

UNIT-III

Signal Analysis:

We can analyse the signals using the concept of vectors.

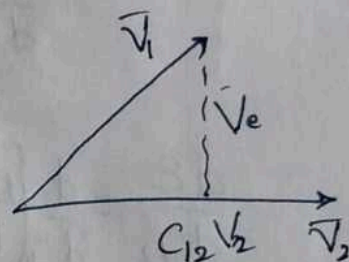
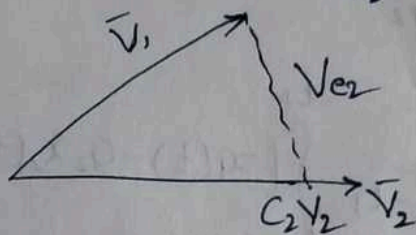
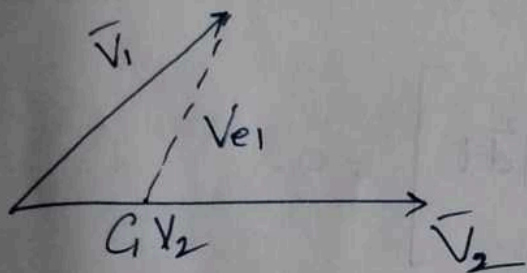
→ Consider two vectors $\bar{V}_1, \bar{V}_2$.

→ The error vector can be found in many ways.

→ It is dependent upon the similarity of the two vectors $(\bar{V}_1, \bar{V}_2)$.

$$\frac{\bar{V}_1 \cdot \bar{V}_2}{\bar{V}_2 \cdot \bar{V}_2} = C_{12} \bar{V}_2 \Rightarrow C_{12} = \frac{\bar{V}_1 \cdot \bar{V}_2}{\bar{V}_2 \cdot \bar{V}_2}$$

The component $\bar{V}_1$ along $\bar{V}_2 = \bar{V}_1 \cos \theta = \frac{\bar{V}_1 \cdot \bar{V}_2}{\bar{V}_2}$ $C_{12} = 0$ ($\perp$)



→ If $C_{12} = 0$, then $\bar{V}_1$ & $\bar{V}_2$ are orthogonal vectors.

→ Orthogonal vectors are independent vectors, $\perp^w$ vectors & no similarities between two vectors.

→ The above concept can be used in signal analysis.

Consider two signals $x_1(t)$ & $x_2(t)$.

$$\Rightarrow x_1(t) \approx C_{12} x_2(t) \text{ for } (t_1 < t < t_2)$$

The error function, $x_e(t)$ is:

$$x_e(t) = x_1(t) - C_{12} x_2(t)$$

To minimise the error, $x_e(t)$, the average of squared error $[x_e^2(t)]$ is taken ($\mathcal{E}$).

$$\mathcal{E} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_e^2(t) dt.$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x_1(t) - G_2 x_2(t)]^2 dt.$$

Find $\frac{d\mathcal{E}}{dG_2}$ to minimize error & equal to zero.

$$\text{i.e. } \frac{d\mathcal{E}}{dG_2} = 0.$$

$$\frac{d\mathcal{E}}{dG_2} = \frac{d}{dG_2} \left[\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x_1(t) - G_2 x_2(t)]^2 dt \right] = 0.$$

$$\frac{d}{dG_2} \left\{ \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} [x_1^2(t) - 2G_2 x_1(t)x_2(t) + G_2^2 x_2^2(t)] dt \right] \right\} = 0.$$

$$\frac{1}{(t_2 - t_1)} \left[\frac{d}{dG_2} \int_{t_1}^{t_2} x_1^2(t) dt - \frac{d}{dG_2} \int_{t_1}^{t_2} G_2 x_1(t)x_2(t) dt + \frac{d}{dG_2} \int_{t_1}^{t_2} G_2^2 x_2^2(t) dt \right] = 0.$$

$$\Rightarrow 0 - \frac{2}{t_2 - t_1} \int_{t_1}^{t_2} x_1(t) x_2(t) dt + \int_{t_1}^{t_2} 2C_{12} x_2^2(t) dt = 0.$$

$$\Rightarrow \int_{t_1}^{t_2} x_1(t) x_2(t) dt = C_{12} \int_{t_1}^{t_2} x_2^2(t) dt.$$

$$\Rightarrow C_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2(t) dt}{\int_{t_1}^{t_2} x_2^2(t) dt}$$

If $C_{12} = 0$, then the signal $x_1(t)$ contains no component of the signal $x_2(t)$. Hence $x_1(t)$ & $x_2(t)$ over (t_1, t_2) are orthogonal.

$$\Rightarrow \int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0.$$

→ Show that the following signals are orthogonal over an interval $[0, 1]$.

$$x_1(t) = 2 \quad ; \quad x_2(t) = \sqrt{3}(1-2t)$$

Sol: If $x_1(t)$ & $x_2(t)$ are orthogonal,

$$\int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0. \quad \begin{matrix} \text{LHS} \\ \text{RHS} \end{matrix}$$

$$\Rightarrow \int_0^1 2 \cdot \sqrt{3}(1-2t) dt = 0$$

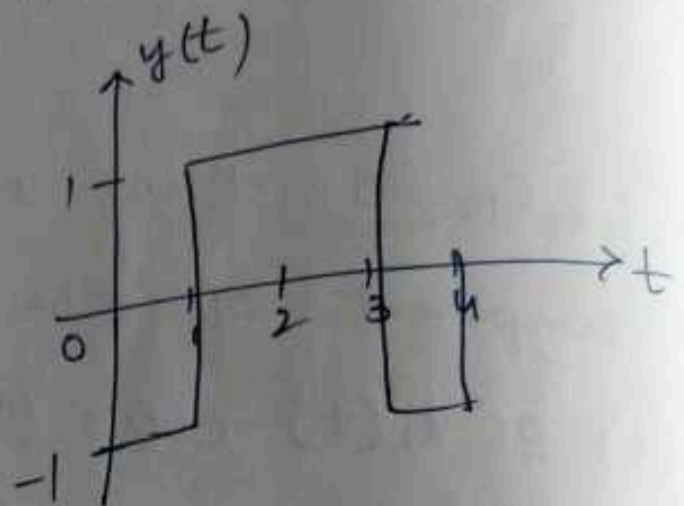
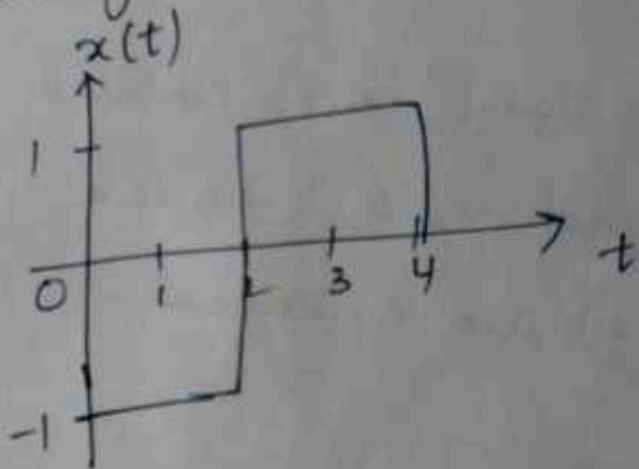
$$\Rightarrow 2\sqrt{3} \left[t \right]_0^1 - \frac{2}{\sqrt{3}} \left[\frac{t^2}{2} \right]_0^1 = 0.$$

$$\Rightarrow 2\sqrt{3} - 2\sqrt{3} = 0$$

$$0 = 0$$

$\therefore x_1(t)$ & $x_2(t)$ are orthogonal.

