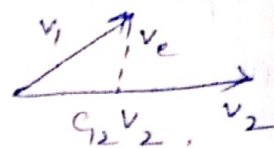


Analogy between vectors and signals:-

A vector is specified by magnitude & direction. Consider two vectors v_1 & v_2 shown in figure.



Let the component of v_1 along v_2 be given by $c_{12}v_2$. The vector v_1 can now be expressed in terms of vector v_2 .

$$v_1 = c_{12}v_2 + v_e.$$

If c_{12} is zero, then the vector has no component along the other vector & hence the two vectors are mutually perpendicular. Such vectors are known as orthogonal vectors.

dot product of two vectors A & B

$$A \cdot B = AB \cos \theta$$

where θ is the angle b/w vectors A & B & also

$$A \cdot B = B \cdot A$$

According to this notation

$$\text{the component of } A \text{ along } B = A \cos \theta = \frac{A \cdot B}{B}$$

$$\text{the component of } B \text{ along } A = B \cos \theta = \frac{A \cdot B}{A}$$

$$\text{The component of } v_1 \text{ along } v_2 = \frac{v_1 \cdot v_2}{v_2}$$

$$c_{12} = \frac{v_1 \cdot v_2}{v_2^2} = \frac{v_1 \cdot v_2}{v_2 \cdot v_2} = c_{12} v_2.$$

If v_1 & v_2 are orthogonal then

$$v_1 \cdot v_2 = 0$$

$$c_{12} = 0.$$

Signals:-

Consider two signals $f_1(t)$ & $f_2(t)$ & if we want to approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1 < t < t_2)$ as follows.

$$f_1(t) \approx c_{12} f_2(t) \text{ for } (t_1 < t < t_2)$$

Now calculate c_{12} such that the error between the actual function & the approximated function is minimum over the interval $(t_1 < t < t_2)$. Let us define an error function $f_e(t)$ as

$$f_e(t) = f_1(t) - c_{12} f_2(t)$$

One approach of minimizing the error $f_e(t)$ is to minimize mean or average error (M) over the interval (t_1, t_2)

$$M = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f_1(t) - c_{12} f_2(t) dt$$

This approach can give false indications that the error is zero, if large +ve & -ve errors are present they may get cancel each other. so the alternative approach is to find the mean square error over the interval (t_1, t_2) which is given by

$$e = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - c_{12} f_2(t)]^2 dt$$

In order to calculate the coefficient c_{12} that will minimize mean square error. & is given by

$$\frac{de}{dc_{12}} = 0$$

$$\frac{d}{dc_{12}} \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f_1(t) - c_{12} f_2(t)]^2 dt \right] = 0.$$

$$\frac{1}{t_2 - t_1} \frac{d}{dc_{12}} \left[\int_{t_1}^{t_2} [f_1^2(t) + c_{12}^2 f_2^2(t) - 2f_1(t) f_2(t) c_{12}] dt \right] = 0.$$

$$\frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} [0 + 2c_{12} f_2^2(t) - 2f_1(t) f_2(t)] dt \right] = 0.$$

$$2c_{12} \int_{t_1}^{t_2} f_2^2(t) dt - 2 \int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0.$$

$$c_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) dt}{\int_{t_1}^{t_2} f_2^2(t) dt}.$$

If $f_1(t)$ & $f_2(t)$ are orthogonal to each other in the interval (t_1, t_2) then

$$c_{12} = 0.$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0.$$

Problems:-

1. show that the functions $\sin n\omega_0 t$ & $\sin m\omega_0 t$ are orthogonal over any interval $(t_0, t_0 + \frac{2\pi}{\omega_0})$ for integral values of n & m .

Sol:- The condition for two functions to be orthogonal is

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0.$$

$$\int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} \sin n\omega_0 t \sin m\omega_0 t dt = \frac{1}{2} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} 2 \sin n\omega_0 t \sin m\omega_0 t dt.$$

$$= \frac{1}{2} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} \cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t \, dt$$

$$= \frac{1}{2} \left[\frac{\sin(n-m)\omega_0 t}{(n-m)\omega_0} - \frac{\sin(n+m)\omega_0 t}{(n+m)\omega_0} \right]_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= 0.$$

② A rectangular function $f(t)$ is defined by $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$. Approximate this function by a waveform $\sin t$ over the interval $(0, 2\pi)$ such that the mean square error is minimum.

