

FOURIER TRANSFORMS

Derivation of Fourier transform from Fourier series:

Fourier series is a powerful tool in dealing with different problems involving periodic functions. For non-periodic functions, Fourier transform is used. Fourier series can be extended to Non-periodic functions.

The Exponential Fourier series for periodic signals can be given as

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \longrightarrow (1)$$

$$F_n = \frac{1}{T_0} \int_0^T \tilde{f}(t) e^{-jn\omega_0 t} dt \longrightarrow (2)$$

where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$.

f_0 is the fundamental frequency of the periodic signal $\tilde{f}(t)$.

For non-periodic signals which exist over infinite duration of time, i.e., for non-periodic signals $\tilde{f}(t)$ is written as $f(t)$.

From Equation (2)

$$T_0 F_n = \int_{-\infty}^{\infty} f(t) e^{-jn\omega_0 t} dt$$

Let $T_0 F_n$ be $F(\omega)$ & $n\omega_0 = \omega$

$$\therefore F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \longrightarrow (4)$$

Therefore, the coefficient $F_n = \frac{1}{T_0} F(\omega)$

$$f_n = \frac{1}{T_0} F(n\omega_0).$$

On substituting the above two Equations in Eq (1).

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} F(n\omega_0) \right] e^{jn\omega_0 t}$$

$$\tilde{f}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t}$$

$$\tilde{f}(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t} \omega_0 \rightarrow (3)$$

In Eq (3), Let ω_0 approaches to zero then approaches infinitely, such that the product $n\omega_0$ approaches a continuous frequency variable ω , & the discrete sum is replaced by continuous integral.

$\tilde{f}(t) \rightarrow f(t)$ as ω_0 approaches zero, then Equation (3) becomes

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \rightarrow (5)$$

Eq. (4) & (5) are Fourier transform pairs

$$F(\omega) = \mathcal{F}[f(t)] \quad \& \quad f(t) = \mathcal{F}^{-1}[F(\omega)]$$

Condition for Existence of Fourier transform of a Signal

The Fourier transform does not exist for all periodic functions. The condition for $x(t)$ to have Fourier transform are:

(i) $f(t)$ is absolutely integrable over $(-\infty, \infty)$ that is

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

(ii) $f(t)$ has finite no. of discontinuities & a finite no. of maxima & minima in every finite time interval.

Fourier Transforms of standard signals:-

1. Unit Impulse function:-

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

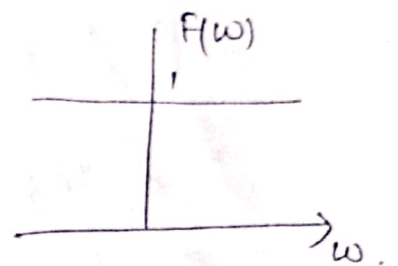
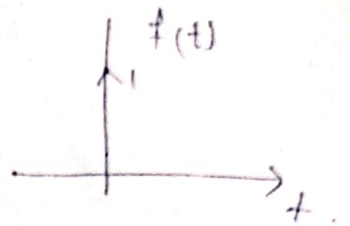
As impulse is defined only at $t=0$

$$F(\omega) = \delta(0) e^{-j\omega(0)}$$

$$= 1.$$

$$\boxed{\mathcal{F}[\delta(t)] = 1}$$

$$\delta(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0. \end{cases}$$



2. Signum function:-

This signal is not absolutely integrable.

Therefore we take the function $e^{-a|t|} \text{sgn}(t)$ & substitute the limit $a \rightarrow 0$.

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

$$\mathcal{F}[\text{sgn}(t)] = F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} \text{sgn}(t) e^{-j\omega t} dt.$$

$$F(\omega) = \lim_{a \rightarrow 0} \left(\int_{-\infty}^0 e^{+at} (-1) e^{-j\omega t} dt + \int_0^{\infty} e^{-at} (1) e^{-j\omega t} dt \right)$$

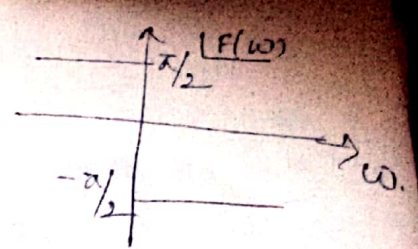
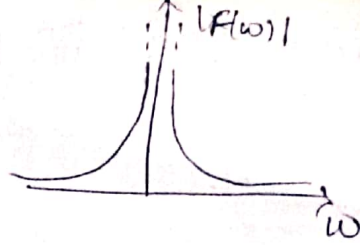
$$= \lim_{a \rightarrow 0} \left[- \int_{-\infty}^0 e^{(a+j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt \right]$$

$$= \lim_{a \rightarrow 0} \left[- \frac{e^{(a+j\omega)t}}{a+j\omega} \Big|_{-\infty}^0 + \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Big|_0^{\infty} \right]$$

$$= \lim_{a \rightarrow 0} \left[\frac{-1}{a+j\omega} + \frac{1}{a+j\omega} \right] = \lim_{a \rightarrow 0} \left[\frac{-a-j\omega + a+j\omega}{a^2 + \omega^2} \right]$$

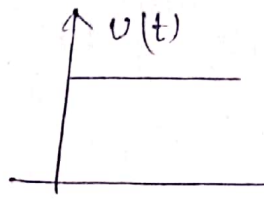
$$= \frac{-2j\omega}{\omega^2} = \frac{2}{j\omega}$$

$$\mathcal{F}[\text{sgn}(t)] = \frac{2}{j\omega}$$



3. Unit step function:-

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



$$\begin{aligned} \mathcal{F}[u(t)] &= \mathcal{F}\left[\frac{1}{2}(1 + \text{sgn}(t))\right] \\ &= \left[\mathcal{F}[1] + \mathcal{F}[\text{sgn}(t)]\right] \frac{1}{2} \\ &= \frac{1}{2} [2\pi \delta(\omega)] + \frac{1}{2} \left[\frac{2}{j\omega}\right] \end{aligned}$$

$$\begin{aligned} \text{sgn}(t) &= -1 + 2u(t) \\ 2u(t) &= 1 + \text{sgn}(t) \end{aligned}$$

$$\mathcal{F}[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega}$$

Note:-

we know know $\mathcal{F}[\delta(t)] = 1$.

$$\mathcal{F}^{-1}[2\pi \delta(\omega)] = 1$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega) e^{j\omega t} d\omega$$

$$= \delta(0) \cdot e^{j(0)t}$$

$$= 1 \cdot 1$$

$$\mathcal{F}[1] = 2\pi \delta(\omega)$$

Fourier transform of constant A

$$\boxed{\mathcal{F}[A] = 2\pi\delta(\omega)}$$

4. Single-sided Exponential Signal:-

$$f(t) = e^{-at} \quad t \geq 0$$
$$= 0 \quad t < 0.$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \int_0^{\infty} e^{-at} \cdot e^{-j\omega t} dt.$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt = \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \Big|_0^{\infty}.$$

$$= \frac{1}{a+j\omega}$$

$$\boxed{\mathcal{F}[e^{-at} u(t)] = \frac{1}{a+j\omega}}$$

5. Double-sided Exponential Signal:-

$$f(t) = e^{-a|t|} = \begin{cases} e^{-at} & t \geq 0. \\ e^{at} & t \leq 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt.$$

$$= \int_{-\infty}^0 e^{+(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt.$$

$$F(\omega) = \left. \frac{e^{(a-j\omega)t}}{a-j\omega} \right|_{-\infty}^0 + \left. \frac{e^{-(a+j\omega)t}}{- (a+j\omega)} \right|_0^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a-j\omega + a+j\omega}{a^2 + \omega^2}$$

$$\boxed{\mathcal{F}[e^{-a|t|}] = \frac{2a}{a^2 + \omega^2}}$$

6. Gate Function :—

$$f(t) = A \text{ Rect}\left(\frac{t}{2\tau}\right) = A \tau \left(\frac{t}{2\tau}\right) = \begin{cases} A & ; |t| < \tau/2 \\ 0 & ; |t| > \tau/2 \end{cases}$$

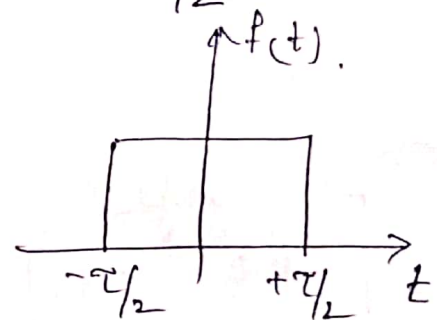
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \int_{-\tau/2}^{+\tau/2} A e^{-j\omega t} dt.$$

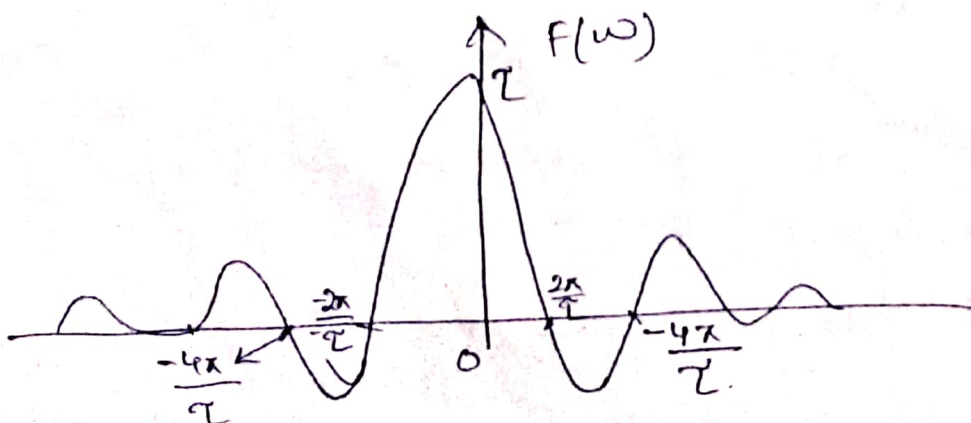
$$= A \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-\tau/2}^{+\tau/2} = \frac{-A}{j\omega} \left[e^{-j\omega \tau/2} - e^{j\omega \tau/2} \right]$$

$$= \frac{2A}{\omega} \sin\left(\frac{\omega \tau}{2}\right) = A \tau \text{ sa}\left(\frac{\omega \tau}{2}\right)$$

$$\boxed{\mathcal{F}\left[A \text{ Rect}\left(\frac{t}{2\tau}\right)\right] = A \tau \text{ sa}\left(\frac{\omega \tau}{2}\right)}$$

