

UNIT-2

What is LTI system

The representation of DT signal in terms of impulses

The representation of CT signal in terms of impulses

The CT unit impulse responses

convolution for CT LTI system

properties of convolution

properties of CT LTI system

memory less systems

stability

invertibility

causality

unit step response

Differential equation models and solutions of differential equations

natural and forced responses

ZIR & ZSR

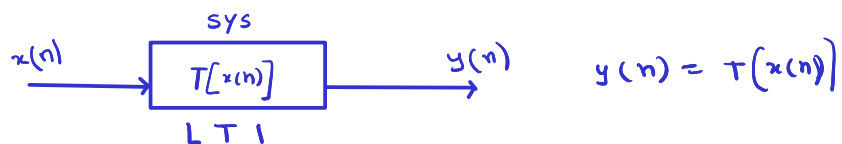
stability in terms of natural responses

system response to complex exponential inputs

LTI (linear time invariant) system:

→ The system which satisfies the both properties of linearity and time invariant is called LTI system.

many physical process possess linearity and time invariant property thus can be modeled as LTI systems.



$\delta(n) \rightarrow h(n) \rightarrow$ unit impulse response

$u(n) \rightarrow y(n) \rightarrow$ unit step response

$r(n) \rightarrow y(n) \rightarrow$ unit ramp response

$\delta(t) \rightarrow h(t) \rightarrow$ unit impulse response

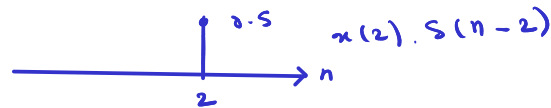
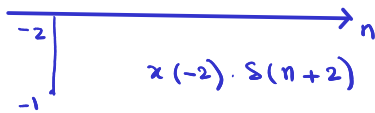
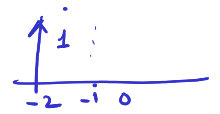
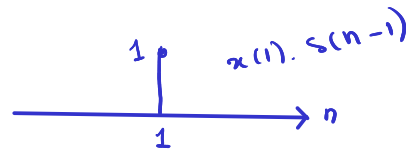
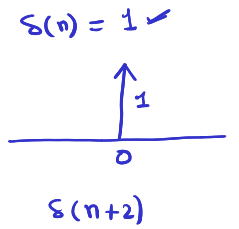
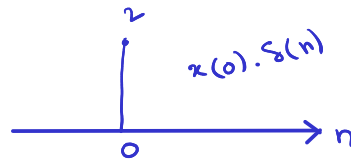
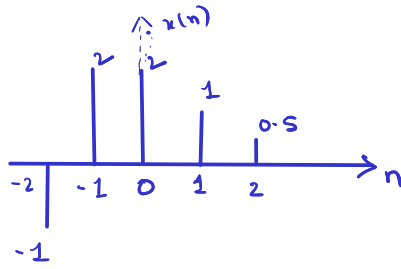
Any LTI system response can be constructed as sum of linear combination of basic signal response

one of the important characteristics of unit impulse is both in DT and CT, is that very general signal can be represented as linear combination of delayed unit impulses

we can know the characterization of any LTI system by using the properties of superposition and time invariance in terms of its response to an unit impulse

such a representation is called convolution sum in DT and convolution integral in CT

Representation of DT signals in terms of impulses



$$x(n) = x(-2) \delta(n+2) + x(-1) \delta(n+1) + x(0) \delta(n) + x(1) \delta(n-1) + x(2) \delta(n-2)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k) \quad \checkmark \rightarrow \text{i/p}$$

$$y(n) = T \left\{ \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k) \right\}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot T \left\{ \delta(n-k) \right\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \rightarrow \text{o/p}$$

convolution sum

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

→ Replace $x(n)$, $y(n)$ by $x(k)$, $h(k)$

→ $h(k) \rightarrow h[-k]$ T.R

→ $h[-k+n]$ or $h[-(k-n)]$

if n +ve shift to RS

n -ve shift to LS

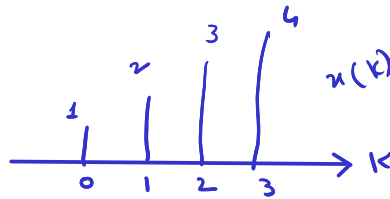
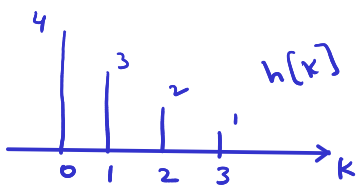
Q] Find the response of DT LTI system having impulse response $\{4, 3, 2, 1\}$ for an i/p $\{1, 2, 3, 4\}$

$\uparrow$
 $h[n]$

$\uparrow$
 $x[n]$

$$h[k] = \{4, 3, 2, 1\}, \quad x[k] = \{1, 2, 3, 4\}$$

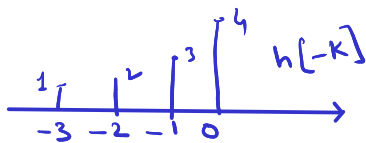
Graphical method



Length of resultant

$$\begin{aligned} \text{seq} &= L + M - 1 \\ &= 4 + 4 - 1 \\ &= 7 \end{aligned}$$

$$h[k] \rightarrow h[-k]$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

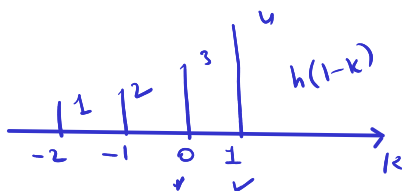
$$y[0] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[0-k]$$

$$\begin{aligned} y[0] &= x[0] \cdot h[0] \\ &= 1 \times 4 \Rightarrow 4 \checkmark \end{aligned}$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[1-k]$$

$\Rightarrow$

4



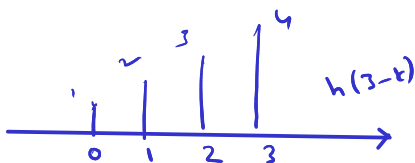
$$\begin{aligned} y[1] &= (3 \times 1) + (4 \times 2) \\ &= 11 \checkmark \end{aligned}$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[2-k]$$



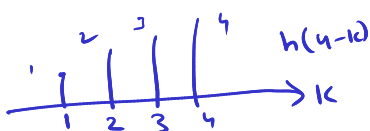
$$\begin{aligned} y[2] &= (2 \times 1) + (3 \times 2) + (4 \times 3) \\ &= 2 + 6 + 12 \\ &= 20 \end{aligned}$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[3-k]$$



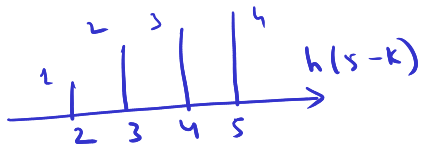
$$\begin{aligned} y[3] &= (1 \times 1) + (2 \times 2) + (3 \times 3) + (4 \times 4) \\ &= 30 \end{aligned}$$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[4-k]$$



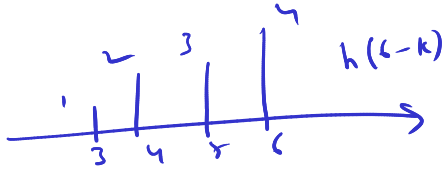
$$\begin{aligned} y[4] &= (2 \times 1) + (2 \times 3) + (3 \times 4) \\ &= 2 + 6 + 12 \\ &= 20 \end{aligned}$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(5-k)$$

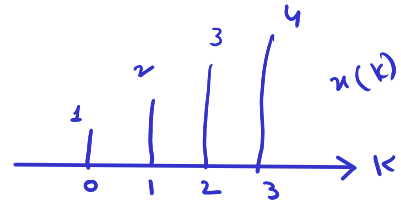


$$y(5) = (3 \times 1) + (4 \times 2) = 11$$

$$y(6) = \sum_{k=-\infty}^{\infty} x(k) h(6-k)$$



$$y(6) = 1 \times 4 = 4$$



$$y(n) = \{4, 11, 20, 30, 20, 11, 4\} \checkmark$$

Tabular method! ✓

	h(n)			
x(n)	4	3	2	1
1	4	3	2	1
2	8	6	4	2
3	12	9	6	3
4	16	12	8	4

$$y(n) = \{4, 11, 20, 30, 20, 11, 4\} \checkmark$$

$$x(n) = \{1, 0, 2\}$$

$$h(n) = \{2, 0, 1, -1\}$$

$$y(n) = \{-, -, -, -\}$$

Q) Find the response of DT LTI system having impulse response $\{1, 2, 3\}$ for an i/p $\{-1, 1, 0, 1, -1\}$

So, $x[n] = \{-1, 1, 0, 1, -1\}$ $h[n] = \{1, 2, 3\}$ $y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

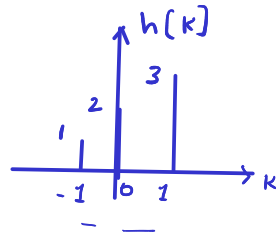
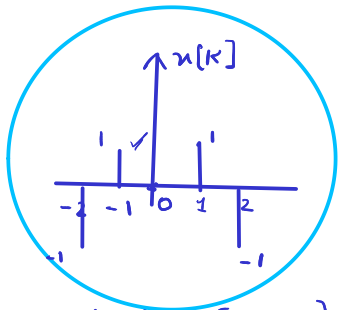
$\uparrow$ $\uparrow$
 $x[-1]$ $x[1]$ $x[2]$ $h[-1]$ $h[0]$ $h[1]$

Length of resultant seq.