Sol: - The function $f(t)$ will be approximated over the interval $(0, 2\pi)$ as $f(t) \simeq c_{12} \sin t$.

$$c_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) \, dt}{\int_{t_1}^{t_2} f_2^2(t) \, dt} = \frac{\int_0^{2\pi} f(t) \sin t \, dt}{\int_0^{2\pi} \sin^2 t \, dt}$$

$$= \frac{\int_0^{\pi} \sin t \, dt - \int_{\pi}^{2\pi} \sin t \, dt}{\int_0^{2\pi} \sin^2 t \, dt}$$

$$= \frac{-\cos t \Big|_0^{\pi} - [-\cos t]_{\pi}^{2\pi}}{\pi}$$

$$= \frac{-[-1-1] + [1+1]}{\pi} = \frac{2+2}{\pi} = \frac{4}{\pi}$$

$$f(t) = \frac{4}{\pi} \sin t$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \frac{1 - \cos 2t}{2} \, dt$$

$$= \frac{1}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2} [2\pi] = \pi$$

Orthogonal Signal Space:-

Let us consider a set of n functions $g_1(t), g_2(t), \dots, g_n(t)$ which are orthogonal to one another over an interval t_1 to t_2 that is

$$\int_{t_1}^{t_2} g_j(t) g_k(t) dt = 0 \quad j \neq k$$

and let

$$\int_{t_1}^{t_2} g_j^2(t) dt = k_j$$

Let an arbitrary function $f(t)$ be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal functions.

$$\begin{aligned} f(t) &\approx c_1 g_1(t) + c_2 g_2(t) + \dots + c_k g_k(t) + \dots + c_n g_n(t) \\ &= \sum_{r=1}^n c_r g_r(t) \end{aligned}$$

For the best approximation, proper values of constants $c_1, c_2, \dots, c_n$ must be found out such that e , the mean square of $f_e(t)$ is minimized. So,

$$f_e(t) = f(t) - \sum_{r=1}^n c_r g_r(t)$$

and

$$e = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^n c_r g_r(t) \right]^2 dt.$$

In order to minimize the mean square error

$$\frac{\partial e}{\partial c_j} = 0.$$

$$\frac{\partial}{\partial c_j} \left[\int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^n c_r g_r(t) \right]^2 dt \right] = 0.$$

$$\frac{\partial}{\partial c_j} \int_{t_1}^{t_2} f^2(t) dt + \frac{\partial}{\partial c_j} \int_{t_1}^{t_2} c_r^2 g_r^2(t) dt - \frac{\partial}{\partial c_j} \left[2 f(t) c_r g_r(t) \right] = 0.$$

$$\frac{\partial}{\partial c_j} \int_{t_1}^{t_2} [-2 c_j f(t) g_j(t) + c_j^2 g_j^2(t)] dt = 0$$

$$2 \int_{t_1}^{t_2} f(t) g_j(t) dt = 2 c_j \int_{t_1}^{t_2} g_j^2(t) dt$$

$$c_j = \frac{\int_{t_1}^{t_2} f(t) g_j(t) dt}{\int_{t_1}^{t_2} g_j^2(t) dt}$$

$$c_j = \frac{1}{k_j} \int_{t_1}^{t_2} f(t) g_j(t) dt$$

Evaluation of Mean Square Error:-

$$E = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) - \sum_{r=1}^n c_r g_r(t) \right]^2 dt$$

$$E = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) dt + \sum_{r=1}^n c_r^2 \int_{t_1}^{t_2} g_r^2(t) dt - 2 \sum_{r=1}^n c_r \int_{t_1}^{t_2} f(t) g_r(t) dt \right] \rightarrow (1)$$