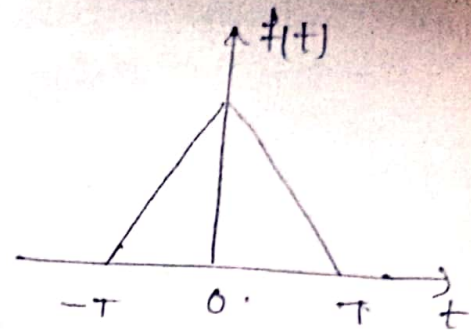


$$\begin{aligned} -t &< \tau/2 \\ t &> -\tau/2 \\ t &< \tau/2 \end{aligned}$$



⑦ Triangular Pulse :-

$$f(t) = \begin{cases} A \left(1 - \frac{|t|}{T}\right) & |t| \leq T \\ 0 & |t| > T \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-T}^0 A \left(1 + \frac{t}{T}\right) e^{-j\omega t} dt + \int_0^T A \left(1 - \frac{t}{T}\right) e^{-j\omega t} dt$$

$$= \int_0^T A \left(1 - \frac{t}{T}\right) e^{j\omega t} dt + \int_0^T A \left(1 - \frac{t}{T}\right) e^{-j\omega t} dt$$

$$= 2A \int_0^T \left(1 - \frac{t}{T}\right) \frac{e^{j\omega t} + e^{-j\omega t}}{2} dt$$

$$= 2A \int_0^T \left(1 - \frac{t}{T}\right) \cos \omega t \cdot dt$$

$$= 2A \left[\left(1 - \frac{t}{T}\right) \frac{\sin \omega t}{\omega} - \frac{1}{\omega^2 T} \cos \omega t \right]_0^T$$

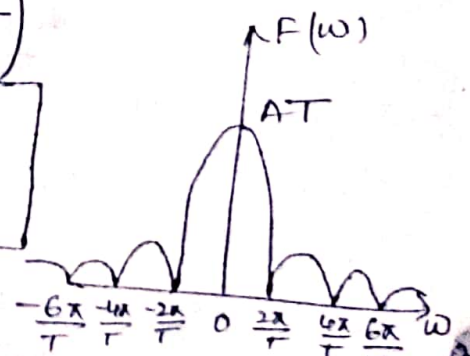
$$= 2A \left[\left(1 - \frac{T}{T}\right) \frac{\sin \omega T}{\omega} - \frac{1}{\omega^2 T} \cos \omega T + \frac{1}{\omega^2 T} \right]$$

$$= \frac{2A}{\omega^2 T} [-\cos \omega T + 1] = \frac{2A}{\omega^2 T} 2 \sin^2 \left(\frac{\omega T}{2} \right)$$

$$= \frac{4A}{\omega^2 T} \sin^2 \left(\frac{\omega T}{2} \right) = \frac{4A}{\omega^2 T} \times \frac{\omega T}{2} \times \frac{\omega T}{2} \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$$

$$= AT \operatorname{sinc}^2 \left(\frac{\omega T}{2\pi} \right)$$

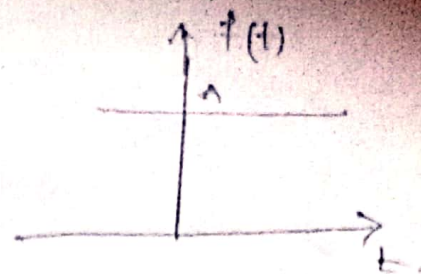
$$\boxed{\mathcal{F} \left[A \left(1 - \frac{|t|}{T}\right) \right] \longleftrightarrow AT \operatorname{sinc}^2 \left(\frac{\omega T}{2\pi} \right)}$$



(8) Dc signal :-

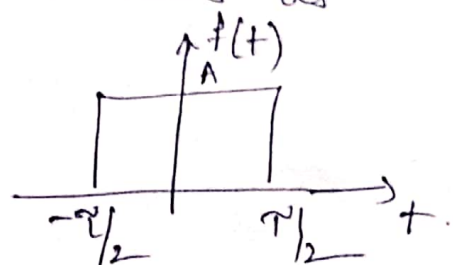
$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \int_{-\infty}^{\infty} A \cdot e^{-j\omega t} dt = \infty.$$



Thus, the above function $f(t)$ does not satisfy the absolute integrability condition. An alternate way of finding its transform is to consider a gate function of height A & width τ secs as shown below.

$$\mathcal{F}[\text{Gate function}] = A\tau \text{sa}\left(\frac{\omega\tau}{2}\right).$$



The Fourier transform of a constant is
F.T of gate function as $\tau \rightarrow \infty$

$$\mathcal{F}[A] = \lim_{\tau \rightarrow \infty} A\tau \text{sa}\left(\frac{\omega\tau}{2}\right)$$

$$= 2\pi A \lim_{\tau \rightarrow \infty} \frac{\tau}{2\pi} \text{sa}\left(\frac{\omega\tau}{2}\right).$$

$$k = \frac{\tau}{2}$$

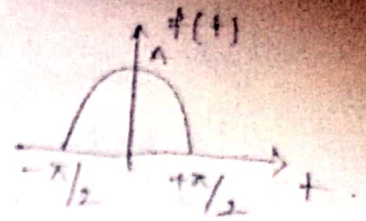
$$\text{But } \lim_{k \rightarrow \infty} \frac{k}{\pi} \text{sa}(kt) = \delta(t).$$

$$\text{So } \boxed{\mathcal{F}[A] = 2\pi A \cdot \delta(\omega)}$$

$$\boxed{\mathcal{F}[1] = 2\pi \delta(\omega)}$$

→ find the F.T of cosine wave at $t=0$.

$$f(t) = A \cos t$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\pi/2}^{+\pi/2} A \cos t e^{-j\omega t} dt$$

$$= A \int_{-\pi/2}^{+\pi/2} \frac{e^{jt} + e^{-jt}}{2} e^{-j\omega t} dt$$

$$= \frac{A}{2} \left[\int_{-\pi/2}^{+\pi/2} e^{-j(\omega-1)t} dt + \int_{-\pi/2}^{+\pi/2} e^{-j(\omega+1)t} dt \right]$$

$$= \frac{A}{2} \left[\frac{e^{-j(\omega-1)t}}{-j(\omega-1)} \right]_{-\pi/2}^{+\pi/2} + \frac{e^{-j(\omega+1)t}}{-j(\omega+1)} \Big|_{-\pi/2}^{+\pi/2}$$

$$= \frac{A}{2} \left[\frac{e^{-j\frac{\pi}{2}(\omega-1)} - e^{j\frac{\pi}{2}(\omega-1)}}{-j(\omega-1)} + \frac{e^{-j(\omega+1)\frac{\pi}{2}} - e^{-j(\omega+1)\frac{\pi}{2}}}{-j(\omega+1)} \right]$$

$$= \frac{A}{2} \left[\frac{-2j \sin\left(\frac{\pi}{2}(\omega-1)\right)}{-j(\omega-1)} + \frac{-2j \sin\left(\frac{\pi}{2}(\omega+1)\right)}{-j(\omega+1)} \right]$$

$$= A \left[\frac{\sin\left(\frac{\pi}{2}(\omega-1)\right)}{(\omega-1)} + \frac{\sin\left(\frac{\pi}{2}(\omega+1)\right)}{\omega+1} \right]$$

$$= \frac{A\pi}{2} \left[\frac{\sin\left(\frac{\pi}{2}(\omega-1)\right)}{\frac{\pi}{2}(\omega-1)} + \frac{\sin\left(\frac{\pi}{2}(\omega+1)\right)}{\frac{\pi}{2}(\omega+1)} \right]$$

$$= \frac{A\pi}{2} \left[\text{sinc}\left(\frac{\omega}{2}\right) + \text{sinc}\left(\frac{\omega}{2}\right) \right]$$

Properties of Fourier transform:-

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

① Linearity property:-

$$\text{If } \mathcal{F}[f_1(t)] = F_1(\omega) \text{ \& } \mathcal{F}[f_2(t)] = F_2(\omega)$$

$$\text{then } \mathcal{F}[a f_1(t) + b f_2(t)] = a F_1(\omega) + b F_2(\omega)$$

Proof:-

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\begin{aligned} \mathcal{F}[a f_1(t) + b f_2(t)] &= \int_{-\infty}^{\infty} [a f_1(t) + b f_2(t)] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} a f_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} b f_2(t) e^{-j\omega t} dt \\ &= a \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} f_2(t) e^{-j\omega t} dt \\ &= a F_1(\omega) + b F_2(\omega) \end{aligned}$$

② Scaling property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Proof:-

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

$$\text{Let } at = x$$

$$t = x/a, \quad dt = \frac{dx}{a}$$

$$\begin{aligned}
 \mathcal{F}[\mathcal{F}(at)] &= \int_{-\infty}^{\infty} \mathcal{F}(x) e^{-j\omega(\frac{x}{a})} \frac{dx}{a} \\
 &= \frac{1}{a} \int_{-\infty}^{\infty} \mathcal{F}(x) e^{-j(\frac{\omega}{a})x} dx \\
 &= \frac{1}{a} f\left(\frac{\omega}{a}\right)
 \end{aligned}$$

Hence

$$\mathcal{F}(at) \leftrightarrow \frac{1}{a} f\left(\frac{\omega}{a}\right)$$

Similarly, it can be shown that if $a < 0$

$$\mathcal{F}(at) \leftrightarrow \frac{1}{a} f\left(\frac{\omega}{a}\right)$$

$$\therefore \mathcal{F}(at) \leftrightarrow \frac{1}{|a|} f\left(\frac{\omega}{a}\right)$$

Significance:—

The func. $\mathcal{F}(at)$ represents func. $\mathcal{F}(t)$ compressed in the time scale by a factor of a . Similarly the func. $F(\frac{\omega}{a})$ represents a func. $F(\omega)$ expanded in the freq. scale by the same factor a .

The scaling property therefore states that compression in the time domain is equivalent to expansion in the freq. domain & vice versa.

③ Symmetry property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(t) \longleftrightarrow 2\pi f(-\omega)$$

Proof:-

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{jxt} dx.$$

$$2\pi f(t) = \int_{-\infty}^{\infty} f(x) e^{jxt} dx.$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} f(x) e^{-jxt} dx.$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} f(x) e^{-jx\omega} dx.$$

Replacing the dummy variable x by another variable t , we get.