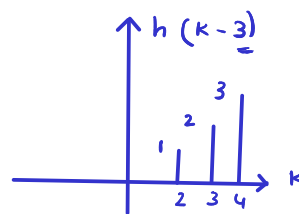
$$L + m - 1 \Rightarrow 5 + 3 - 1 \Rightarrow 7$$

$$n = -3, -2, -1, 0, 1, 2, 3$$

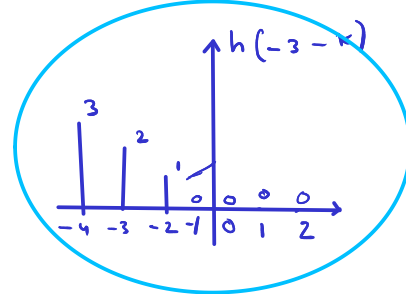
$$y[-3] = \sum_{k=-\infty}^{\infty} x[k] h[-3-k]$$



$\Rightarrow$



$\Rightarrow$



$$y[-3] = [(-1) \times 1] + [(1) \times 0] + [0 \times 0] + [(1) \times 0] + [(-1) \times 0]$$

$$y[-3] = -1$$

$$y[-2] = \sum_{k=-\infty}^{\infty} x[k] h[-2-k]$$

$$y[-2] = (1 \times 1) + (-1 \times 2) = 1 - 2$$

$$y[-2] = -1$$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k] h[-1-k]$$

$$y[-1] = (1 \times 2) + (-1 \times 3) = 2 - 3 = -1$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k] h[0-k]$$

$$y[0] = (1 \times 3) + (1 \times 1) = 4$$

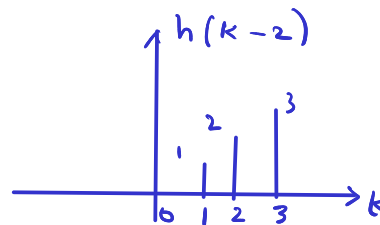
$$y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1-k]$$

$$y[1] = (1 \times 2) + (-1 \times 1) = 1$$

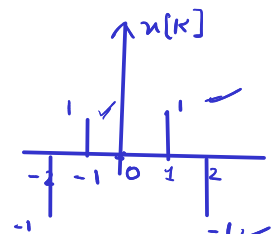
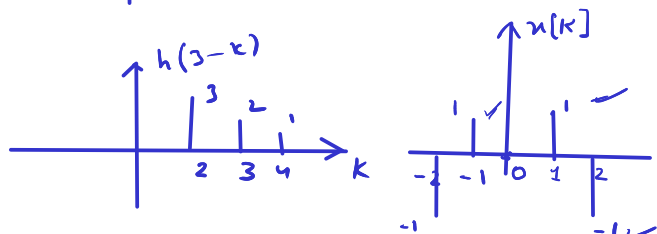
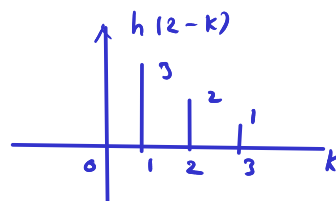
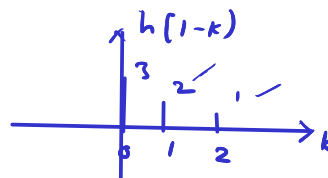
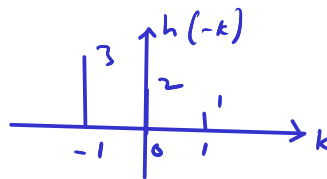
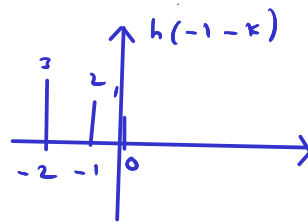
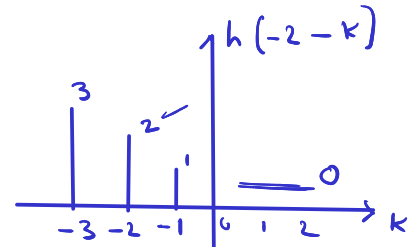
$$y[2] = \sum_{k=-\infty}^{\infty} x[k] h[2-k]$$

$$y[2] = (1 \times 3) + (-1 \times 2) = 3 - 2 = 1$$

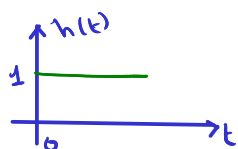
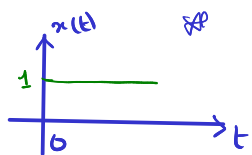
$$y[3] = \sum_{k=-\infty}^{\infty} x[k] h[3-k]$$



$\Rightarrow$



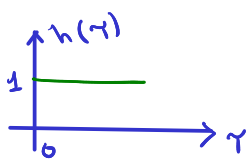
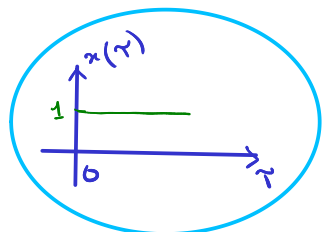
Ex ① $x(t) = u(t)$, $h(t) = u(t)$, find $y(t) = ?$



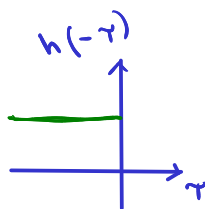
→ Replace t with τ

→ $h(\tau) \rightarrow h(-\tau)$

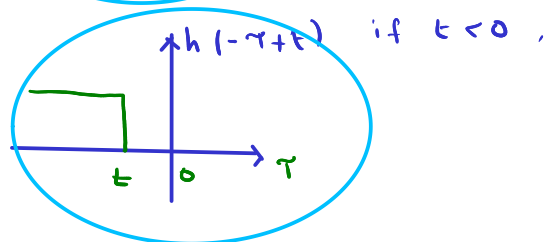
→ $h(-\tau + t)$



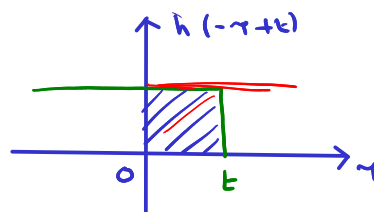
⇒



$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$



$h(-\tau + t)$ for $t > 0$



no overlapping area b/w $x(\tau), h(-\tau + t)$

∴ $y(t) = 0$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$

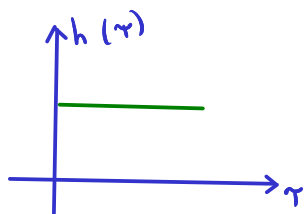
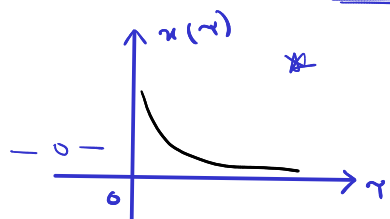
$$y(t) = \int_0^t u(\tau) \cdot u(t-\tau) d\tau$$

$$\therefore y(t) = \begin{cases} t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

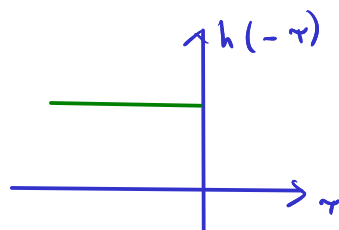
$$y(t) = \int_0^t 1 \cdot d\tau \Rightarrow [\tau]_0^t \Rightarrow t$$

