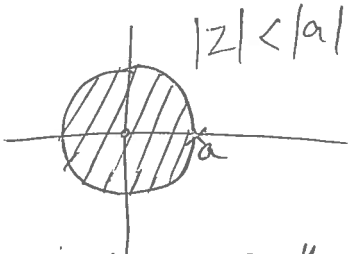
$$Z(x(n)) = X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}$$

ROC:- $|z| > |a|$

(b) $x(n) = -a^n u(-n-1)$ (Anticausal Signal)

$$X(z) = \sum_{n=-\infty}^{-1} (a^n) z^{-n} = \sum_{n=1}^{\infty} -a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n = +1 - \sum_{n=0}^{\infty} a^{-1} z^{-1}$$

$$X(z) = -\left(\left(\frac{1}{1 - a^{-1} z}\right) - 1\right) = 1 - \frac{1}{1 - a^{-1} z}$$




$$X(z) = \frac{z}{z-a} \qquad \text{ROC; } |z| < |a|.$$

In General the ROC of a Causal Signal is $|z| > a$ & the ROC of a Non-Causal Signal is $|z| < a$, where 'a' is some constant.

Ex:- Find the Z-Transform & ROC of $X(z)$ for.

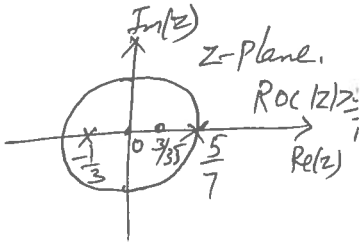
$$x(n) = 3\left(\frac{5}{7}\right)^n u(n) + 2\left(\frac{-1}{3}\right)^n u(n).$$

$$X(z) = \sum_{n=0}^{\infty} 3\left(\frac{5}{7}\right)^n z^{-n} + \sum_{n=0}^{\infty} 2\left(\frac{-1}{3}\right)^n z^{-n} = 3\sum_{n=0}^{\infty}\left(\frac{5}{7} z^{-1}\right) + 2\sum_{n=0}^{\infty}\left(\frac{-1}{3} z^{-1}\right)^n$$

The above summation converges if $\left|\frac{5}{7} z^{-1}\right| < 1$ & $\left|\frac{-1}{3} z^{-1}\right| < 1$

i.e., $|z| > \frac{5}{7}$ & $|z| > \frac{1}{3}$

$\therefore$ $X(z)$ converges for $|z| > \frac{5}{7}$.




$$X(z) = 3\left(\frac{1}{1 - \frac{5}{7} z^{-1}}\right) + 2\left(\frac{1}{1 + \frac{1}{3} z^{-1}}\right) = \frac{5z(z - 3/35)}{(z - 5/7)(z + 1/3)}$$

Ex:- Find the Z-transform of the following

(a) $n\delta(n)$ (b) $\delta(n-4)$ (c) $\delta(n+3)$ (d) $n\delta(n-2)$ (e) $y(n)=x(n-1)u(n)$

(f) $y(n)=x(n+1)u(n)$

Sol:-

(a) $n\delta(n)$

$x(n)=n\delta(n)$

$$X(z)=Z(n\delta(n))=\sum_{n=-\infty}^{\infty} n\delta(n)z^{-n}=0\,\delta(0)z^{0}=0.$$

(b) $x(n)=\delta(n-4)$

$$X(z)=Z(\delta(n-4))=\sum_{n=-\infty}^{\infty}\delta(n-4)z^{-n}=\delta(4-4)z^{-4}=z^{-4}$$

ROC:- Entire z-plane except $z=0$

(c) $x(n)=\delta(n+3)$

$X(z)=z^{3}$. ROC:- Entire Z-plane except at $z=\infty$.