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \mathcal{F}[f(t)]$$

$$\therefore \boxed{f(t) \longleftrightarrow 2\pi f(-\omega)}$$

④ Time-shifting property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(t-t_0) \longleftrightarrow F(\omega) e^{-j\omega t_0}.$$

Proof:-

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0) e^{-j\omega t} dt.$$

$$\text{let } t - t_0 = x.$$

$$\begin{aligned}\mathcal{F}[f(t-t_0)] &= \int_{-\infty}^{\infty} f(x) e^{-j\omega(x+t_0)} dx \\&= \int_{-\infty}^{\infty} f(x) e^{-j\omega x} e^{-j\omega t_0} dx \\&= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \\&= e^{-j\omega t_0} F(\omega).\end{aligned}$$

This theorem states that if a function is shifted in the time domain by t_0 sec's then its magnitude spectrum $F(\omega)$ remains unchanged, but the phase spectrum is changed by an amount $-\omega t_0$.

Ex:-

$$\mathcal{F}[\text{rect}(t - \tau/2)] = ?$$

$$\mathcal{F}[\text{rect}(\frac{t}{\tau})] = \tau \text{sinc}(\frac{\omega\tau}{2}).$$

$$\mathcal{F}[\text{rect}(t - \tau/2)] = \tau \text{sinc}(\frac{\omega\tau}{2}) e^{-j\omega\tau/2}$$

⑤ Frequency-shifting property:- (or Frequency-translation (or) modulation theorem)

If $f(t) \longleftrightarrow F(\omega)$

then $f(t) e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$

Proof:-

$$\begin{aligned}\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t) e^{j\omega_0 t} e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_0)t} dt \\ &= F(\omega - \omega_0)\end{aligned}$$

The theorem states that a shift of ω_0 in the freq. domain is equivalent to multiplication by $e^{j\omega_0 t}$ in the time domain.

This is known as freq. translation theorem since the signal multiplied by a factor $e^{j\omega_0 t}$ translates the whole freq. spectrum $F(\omega)$ by an amount ω_0 .

Ex:-

If a signal of freq. ω_0 ~~can~~ is expressed as the sum of exponentials, then the multiplication of the sig. $f(t)$ by a sinusoidal sig. (modulation) will translate the whole freq. spectrum. That is

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2} [f(t)e^{j\omega_0 t} + f(t)e^{-j\omega_0 t}]$$

Using freq. shift theorem

$$f(t) \longleftrightarrow F(\omega)$$

then

$$f(t) \cos \omega_0 t \longleftrightarrow \frac{1}{2} [F(\omega + \omega_0) + F(\omega - \omega_0)]$$

similarly

$$f(t) \sin \omega_0 t \longleftrightarrow \frac{j}{2} [F(\omega + \omega_0) - F(\omega - \omega_0)]$$

⑥ Time Differentiation property:-

$$f(t) \longleftrightarrow F(\omega)$$

$$\frac{d f}{d t} \longleftrightarrow j \omega F(\omega)$$

Proof:-

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\frac{d f}{d t} = \frac{1}{2\pi} \frac{d}{d t} \left[\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} F(\omega) j\omega e^{j\omega t} d\omega \right]$$

$$= \cancel{\frac{1}{2\pi}} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega F(\omega)) e^{j\omega t} d\omega \right]$$

$$= \cancel{\frac{1}{2\pi}} [j\omega F(\omega)]$$

$$\Rightarrow \frac{d f}{d t} \longleftrightarrow j\omega F(\omega)$$

Similarly

$$\frac{d^n f}{d t^n} \longleftrightarrow (j\omega)^n F(\omega)$$

④ Frequency Differentiation Property:-

$$\mathcal{F}\{f(t)\} \longleftrightarrow F(\omega)$$

then

$$-jt f(t) \longleftrightarrow \frac{dF}{d\omega}$$

Proof:-

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\frac{dF}{d\omega} = \frac{d}{d\omega} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right]$$

$$= \int_{-\infty}^{\infty} f(t) (-jt) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} (-jt) (f(t)) e^{-j\omega t} dt$$

$$-jt f(t) \longleftrightarrow \frac{dF}{d\omega}$$

Similarly

$$(-jt)^n f(t) = \frac{d^n F}{d\omega^n}$$

⑧ Time - Convolution property:-

$$\mathcal{F}\{f_1(t)\} \longleftrightarrow F_1(\omega)$$

$$\mathcal{F}\{f_2(t)\} \longleftrightarrow F_2(\omega)$$

then

$$f_1(t) * f_2(t) \longrightarrow F_1(\omega) \cdot F_2(\omega)$$

Proof:-

According to the Convolution property of signals

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$\mathcal{F}[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [f_1(t) * f_2(t)] e^{-j\omega t} dt.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_1(\tau) f_2(t-\tau) d\tau] e^{-j\omega t} dt.$$

According to the time shifting property

$$\mathcal{F}[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau) F_2(\omega) e^{-j\omega \tau} d\tau.$$

$$= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega \tau} d\tau.$$

$$= F_2(\omega) \cdot F_1(\omega).$$

$$= F_1(\omega) \cdot F_2(\omega).$$

⑨ Frequency Convolution Property:-

Q#

$$F_1(\omega) \xrightarrow{\mathcal{F}^{-1}} f_1(t)$$

$$F_2(\omega) \xrightarrow{\mathcal{F}^{-1}} f_2(t)$$

then

$$F_1(\omega) * F_2(\omega) \xrightarrow{\mathcal{F}^{-1}} 2\pi f_1(t) f_2(t)$$

Proof:-

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F_1(\omega) * F_2(\omega)] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(\omega_0) F_2(\omega - \omega_0) e^{j\omega t} d\omega d\omega_0$$

$$= \int_{-\infty}^{\infty} F_1(\omega_0) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(\omega - \omega_0) e^{j\omega t} d\omega \right] d\omega_0$$

According to the frequency shifting property

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = \int_{-\infty}^{\infty} F_1(\omega_0) f_2(t) e^{j\omega_0 t} d\omega_0$$

On Multiplying & dividing by 2π .

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = 2\pi f_2(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega_0) e^{j\omega_0 t} d\omega_0 \right]$$

$$= 2\pi f_2(t) f_1(t)$$

$$= 2\pi f_1(t) f_2(t)$$

(10) Time-Integration Property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } \int_{-\infty}^t f(t) dt \longrightarrow \frac{F(\omega)}{j\omega}$$

Proof:-

$$\text{Let } \int_{-\infty}^t f(t) dt = \phi(t) \rightarrow (1)$$

Differentiating on both sides

$$\frac{d}{dt} [f(t)] = \frac{d}{dt} [\phi(t)]$$

According to time differentiation property

$$\frac{d}{dt} [f(t)] \rightarrow j\omega f(\omega)$$

$$\text{i.e. } \frac{d}{dt} [\phi(t)] \rightarrow j\omega \phi(\omega) \rightarrow (2)$$

On substituting Eq. (1) in Eq. (2)

$$\frac{d}{dt} \left[\int_{-\infty}^t f(t) dt \right] \rightarrow j\omega \phi(\omega) \rightarrow (3)$$

$$\text{But as } \mathcal{F}[f(t)] \rightarrow f(\omega) \rightarrow (4)$$

Equating Eq. (3) & Eq. (4)

$$f(\omega) = j\omega \phi(\omega)$$

$$\phi(\omega) = \frac{f(\omega)}{j\omega}$$

$$\mathcal{F}[\phi(t)] = \frac{f(\omega)}{j\omega}$$

$$\mathcal{F} \left[\int_{-\infty}^t f(t) dt \right] \rightarrow \frac{f(\omega)}{j\omega}$$

II) Frequency - Integration Property:-

$$\mathcal{F}[f(t)] \leftrightarrow f(\omega)$$

$$\text{Then } \int_{-\infty}^{\infty} f(\omega) d\omega \rightarrow \frac{f(t)}{-jt}$$

Proof:—

$$\text{Let } \int_{-\infty}^{\infty} F(\omega) d\omega = \psi(\omega) \longrightarrow (1)$$

According to freq. differentiation property

$$\frac{d}{d\omega} [F(\omega)] \longrightarrow -j\omega f(t)$$

$$\text{i.e. } \frac{d}{d\omega} [\psi(\omega)] \longrightarrow -j\omega \psi(t) \longrightarrow (2)$$

Substituting Eq (1) in Eq (2)

$$\frac{d}{d\omega} \left[\int_{-\infty}^{\infty} F(\omega) d\omega \right] \longrightarrow -j\omega \psi(t)$$

$$F(\omega) d\omega \longrightarrow -j\omega \psi(t) \longrightarrow (3)$$

$$\text{But } \mathcal{F}^{-1}[F(\omega)] = f(t) \longrightarrow (4)$$

Equating Eq. (3) & (4)

$$f(t) = -j\omega \psi(t)$$

$$\psi(t) = \frac{f(t)}{-j\omega}$$

$$\int_{-\infty}^{\infty} F(\omega) d\omega \longrightarrow \frac{f(t)}{-j\omega}$$

(12) Parseval's Theorem:— (Energy density spectrum)

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(\omega)$$

then According to Parseval's Theorem

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Proof:-

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} f(t) f^*(t) dt \longrightarrow (1)$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\mathcal{F}^{-1}[F^*(\omega)] = f^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{j\omega t} d\omega \longrightarrow (2)$$

On substituting Eq. (2) in Eq. (1).

$$E = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Problems:-

①. Find the F.T of a periodic function.

Sol:-

The Fourier transform of a periodic function does not exist, since it does not satisfy the absolute integrability condition i.e.

$\int_{-\infty}^{\infty} f(t) dt$ is not finite for a periodic function, but in limiting case it exists.

Any periodic function can be expressed in terms of exponential Fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{jn\omega_0 t}$$

Taking Fourier transform on both sides

$$\begin{aligned} \mathcal{F}[f(t)] &= \mathcal{F}\left[\sum_{n=-\infty}^{\infty} f_n e^{jn\omega_0 t}\right] \\ &= \sum_{n=-\infty}^{\infty} f_n \mathcal{F}[e^{jn\omega_0 t}] \\ &= \sum_{n=-\infty}^{\infty} f_n 2\pi \delta(\omega - n\omega_0) \end{aligned}$$

$$\boxed{\mathcal{F}[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} f_n \delta(\omega - n\omega_0)}$$

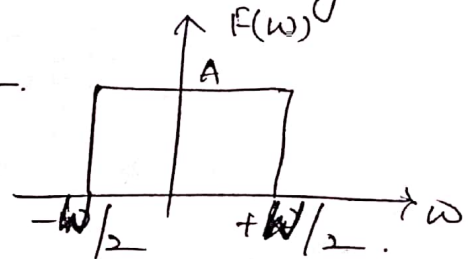
The above relation states that "the spectral density function or Fourier transform of a

periodic signal consists of impulses located at the harmonic freq's of the signal, at that the strength of the impulse is same as 2π times the value of corresponding coefficient of the Exponential Fourier series.

② Find & sketch the Inverse Fourier transform of the waveform shown in Fig.