∴ $u(t) * u(t) = r(t)$ ✓

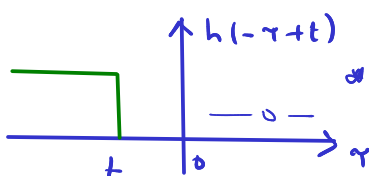
Ex ②: $x(t) = \underline{\underline{e^{-t}} \cdot u(t)}$, $h(t) = \underline{\underline{u(t)}}$, $y(t) = ?$



⇒

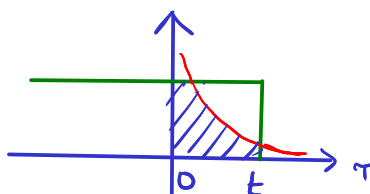


$h(-\tau + t)$, if $t < 0$



$y(t) = 0$

$h(-\tau + t)$, if $t \geq 0$



$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$y(t) = \int_0^t e^{-\tau} u(\tau) \cdot u(t-\tau) d\tau \Rightarrow \int_0^t e^{-\tau} d\tau \Rightarrow \left[\frac{e^{-\tau}}{-1} \right]_0^t \Rightarrow -[e^{-\tau}]_0^t$$

$$y(t) \Rightarrow [1 - e^{-t}] \cdot u(t)$$

$$y(t) = \begin{cases} 1 - e^{-t} & , t \geq 0 \checkmark \\ 0 & , t < 0 \checkmark \end{cases}$$

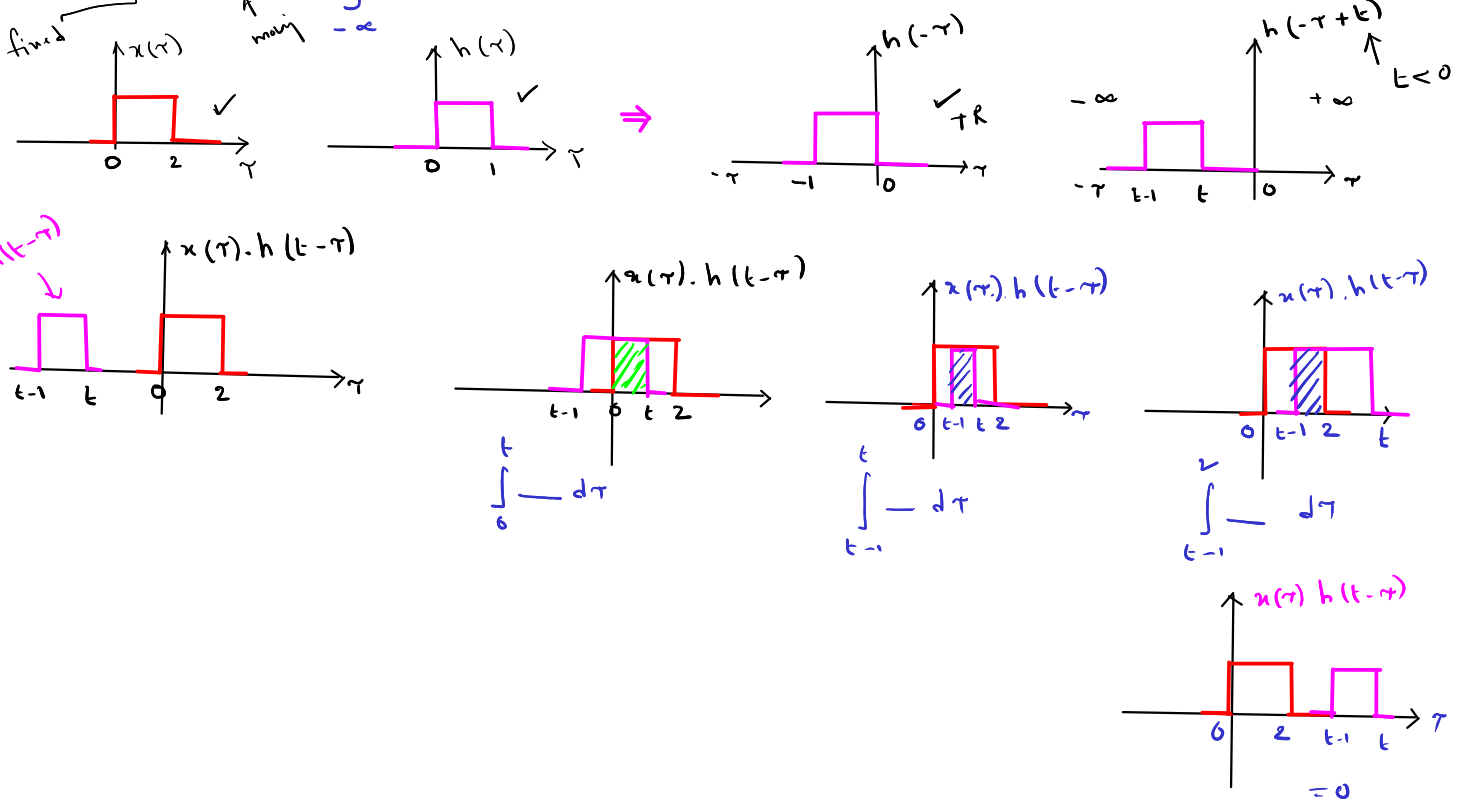
→ convolution of any signal with $u(t)$, is the integral of corresponding signal

$$y(t) = \underline{\bar{e}^t} \cdot u(t) * u(t)$$

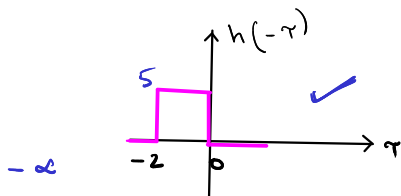
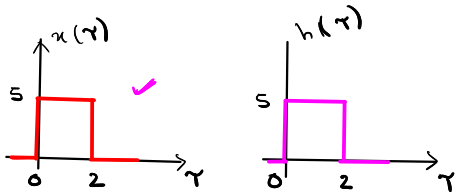
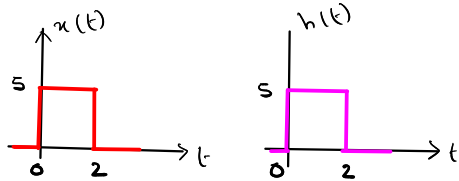
$$y(t) = \int_0^t \bar{e}^t \cdot dt \Rightarrow \left(\frac{\bar{e}^t}{-1} \right)_0^t \Rightarrow (1 - \bar{e}^t) \cdot u(t) \checkmark$$

⇒ A convolution is an integral that express the amount of overlap of one function when it is shifted over another function

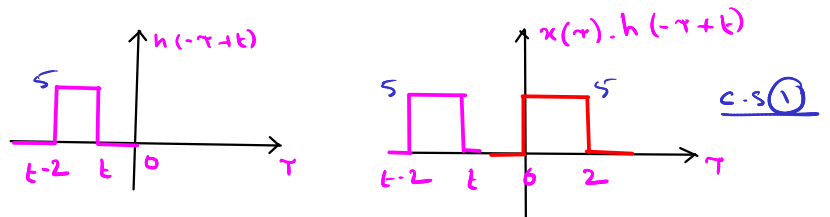
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau$$