But $c_r = \frac{1}{k_r} \int_{t_1}^{t_2} f(t) g_r(t) dt$

$$\int_{t_1}^{t_2} f(t) g_r(t) dt = c_r k_r \quad \& \quad k_r = \int_{t_1}^{t_2} g_r^2(t) dt \rightarrow (2) \quad (3)$$

Substituting Eq. (2) & (3) in Eq. (1).

$$E = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) dt + \sum_{r=1}^n c_r^2 k_r - 2 \sum_{r=1}^n c_r^2 k_r \right]$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) dt - \sum_{r=1}^n c_r^2 k_r \right]$$

$$E = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) dt - (c_1^2 k_1 + c_2^2 k_2 + \dots + c_n^2 k_n) \right]$$

Problem :-

A rectangular function is defined by $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$.

(a) Approximate the above rectangular function by an orthogonal set $\sin t$.

(b) Evaluate the mean square error in this approximation.

Sol :-

$$C_r = \frac{\int_0^{2\pi} f(t) \sin rt \, dt}{\int_0^{2\pi} \sin^2 rt \, dt}.$$

$$\begin{aligned} \int_0^{2\pi} f(t) \sin rt \, dt &= \int_0^{\pi} \sin rt \, dt - \int_{\pi}^{2\pi} \sin rt \, dt \\ &= \left(-\frac{\cos rt}{r} \right)_0^{\pi} + \left(\frac{\cos rt}{r} \right)_{\pi}^{2\pi} \\ &= \frac{-\cos r\pi}{r} + \frac{1}{r} + \frac{\cos 2\pi r}{r} - \frac{\cos r\pi}{r} \end{aligned}$$

$$r \text{ odd} = \frac{1+1+1-1}{r} = \frac{4}{r}.$$

$$r \text{ even} = \frac{-1+1+1-1}{r} = 0.$$

$$C_r = \begin{cases} \frac{4}{\pi r} & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases}$$

$f(t)$ is approximated by

$$f(t) = \frac{4}{\pi} \left[\sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \dots \right].$$

(b) Mean Square Error is given by

$$E = \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} f^2(t) \, dt - \sum_{r=1}^n C_r^2 K_r \right].$$

$$t_2 - t_1 = 2\pi - 0 = 2\pi.$$

$$K_r = \int_0^{2\pi} \sin^2 rt \, dt = \pi.$$

$$\int_0^{2\pi} 1^2 \, dt = t \Big|_0^{2\pi} = 2\pi.$$

$$\begin{aligned} &\int_0^{2\pi} \sin^2 rt \, dt \\ &= \int_0^{2\pi} \frac{1 - \cos 2rt}{2} \, dt \\ &= \frac{1}{2} \left[t - \frac{\sin 2rt}{2r} \right]_0^{2\pi} \\ &r \text{ odd} \Rightarrow \frac{1}{2} [2\pi - 0] \\ &= \pi \\ &r \text{ even} \Rightarrow \frac{1}{2} [2\pi - 0] \\ &= \pi. \end{aligned}$$

$$E = \frac{1}{2\pi} \left[2\pi - \pi \sum_{i=1}^n C_i^2 \right]$$

One term approximation

$$E_1 = \frac{1}{2\pi} \left[2\pi - \pi \left(\frac{4}{\pi} \right)^2 \right] = 0.19$$

Two term approximation

$$E_2 = \frac{1}{2\pi} \left[2\pi - \left(\frac{4}{\pi} \right)^2 \pi - \left(\frac{4}{3\pi} \right)^2 \pi \right]$$

$$= 0.1$$

Three term approximation

$$E_3 = \frac{1}{2\pi} \left[2\pi - \left(\frac{4}{\pi} \right)^2 \pi - \left(\frac{4}{3\pi} \right)^2 \pi - \left(\frac{4}{5\pi} \right)^2 \pi \right]$$

$$= 0.0675$$

$$E_4 = \frac{1}{2\pi} \left[2\pi - \left(\frac{4}{\pi} \right)^2 \pi - \left(\frac{4}{3\pi} \right)^2 \pi - \left(\frac{4}{5\pi} \right)^2 \pi - \left(\frac{4}{7\pi} \right)^2 \pi \right]$$

$$= 0.051$$