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t)$$



$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-W/2}^{W/2} A \cdot e^{j\omega t} d\omega$$

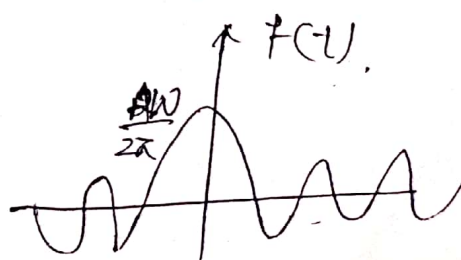
$$= \frac{A}{2\pi} \left[\frac{e^{j\omega t}}{jt} \right]_{-W/2}^{W/2}$$

$$= \frac{A}{2\pi jt} \left[e^{j\frac{W}{2}t} - e^{-j\frac{W}{2}t} \right]$$

$$= \frac{A}{\pi t} \left[\frac{e^{j\frac{W}{2}t} - e^{-j\frac{W}{2}t}}{2j} \right]$$

$$= \frac{A}{\pi t} \left[\sin \frac{Wt}{2} \right]$$

$$= \frac{AW}{2\pi} \left[\text{sinc} \left(\frac{Wt}{2} \right) \right]$$



③ Find FT of the following time function

$$x(t) = e^{-3t} [u(t+2) - u(t+3)]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-3t} [u(t+2) - u(t+3)] e^{-j\omega t} dt$$

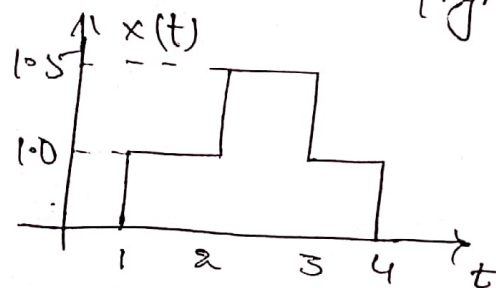
$$= \int_{-2}^3 e^{-3t} e^{-j\omega t} dt = \int_{-2}^3 e^{-(3+j\omega)t} dt$$

$$= \left[\frac{e^{-(3+j\omega)t}}{-(3+j\omega)} \right]_{-2}^3$$

$$= \frac{e^{-2(3+j\omega)} - e^{-3(3+j\omega)}}{3+j\omega}$$

④ Find the Fourier Transform of the signal shown in figure

$$f(t) = \begin{cases} 1.0 & 1 < t < 2 \\ 1.5 & 2 < t < 3 \\ 1.0 & 3 < t < 4 \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_1^2 1.0 e^{-j\omega t} dt + \int_2^3 1.5 e^{-j\omega t} dt + \int_3^4 1.0 e^{-j\omega t} dt$$

$$F(\omega) = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_1^2 + 1.5 \left[\frac{e^{-j\omega t}}{-j\omega} \right]_2^3 + \left[\frac{e^{-j\omega t}}{-j\omega} \right]_3^4$$

$$= \frac{-1}{j\omega} [e^{-2j\omega} - e^{-j\omega}] + \frac{1.5}{-j\omega} [e^{-3j\omega} - e^{-2j\omega}] - \frac{1}{j\omega} [e^{-4j\omega} - e^{-3j\omega}]$$

$$= \frac{-1}{j\omega} [e^{-2j\omega} - e^{-j\omega} + 1.5e^{-3j\omega} - 1.5e^{-2j\omega} + e^{-4j\omega} - e^{-3j\omega}]$$

$$= \frac{-1}{j\omega} [(1-1.5)e^{-2j\omega} - e^{-j\omega} + (1.5-1)e^{-3j\omega} + e^{-4j\omega}]$$

$$= \frac{-1}{j\omega} [-0.5e^{-2j\omega} - e^{-j\omega} + 0.5e^{-3j\omega} + e^{-4j\omega}]$$

$$= \frac{1}{j\omega} [0.5e^{-2j\omega} + e^{-j\omega} - 0.5e^{-3j\omega} - e^{-4j\omega}]$$

$$= \frac{1}{j\omega} [e^{-j\omega} + 0.5e^{-2j\omega} - 0.5e^{-3j\omega} - e^{-4j\omega}]$$

$$X(\omega) = \frac{1}{j\omega} [e^{-j\omega} + 0.5e^{-2j\omega} - 0.5e^{-3j\omega} - e^{-4j\omega}]$$

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(6) Obtain the F.T of the following

(i) $x(t) = A \sin(2\pi f_c t) u(t)$

(ii) $x(t) = f(t) \cos(2\pi f_c t + \phi)$

Sol:-

(i) $x(t) = A \sin(2\pi f_c t) u(t)$

Taking F.T on both sides

$$X(\omega) = \mathcal{F}[A \sin(2\pi f_c t) u(t)]$$

$$\mathcal{F}[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega}$$

$$\sin 2\pi f_c t = \sin \omega_c t = \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j}$$

$$\begin{aligned} \mathcal{F}[A \sin \omega_c t u(t)] &= \mathcal{F}\left[A \frac{e^{j\omega_c t} - e^{-j\omega_c t}}{2j} u(t)\right] \\ &= \frac{A}{2j} \left[\mathcal{F}[e^{j\omega_c t} u(t)] - \mathcal{F}[e^{-j\omega_c t} u(t)] \right] \end{aligned}$$

On Applying freq. shifting property

$$\mathcal{F}[e^{j\omega_0 t} f(t)] = F(\omega - \omega_0)$$

$$\begin{aligned} X(\omega) &= \frac{A}{2j} \left[\mathcal{F}[e^{j\omega_c t} u(t)] - \mathcal{F}[e^{-j\omega_c t} u(t)] \right] \\ &= \frac{A}{2j} \left[\pi \delta(\omega - \omega_c) + \frac{1}{j(\omega - \omega_c)} - \pi \delta(\omega + \omega_c) + \frac{1}{j(\omega + \omega_c)} \right] \end{aligned}$$

$$X(\omega) = \frac{A}{2j} \left[\pi \delta(\omega - \omega_c) - \pi \delta(\omega + \omega_c) \right] \\ + \frac{A}{2j} \left[\frac{1}{j(\omega - \omega_c)} - \frac{1}{j(\omega + \omega_c)} \right]$$

$$X(\omega) = \frac{A j \pi}{2} \left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c) \right] \\ + \frac{A}{2} \left[\frac{1}{\omega + \omega_c} - \frac{1}{\omega - \omega_c} \right] \\ = \frac{A}{2} \left[j \pi (\delta(\omega + \omega_c) - \delta(\omega - \omega_c)) - \frac{2\omega_c}{\omega^2 - \omega_c^2} \right]$$

$$(ii) x(t) = f(t) \cos(2\pi f_c t + \phi)$$

$$\cos(2\pi f_c t + \phi) = \cos(\omega_c t + \phi) \\ = \frac{e^{j(\omega_c t + \phi)} + e^{-j(\omega_c t + \phi)}}{2} \\ = \frac{e^{j\omega_c t} \cdot e^{j\phi} + e^{-j\omega_c t} \cdot e^{-j\phi}}{2}$$

$$f[x(t)] = \mathcal{F}[f(t) \cos 2\pi f_c t + \phi] \\ = \mathcal{F}[f(t) \cos(\omega_c t + \phi)] \\ = \mathcal{F}\left[f(t) \left[\frac{e^{j\omega_c t} \cdot e^{j\phi} + e^{-j\omega_c t} \cdot e^{-j\phi}}{2} \right]\right]$$

$$\begin{aligned} \mathcal{F}[x(t)] &= \mathcal{F}\left[f(t) e^{j\omega_c t} \cdot e^{j\phi} \frac{1}{2} + \frac{1}{2} f(t) e^{-j\omega_c t} e^{-j\phi}\right] \\ &= \frac{1}{2} \mathcal{F}\left[f(t) e^{j\omega_c t} e^{j\phi} + f(t) e^{-j\omega_c t} e^{-j\phi}\right] \\ &\rightarrow \textcircled{1} \end{aligned}$$

By using freq. shifting property:—

$$f(t) e^{j2\pi f_c t} \longleftrightarrow F(\omega - \omega_c)$$

$$f(t) e^{-j2\pi f_c t} \longleftrightarrow F(\omega + \omega_c)$$

On Applying the above two Equations
to Equation $\textcircled{1}$

$$X(\omega) = \frac{1}{2} e^{j\phi} F(\omega - \omega_c) + \frac{1}{2} e^{-j\phi} F(\omega + \omega_c)$$

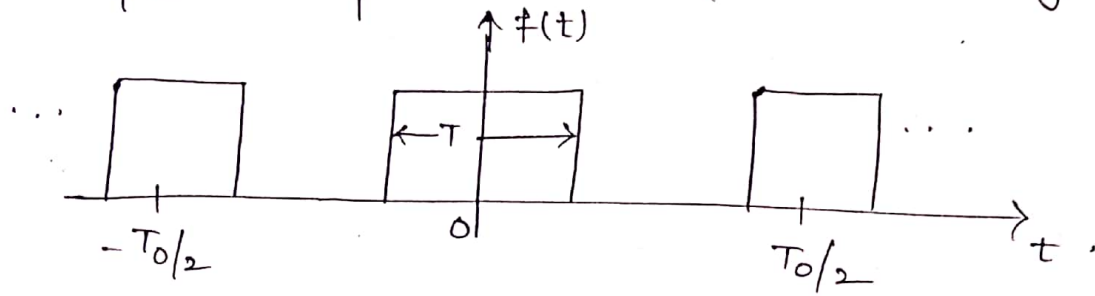
$$= \frac{1}{2} [\cos\phi + j\sin\phi] F(\omega - \omega_c) +$$

$$\frac{1}{2} [\cos\phi - j\sin\phi] F(\omega + \omega_c)$$

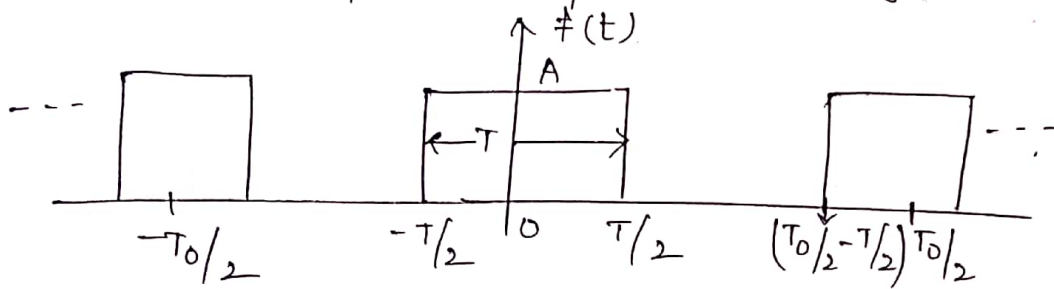
$$= \frac{1}{2} \cos\phi [F(\omega - \omega_c) + F(\omega + \omega_c)] +$$

$$\frac{j\sin\phi}{2} [F(\omega - \omega_c) - F(\omega + \omega_c)]$$

⑦ Find out the Fourier transform of the periodic pulse train shown in figure:-



The given periodic pulse train is.