Ex ①



$h(-\tau+t)$ if $t < 0$

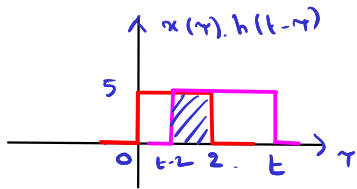


$$y(t) = 0$$

$h(-\tau+t)$ if $t \geq 0$, $t-2 < 0 \Rightarrow 0 \leq t \leq 2$

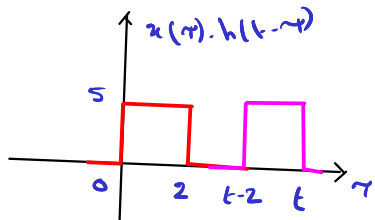
$$\begin{aligned} & \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau \\ & \Rightarrow \int_0^t (5)(5) d\tau \\ & \Rightarrow \int_0^t 25 d\tau \Rightarrow 25[\tau]_0^t \Rightarrow 25t \end{aligned}$$

if $t > 2$ & $t-2 \leq 2$

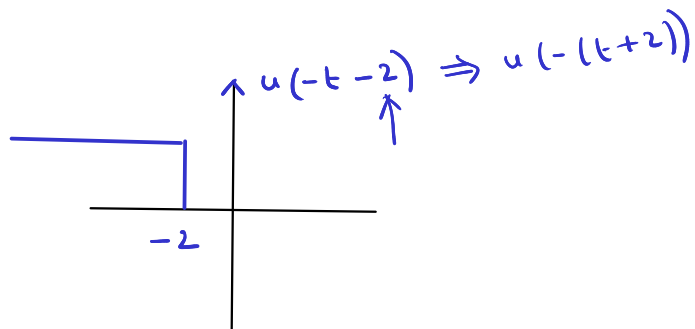
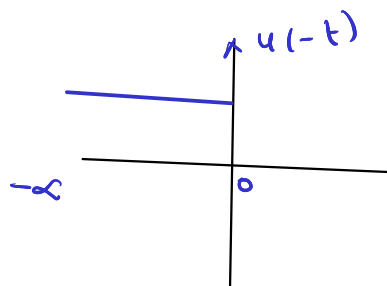
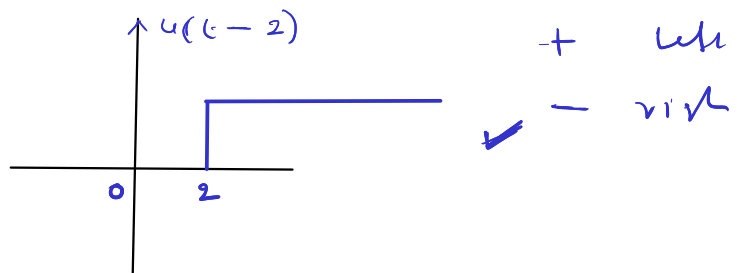
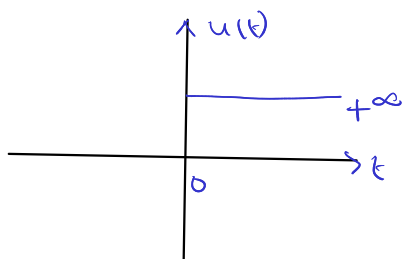


$$y(t) = \int_{t-2}^2 (5)(5) d\tau \Rightarrow 25[\tau]_{t-2}^2 \Rightarrow 25(2-t+2) \Rightarrow 25(4-t) \leftarrow$$

if $t-2 > 2 \Rightarrow t > 4$



$$y(t) = 0 \quad y(t) = \begin{cases} 0 & , t < 0 \\ 25t & , 0 \leq t \leq 2 \\ 5(4-t) & , 2 \leq t \leq 4 \\ 0 & , t > 4 \end{cases}$$



$$\textcircled{1} \quad x(t) = e^{-3t} \{u(t) - u(t-2)\} \quad , \quad h(t) = e^{-t} u(t) \quad = \int_{-\infty}^{\infty} \cancel{x(\tau) \cdot h(t-\tau)}$$

$$\textcircled{2} \quad x(t) = u(t) - u(t-2) \quad , \quad h(t) = t \cdot [u(t) - u(t-1)]$$

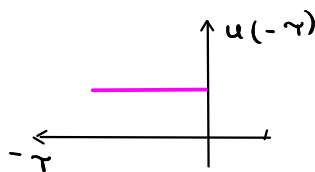
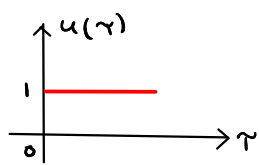
Ex: $x(t) = e^{-3t} [u(t) - u(t-2)]$, $h(t) = e^{-t} \cdot u(t)$, $y(t) = ?$

$x(\tau) = e^{-3\tau} [u(\tau) - u(\tau-2)]$, $h(\tau) = e^{-\tau} \cdot u(\tau)$

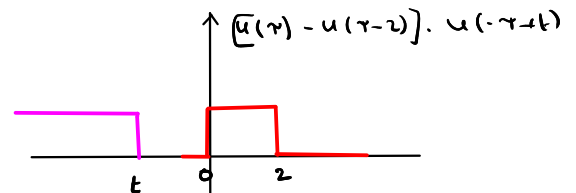
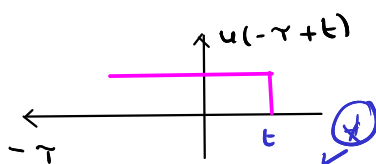
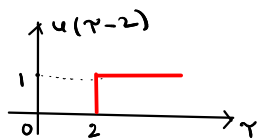
$y(t) = \int_{-\infty}^{\infty} x(\tau) \underline{h(t-\tau)} d\tau \checkmark$

$y(t) = \int_{-\infty}^{\infty} e^{-3\tau} (u(\tau) - u(\tau-2)) e^{-(t-\tau)} \cdot u(t-\tau) d\tau$

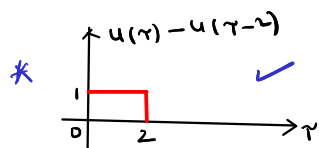
$y(t) = e^{-t} \int_{-\infty}^{\infty} \underbrace{e^{-2\tau}}_{x(\tau)} \underbrace{(u(\tau) - u(\tau-2)) u(t-\tau)}_{h(t-\tau)} d\tau$



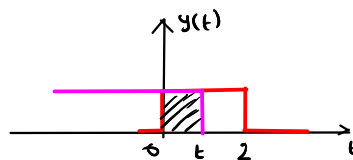
case(1) :- when $t < 0$



no overlapping $\therefore y(t) = 0$



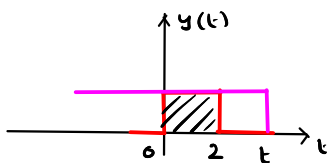
case(2) :- when $0 \leq t \leq 2$



$$y(t) = e^{-t} \int_0^t e^{-2\tau} d\tau \Rightarrow e^{-t} \left[\frac{e^{-2\tau}}{-2} \right]_0^t$$

$$\Rightarrow \frac{e^{-t}}{-2} [e^{-2t} - 1] \Rightarrow \frac{1}{2} [e^{-t} - e^{-3t}]$$

case(3) :- $t > 2$



$y(t) = e^{-t} \int_0^2 e^{-2\tau} d\tau$

$= e^{-t} \left[\frac{e^{-2\tau}}{-2} \right]_0^2$

$= \frac{e^{-t}}{-2} [e^{-4} - 1]$

$y(t) = \frac{1}{2} e^{-t} (1 - e^{-4}) \checkmark$

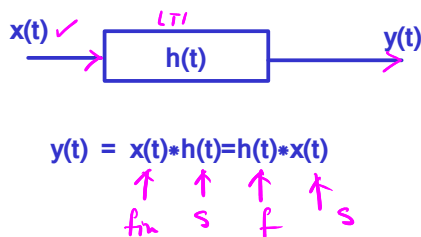
$\therefore y(t) = \begin{cases} 0 & , t < 0 \\ \frac{1}{2} (e^{-t} - e^{-3t}) & , 0 \leq t \leq 2 \\ \frac{1}{2} e^{-t} (1 - e^{-4}) & , t > 2 \end{cases}$

Properties of convolution

1. Commutative:

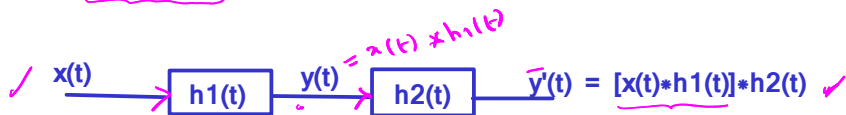
$$x_1(t) * x_2(t) = x_2(t) * x_1(t) \quad \checkmark$$

↑ fixed ↑ $x_2(t-\tau)$ ↑ fixed ↑ $x_1(t-\tau)$



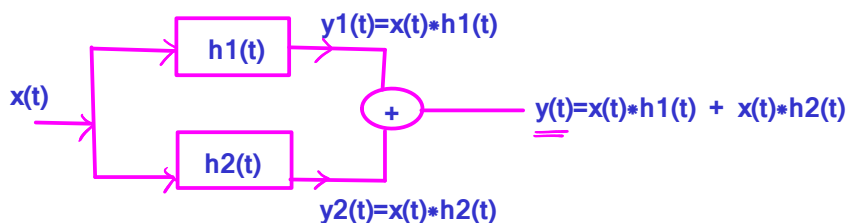
2. Associative :

$$[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)] \quad \checkmark$$



3. Distributive:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$



5. Derivative

$$y(t) = x(t) * h(t)$$

$$\begin{aligned} \frac{d}{dt} y(t) &= \frac{d}{dt} [x(t) * h(t)] \\ &= x(t) * \frac{d}{dt} h(t) \quad \checkmark \\ &\quad (\text{or}) \\ &= \frac{d}{dt} x(t) * h(t) \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{Ex: } y(t) &= r(t) * u(t) \\ \frac{d}{dt} y(t) &= r(t) * \frac{d}{dt} u(t) \\ &= r(t) * \delta(t) \Rightarrow r(t) \end{aligned}$$