HW $\rightarrow$ Prove that the signals $x(t)$ & $y(t)$ are orthogonal over the interval $[0, 4]$.



$$x(t) = \begin{cases} -1 & \text{for } 0 \leq t < 2 \\ 1 & \text{for } 2 \leq t \leq 4 \end{cases}$$

$$y(t) = \begin{cases} -1 & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < 3 \\ -1 & \text{for } 3 \leq t \leq 4 \end{cases}$$

EXAMPLE 3.2 Show that the functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ are orthogonal over any interval $\{t_0 \text{ to } [t_0 + (2\pi/\omega_0)]\}$ for integral values of n and m .

Solution: To show that the functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ are orthogonal over any interval $\{t_0 \text{ to } [t_0 + (2\pi/\omega_0)]\}$ for integral values of n and m , we have to show that the integral

$$I = \int_{t_0}^{t_0 + (2\pi/\omega_0)} \sin n\omega_0 t \cos m\omega_0 t \, dt = 0$$

$$I = \int_{t_0}^{t_0 + (2\pi/\omega_0)} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

$$= \int_{t_0}^{t_0 + (2\pi/\omega_0)} \frac{1}{2} [\sin (n+m) \omega_0 t + \sin (n-m) \omega_0 t] \, dt$$

$$= \frac{1}{2} \left[-\frac{\cos(n+m) \omega_0 t}{n+m} - \frac{\cos(n-m) \omega_0 t}{n-m} \right]_{t_0}^{t_0 + (2\pi/\omega_0)}$$

$$= -\frac{1}{2} \left\{ \left[\frac{\cos(n+m) \omega_0 [t_0 + (2\pi/\omega_0)] - \cos(n+m) \omega_0 t_0}{n+m} \right] + \left[\frac{\cos(n-m) \omega_0 [t_0 + (2\pi/\omega_0)] - \cos(n-m) \omega_0 t_0}{n-m} \right] \right\}$$

$$= -\frac{1}{2} \left\{ \left[\frac{\cos(n+m) \omega_0 t_0 - \cos(n+m) \omega_0 t_0}{n+m} \right] + \left[\frac{\cos(n-m) \omega_0 t_0 - \cos(n-m) \omega_0 t_0}{n-m} \right] \right\}$$

$$= 0$$

This shows that the functions $\sin n\omega_0 t$ and $\cos m\omega_0 t$ are mutually orthogonal.

EXAMPLE 3.3 Prove that the complex exponential signals are orthogonal functions.

Solution: Consider two complex exponential signals,

$$x_1(t) = e^{jn\omega_0 t} \quad \text{and} \quad x_2(t) = e^{jm\omega_0 t}$$

Let the interval be t_0 to $t_0 + T$, i.e. from t_0 to $t_0 + (2\pi/\omega_0)$. $x_1(t)$ and $x_2(t)$ are orthogonal over

the interval t_0 to $t_0 + (2\pi/\omega_0)$, if $I = \int_{t_0}^{t_0 + (2\pi/\omega_0)} x_1(t) x_2^*(t) dt = 0$.

Here $x_1(t) = e^{jn\omega_0 t}$ and $x_2^*(t) = [e^{jm\omega_0 t}]^* = e^{-jm\omega_0 t}$

$$\begin{aligned} \therefore I &= \int_{t_0}^{t_0 + (2\pi/\omega_0)} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt \\ &= \int_{t_0}^{t_0 + (2\pi/\omega_0)} e^{j(n-m)\omega_0 t} dt = \left[\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0} \right]_{t_0}^{t_0 + (2\pi/\omega_0)} \\ &= \left[\frac{e^{j(n-m)\omega_0 [t_0 + (2\pi/\omega_0)]} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0} \right] = \frac{e^{j(n-m)\omega_0 t_0} e^{j(n-m)2\pi} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0} \\ &= \frac{e^{j(n-m)\omega_0 t_0} - e^{j(n-m)\omega_0 t_0}}{j(n-m)\omega_0} \quad \boxed{e^{j(n-m)2\pi} = \cos(n-m)2\pi + j \sin(n-m)2\pi = 1} \\ &= 0 \end{aligned}$$

$$\therefore \int_{t_0}^{t_0+T} e^{jn\omega_0 t} (e^{jm\omega_0 t})^* dt = 0$$

This proves that the complex exponential signals are orthogonal functions.

EXAMPLE 3.4 Prove that the functions $x_p(t)$ and $x_q(t)$ where $x_r(t) = \frac{1}{\sqrt{T}} (\cos r\omega_0 t + \sin r\omega_0 t)$, $T = (2\pi/\omega_0)$ are orthogonal over the period (0 to T).

Solution: Given $x_r(t) = \frac{1}{\sqrt{T}} (\cos r\omega_0 t + \sin r\omega_0 t)$

where $T = \frac{2\pi}{\omega_0}$, i.e. $\omega_0 = \frac{2\pi}{T}$

$$\therefore x_p(t) = \frac{1}{\sqrt{T}} (\cos p\omega_0 t + \sin p\omega_0 t)$$

and $x_q(t) = \frac{1}{\sqrt{T}} (\cos q\omega_0 t + \sin q\omega_0 t)$

We know that the signals $x_p(t)$ and $x_q(t)$ are orthogonal over the period (0 to T), if

$$\int_0^T x_p(t) x_q(t) dt = 0$$

In this case,

$$\begin{aligned} x_p(t) x_q(t) &= \frac{1}{\sqrt{T}} [\cos p\omega_0 t + \sin p\omega_0 t] \frac{1}{\sqrt{T}} [\cos q\omega_0 t + \sin q\omega_0 t] \\ &= \frac{1}{T} [\cos p\omega_0 t \cos q\omega_0 t + \sin p\omega_0 t \sin q\omega_0 t + \cos p\omega_0 t \sin q\omega_0 t + \sin p\omega_0 t \cos q\omega_0 t] \\ &= \frac{1}{T} [\cos(p-q)\omega_0 t + \sin(p+q)\omega_0 t] \end{aligned}$$

$$\begin{aligned} \therefore I &= \int_{t_0}^{t_0+T} x_p(t) x_q(t) dt \\ &= \int_0^T \frac{1}{T} [\cos(p-q)\omega_0 t + \sin(p+q)\omega_0 t] dt \\ &= \frac{1}{T} \left[\frac{\sin(p-q)\omega_0 t}{(p-q)\omega_0} - \frac{\cos(p+q)\omega_0 t}{(p+q)\omega_0} \right]_0^T \\ &= \frac{1}{T} \left[\frac{\sin(p-q)\omega_0 T - \sin 0}{(p-q)\omega_0} - \frac{\cos(p+q)\omega_0 T - \cos 0}{(p+q)\omega_0} \right] \\ &= \frac{1}{T} \left[\frac{\sin(p-q)\omega_0(2\pi/\omega_0) - \sin 0}{(p-q)\omega_0} - \frac{\cos(p+q)\omega_0(2\pi/\omega_0) - \cos 0}{(p+q)\omega_0} \right] \\ &= \frac{1}{T} \left[\frac{\sin(p-q)2\pi - \sin 0}{(p-q)\omega_0} - \frac{\cos(p+q)2\pi - \cos 0}{(p+q)\omega_0} \right] \\ &= \frac{1}{T} \left[\frac{0-0}{(p-q)\omega_0} - \frac{1-1}{(p+q)\omega_0} \right] \\ &= 0 \end{aligned}$$

$$\therefore \int_0^T x_p(t) x_q(t) dt = 0$$

Thus, the functions $x_p(t)$ and $x_q(t)$ are orthogonal over the period [0 to T].

EXAMPLE 3.5 Prove that the signals $x(t)$ and $y(t)$ shown in Figure 3.2 are orthogonal over the interval [0, 4].