Then, $f(t)$ can be defined as

$$f(t) = \begin{cases} A & -T/2 < t < T/2 \\ 0 & T/2 < t < (T_0/2 - T/2) \end{cases}$$

The Fourier transform of periodic pulse train $f(t)$ is given by

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$$

where

$$F_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$$

$$T = T_0/2$$

$$F_n = \frac{1}{T_0/2} \int_{-T/2}^{+T/2} A e^{-jn\omega_0 t} dt$$

$$F_n = \frac{2A}{T_0} \int_{-T/2}^{T/2} e^{-jn\omega_0 t} dt$$

$$= \frac{2A}{T_0} \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right]_{-T/2}^{T/2}$$

$$= \frac{-2A}{jn\omega_0 T_0} \left[e^{-jn\omega_0 T/2} - e^{jn\omega_0 T/2} \right]$$

$$= \frac{4A}{n\omega_0 T_0} \left[\frac{e^{jn\omega_0 T/2} - e^{-jn\omega_0 T/2}}{2j} \right]$$

$$= \frac{4A}{n\omega_0 T_0} \left[\sin \left[\frac{n\omega_0 T}{2} \right] \right]$$

$$= \frac{2AT}{T_0} \frac{\sin \left[\frac{n\omega_0 T}{2} \right]}{\left[\frac{n\omega_0 T}{2} \right]}$$

$$= \frac{2AT}{T_0} \text{sinc} \left[\frac{n\omega_0 T}{2} \right]$$

But $\omega_0 = \frac{2\pi}{T_0}$

Hence

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0)$$

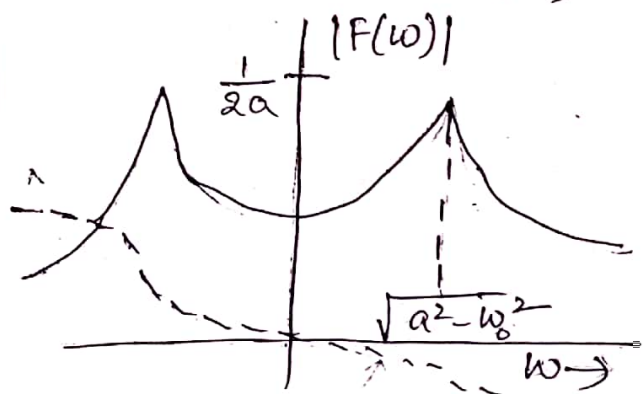
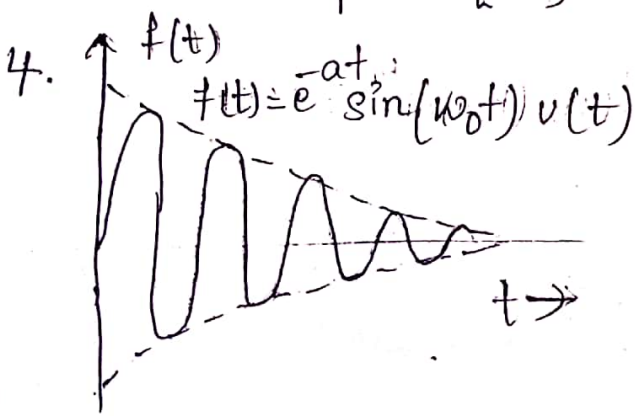
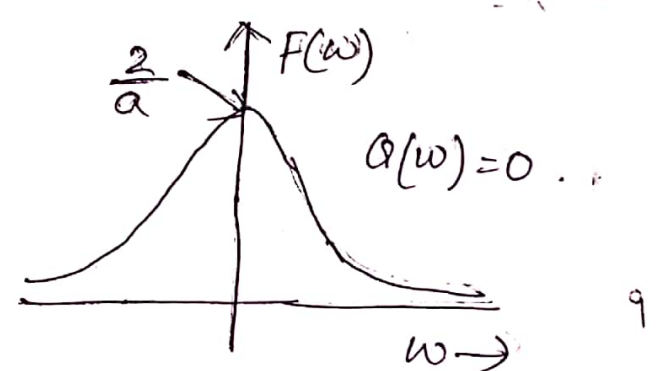
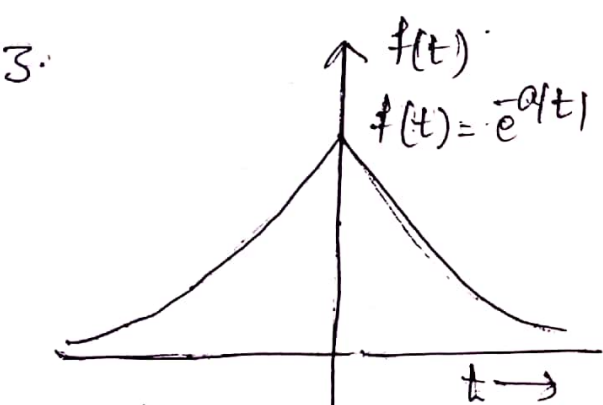
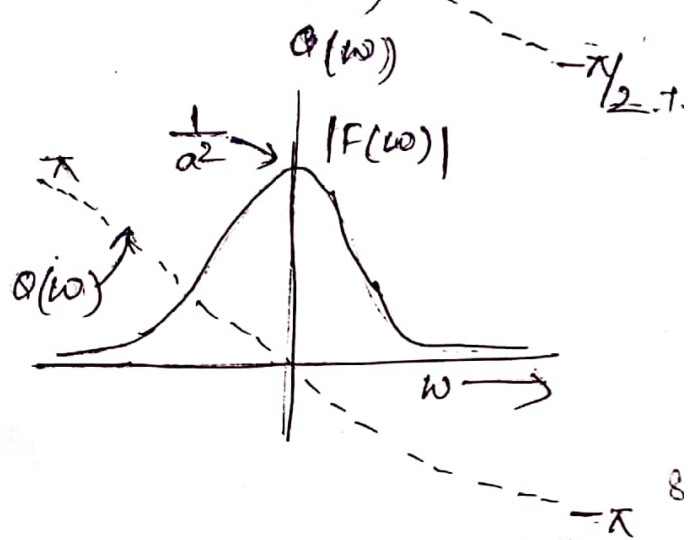
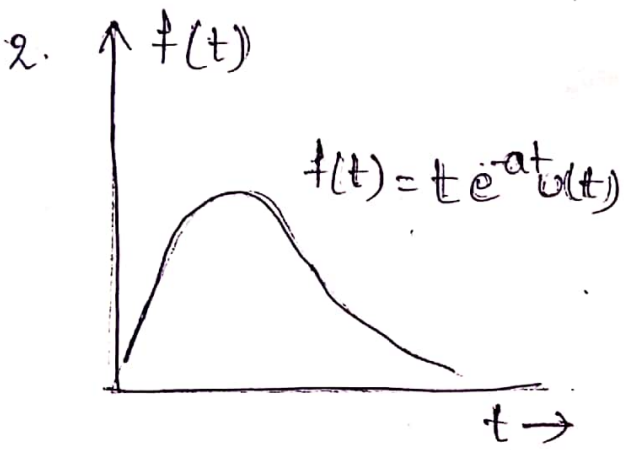
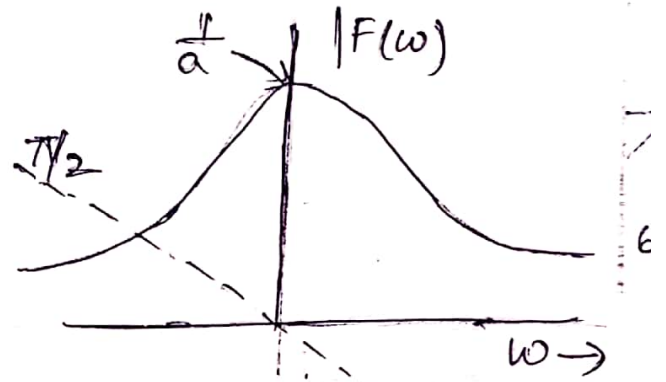
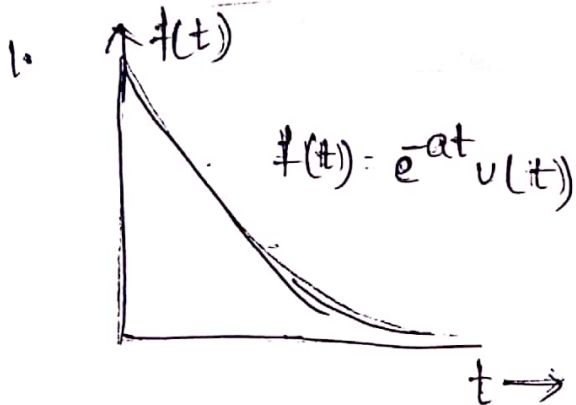
$$= 2\pi \sum_{n=-\infty}^{\infty} \frac{2AT}{T_0} \text{sinc} \left[\frac{n\pi T}{T_0} \right] \delta(\omega - n\omega_0)$$

$$F(\omega) = \frac{4\pi AT}{T_0} \sum_{n=-\infty}^{\infty} \text{sinc} \left(\frac{n\pi T}{T_0} \right) \delta(\omega - n\omega_0)$$