4. properties of impulse signals

$$x(t) * \delta(t - t_1) = x(t - t_1)$$

$$x(t) * \delta(t) = x(t)$$

$$x(t) * A \delta(t - t_1) = A x(t - t_1)$$

$$x(t) * A \delta(t) = A \cdot x(t)$$

$$\text{Ex (i)} \quad x(t) = r(t)$$

$$\frac{r(t)}{\delta} * \delta(t-2) = r(t-2) \quad \checkmark$$

$$\mathcal{I}[\mathcal{D}[y(t)]] = y(t) \quad \checkmark$$

$$6. \quad x(t) * h(t) = \int_{-\infty}^t x(\tau) d\tau$$

Ex

$$y(t) = x(t) * u(t)$$

$$\frac{d}{dt} y(t) = x(t) * \frac{d}{dt} u(t)$$

$$\begin{aligned} \frac{d}{dt} y(t) &= x(t) * \delta(t) \\ \int_{-\infty}^t \frac{d}{dt} y(\tau) &= \int_{-\infty}^t x(\tau) * \delta(\tau) \end{aligned}$$

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

7. Time delay

$$x_1(t) * x_2(t) = y(t)$$

$$\downarrow t_1 \quad \downarrow t_2$$

$$x_1(t-t_1) * x_2(t-t_2) = y(t-(t_1+t_2))$$

Ex: $u(t-1) * u(t-2) = ?$

$$u(t) * u(t) = \int_{-\infty}^t u(\tau) d\tau \Rightarrow r(t)$$

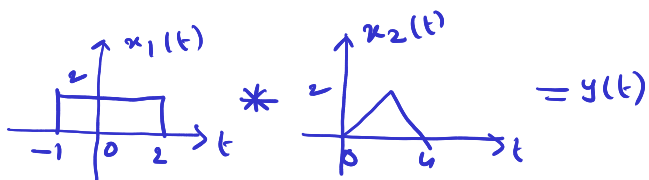
$$u(t-1) * u(t-2) = r(t-3)$$

⑨ Area:

$$x_1(t) * x_2(t) = y(t)$$

$$A_1 \times A_2 = A$$

Ex:



$$A_1 = 3 \times 2 = 6$$

$$\uparrow \quad \uparrow$$

$$\omega \times h$$

$$A_2 = \frac{1}{2} \times 4 \times 2 = 4$$

$$A = A_1 \times A_2 \Rightarrow 6 \times 4 \Rightarrow 24 \checkmark$$

8. Time scaling

$$x_1(t) * x_2(t) = y(t)$$

$$\downarrow a \quad \downarrow a$$

$$x_1(at) * x_2(at) = \frac{1}{|a|} y(at), \quad a \neq 0$$

Ex: $(t^2+t) * t = y(t) \checkmark$

$$? * ? = \frac{1}{3} y(3t)$$

$$a=3$$

$$\begin{array}{l} x_1(3t) * x_2(3t) \\ x_1(t) = t^2 + t \checkmark \\ x_1(3t) = 9t^2 + 3t \checkmark \\ x_2(t) = t \checkmark \\ x_2(3t) = 3t \checkmark \end{array}$$

$$\therefore (9t^2 + 3t) * 3t = \frac{1}{3} y(3t)$$

⑩ Duration/Extension

$$x_1(t), t_1 \leq t \leq t_2$$

$$x_2(t), t_3 \leq t \leq t_4$$

$$\underline{y(t)}, t_1+t_3 \leq t \leq t_2+t_4$$

- The upper and lower limits of convolution integral are the individual sum of upper and lower limits of input and $h(t)$
- Length of convolution integral is the sum of the individual input(L) and impulse response(M), $L+M$
- Area under convolution integral is the product of individual areas ✓ $A = A_1 \times A_2$
- Convolution of two rectangular pulses of equal width is an triangular pulse and slope of raising equals multiplication of individual amplitudes ✓
- Convolution of two rectangular pulses of unequal width results in a trapezoidal whose width is the sum of widths of the pulses ✓

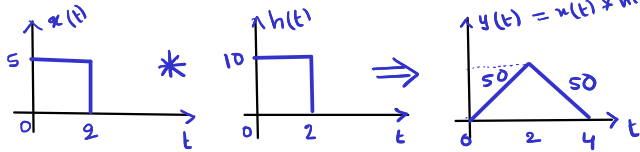
lower limit = sum of lower limits

upper limit = sum of upper limits

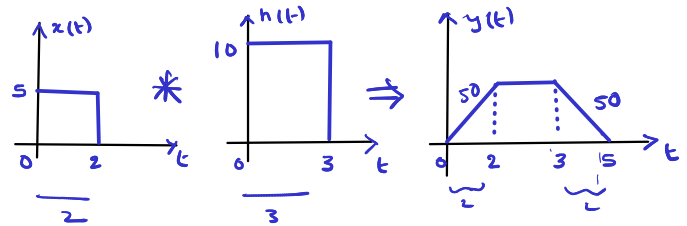
slope of raising arms = multiplication of individual amplitudes

width of raising and falling arms are same and equal to the width of smaller width of individual

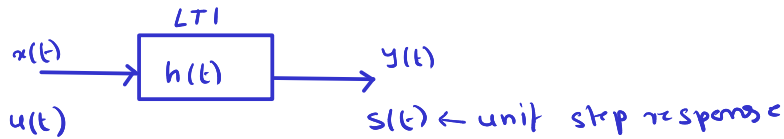
①



②



Unit step response of an LTI system



$$s(t) = u(t) * h(t)$$

$$s(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau$$

By using commutative prop.

$$s(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \quad \begin{cases} u(t-\tau) = 1, & t-\tau \geq 0, \quad t \geq \tau, \quad \underline{\tau \leq t} \\ = 0 & t-\tau < 0, \quad t < \tau, \quad \tau > t \end{cases}$$

$$s(t) = \int_{\tau=-\infty}^t h(\tau) d\tau \quad \checkmark$$

if we take first order derivative

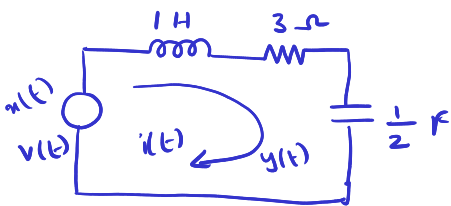
$$h(\tau) = \frac{d}{d\tau} s(\tau)$$

$$u(t) = 1, \quad t \geq 0$$

Ex ① $h(t) = t \cdot u(t)$

$$s(t) = \int_{-\infty}^t h(\tau) d\tau \Rightarrow \int_{-\infty}^t \tau \cdot u(\tau) d\tau \Rightarrow \int_0^t \tau d\tau \Rightarrow \left[\frac{\tau^2}{2} \right]_0^t \Rightarrow \frac{t^2}{2} \quad \checkmark$$

system model:



taking $v(t) = x(t) \quad \checkmark$

$i(t) = y(t) \quad \checkmark$

$$v(t) = v_L(t) + v_R(t) + v_C(t)$$

$$x(t) = L \frac{d}{dt} i(t) + i(t) \cdot R + \frac{1}{C} \int i(t) dt$$

$$x(t) = L \frac{d}{dt} y(t) + R y(t) + \frac{1}{C} \int y(t) dt$$

by taking diff on both sides

$$\checkmark \quad \frac{dx(t)}{dt} = L \frac{d^2 y(t)}{dt^2} + R \frac{dy(t)}{dt} + \frac{1}{C} y(t)$$

consider $\frac{d}{dt} = D$

$$D x(t) = L D^2 y(t) + R D y(t) + \frac{1}{C} y(t)$$

$$D x(t) = D^2 y(t) + 3 D y(t) + 2 y(t)$$

$$D x(t) = y(t) [D^2 + 3D + 2]$$

$$\frac{y(t)}{x(t)} = \frac{D}{D^2 + 3D + 2} \quad \checkmark$$

→ A LTI-CT system can be represented in the form of differential equation

$$\frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

→ ①

→ for a given i/p, the o/p of the system can be obtained by solving the differential equation