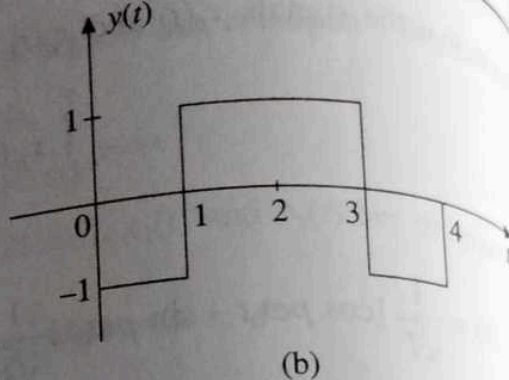
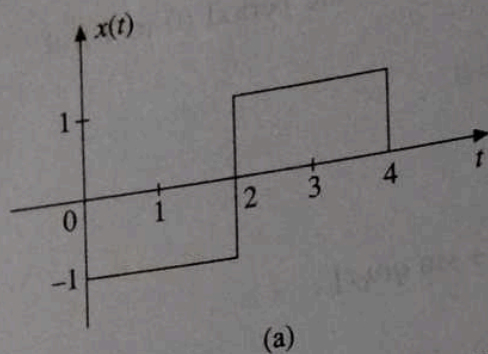


Figure 3.2 Waveforms for Example 3.5.

Solution: The signals $x(t)$ and $y(t)$ shown in Figure 3.2 can be mathematically expressed

$$x(t) = \begin{cases} -1 & \text{for } 0 < t < 2 \\ 1 & \text{for } 2 < t < 4 \end{cases}$$

$$y(t) = \begin{cases} -1 & \text{for } 0 < t < 1 \\ 1 & \text{for } 1 < t < 3 \\ -1 & \text{for } 3 < t < 4 \end{cases}$$

We know that $x(t)$ and $y(t)$ are orthogonal over the interval $[0, 4]$ if $\int_0^4 x(t) y(t) dt = 0$.

In this case,

$$\begin{aligned} \int_0^4 x(t) y(t) dt &= \int_0^1 x(t) y(t) dt + \int_1^2 x(t) y(t) dt + \int_2^3 x(t) y(t) dt + \int_3^4 x(t) y(t) dt \\ &= \int_0^1 (-1)(-1) dt + \int_1^2 (-1)(1) dt + \int_2^3 (1)(1) dt + \int_3^4 (1)(-1) dt \\ &= \int_0^1 dt - \int_1^2 dt + \int_2^3 dt - \int_3^4 dt \\ &= [t]_0^1 - [t]_1^2 + [t]_2^3 - [t]_3^4 \\ &= [1 - 0] - [2 - 1] + [3 - 2] - [4 - 3] \\ &= 1 - 1 + 1 - 1 \\ &= 0 \end{aligned}$$

$$\therefore \int_0^4 x(t) y(t) dt = 0$$

This proves that the given signals are orthogonal over the interval $[0, 4]$.

EXAMPLE 3.6 A rectangular function is defined as:

$$x(t) = \begin{cases} A & \text{for } 0 < t < \frac{\pi}{2} \\ -A & \text{for } \frac{\pi}{2} < t < \frac{3\pi}{2} \\ A & \text{for } \frac{3\pi}{2} < t < 2\pi \end{cases}$$

Approximate the above function by $A \cos t$ between the intervals $(0, 2\pi)$ such that the mean square error is minimum.

Solution: The given function $x(t)$ can be approximated by $A \cos t$ over the interval $(0, 2\pi)$ as:

$$x(t) \approx C_{12} A \cos t$$

C_{12} has to be found such that the mean square error is minimum over the interval $(0, 2\pi)$ as:

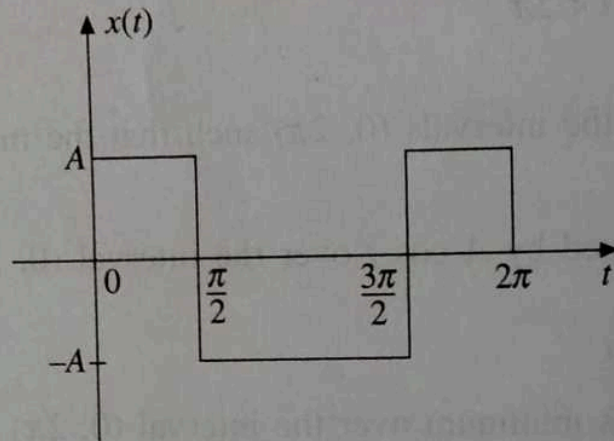
$$\begin{aligned} C_{12} &= \frac{\int_0^{2\pi} x(t) A \cos t dt}{\int_0^{2\pi} [A \cos t]^2 dt} \\ &= \frac{\int_0^{\pi/2} (A) A \cos t dt + \int_{\pi/2}^{3\pi/2} (-A) A \cos t dt + \int_{3\pi/2}^{2\pi} (A) A \cos t dt}{\int_0^{2\pi} A^2 \frac{[1 + \cos 2t]}{2} dt} \\ &= \frac{A^2 \left[\int_0^{\pi/2} \cos t dt - \int_{\pi/2}^{3\pi/2} \cos t dt + \int_{3\pi/2}^{2\pi} \cos t dt \right]}{\frac{A^2}{2} \left[\int_0^{2\pi} (1 + \cos 2t) dt \right]} = \frac{[\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{3\pi/2} + [\sin t]_{3\pi/2}^{2\pi}}{\frac{1}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}} \\ &= \frac{\left[\sin \frac{\pi}{2} - \sin 0 \right] - \left[\sin \frac{3\pi}{2} - \sin \frac{\pi}{2} \right] + \left[\sin 2\pi - \sin \frac{3\pi}{2} \right]}{\frac{1}{2} \left[(2\pi - 0) + \frac{\sin 4\pi - \sin 0}{2} \right]} \\ &= \frac{(1 - 0) - (-1 - 1) + [0 - (-1)]}{\pi + \frac{0 - 0}{2}} = \frac{4}{\pi} \end{aligned}$$

$$\therefore C_{12} = \frac{4}{\pi}$$

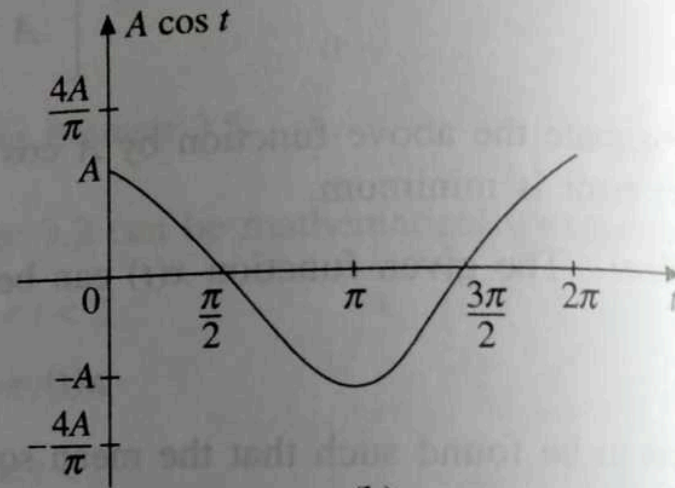
∴

$$x(t) \approx \frac{4}{\pi} A \cos t$$

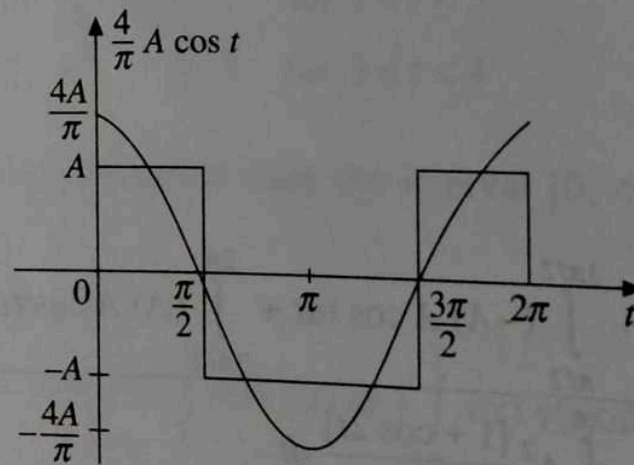
So the given function $x(t)$ can be approximated by $A \cos t$ as $x(t) \approx (4/\pi) A \cos t$ over the interval $(0, 2\pi)$ such that the mean square error is minimum. The rectangular function $x(t)$, the cosine waveform $A \cos t$, and the approximation of the rectangular function in terms of the cosine waveform are shown in Figure 3.3.