Various Functions of Time & Their Spectral-Density Functions:-

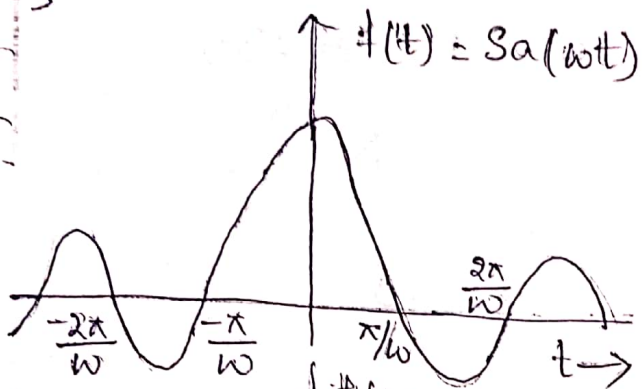
Function in time

Spectral Density Function

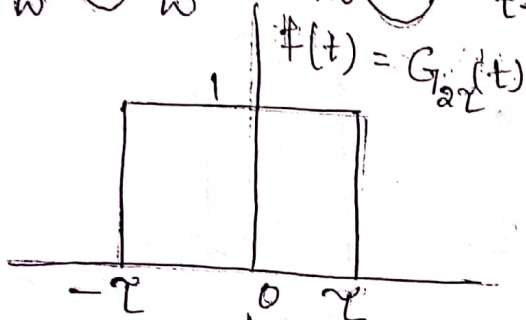


Function in time

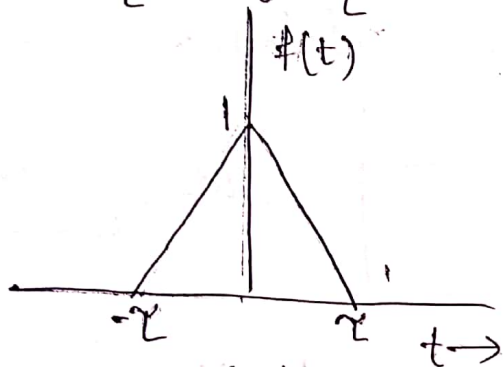
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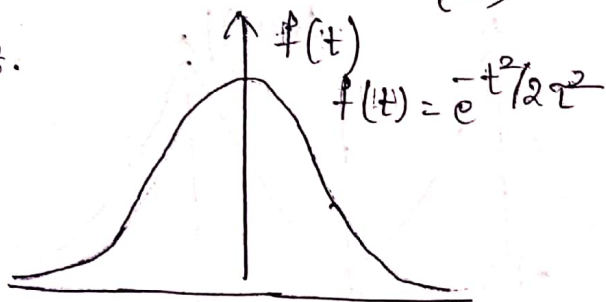
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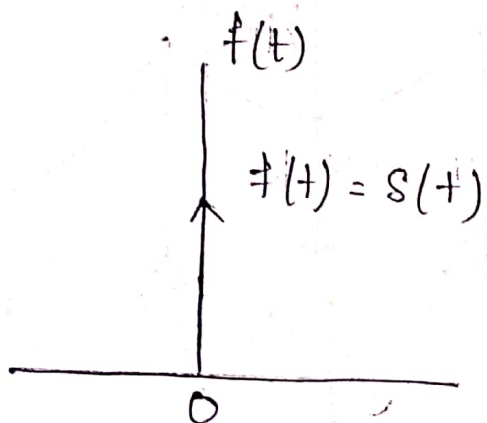
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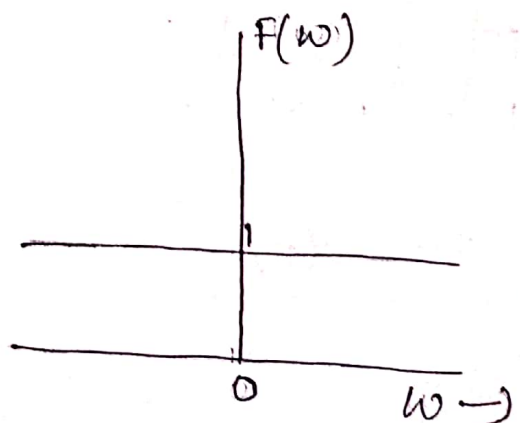
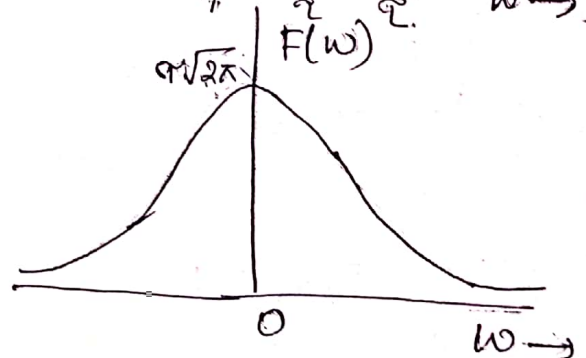
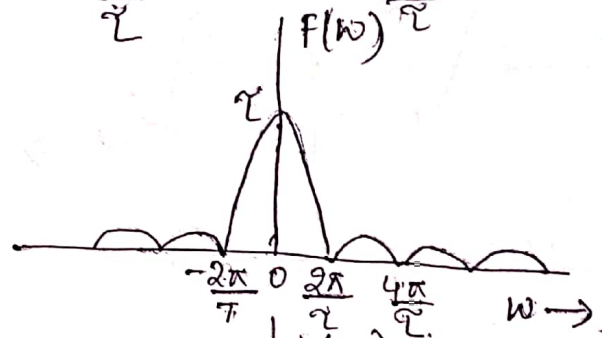
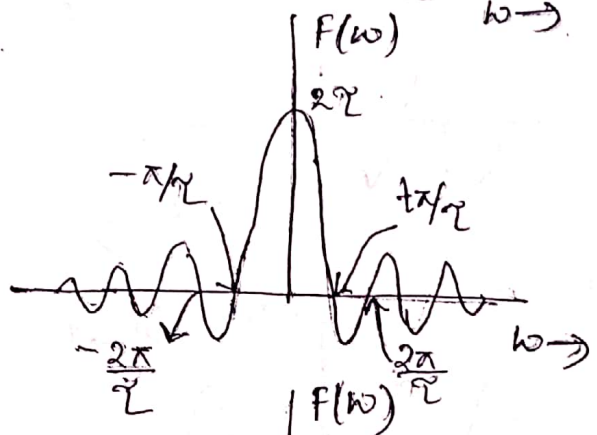
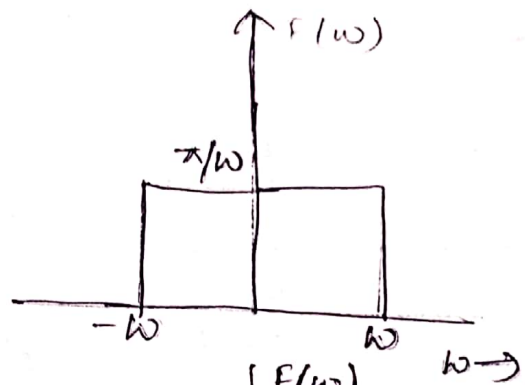


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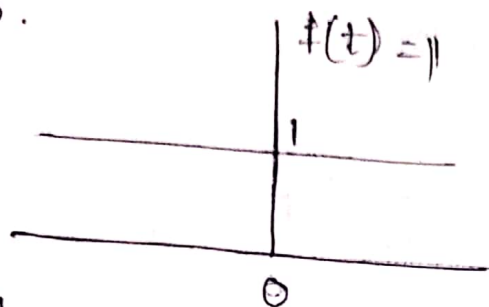
Spectral Density Function

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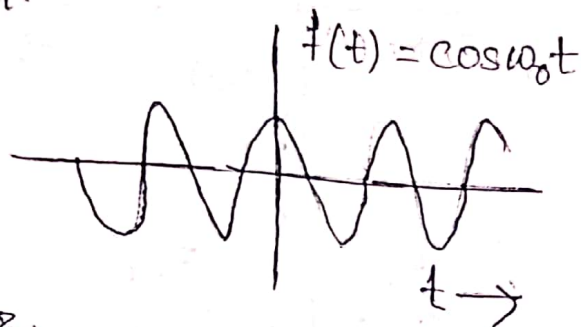


Function in time

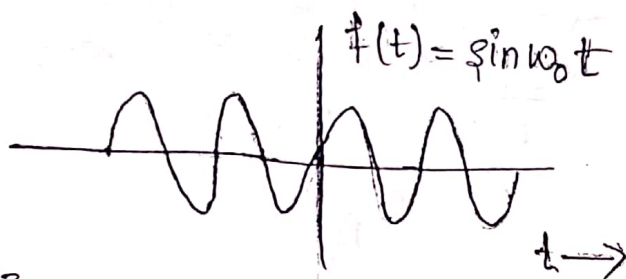
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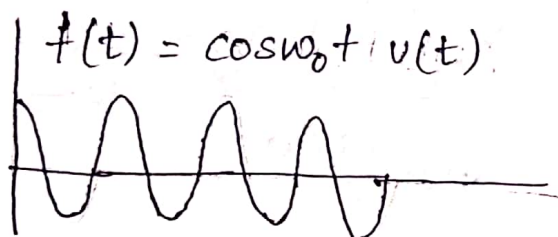
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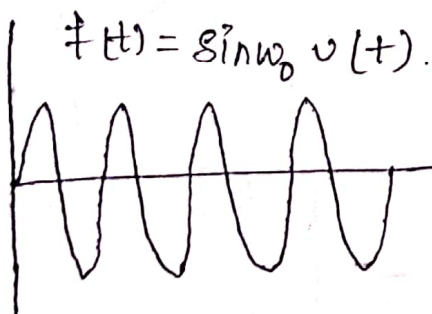
12.



13.

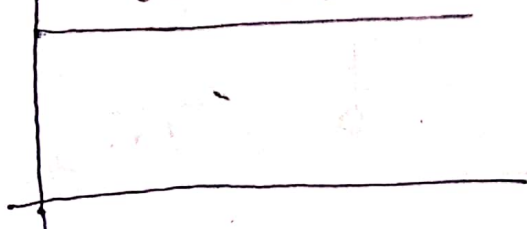


14.