→ the solution of diff eq consist of two parts

① Zero state response (ZSR):

② Zero input response (ZIR)

$$\text{Total response} = \underset{\substack{\text{(or)} \\ \text{forced} \\ \text{response}}}{\text{ZSR}} + \underset{\substack{\text{(or)} \\ \text{natural response}}}{\text{ZIR}}$$

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

① ZSR / forced response / particular solution

→ ZSR is the response of the system when $x(t)$ is non zero (all the initial values (or) condition are zero)

② ZIR / natural response / homogeneous solution $y_h(t)$

→ the ZIR is the solution of the differential equation

when $x(t) = 0$ (result due to internal system conditions such as energy storing elements or initial values)

from eq ①

$$\frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0 \rightarrow (2)$$

Let the solution to the above equation

$$y_h(t) = c \cdot e^{\lambda t}$$

$$\frac{dy_h(t)}{dt} = c \cdot \lambda e^{\lambda t}$$

$$\frac{d^2 y_h(t)}{dt^2} = c \cdot \lambda^2 e^{\lambda t} \quad \dots \quad \frac{d^N y_h(t)}{dt^N} = c \cdot \lambda^N e^{\lambda t}$$

$$c \lambda^N e^{\lambda t} + a_{N-1} c \lambda^{N-1} e^{\lambda t} + \dots + a_1 c \lambda e^{\lambda t} + a_0 c e^{\lambda t} = 0$$

$$c e^{\lambda t} \left[\lambda^N + a_{N-1} \lambda^{N-1} + \dots + a_1 \lambda + a_0 \right] = 0$$

cancel

$$\lambda^N + a_{N-1} \lambda^{N-1} + \dots + a_1 \lambda + a_0 = 0$$

This polynomial is called the characteristic equation of the system.

This equation can be represented in factorized form as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0$$

here $\lambda_1, \lambda_2 \dots \lambda_N$ are roots of the characteristic eq.
also called characteristic roots

① Distinct roots

If the roots $\lambda_1, \lambda_2 \dots \lambda_N$ are distinct, then the solution is

$$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_N e^{\lambda_N t}$$

$c_1, c_2 \dots c_N \rightarrow$ arbitrary constant

Ex: $\lambda_1 = 2, \lambda_2 = 3 \quad y_h(t) = c_1 e^{2t} + c_2 e^{3t}$

② Repeated roots

If roots are repeated, then

$$y_h(t) = (c_1 + c_2 t + c_3 t^2 + \dots + c_m t^{m-1}) e^{\lambda_1 t} + c \cdot e^{\lambda_2 t} + \dots$$

Ex: If $\lambda_1 = 5$, $\lambda_2 = 5$, $\lambda_3 = 3$

$$y_h(t) = (c_1 + c_2 t) e^{5t} + c_3 e^{3t}$$

③ Complex roots :-

If the roots are complex

λ & λ^* then

$$\lambda_1 = a + jb$$

$$\lambda_2 = a - jb$$

$$y_h(t) = c e^{at} [\cos(bt + \theta)]$$

particular solution ($y_p(t)$) :-

→ The forced response of an LTI system is the response when initial condition is zero ✓

$y_p(t)$ can have different solutions

$$\begin{aligned} \text{① If } x(t) &= A \checkmark \\ y_p(t) &= K \checkmark \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{① If } x(t) &= A \checkmark \\ y_p(t) &= K \checkmark \end{aligned}} \right\} \checkmark \quad \begin{aligned} x(t) &= u(t) \checkmark \\ y_p(t) &= c \checkmark \end{aligned}$$

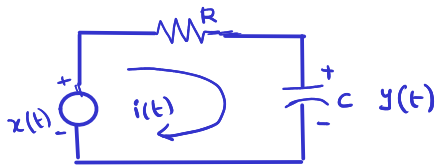
$$\begin{aligned} \text{② If } x(t) &= e^{at} \quad (a \neq \lambda_i) \\ y_p(t) &= A \cdot e^{at} \end{aligned}$$

$$\begin{aligned} \text{③ If } x(t) &= e^{at} \quad (a = \lambda_i) \\ y_p(t) &= A t e^{at} \end{aligned}$$

$$\begin{aligned} \text{④ If } x(t) &= \cos(\omega t + \theta) \\ y_p(t) &= A \cos(\omega t + \theta) \end{aligned}$$

Example problems:

1. Determine the ZIR of the circuit



$$-x(t) + i(t)R + y(t) = 0$$

$$x(t) = i(t)R + y(t) \rightarrow (1)$$

$$y(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau$$

$$\frac{dy(t)}{dt} = \frac{1}{C} i(t)$$

$$i(t) = C \frac{dy(t)}{dt} \rightarrow (2)$$

Sub eq (2) in eq (1)

$$x(t) = C \frac{dy(t)}{dt} R + y(t)$$

$$x(t) = RC \frac{dy(t)}{dt} + y(t) \rightarrow (3)$$

To obtain $y_{zi}(t)$,

$$x(t) = 0$$

from eq (3)

$$RC \frac{dy(t)}{dt} + y(t) = 0 \rightarrow (4)$$

$$y_{zi}(t) = c_1 e^{\lambda_1 t}$$

$$\frac{dy_{zi}(t)}{dt} = c_1 \lambda_1 e^{\lambda_1 t}$$

Sub above eq in eq (4)

$$RC(c_1 \lambda_1 e^{\lambda_1 t}) + c_1 e^{\lambda_1 t} = 0$$

$$c_1 e^{\lambda_1 t} (RC \lambda_1 + 1) = 0$$

$$RC \lambda_1 + 1 = 0$$

$$\lambda_1 = -1/RC$$

$$y_{zi}(t) = c_1 e^{(-1/RC)t} \checkmark$$

2. Find the solution of given differential equation

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt} \checkmark \quad \text{or} \quad (D^2 + 3D + 2)y(t) = x(t)(1 + D) \checkmark$$

with initial conditions

$$y_{zi}(0) = 0 \quad \& \quad y'_{zi}(0) = -5$$

Sol to find ZIR, we make $x(t) = 0$

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = 0$$

The characteristic polynomial equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda^2 + 2\lambda + \lambda + 2 = 0$$

$$\lambda(\lambda + 2) + (\lambda + 2) = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$\lambda = -1, -2$ roots are different

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$y_{zi}(t) = c_1 e^{-t} + c_2 e^{-2t} \rightarrow (1)$$

$$y_{zi}(t) = 0 \quad \text{at } t = 0$$

$$0 = c_1 e^{-0} + c_2 e^{-2(0)}$$

$$0 = c_1 + c_2$$

$$c_1 = -c_2$$

$$y'_{zi}(t) = -5, \quad \text{at } t = 0$$

$$\frac{d}{dt} y_{zi}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

$$-5 = -c_1 - 2c_2$$

$$c_1 + 2c_2 = 5$$

$$c_1 - 2c_1 = 5$$

$$c_1 = -5, \quad c_2 = 5$$

$$y_{zi}(t) = -5e^{-t} + 5e^{-2t} \checkmark$$

③ $(D^2 + 6D + 9)y(t) = (3D + 5)x(t)$ with initial conditions $y_{21}(0) = 3$ ✓
 $y'_{21}(0) = -7$

so/ $(D^2 + 6D + 9)y(t) = 0$

the characteristic polynomial

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\lambda^2 + 3\lambda + 3\lambda + 9 = 0$$