(a)



(b)



(c)

Figure 3.3 (a) Rectangular function, (b) Cosine wave, (c) Approximation.

3.3 GRAPHICAL EVALUATION OF

Correlation of functions:

Signal comparison or correlation (similarities) of two signals can be noticed.

Let $x_1(t)$ & $x_2(t)$ are two waveforms.

The waveform $x_1(t)$ contains an amount $C_{12}x_2(t)$ of that particular waveform $x_2(t)$ $[t_1, t_2]$.

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t) x_2(t) dt}{\int_{t_1}^{t_2} x_2^2(t) dt}$$

If $\int_{t_1}^{t_2} x_1(t) x_2(t) dt = 0$, the two signals have no similarity over (t_1, t_2) .

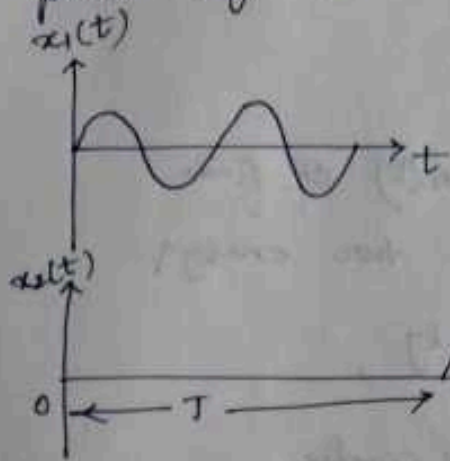
Let the interval is $(-\infty, \infty)$,

$$\Rightarrow \int_{-\infty}^{\infty} x_1(t) x_2(t) dt$$

But, there is a difficulty with this test integral.

Consider, a radar pulse,

Suppose received pulse is a delayed one of Transmitted pulse by T sec.



$x_1(t)$ = Transmitted pulse

$x_2(t)$ = Received pulse

For the above example also, similarities doesn't exist.

But it doesn't mean they are orthogonal.

Hence, to find similarities for such type of waveforms, consider a s/g with some shift.

$$\int_{-\infty}^{\infty} x_1(t) x_2(t-\tau) dt \quad \tau = \text{searching parameter.}$$

The above integral is called cross correlation between two signals $x_1(t)$ & $x_2(t)$.

$$\Rightarrow R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau) x_2(t) dt$$

There are two types of correlation.

- (1) Auto-correlation
- (2) Cross-Correlation.

7.6.1 Cross Correlation

The cross correlation between two different waveforms or signals is a measure of similarity or match or relatedness or coherence between one signal and the time delayed version of another signal. That means the cross correlation between two signals indicates how much one signal is related to the time delayed version of another signal.

Cross correlation of energy signals

Consider two general complex signals $x_1(t)$ and $x_2(t)$ of finite energy. The cross correlation of these two energy signals denoted by $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2^*(t - \tau) dt = \int_{-\infty}^{\infty} x_1(t + \tau) x_2^*(t) dt$$

If the two signals $x_1(t)$ and $x_2(t)$ are real, then

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2(t - \tau) dt = \int_{-\infty}^{\infty} x_1(t + \tau) x_2(t) dt$$

If $x_1(t)$ and $x_2(t)$ have some similarity, then the cross correlation $R_{12}(\tau)$ will have some finite value over the range of τ . Also if

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$$

i.e. if

$$R_{12}(0) = 0$$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as:

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) x_1^*(t - \tau) dt$$

In the above equations, the cross correlation function $R_{12}(\tau)$ is a function of the variable τ . The variable τ is called the *delay parameter* or the *scanning parameter* or the *searching parameter*. It is time delay or time shift of one of the two signals. The delay parameter τ determines the correlation between two signals. Two signals with no cross correlation at $\tau = 0$ can have significant cross correlation by adjusting the parameter τ . Two signals for which the cross correlation is zero for all values of τ are called *uncorrelated* or *incoherent signals*.

Properties of cross correlation function for energy signals

Following are the properties of cross correlation for energy signals:

1. The cross correlation functions exhibit conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

That is unlike convolution, cross correlation is not in general commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(-\tau)$$

2. If $R_{12}(0) = 0$

i.e. if

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = 0$$

then the two signals are said to be orthogonal over the entire time interval.

3. The cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of second signal.

i.e.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

This is known as *correlation theorem*.

Cross correlation of power (periodic) signals

The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ and $x_2(t)$ may be defined with the help of average form of correlation. If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T , then cross correlation is defined as:

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t - \tau) dt$$

$$R_{21}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_2(t) x_1^*(t - \tau) dt$$

Properties of cross correlation function for power (periodic) signals

Following are the properties of cross correlation for power signals:

1. The Fourier transform of the cross correlation of two signals is equal to the multiplication of Fourier transform of one signal and complex conjugate of Fourier transform of other signal.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

2. If $R_{12}(0) = 0$

i.e. if
$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t) dt = 0$$

then the signals are said to be orthogonal over the entire time interval.

3. The cross correlation exhibits conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

4. Unlike convolution, the cross correlation is not commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

EXAMPLE 7.12 Prove that $R_{12}(\tau) = R_{21}^*(-\tau)$ i.e. the cross correlation exhibits conjugate symmetry.

Solution: The cross correlation of two signals $x_1(t)$ and $x_2(t)$ is given as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) x_2^*(t - \tau) dt$$

Let $t - \tau = n$ in the above equation for $R_{12}(\tau)$,

$$\therefore R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(n + \tau) x_2^*(n) dn$$

Also we know that

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t) x_1^*(t - \tau) dt$$

Let $t = n$ in the above equation for $R_{21}(\tau)$.

$$\therefore R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(n) x_1^*(n - \tau) dn$$

$$R_{21}^*(\tau) = \int_{-\infty}^{\infty} x_2^*(n) x_1(n - \tau) dn$$

$$R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_2^*(n) x_1(n + \tau) dn$$

Comparing the above two equations for $R_{12}(\tau)$ and $R_{21}^*(-\tau)$, we can write

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

7.6.2 Autocorrelation

The autocorrelation function gives the measure of match or similarity or relatedness or coherence between a signal and its time delayed version. This means that the autocorrelation function is a special form of cross correlation function. It is defined as the correlation of a signal with itself.

The autocorrelation is defined separately for energy signals and power signals.

Autocorrelation for energy signals

The autocorrelation of an energy signal $x(t)$ is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t - \tau) dt$$

where τ is called the delay parameter and the signal $x(t)$ is shifted by τ in positive direction.

If $x(t)$ is shifted by τ in negative direction, then

$$R(\tau) = \int_{-\infty}^{\infty} x(t + \tau) x^*(t) dt$$

Properties of autocorrelation function of energy signals

Following are the properties of autocorrelation for energy signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Thus, it states that the real part of $R(\tau)$ is an even function of τ and the imaginary part of $R(\tau)$ is an odd function of τ .

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t - \tau) dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t) x(t - \tau) dt$$

$$R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t) x(t+\tau) dt = R(\tau)$$

$$R(\tau) = R^*(-\tau)$$

2. The value of autocorrelation function of an energy signal at origin (i.e. at $\tau = 0$) is equal to the total energy of that signal, i.e.