(d) $x(n)=n\delta(n-2)$

$\delta(n-2)=1$ for $n=2$

$=0$ otherwise

$$X(z)=\sum_{n=-\infty}^{\infty} n\delta(n-2)z^{-n}=2z^{-2}.$$ ROC:- Except at $z=0$

(e) $y(n)=x(n-1)u(n)$

$$Y(z)=\sum_{n=-\infty}^{\infty}x(n-1)u(n)z^{-n}=\sum_{n=0}^{\infty}x(n-1)z^{-n}$$

$n-1=K$ $\quad K=-1$ to $\infty$

$$Y(z)=\sum_{K=-1}^{\infty}x(K)z^{-(K+1)}=z^{-1}\sum_{K=-1}^{\infty}x(K)z^{-K}=z^{-1}\left(\sum_{K=-1}^{-1}x(K)z^{-K}+\sum_{K=0}^{\infty}x(K)z^{-K}\right)$$

$$Y(z)=z^{-1}\left(x(-1)z^{+1}+X(z)\right)=x(-1)+z^{-1}X(z).$$

(f) $y(n)=x(n+1)u(n)$

$$Y(z)=\sum_{n=0}^{\infty}x(n+1)z^{-n}$$ $\quad n+1=K$, $\quad n=K-1$

$$Y(z)=\sum_{K=1}^{\infty}x(K)z^{-(K-1)}=z\sum_{K=1}^{\infty}x(K)z^{-K}=zX(z)-x(0)$$

Ex:- Find the Convolution of the sequences

$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ & $x_2(n) = \left(\frac{1}{3}\right)^{n-2} u(n-2)$.

⇒ Using ⓐ .. Convolution property of Z-Transforms & ⓑ Time domain meth

⇒ ⓐ Convolution property of Z-Transform :-

$$X_1(z) = \frac{z}{z-\frac{1}{2}}$$

$$X_2(z) = z^{-2}\left(\frac{z}{z-\frac{1}{3}}\right) = \frac{z^{-1}}{z-\frac{1}{3}}$$

$$x(n) = x_1(n) * x_2(n)$$

$$X(z) = X_1(z)X_2(z) = \frac{z}{z-\frac{1}{2}} \times \frac{z^{-1}}{z-\frac{1}{3}} = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$X(z) = Z^{-1}\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{3}}\right)6 = IZT\left(6\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{3}}\right)\right)$$

$$x(n) = 6\left(\left(\frac{1}{2}\right)^{n-1}u(n-1) - \left(\frac{1}{3}\right)^{n-1}u(n-1)\right)$$

ⓑ Time Domain :-

$$x_1(n) * x_2(n) = \sum_{K=0}^{n} x_1(K)\, x_2(n-K)$$

$$= \sum_{K=0}^{n-2} \left(\frac{1}{2}\right)^K u(K) \left(\frac{1}{3}\right)^{n-2-K} u(n-2-K)$$

$$= \sum_{K=0}^{n-2} \left(\frac{1}{2}\right)^K \left(\frac{1}{3}\right)^{n-2-K} = \sum_{K=0}^{n-2} \left(\frac{1}{2}\right)^K \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^{-2} \left(\frac{1}{3}\right)^{-K}$$

$$= 9\left(\frac{1}{3}\right)^n \left(\sum_{K=0}^{n-2} \left(\frac{3}{2}\right)^K\right) = 9\left(\frac{1}{3}\right)^n \left(\frac{1-\left(\frac{3}{2}\right)^{n-1}}{1-\frac{3}{2}}\right)$$

$$= -18\left(\frac{1}{3}\right)^n \left(1-\left(\frac{3}{2}\right)^{n-1}\right) = -6\left(\left(\frac{1}{3}\right)^{n-1}u(n-1) - \left(\frac{1}{3}\right)^{n-1}\left(\frac{3}{2}\right)^{n-1}u(n-1)\right)$$

$$= +6\left(\left(\frac{1}{2}\right)^{n-1}u(n-1) - \left(\frac{1}{3}\right)^{n-1}u(n-1)\right)$$

Ex $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ $x_2(n) = \left(\frac{1}{5}\right)^n u(n)$

Correlation Property:-

* $R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z^{-1})$

Proof:-

$$Z[x_1(n) \otimes x_2(n)] = \sum_{n=-\infty}^{\infty} (x_1(n) \otimes x_2(n))\, z^{-n}.$$

$$x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n) \quad \text{or} \quad x_1(n) \otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l-n)\, x_2(l)$$

$$Z(x_1(n) \otimes x_2(n)) = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n)\, z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x_1(l) \sum_{n=-\infty}^{\infty} x_2(l-n)\, z^{-n}.$$