$$\lambda(\lambda + 3) + 3(\lambda + 3) = 0$$

$$(\lambda + 3)(\lambda + 3) = 0$$

$$\lambda_1 = -3, \lambda_2 = -3 \text{ roots repeated}$$

$$y_{21}(t) = (c_1 + c_2 t) e^{-3t}$$

$$y'_{21}(0) = -7$$

$$y_{21}(0) = 3$$

$$3 = (c_1 + c_2(0)) e^{-3(0)}$$

$$c_1 = 3$$

$$\frac{d}{dt} y_{21}(t) = c_1(-3) e^{-3t} + t(-3) e^{-3t} + e^{-3t} c_2$$

$$-7 = -3c_1 e^{-3(0)} - 3(0) e^{-3(0)} + c_2 e^{-3(0)}$$

$$-7 = -3c_1 - 0 + c_2$$

$$c_2 = -7 + 3c_1$$

$$c_2 = -7 + 3(3)$$

$$c_2 = 2$$

$$y_{21}(t) = (3 + 2t) e^{-3t}$$

④ $(D^2 + 4D + 40)y(t) = (D + 2)x(t)$, $y_{21}(0) = 2$ & $y'_{21}(0) = 16.78$

so/ $(D^2 + 4D + 40)y(t) = 0$

the characteristic eq

$$\lambda^2 + 4\lambda + 40 = 0$$

$$\lambda_1 = -2 + 6j, \lambda_2 = -2 - 6j$$

roots are complex

$$y_{21}(t) = c \cdot e^{at} \cos(bt + \theta)$$

$$y_{21}(t) = c e^{-2t} \cos(6t + \theta) \quad \checkmark$$

$$y_{21}(0) = 2$$

$$2 = c e^{-2(0)} \cos(6(0) + \theta)$$

$$2 = c \cos \theta \Rightarrow c = \frac{2}{\cos \theta}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 4(1)(40)}}{2}$$

$$= \frac{-4 \pm \sqrt{-144}}{2}$$

$$= \frac{-4 \pm 12j}{2}$$

$$= -2 \pm 6j \quad \checkmark$$

$$y'_{21}(0) = 16.78$$

$$y_{21}(t) = c \left[e^{-2t} \cos(6t + \theta) \right]$$

$$y'_{21}(t) = c \left[\cos(6t + \theta) e^{-2t} (-2) + e^{-2t} (-\sin(6t + \theta)) 6 \right]$$

$$y'_{21}(t) = c \left[-2 \cos(6t + \theta) e^{-2t} - 6 e^{-2t} \sin(6t + \theta) \right] \checkmark$$

$$\text{at } t=0, y'_{21}(t) = 16.78$$

$$16.78 = -2c \cos \theta - 6c \sin \theta$$

$$16.78 = -2 \left(\frac{2}{\cancel{\cos \theta}} \right) \cos \theta - 6 \left(\frac{2}{\cos \theta} \right) \sin \theta$$

$$16.78 = -4 - 12 \tan \theta$$

$$12 \tan \theta = -4 - 16.78$$

$$\tan \theta = \frac{-20.78}{12} = -1.73$$

$$\theta = \tan^{-1}(-1.73)$$

$$\theta = -\pi/3 \checkmark$$

$$\therefore c = \frac{2}{\cos \theta} = \frac{2}{\cos(-\pi/3)} \Rightarrow 4 \checkmark$$

$$\boxed{c=4}$$

$$y(t) = 4 e^{-2t} \cos(6t - \pi/3) \checkmark$$

Ex ① Find the particular solution of given diff with $x(t) = e^{-t}$

$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6 y(t) = 2 \cdot x(t) + \frac{d}{dt} x(t)$$

$$\text{sol} \quad \frac{d^2 y_p(t)}{dt^2} + 5 \frac{dy_p(t)}{dt} + 6 y_p(t) = 2 \cdot e^{-t} + \frac{d}{dt} e^{-t}$$

$$\text{if } x(t) = e^{-t} \text{ then } y_p(t) = c e^{-t}$$

$$\frac{d^2}{dt^2} c \cdot e^{-t} + 5 \frac{d}{dt} c \cdot e^{-t}(t) + 6 c \cdot e^{-t} = 2 e^{-t} + \frac{d}{dt} e^{-t}$$

$$c \cdot e^{-t} - 5 c e^{-t} + 6 c e^{-t} = 2 e^{-t} - e^{-t}$$

$$e^{-t} (c - 5c + 6c) = e^{-t}$$

$$7c - 5c = 1$$

$$2c = 1$$

$$c = 1/2$$

$$y_p(t) = c e^{-t}$$

$$y_p(t) = \frac{1}{2} e^{-t} \checkmark$$

$$\begin{array}{l} c(-e^{-t}) \\ -c e^{-t} \\ -c(-e^{-t}) \\ c \cdot e^{-t} \end{array}$$

Ex:- Find the response of the given diff eq with i/p $x(t) = K \cdot e^{3t} \cdot u(t)$
 Assume initial condition as $y(0) = 0$

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

So $y_T(t) = y_h(t) + y_p(t) \quad \text{or} \quad y_n(t) + y_f(t)$

$y_h(t)$:- $x(t) = 0$

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0$$

The characteristic eqn is

$$\lambda + 2 = 0$$

$$\lambda = -2$$

$$y_h(t) = c_1 e^{\lambda t}$$

$$\boxed{y_h(t) = c_1 e^{-2t}}$$

$$y_T(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = c_1 e^{-2t} + \frac{K}{5} e^{3t}} \quad \checkmark$$

$$y(0) = 0, t = 0$$

$$y(0) = c_1 e^{-2(0)} + \frac{K}{5} e^{3(0)}$$

$$0 = c_1 + \frac{K}{5}$$

$$c_1 = -\frac{K}{5} \quad \therefore y(t) = -\frac{K}{5} e^{-2t} + \frac{K}{5} e^{3t} \quad \checkmark$$

$$y(t) = \frac{K}{5} (e^{3t} - e^{-2t}) u(t) \quad \checkmark$$

$y_p(t)$:-

$$\frac{d}{dt} y_p(t) + 2y_p(t) = K e^{3t} u(t)$$

$$\text{for } x(t) = K e^{3t} \quad t > 0$$

$$\text{then } y_p(t) = A e^{3t} \quad \checkmark$$

$$\frac{dy_p(t)}{dt} = 3A e^{3t}$$

$$3A e^{3t} + 2A e^{3t} = K e^{3t}$$

$$\text{let } t = 0$$

$$3A + 2A = K$$

$$5A = K$$

$$\boxed{A = K/5}$$

$$\boxed{y_p(t) = \frac{K}{5} e^{3t}}$$

→ Natural response

$$y_n(t) \left\{ \begin{array}{l} x(t) = 0 \\ \text{Given} \\ \text{initial} \\ \text{condn} \end{array} \right.$$

$$y_{21}(t)$$

forced response

$$x(t) \neq 0$$

$$y_f(t)$$

$$y_{2s}(t) = y_n(t) + y_f(t) \Big|_0 \text{ initial condition}$$

$$T = y_{21}(t) + y_{2s}(t)$$

Ex 1:- $\frac{d}{dt} y(t) + 5 y(t) = x(t)$, $x(t) = \underline{u(t)}$, $y(0) = 2$, ZIR, ZSR, TR

Step 1. Natural response, $x(t) = 0$

$$\frac{d}{dt} y(t) + 5 y(t) = 0$$

$$\lambda + 5 = 0$$

$\lambda = -5$ roots distinct

$$y_n(t) = c_1 e^{\lambda t}$$

$$y_n(t) = c_1 e^{-5t} \checkmark$$

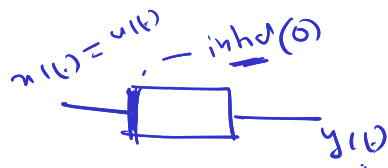
$$y_{z1}(0) = c_1 e^{-5(0)}$$

$$2 = c_1$$

$$y_{z1}(t) = 2 e^{-5t} \checkmark$$

$$y_f(t) = y_{z1}(t) + y_{zs}(t)$$

$$y_f(t) = 2 e^{-5t} + \left(-\frac{1}{5} e^{-5t} + \frac{1}{5}\right)$$



ZIR ZSR
✓

H.W $\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6 y(t) = \underbrace{2 e^{-t} u(t)}_{x(t)}$
 $y(0) = 0, y'(0) = 1$