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

3. If τ is increased in either direction, the autocorrelation $R(\tau)$ reduces. As τ reduces autocorrelation, $R(\tau)$ increases and it is maximum at $\tau = 0$, i.e. at the origin. Therefore,

$$|R(\tau)| \leq R(0) \quad (\text{for all } \tau)$$

Proof: Consider the functions $x(t)$ and $x(t+\tau)$. $[x(t) \pm x(t+\tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)x(t+\tau) \geq 0$$

or

$$x^2(t) + x^2(t+\tau) \geq \pm 2x(t)x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2 dt \geq 2 \int_{-\infty}^{\infty} x(t)x(t+\tau) dt$$

$$\therefore E + E \geq 2R(\tau) \quad [\text{If } x(t) \text{ is real valued function}]$$

$$\therefore E \geq R(\tau)$$

or

$$R(0) \geq |R(\tau)| \quad (\text{Since } R(0) = E)$$

4. The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ of energy signal form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Autocorrelation theorem

The autocorrelation theorem states that the Fourier transform of autocorrelation function $R(\tau)$ yields the energy density function of signal $x(t)$, i.e.

$$F[R(\tau)] = |X(\omega)|^2 = \psi(\omega)$$

Proof: The Fourier transform of $R(\tau)$ is

$$\begin{aligned} F[R(\tau)] &= \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) x(t-\tau) e^{-j\omega\tau} dt d\tau \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \int_{-\infty}^{\infty} x(t-\tau) e^{j\omega(t-\tau)} d\tau \\ &= X(\omega) \int_{-\infty}^{\infty} x(t-\tau) e^{j\omega(t-\tau)} d\tau \end{aligned}$$

Letting $t - \tau = n$ in the second integral, we have

$$\begin{aligned} F[R(\tau)] &= X(\omega) \int_{-\infty}^{\infty} x(n) e^{j\omega n} dn \\ &= X(\omega) X(-\omega) = |X(\omega)|^2 \\ &= \psi(\omega) \end{aligned}$$

For a given function $x(t)$, there is a unique autocorrelation function but the reverse is not true. A given autocorrelation function may correspond to an infinite variety of waveforms.

Autocorrelation function for power (periodic) signals

The autocorrelation function of a periodic signal with any period T is given by

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t-\tau) dt$$

Properties of autocorrelation function for power signals

Following are the properties of autocorrelation function for power signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Proof: We have

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t-\tau) dt$$

$\therefore$

$$R^*(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) x(t-\tau) dt$$

$\therefore$

$$R^*(-\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t) x(t+\tau) dt = R(\tau)$$

$\therefore$

$$R(\tau) = R^*(-\tau)$$

2. The autocorrelation function at origin is equal to the average power of that signal, i.e.

$$R(0) = P$$

3. The autocorrelation function $R(\tau)$ has maximum value at the origin, i.e.

$$|R(\tau)| \leq R(0)$$

The value of autocorrelation reduces as τ increases from origin.

4. The autocorrelation function $R(\tau)$ and power spectral density $S(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

5. The autocorrelation function is periodic with the same period as the periodic signal itself, i.e.

$$R(\tau) = R(\tau \pm nT), \quad n = 1, 2, 3, \dots$$

Fourier Series Representation of Periodic Signals:

The analysis of a s/g is more easy in freq. domain.
So, we have to transform the s/gs from time-domain to frequency domain.

There are 3 transformation methods present for CTS. They are:

- (1) Fourier Series
- (2) Fourier Transform
- (3) Laplace Transform.

(1) Fourier Series:

→ Fourier Series is applicable only to periodic signals i.e. signals which repeat periodically over $-s < t < s$.

→ They are represented as a linear combination of orthogonal functions called Fourier Series.

→ There are 3 methods of Fourier Series. They are:

- (1) Trigonometric form
- (2) Cosine form
- (3) Exponential form.

For the Fourier series to exist, it must satisfy certain conditions. ^(periodic s/gs) They are called Dirichlet's conditions. They are (for each period):

- (1) $x(t)$ must be a single valued function
- (2) $x(t)$ has only a finite no. of maxima & minima.
- (3) $x(t)$ has a finite number of discontinuities.
- (4) $x(t)$ has to be absolutely integrable over one period, i.e., $\int_0^T |x(t)| dt < \infty$.

① Trigonometric Form of Fourier Series:

A sinusoidal s/g,

$x(t) = A \sin \omega_0 t$ is a periodic s/g.

Let T = Time period

$$T = \frac{2\pi}{\omega_0}$$

Let,

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_k \cos k\omega_0 t \\ + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_k \sin k\omega_0 t$$

$$\text{i.e. } x(t) = a_0 + \sum_{n=1}^k [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

where $a_1, a_2, \dots, a_n$ & $b_1, b_2, \dots, b_n$ are constants

ω_0 = fundamental frequency.

Prove that $x(t)$ is a periodic signal with period T where $T = \frac{2\pi}{\omega_0}$.

The condition for a periodic signal is,

$$x(t) = x(t+T) \quad \forall t.$$

$$\Rightarrow x(t+T) = a_0 + \sum_{n=1}^K \left[a_n \cos n\omega_0(t+T) + b_n \sin n\omega_0(t+T) \right]$$

$$= a_0 + \sum_{n=1}^K \left[a_n \cos n\omega_0 \left[t + \frac{2\pi}{\omega_0} \right] + b_n \sin n\omega_0 \left[t + \frac{2\pi}{\omega_0} \right] \right]$$

$$= a_0 + \sum_{n=1}^K \left[a_n \cos(n\omega_0 t + 2n\pi) + b_n \sin(n\omega_0 t + 2n\pi) \right]$$

$$= a_0 + \sum_{n=1}^K (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$= x(t)$$

→ Now, from the above,

frequencies $0, \omega_0, 2\omega_0, \dots, k\omega_0$ is a periodic signal with period T .

→ If $K \rightarrow \infty$, the above $x(t)$ becomes infinite

fourier series with frequency terms $0, \omega_0, 2\omega_0, \dots$ is called as trigonometric form of Fourier Series.

$$x(t) = \sum_{n=0}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t.$$

a_0 = dc component ; a_n & b_n = constants
 $b_0 = 0$ for $n=0$

The first harmonic: $a_1 \cos \omega_0 t + b_1 \sin \omega_0 t$

The second harmonic: $a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$

The n^{th} harmonic: $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$.

The constants $a_0, a_1, \dots, a_n; b_1, \dots, b_n$ are called Fourier Coefficients.

$a_0 = ?$
Evaluate a_0 for $x(t)$ over $(t_0 \text{ to } t_0 + T)$:

We have:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

$$\int_{t_0}^{t_0+T} x(t) dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t] \right] dt$$

$$= a_0 [t_0 + T - t_0] + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t dt$$

$$= a_0 T + \sum_{n=1}^{\infty} a_n \left[-\frac{1}{n} \sin n\omega_0 t \right]_{t_0}^{t_0+T} +$$

$$\sum_{n=1}^{\infty} b_n \left[\frac{1}{n} \cos n\omega_0 t \right]_{t_0}^{t_0+T}$$

$$= a_0 T + \sum_{n=1}^{\infty} \left[-\frac{a_n}{n} [\sin n\omega_0 (t_0 + T) - \sin n\omega_0 t_0] \right]$$

$$+ \sum_{n=1}^{\infty} \left[\frac{b_n}{n} [\cos n\omega_0 (t_0 + T) - \cos n\omega_0 t_0] \right]$$

$$\int_{t_0}^{t_0+T} x(t) dt = a_0 T \Rightarrow a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

The 1st component is a_0 (dc component)

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

Find the Fourier Coefficients a_0 & b_n .