$l-n=m$

$n = l-m$

$$= \sum_{l=-\infty}^{\infty} x_1(l) \sum_{m=-\infty}^{\infty} x_2(m)\, z^{-(l-m)}$$

$$= \left(\sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l}\right)\left(\sum_{m=-\infty}^{\infty} x_2(m)\, z^{m}\right)$$

$$= X_1(z)\, X_2(z^{-1})$$

$$R_{12}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z^{-1})$$

Ex:- Determine the Z-Transform of $x(n) = \frac{1}{3}(n^2+n)\left(\frac{1}{2}\right)^{n-1} u(n-1)$

$\Rightarrow$ $Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{z}{z-\frac{1}{2}}; \quad |z| > \frac{1}{2}.$

Time shifting

$$Z\left\{\left(\frac{1}{2}\right)^{n-1} u(n-1)\right\} = \frac{z^{-1} \times z}{z-\frac{1}{2}} = \frac{1}{z-\frac{1}{2}}$$

Multiplication by n

$$Z\left\{n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right\} = -z\frac{d}{dz}\left\{Z\left(\frac{1}{2}\right)^{n-1} u(n-1)\right\} = -z\frac{d}{dz}\left(\frac{1}{z-\frac{1}{2}}\right) = \frac{z}{\left(z-\frac{1}{2}\right)^2}$$

Multiplication by $n^2$.

$$Z\left\{n^2\left(\frac{1}{2}\right)^{n-1} u(n-1)\right\} = z^2\frac{d^2}{dz^2}\left(\frac{1}{z-\frac{1}{2}}\right) = \frac{z\left(z+\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)^3}.$$

$$\Rightarrow X(z) = (2/3)\, z^2/\left(z-\frac{1}{2}\right)^3$$

$$X(z) = \frac{1}{3}\left(\frac{z\left(z+\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)^3} + \frac{z}{\left(z-\frac{1}{2}\right)^2}\right)$$

# Trignometry Formulae

| θ | Radians | Sinθ | Cosθ | tanθ |
|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | $\infty$ |
| 180° | $\pi$ | 0 | -1 | 0 |
| 360° | $2\pi$ | 0 | 1 | 0 |

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta = 1 + \sec^2\theta$

$\cot^2\theta = 1 + \operatorname{cosec}^2\theta$

$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$

$\cos A \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$

$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$\cos A \sin B \quad \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$ ; $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos(A-B) = \cos A\cos B + \sin A \sin B$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin(A-B) = \sin A\cos B - \cos A \sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\sin(2\pi n + \theta) = \sin\theta$

$\cos(2\pi n + \theta) = \cos\theta$

$\tan(2\pi n + \theta) = \tan\theta$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

I-Q :-

$\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

$\tan(90-\theta) = \cot\theta$

II-Q :-

$\sin(180-\theta) = \sin\theta$

$\cos(180-\theta) = -\cos\theta$

$\tan(180-\theta) = -\tan\theta$

III-Q :-

$\sin(180+\theta) = -\sin\theta$

$\cos(180+\theta) = -\cos\theta$

$\tan(180+\theta) = \tan\theta$

IV-Q :-

$\sin(360-\theta) = -\sin\theta$

$\cos(360-\theta) = \cos\theta$

$\tan(360-\theta) = -\tan\theta$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$.

## Integration Formulas :-

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int e^x dx = e^x$$

$$\int \frac{1}{x} dx = \ln x$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$$

$$\int dx = x$$

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\int \tan x \, dx = \log|\sec x|$$

$$\int \cot x \, dx = \log|\sin x|.$$

$$\int \sec x \, dx = \log|\sec x + \tan x|$$

$$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x|$$

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int f'(x) \left(\int g(x) dx\right) dx$$

## Differentiation formulas :-

$$\frac{d}{dx} a^x = a^x \ln a.$$

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(cy) = c\frac{dy}{dx}.$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x.$$