forced response

$$\frac{d}{dt} y(t) + 5 y(t) = u(t) \rightarrow \textcircled{1}$$

if $x(t) = \underline{u(t)}$ then $y_f(t) = c \rightarrow \textcircled{2}$

$$\frac{d}{dt} y_f(t) = 0 \rightarrow \textcircled{3}$$

$$0 + 5c = 1 \checkmark$$

$$5c = 1 \Rightarrow c = \frac{1}{5}$$

$$y_f(t) = \frac{1}{5}$$

$$y_{zs}(t) = y_n(t) + y_f(t)$$

$$y_{zs}(t) = c_1 e^{-5t} + \frac{1}{5} \checkmark$$

Take initial conditions as 0

$$t=0 \quad y(0) = 0 \checkmark$$

$$y_{zs}(0) = c_1 e^{-5(0)} + \frac{1}{5}$$

$$0 = c_1 + \frac{1}{5}$$

$$c_1 = -\frac{1}{5}$$

$$y_{zs}(t) = -\frac{1}{5} e^{-5t} + \frac{1}{5}$$

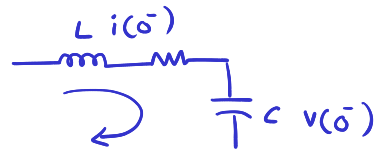
$$\textcircled{2} \quad \frac{d^2}{dt^2} y(t) + 3 \frac{d}{dt} y(t) + 2 y(t) = x(t) + \frac{d}{dt} x(t)$$

$$x(t) = 5 \cdot u(t) \checkmark \quad (\underline{\underline{ZSR}})$$

Practical Initial conditions

$$y_{zi}(0) = 0, \quad y'_{zi}(0) = -5$$

$$t=0^- \quad t=0^-, \quad t=0^+$$



$t=0$ is the reference point

$t=0^-$ Just before the input is applied

$t=0^+$ Just after the input is applied

total response

$$y(t) = y_{zi}(t) + \underbrace{y_{zs}(t)}_0 \quad \therefore x(t) = 0$$

$$y(t) = y_{zi}(t) \quad y_{zs}(t) = 0$$

at $t=0^-$

$$y(0^-) = y_{zi}(0^-)$$

$$y'(0^-) = y'_{zi}(0^-)$$

$$y_{zi}(0^-) = y_{zi}(0^+)$$

$$y'_{zi}(0^-) = y'_{zi}(0^+)$$

$\therefore y(0^-) = y_{zi}(0^-) = y_{zi}(0^+)$

 ✓

total response

$$y(0^-) \neq y(0^+) \quad \checkmark$$

$$y'(0^-) \neq y'(0^+) \quad -$$

The unit Impulse Response $h(t)$

The unit impulse response of an LTIC system described by the N^{th} order differential equation

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \dots + b_{N-1} D + b_N) x(t)$$

Practically $M < N$, in general $M=N$

$$\begin{array}{c} \uparrow \\ (D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^N + b_1 D^{N-1} + \dots + b_{N-1} D + b_N) x(t) \end{array} \quad \begin{array}{c} \swarrow \delta(t) \\ \uparrow h(t) \end{array} \quad \longrightarrow \textcircled{1}$$

→ Before deriving the general expression for the unit impulse response $h(t)$, let understand the nature of $h(t)$

→ The impulse response $h(t)$ is the system response to an impulse input $\delta(t)$ applied at $t=0$ with all initial conditions zero at $t=0^-$

→ $\delta(t)$ is like lightning, which strikes instantaneously at $t=0$ and then vanishes, but at $t=0$ it generates energy, means it create non zero initial conditions with in the system at $t=0^+$

→ Although the impulse input $\delta(t)$ vanishes for $t > 0$ so that the system has no input after the impulse has been applied

→ The system will still have a response generated by these newly created initial conditions ✓

→ The impulse response $h(t)$, therefore, must consist of the systems characteristic mode for $t \geq 0^+$

$$h(t) = \text{characteristic mode terms} \quad \text{for } t \geq 0^+$$

This response is valid for $t > 0$, but at $t=0$?

$$h(t) = A_0 \delta(t) + \text{characteristic mode term} \quad t \geq 0$$



in eq 1 put $x(t)=\delta(t)$ and $y(t)=h(t)$ and compare N^{th} order terms

$$D^N y(t) = b_0 D^N x(t)$$

$$D^N h(t) = b_0 D^N \delta(t)$$

$$\boxed{h(t) = b_0 \delta(t)}$$

$$\therefore \boxed{h(t) = b_0 \delta(t) + \text{characteristic modes}}$$

if $M=N$ ✓ then we are having $b_0 \delta(t)$

if $M < N$ then $b_0 = 0$

The unknown coefficient of the N characteristic modes in $h(t)$ can be determined by using the technique of impulse matching

The unit impulse response $h(t)$ is given by

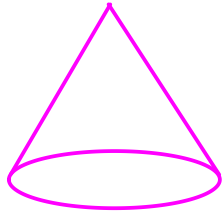
$$\boxed{h(t) = b_0 \delta(t) + [p(t) y_n(t)] u(t)} \quad \checkmark$$

where $y_n(t)$ is a linear combination of the characteristic modes of the system subjected to the following initial conditions

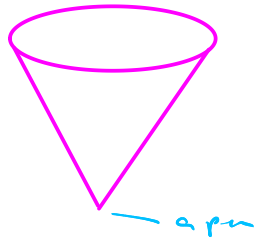
$$y_n(0) = y_n'(0) = y_n''(0) = \dots = y_n^{(N-2)}(0) = 0 \quad \& \quad y_n^{(N-1)}(0) = 1 \quad \checkmark$$

System stability

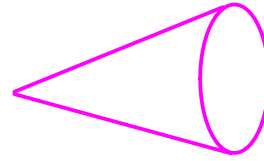
lets consider right circular cone



stable equilibrium



unstable equilibrium



neutral equilibrium

BIBO



BIBO stability

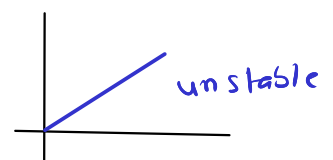
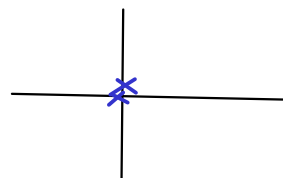
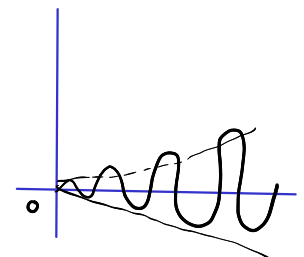
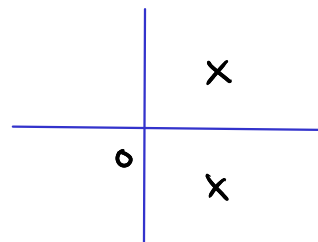
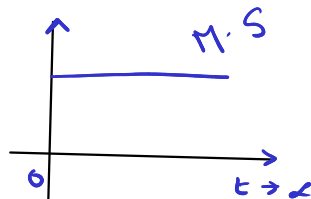
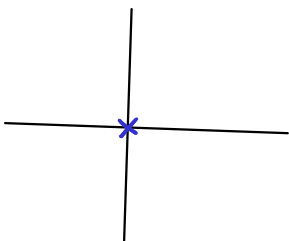
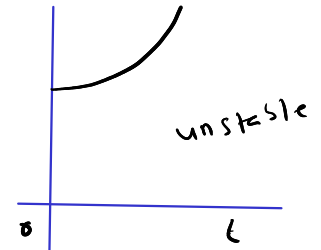
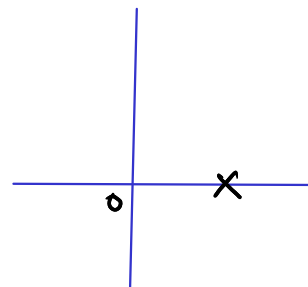
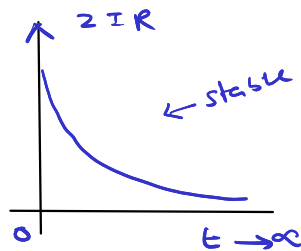
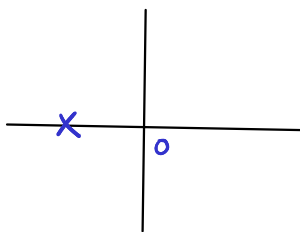
$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

internal ✓
external ✓

Inter (asymptotic) stability:

1. An LTIC system is asymptotically stable if all the characteristic roots are in LHP, the roots may be simple (unrepeated) or repeated
2. An LTIC system is unstable if one or both of the following conditions exist
 - i) at least one root is in RHP
 - ii) there are repeated roots on the imaginary axis
3. An LTIC system is marginally stable, if there are no roots in the RHP and there are some unrepeated roots on the imaginary axis

root location



Relationship b/w BIBO and asymptotic stability

absolutely integrable $\rightarrow$ BIBO stable

Asymptotic stable $\rightarrow$ BIBO stable

marginally stable $\rightarrow$ BIBO unstable

Asymptotic unstable $\rightarrow$ BIBO unstable