To evaluate a_0 & b_n ,

multiply $x(t)$ by $\cos m\omega_0 t$ & $\sin m\omega_0 t$.

& Integrate over one period.

Find a_0 :

We have,

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Multiply $x(t)$ by $\cos m\omega_0 t$ & Integrate.

$$\begin{aligned} \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t dt &= a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t dt \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cos m\omega_0 t dt. \end{aligned}$$

1st term:

$$a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t dt = \frac{a_0}{m\omega_0} \sin m\omega_0 t \Big|_{t_0}^{t_0+T}$$

$$= \frac{a_0}{m\omega_0} \left[\sin m\omega_0 (t_0+T) - \sin m\omega_0 t_0 \right]$$

$$= \frac{a_0}{m\omega_0} \left[\sin m\omega_0 t_0 - \sin m\omega_0 t_0 \right]$$

2nd term: (i) If $n \neq m$.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n \omega_0 t \cos m \omega_0 t dt &= \frac{a_n}{2} \sum_{n=1}^{\infty} \int_{t_0}^{t_0+T} 2 \cos n \omega_0 t \cos m \omega_0 t dt \\ &= \frac{1}{2} \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} [\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t] dt \\ &= \frac{1}{2} \sum_{n=1}^{\infty} a_n \left[\frac{1}{(n+m)\omega_0} \sin(n+m)\omega_0 t \right]_{t_0}^{t_0+T} + \\ &\quad \frac{1}{n-m} \sin(n-m)\omega_0 t \Big|_{t_0}^{t_0+T} \\ &= 0 \end{aligned}$$

If $n = m$.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos^2 n \omega_0 t dt &= \frac{1}{2} a_n \int_{t_0}^{t_0+T} 2 \cos^2 n \omega_0 t dt \\ &= \frac{1}{2} a_n \int_{t_0}^{t_0+T} (\cos 2n \omega_0 t + 1) dt \\ &= \frac{1}{2} a_n [t_0+T - t_0] \\ &= \frac{1}{2} a_n \cdot T. \end{aligned}$$

3rd term:

$$\Rightarrow \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n \omega_0 t \cos m \omega_0 t dt = 0.$$

$$\therefore \int_{t_0}^{t_0+T} x(t) \cos m \omega_0 t dt = 0 + \frac{1}{2} \cdot a_n \cdot T + 0.$$

$$\Rightarrow \int_{t_0}^{t_0+T} x(t) \cos n \omega_0 t dt = \frac{T}{2} a_n.$$

$$\Rightarrow \int_{t_0}^{t_0+T} x(t) \cos n \omega_0 t dt = \frac{a_n}{2} \cdot T \Rightarrow a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n \omega_0 t dt$$

Similarly, to find b_n ,
multiply $x(t)$ with $\sin n\omega_0 t$ & integrate over
($t_0, t_0 + T$).

$$\Rightarrow \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t dt = \frac{b_n}{2} \cdot T \quad \left(\text{After analysis, we obtain } \frac{b_n T}{2} \right)$$

$$\Rightarrow \boxed{b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t dt}$$

$\therefore$ The Fourier coefficients of Trigonometric Fourier

Series are:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t dt.$$

Cosine Representation (Alternate form of Trigonometric Representation)

We have,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Now multiply & divide a_n with $\sqrt{a_n^2 + b_n^2}$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t \right]$$

$$\text{Let } a_0 = A_0 \quad \cos \theta_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}$$

$$\sqrt{a_n^2 + b_n^2} = A_n \quad \sin \theta_n = \frac{-b_n}{\sqrt{a_n^2 + b_n^2}}$$

Sub. as above in $x(t)$

$$\Rightarrow x(t) = A_0 + \sum_{n=1}^{\infty} A_n [\cos \theta_n \cos n\omega_0 t - \sin \theta_n \sin n\omega_0 t]$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n [\cos(n\omega_0 t + \theta_n)]$$

The above $x(t)$ is called cosine representation of Trigonometric Fourier Series.

where A_0 = DC component; A_n = Amplitude coeff.

$A_1 \cos(\omega_0 t + \theta_1)$ = first harmonic

$\vdots$

$A_n \cos(n\omega_0 t + \theta_n)$ = n^{th} harmonic.

$\therefore$ The cosine form is also called the Harmonic form Fourier Series (or) Polar form Fourier Series.

Exponential Fourier Series

Here, $x(t)$ is represented as weighted sum of the complex exponential functions. It is also a more general, more convenient form of representation. It is extensively used in communication theory.

Complex exponential function is $e^{jn\omega_0 t}$, $n=0, \pm 1, \pm 2, \dots$

From cosine representation,

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

Consider n^{th} harmonic,

$$A_n \cos(n\omega_0 t + \theta_n) = A_n \left[\frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2} \right]$$

Sub. the above in $x(t)$.

$$\Rightarrow x(t) = A_0 + \sum_{n=1}^{\infty} \left[\frac{A_n}{2} e^{j(n\omega_0 t + \theta_n)} + \frac{A_n}{2} e^{-j(n\omega_0 t + \theta_n)} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{jn\omega_0 t} e^{j\theta_n} + e^{-jn\omega_0 t} e^{-j\theta_n} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \left[\frac{A_n}{2} e^{jn\omega_0 t} e^{j\theta_n} \right] + \sum_{n=1}^{\infty} \left[\frac{A_n}{2} e^{-jn\omega_0 t} e^{-j\theta_n} \right]$$

Let $n = -k$; sub. ~~n~~ with k in 2nd term.

$$\Rightarrow x(t) = A_0 + \sum_{n=1}^{\infty} \left[\frac{A_n}{2} e^{j\theta_n} \right] e^{jn\omega_0 t} + \sum_{k=1}^{\infty} \left[\frac{A_k}{2} e^{j\theta_k} \right] e^{-jk\omega_0 t}$$

Let $A_k = A_n$, $+ \omega_k = \omega_n$

Now let $A_0 = C_0$, $\frac{A_n}{2} e^{jn\omega_0} = C_n$, $n > 0$

Sub. as above in $x(t)$.

$$\Rightarrow x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{jn\omega_0} e^{jn\omega_0 t} + \sum_{n=-1}^{\infty} \frac{A_n}{2} e^{jn\omega_0} e^{jn\omega_0 t}$$

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} \frac{A_n}{2} e^{jn\omega_0} e^{jn\omega_0 t}$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$$

where $C_0 = A_0$

$$C_n = \frac{A_n}{2} e^{jn\omega_0}$$

$$\omega_0 = \frac{2\pi}{T}$$

Find the Fourier Coefficients of exponential Fourier Series C_0, C_n :

We have , $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$; $\omega_0 = \frac{2\pi}{T}$

Multiply $x(t)$ with $e^{-jk\omega_0 t}$ and integrating over one period ,

$$\Rightarrow \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = \int_{t_0}^{t_0+T} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} e^{-jk\omega_0 t} dt$$

We know that , $\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} 0 & \text{for } k \neq n \\ T & \text{for } k = n \end{cases}$

$$\Rightarrow \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = T \cdot C_k$$

$$C_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$

$$C_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$$C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

Derivation of Trigonometric Fourier Series from Exponential Fourier Series:

We have complex exponential Fourier Series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_{-n} e^{-jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} (C_{-n} e^{-jn\omega_0 t} + C_n e^{jn\omega_0 t})$$

$$= C_0 + \sum_{n=1}^{\infty} [C_{-n} (\cos n\omega_0 t - j \sin n\omega_0 t) + C_n (\cos n\omega_0 t + j \sin n\omega_0 t)]$$

$$\Rightarrow x(t) = C_0 + \sum_{n=1}^{\infty} (C_n + C_{-n}) \cos n\omega_0 t + j (C_n - C_{-n}) \sin n\omega_0 t$$