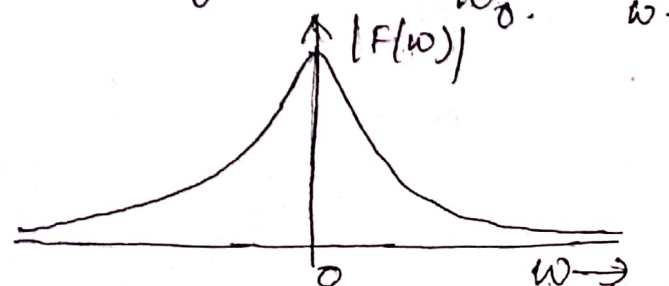
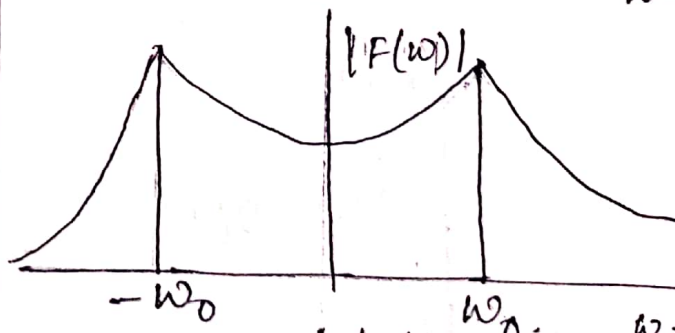
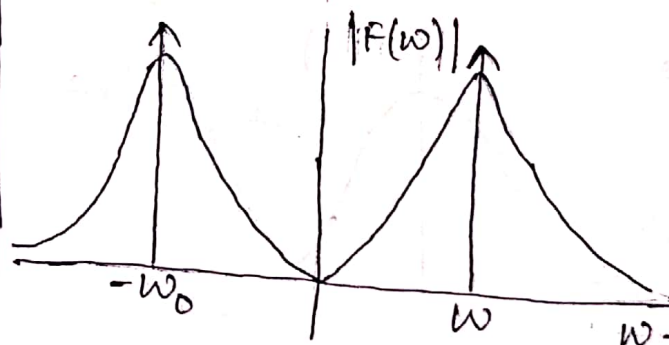
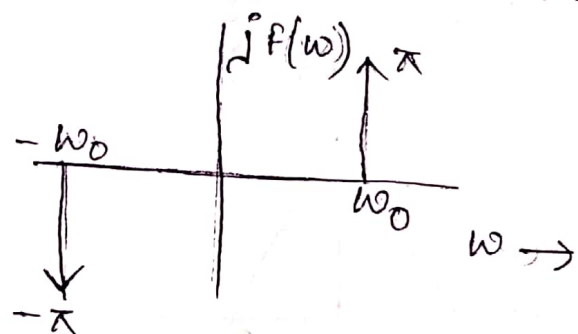
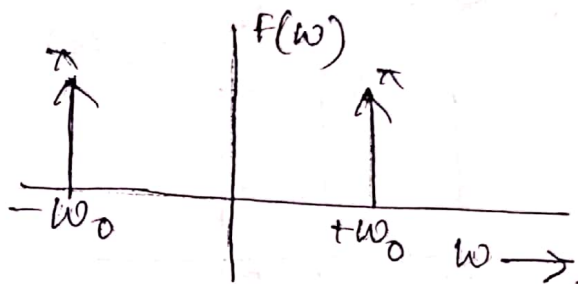
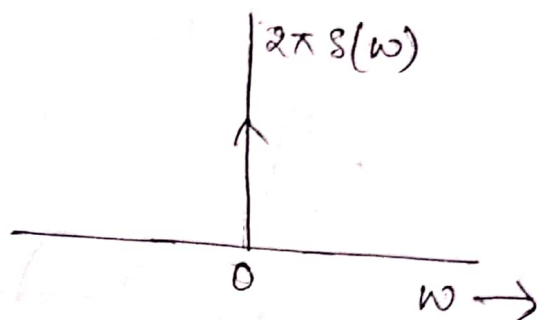


15.

$$f(t) = v(t).$$



Spectral Density Function



Function in time

Spectral Density Function.

16

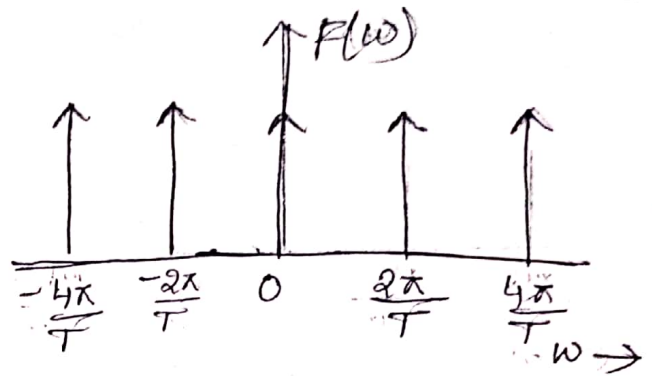
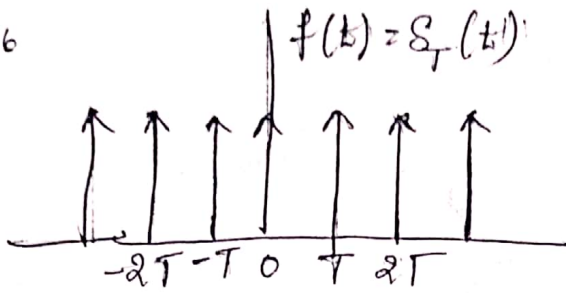


Table of Fourier Transform

$f(t)$	$F(\omega)$
1. $e^{-at} u(t)$	$1/(a+j\omega)$
2. $t e^{-at} u(t)$	$1/(a+j\omega)^2$
3. $ t $	$-2/\omega^2$
4. $\delta(t)$	1
5. 1	$2\pi \delta(\omega)$
6. $u(t)$	$\pi \delta(\omega) + 1/j\omega$
7. $\cos \omega_0 t u(t)$	$\pi/2 [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
8. $\sin \omega_0 t u(t)$	$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
9. $\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
10. $\sin \omega_0 t$	$j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
11. $e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(a+j\omega)^2 + \omega_0^2}$
12. $\frac{\omega}{2\pi} \text{Sa}(\frac{\omega T}{2})$	$G_{\omega}(w)$
13. $G_T(t)$	$T \text{Sa}(\frac{\omega T}{2})$

$$14. \quad 1 - \frac{|t|}{\tau} \dots |t| < \tau$$

$$0 \dots |t| > \tau$$

$$15. \quad e^{-a|t|}$$

$$16. \quad e^{-(t/2\tau)^2}$$

$$17. \quad \delta_T(t)$$

$$\tau \left[\text{sinc}\left(\frac{\omega\tau}{2}\right) \right]^2$$

$$\frac{2a}{a^2 + \omega^2}$$

$$\tau \sqrt{2\pi} e^{-t^2/2\tau^2}$$

$$\omega_0 \delta_{\omega_0}(\omega) \quad (\omega_0 = \frac{2\pi}{T})$$

⑧ Find the FT of $(\cos \omega_0 t) u(t)$

Sol:- $\mathcal{F}[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega}$

$$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\mathcal{F}[\cos \omega_0 t u(t)] = \frac{1}{2} \mathcal{F}[(e^{j\omega_0 t} + e^{-j\omega_0 t}) u(t)]$$

$$= \frac{1}{2} \mathcal{F}[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t)] \quad \text{--- (1)}$$

On using the freq. shifting property.

$$\mathcal{F}[e^{j\omega_0 t} u(t)] = \pi \delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)}$$

$$\mathcal{F}[e^{-j\omega_0 t} u(t)] = \pi \delta(\omega + \omega_0) + \frac{1}{j(\omega + \omega_0)}$$

Substituting the above two Equations in Eq. (1).

$$\mathcal{F}[\cos \omega_0 t \cdot v(t)]$$

$$= \frac{1}{2} \left[\pi \delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)} + \pi \delta(\omega + \omega_0) + \frac{1}{j(\omega + \omega_0)} \right]$$

$$= \frac{\pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] + \frac{1}{2j} \left[\frac{1}{\omega - \omega_0} + \frac{1}{\omega + \omega_0} \right]$$

$$= \frac{\pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] + \frac{1}{2j} \left[\frac{\omega}{\omega^2 - \omega_0^2} \right]$$

$$= \frac{\pi}{2} \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

⑨ find the f.T of $(\sin \omega_0 t) v(t)$.

$$\mathcal{F}[v(t)] = \pi \delta(\omega) + \frac{1}{j\omega}$$

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\mathcal{F}[\sin \omega_0 t \cdot v(t)] = \frac{1}{2j} \mathcal{F}[e^{j\omega_0 t} v(t) + e^{-j\omega_0 t} v(t)]$$

On Using the freq. shifting property

$$\mathcal{F}[e^{j\omega_0 t} v(t)] = \pi \delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)}$$

$$\mathcal{F}[e^{-j\omega_0 t} v(t)] = \pi \delta(\omega + \omega_0) + \frac{1}{j(\omega + \omega_0)}$$

Substituting the above two equations in Eq. ①.

$$\mathcal{F}[\sin \omega_0 t \, u(t)]$$

$$= \frac{1}{2j} \left[\pi \delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)} + \pi \delta(\omega + \omega_0) + \frac{1}{j(\omega + \omega_0)} \right]$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{2j} \left[\frac{1}{j(\omega - \omega_0)} + \frac{1}{j(\omega + \omega_0)} \right]$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{2} \left[\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega} \right]$$

$$= \frac{\pi}{2j} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$$

(10) Find the F.T of $f(t) = t e^{-at} u(t)$

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} t e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} t e^{-at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} t e^{-(a+j\omega)t} dt$$

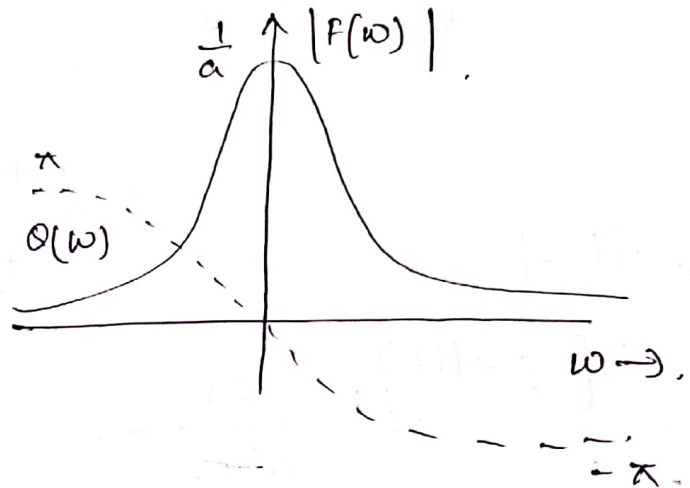
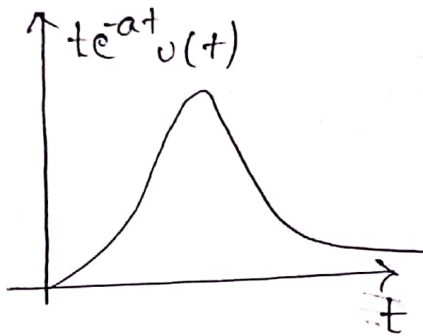
Integration by parts yields

$$F(\omega) = \mathcal{F}[t e^{-at} u(t)] = \frac{1}{(a+j\omega)^2} \quad a > 0.$$

$$\mathcal{F}^{-1} \left[\frac{1}{(a+j\omega)^2} \right] = t e^{-at} u(t).$$

The spectral density function $F(\omega)$ can be expressed in terms of magnitude & phase-density functions.

$$F(\omega) = \frac{1}{(a+j\omega)^2} = \frac{1}{a^2 + \omega^2} e^{-j2 \tan^{-1}(\omega/a)}.$$



(11) From the properties of Fourier transform, find the Fourier transform of the following.