Comparing ① with standard equation,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = j(C_n - C_{-n})$$

Find C_0, C_n, C_{-n}

$$C_0 = \frac{1}{T} \int_0^T x(t) dt = a_0$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T x(t) [\cos n\omega_0 t - j \sin n\omega_0 t] dt$$

$$= \frac{1}{T} \left[\int_0^T x(t) \cos n\omega_0 t dt - j \int_0^T x(t) \sin n\omega_0 t dt \right]$$

$$= \frac{1}{2} \left[\frac{2}{T} \int_0^T x(t) \cos n\omega_0 t dt - j \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t dt \right]$$

$$C_n = \frac{1}{2} [a_n - j b_n]$$

$$C_{-n} = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T x(t) [\cos n\omega_0 t + j \sin n\omega_0 t] dt$$

$$= \frac{1}{2} \left[\frac{2}{T} \int_0^T x(t) \cos n\omega_0 t dt + j \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t dt \right]$$

$$C_{-n} = \frac{1}{2} [a_n + j b_n]$$

Cosine Fourier Series from Exponential Fourier Series

$$A_0 = a_0 = C_0$$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$= \sqrt{(C_n + C_{-n})^2 + \left[\frac{j}{2} (C_n - C_{-n}) \right]^2}$$

$$= \sqrt{(C_n^2 + C_{-n}^2 + 2C_n C_{-n}) - (C_n^2 + C_{-n}^2 - 2C_n C_{-n})}$$

$$= \sqrt{4C_n C_{-n}}$$

$$A_n = 2|C_n|$$

$$\therefore A_0 = C_0 ; A_n = 2|C_n|$$

EXAMPLE 4.1 Find the Fourier series expansion of the half wave rectified sine wave shown in Figure 4.1.

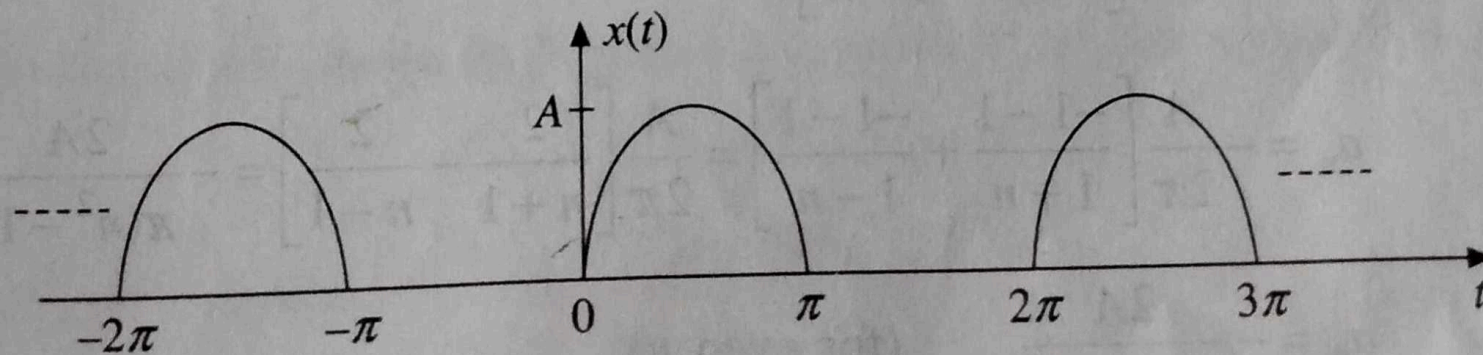


Figure 4.1 Waveform for Example 4.1.

Solution: The periodic waveform shown in Figure 4.1 with period 2π is half of a sine wave with period 2π .

$$x(t) = \begin{cases} A \sin \omega t = A \sin \frac{2\pi}{2\pi} t = A \sin t & 0 \leq t \leq \pi \\ 0 & \pi \leq t \leq 2\pi \end{cases}$$

Now the fundamental period $T = 2\pi$

$$\text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Let

$$t_0 = 0, \quad t_0 + T = T = 2\pi$$

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T x(t) dt = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt = \frac{1}{2\pi} \int_0^{\pi} A \sin t dt \\ &= \frac{A}{2\pi} [-\cos t]_0^{\pi} = \frac{A}{2\pi} [-(\cos \pi - \cos 0)] = \frac{2A}{2\pi} = \frac{A}{\pi} \end{aligned}$$

$$\therefore a_0 = \frac{A}{\pi}$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t dt = \frac{2}{2\pi} \int_0^{2\pi} x(t) \cos nt dt \\ &= \frac{1}{\pi} \int_0^{\pi} A \sin t \cos nt dt = \frac{A}{\pi} \int_0^{\pi} \sin t \cos nt dt \\ &= \frac{A}{2\pi} \int_0^{\pi} \sin(1+n)t + \sin(1-n)t = \frac{A}{2\pi} \left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n} \right] \\ &= -\frac{A}{2\pi} \left[\frac{\cos(1+n)\pi - \cos 0}{1+n} + \frac{\cos(1-n)\pi - \cos 0}{1-n} \right] \\ &= -\frac{A}{2\pi} \left\{ \left[\frac{(-1)^{n+1} - 1}{1+n} \right] + \left[\frac{(-1)^{n-1} - 1}{1-n} \right] \right\} \end{aligned}$$

$$\text{For odd } n, \quad a_n = -\frac{A}{2\pi} \left[\frac{1-1}{1+n} + \frac{1-1}{1-n} \right] = 0$$

$$\text{For even } n, \quad a_n = -\frac{A}{2\pi} \left[\frac{-1-1}{1+n} + \frac{-1-1}{1-n} \right] = \frac{A}{2\pi} \left[\frac{2}{n+1} - \frac{2}{n-1} \right] = -\frac{2A}{\pi(n^2-1)}$$

$$\therefore a_n = -\frac{2A}{\pi(n^2-1)} \quad (\text{for even } n)$$

$$\begin{aligned}
 b_n &= \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t \, dt = \frac{2}{2\pi} \int_0^{2\pi} x(t) \sin nt \, dt \\
 &= \frac{1}{\pi} \int_0^{\pi} A \sin t \sin nt \, dt = \frac{A}{\pi} \int_0^{\pi} \sin t \sin nt \, dt \\
 &= \frac{A}{2\pi} \int_0^{\pi} [\cos(n-1)t - \cos(n+1)t] \, dt \\
 &= \frac{A}{2\pi} \left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1} \right]_0^{\pi}
 \end{aligned}$$

This is zero for all values of n except for $n = 1$.

For $n = 1$,

$$b_1 = \frac{A}{2\pi}$$

Therefore, the trigonometric Fourier series is:

$$\begin{aligned}
 x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \\
 &= a_0 + b_1 \sin t + \sum_{n=1}^{\infty} a_n \cos nt \\
 &= \frac{A}{\pi} + \frac{A}{2\pi} \sin t - \sum_{n=\text{even}}^{\infty} \frac{2A}{\pi(n^2 - 1)} \cos nt
 \end{aligned}$$

EXAMPLE 4.2 Obtain the trigonometric Fourier series for the waveform shown in Figure 4.2

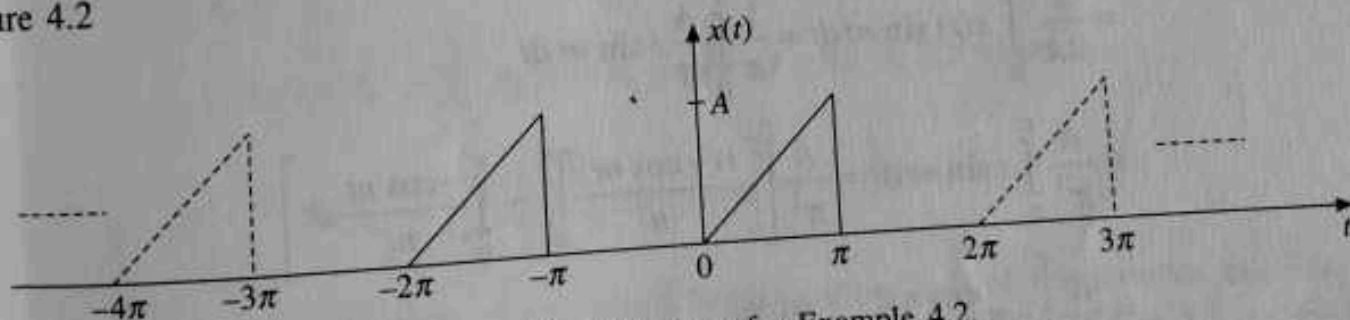


Figure 4.2 Waveform for Example 4.2.

Solution: The waveform shown in Figure 4.2 is periodic with a period $T = 2\pi$.

Let

$$t_0 = 0, t_0 + T = 2\pi$$

Then, Fundamental frequency

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

The waveform is described by

$$x(t) = \begin{cases} (A/\pi)t & \text{for } 0 \leq t \leq \pi \\ 0 & \text{for } \pi \leq t \leq 2\pi \end{cases}$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt = \frac{1}{2\pi} \int_0^{2\pi} x(t) dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{A}{\pi} t dt = \frac{A}{2\pi^2} \left[\frac{t^2}{2} \right]_0^{\pi} = \frac{A}{4}$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t dt$$

$$= \frac{2}{2\pi} \int_0^{\pi} \frac{A}{\pi} t \cos nt dt = \frac{A}{\pi^2} \int_0^{\pi} t \cos nt dt$$

$$= \frac{A}{\pi^2} \left[\left[\frac{t \sin nt}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nt}{n} dt \right] = \frac{A}{\pi^2} \left[\frac{0-0}{n} + \left(\frac{\cos nt}{n^2} \right)_0^{\pi} \right]$$

$$= \frac{A}{\pi^2 n^2} (\cos n\pi - \cos 0)$$

$$\therefore a_n = \begin{cases} -(2A/\pi^2 n^2) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t dt$$

$$= \frac{2}{2\pi} \int_0^{\pi} x(t) \sin nt dt = \frac{1}{\pi} \int_0^{\pi} \frac{A}{\pi} t \sin nt dt$$

$$= \frac{A}{\pi^2} \int_0^{\pi} t \sin nt dt = \frac{A}{\pi^2} \left[\left[\frac{t(-\cos nt)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nt}{n} dt \right]$$

$$= \frac{A}{\pi^2} \left[-\pi \frac{\cos n\pi}{n} + \left(\frac{\sin nt}{n^2} \right)_0^{\pi} \right]$$

$$= -\frac{A}{n\pi} \cos n\pi = \frac{A}{n\pi} (-1)^{n+1}$$

$$\therefore b_n = \begin{cases} A/n\pi & \text{for odd } n \\ -(A/n\pi) & \text{for even } n \end{cases}$$

The trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \frac{A}{4} - \frac{2A}{\pi^2} \sum_{n=\text{odd}}^{\infty} \frac{\cos nt}{n^2} + \frac{A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

$$= \frac{A}{4} - \frac{2A}{\pi^2} \left[\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \dots \right]$$

$$+ \frac{A}{\pi} \left[\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \frac{1}{4} \sin 4t + \dots \right]$$

EXAMPLE 4.3 Find the cosine Fourier series for the waveform shown in Figure 4.3.

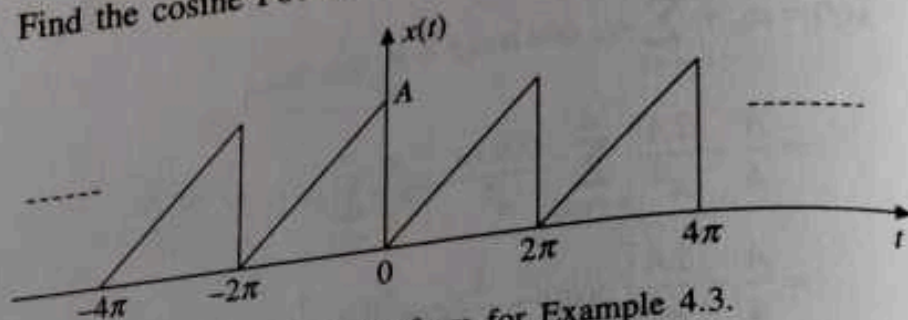


Figure 4.3 Waveform for Example 4.3.

Solution: The waveform shown in Figure 4.3 is given by

$$x(t) = \frac{A}{2\pi} t, \quad \text{for } 0 \leq t \leq 2\pi$$

$$t_0 = 0$$

Let

$$t_0 + T = T = 2\pi$$

$\therefore$

and

$$\text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Trigonometric Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

The coefficients are evaluated as follows:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt = \frac{1}{T} \int_0^{2\pi} x(t) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t dt = \frac{A}{(2\pi)^2} \left[\frac{t^2}{2} \right]_0^{2\pi} = \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t dt = \frac{2}{2\pi} \int_0^{2\pi} \left(\frac{A}{2\pi} t \right) \cos nt dt$$

$$= \frac{2A}{(2\pi)^2} \int_0^{2\pi} t \cos nt dt = \frac{2A}{(2\pi)^2} \left\{ \left[\frac{t(\sin nt)}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nt}{n} dt \right\}$$

$$= \frac{2A}{(2\pi)^2} \left[0 - 0 - \frac{1}{n} \left(-\frac{\cos nt}{n} \right) \right]_0^{2\pi} = \frac{2A}{(2\pi)^2} \left[0 - 0 + \frac{1}{n} \frac{(1-1)}{n} \right] = 0$$

$$\begin{aligned}
 b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin n\omega_0 t \, dt = \frac{2}{2\pi} \int_0^{2\pi} \left(\frac{A}{2\pi} t \right) \sin nt \, dt \\
 &= \frac{2A}{(2\pi)^2} \int_0^{2\pi} t \sin nt \, dt = \frac{2A}{(2\pi)^2} \left\{ \left[\frac{t(-\cos nt)}{n} \right]_0^{2\pi} - \int_0^{2\pi} \left(-\frac{\cos nt}{n} \right) dt \right\} \\
 &= \frac{2A}{(2\pi)^2} \left\{ \left[\frac{t(-\cos nt)}{n} \right]_0^{2\pi} + \frac{1}{n} \left(\frac{\sin nt}{n} \right)_0^{2\pi} \right\} = \frac{2A}{(2\pi)^2} \left[-2\pi \frac{[1]}{n} - 0 + \frac{1}{n} \frac{(0-0)}{n} \right] \\
 &= \frac{2A}{(2\pi)^2} \left(-\frac{2\pi}{n} \right) = -\frac{A}{\pi n}
 \end{aligned}$$

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$$\therefore a_0 = \frac{A}{2}; a_n = 0 \text{ and } b_n = -\frac{A}{\pi n}$$

are the trigonometric Fourier series coefficients.

$$\therefore x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} -\frac{A}{n\pi} \sin nt = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n} \quad 0 \leq t \leq 2\pi$$

is the trigonometric Fourier series representation.

Cosine form of representation

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

$$A_0 = a_0 = \frac{A}{2}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0^2 + \left(-\frac{A}{\pi n} \right)^2} = \frac{A}{n\pi}$$

$$\theta_n = -\tan^{-1} \frac{b_n}{a_n} = -\tan^{-1} \left(\frac{-A/n\pi}{0} \right) = -\frac{\pi}{2}$$

$$\therefore x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \cos \left(nt - \frac{\pi}{2} \right) \text{ is the cosine representation}$$