- (i) $e^{j2t} u(t)$ (ii) $e^{j\omega_0 t}$ (iii) $e^{-3t} u(t-2)$ (iv) $e^{-a/t-2}$
 (v) $\delta(t+2) + \delta(t+1) + \delta(t-1) + \delta(t-2)$ (vi) $\cos \omega_0 t$
 (vii) $\sin(\omega_0 t)$ (viii) $u(-t)$

sol:- (i) $e^{j2t} u(t)$.

$$\mathcal{F}[u(t)] = \pi \delta(\omega) + \frac{1}{j\omega}.$$

$$\mathcal{F}[e^{j\omega_0 t} f(t)] = F(\omega - \omega_0).$$

$$\mathcal{F}[e^{j2t} u(t)] = \pi \delta(\omega - 2) + \frac{1}{j(\omega - 2)}.$$

(ii) $e^{j\omega_0 t}$

$$\mathcal{F}[1] = 2\pi \delta(\omega).$$

$$\mathcal{F}[e^{j\omega_0 t} f(t)] = F(\omega - \omega_0)$$

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi \delta(\omega - \omega_0).$$

$$(ii) e^{-3t} v(t-2) = e^{-3(t-2)} v(t-2) e^{-6}.$$

$$\mathcal{F}[e^{-3t} v(t)] = \frac{1}{3+j\omega}$$

$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0} F(\omega).$$

$$\begin{aligned} \mathcal{F}[e^{-3(t-2)} v(t-2) e^{-6}] &= e^{-6} e^{-j\omega 2} \frac{1}{3+j\omega} \\ &= \frac{e^{-2(3+j\omega)}}{3+j\omega} \end{aligned}$$

$$(iv) e^{-a|t-2|}$$

$$\mathcal{F}[e^{-a|t|}] = \frac{2a}{a^2 + \omega^2}$$

$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0} F(\omega).$$

$$\mathcal{F}[e^{-a|t-2|}] = e^{-j2\omega} \frac{2a}{a^2 + \omega^2}.$$

$$(v) s(t+2) + s(t+1) + s(t-1) + s(t-2).$$

$$\mathcal{F}[s(t)] = 1.$$

$$\mathcal{F}[f(t-t_0)] = e^{-j\omega t_0} F(\omega).$$

$$\begin{aligned} \mathcal{F}[s(t+2) + s(t+1) + s(t-1) + s(t-2)] &= e^{2j\omega} + e^{j\omega} + e^{-j\omega} + e^{-2j\omega} \\ &= 2 \left[\frac{e^{j\omega} + e^{-j\omega}}{2} + \frac{e^{2j\omega} + e^{-2j\omega}}{2} \right] = 2[\cos \omega + \cos 2\omega]. \end{aligned}$$

$$(vi) \cos \omega_0 t = 1 \cdot \cos \omega_0 t.$$

$$\mathcal{F}[1] = 2\pi \delta(\omega)$$

$$\mathcal{F}[\cos \omega_0 t f(t)] = \frac{1}{2} [F(\omega + \omega_0) + F(\omega - \omega_0)]$$

$$\mathcal{F}[1 \cdot \cos \omega_0 t] = \frac{1}{2} \left[2\pi \delta(\omega + \omega_0) + 2\pi \delta(\omega - \omega_0) \right]$$

$$= \pi \left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right].$$

(vii) $\sin \omega_0 t = 1 \cdot \sin \omega_0 t$

$$\mathcal{F}[1] = 2\pi \delta(\omega).$$

$$\mathcal{F}[\sin \omega_0 t \cdot f(t)] = \frac{j}{2} \left[F(\omega + \omega_0) - F(\omega - \omega_0) \right]$$

$$= \frac{j}{2} \left[2\pi \delta(\omega + \omega_0) - 2\pi \delta(\omega - \omega_0) \right]$$

$$= j\pi \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0) \right].$$

(viii) $\mathcal{F}[v(-t)]$

From time reversal property $f(-t) \longleftrightarrow F(-\omega)$.

$$\mathcal{F}[v(-t)] = F[v(t)] / \omega = -\omega$$

$$= \pi \delta(\omega) + \frac{1}{j\omega} / \omega = -\omega.$$

$$= \pi \delta(-\omega) + \frac{1}{j(-\omega)} = \pi \delta(\omega) - \frac{1}{j\omega}.$$

(12) Show that $\frac{j}{\pi t} \longleftrightarrow \text{sgn}(\omega)$.

$$\mathcal{F}[\text{sgn}(t)] = \frac{2}{j\omega}.$$

From symmetry property $\mathcal{F}[F(t)] = 2\pi f(-\omega)$.

$$\mathcal{F}\left[\frac{2}{j t}\right] = 2\pi \text{sgn}(-\omega).$$

$$\frac{2}{j t} = -2\pi \text{sgn}(\omega).$$

$$\frac{j}{\pi t} \longleftrightarrow \text{sgn}(\omega).$$

13) find the F.T of the signal

(i) $f(t) = \frac{1}{a^2 + t^2}$ (ii) $f(t) = \frac{1}{t}$

Sol:-

(i) $\mathcal{F}[e^{-a|t|}] = \frac{2a}{a^2 + \omega^2}$

From Symmetry property $\mathcal{F}[f(t)] = 2\pi f(-\omega)$

$$\mathcal{F}\left[\frac{2a}{a^2 + t^2}\right] = 2\pi e^{-a|-\omega|}$$

$$\mathcal{F}\left[\frac{1}{a^2 + t^2}\right] = \frac{1}{2a} \cancel{2\pi} e^{-a|-\omega|}$$
$$= \frac{\pi}{a} e^{-a|\omega|}$$

(ii) $f(t) = \frac{1}{t}$

$$\mathcal{F}[\text{sgn}(t)] = \frac{2}{j\omega}$$

$$\frac{j}{2} \text{sgn}(t) = \frac{1}{\omega}$$

From symmetry property, $\mathcal{F}[f(t)] = 2\pi f(-\omega)$

$$\mathcal{F}\left[\frac{1}{t}\right] = \cancel{2\pi} \frac{j}{2} \text{sgn}(-\omega)$$

$$= -j\pi \text{sgn}(\omega)$$

14) find the inverse F.T of $x(\omega) = \frac{j\omega}{(2+j\omega)^2}$

$$\mathcal{F}[te^{-at}u(t)] = \frac{1}{(a+j\omega)^2}$$

$$\mathcal{F}[te^{-2t}u(t)] = \frac{1}{(2+j\omega)^2}$$

$$\mathcal{F}\left[\frac{d}{dt} f(t)\right] = j\omega f(\omega)$$

$$\mathcal{F}^{-1}\left[j\omega \cdot \frac{1}{(2+j\omega)^2}\right] = \frac{d}{dt} [te^{-2t}u(t)] =$$

(15) By using partial fraction expansion, find the inverse F.T of the following

$$(i) F(\omega) = \frac{4j\omega + 6}{(j\omega)^2 + 6j\omega + 8}$$

$$(ii) F(\omega) = \frac{1+3j\omega}{(j\omega+6)^2}$$

Sol:-

$$(i) F(\omega) = \frac{4j\omega + 6}{(j\omega)^2 + 6j\omega + 8}$$

$$= \frac{4j\omega + 6}{(j\omega + 2)(j\omega + 4)} = \frac{A}{j\omega + 2} + \frac{B}{j\omega + 4}$$

$$A = \frac{4j\omega + 6}{j\omega + 4} \Big|_{j\omega = -2} = \frac{-8 + 6}{-2 + 4} = -1$$

$$B = \frac{4j\omega + 6}{j\omega + 2} \Big|_{j\omega = -4} = \frac{-16 + 6}{-4 + 2} = \frac{-10}{-2} = 5$$

$$F(\omega) = \frac{-1}{2 + j\omega} + \frac{5}{j\omega + 4}$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = -e^{-2t}u(t) + 5e^{-4t}u(t)$$

$$(ii) F(\omega) = \frac{1+3j\omega}{(j\omega+3)^2} = \frac{A}{3+j\omega} + \frac{B}{(3+j\omega)^2}$$

$$= \frac{3}{3+j\omega} - \frac{8}{(j\omega+3)^2}$$

$$= 3e^{-3t}u(t) - 8te^{-3t}u(t)$$

(16) Using Fourier transform, find the convolution of the signals. $x_1(t) = e^{-2t} u(t)$ & $x_2(t) = e^{-3t} u(t)$.

Sol:- From convolution properties of Fourier transforms,

$$\mathcal{F}[x_1(t) * x_2(t)] = x_1(\omega) x_2(\omega)$$

$$\mathcal{F}[e^{-at} u(t)] = \frac{1}{a+j\omega}$$

$$\mathcal{F}[e^{-2t} u(t)] = \frac{1}{2+j\omega} = x_1(\omega) \quad \mathcal{F}[e^{-3t} u(t)] = \frac{1}{3+j\omega} = x_2(\omega)$$

$$\mathcal{F}[x_1(t) * x_2(t)] = x_1(\omega) x_2(\omega)$$

$$\Rightarrow x_1(t) * x_2(t) = \mathcal{F}^{-1}[x_1(\omega) x_2(\omega)]$$

$$= \mathcal{F}^{-1}\left[\frac{1}{(2+j\omega)(3+j\omega)}\right] = \mathcal{F}^{-1}\left[\frac{A}{2+j\omega} + \frac{B}{3+j\omega}\right]$$

$$= \mathcal{F}^{-1}\left[\frac{1}{2+j\omega} - \frac{1}{3+j\omega}\right] = e^{-2t} u(t) - e^{-3t} u(t)$$

(17) The i/p & o/p of a causal LTI system are related by the differential equation.

$$\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = x(t)$$

(a) Find the impulse response of the system.

(b) What is the response of this system if

$$x(t) = e^{-4t} u(t)$$

Sol:- $\frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 6y(t) = x(t)$

Taking F.T on both sides

$$(j\omega)^2 y(\omega) + 5j\omega y(\omega) + 6y(\omega) = x(\omega)$$

$$[(j\omega)^2 + 5j\omega + 6] y(\omega) = x(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 5j\omega + 6}$$

The impulse response $h(t)$ of the system is the inverse F.T of the transfer function $H(\omega)$ of the system

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}\left[\frac{1}{(j\omega)^2 + 5j\omega + 6}\right]$$

$$H(\omega) = \frac{1}{(j\omega)^2 + 5j\omega + 6} = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 3}$$

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}\left[\frac{1}{j\omega + 2}\right] - \mathcal{F}^{-1}\left[\frac{1}{j\omega + 3}\right]$$

$$= e^{-2t}u(t) - e^{-3t}u(t)$$

$$(b). x(t) = e^{-4t}u(t) \Rightarrow x(\omega) = \frac{1}{j\omega + 4}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(j\omega + 4)} \frac{1}{(j\omega + 3)(j\omega + 2)}$$

$$= \frac{1/2}{j\omega + 2} - \frac{1}{j\omega + 3} + \frac{1/2}{j\omega + 4}$$

Taking inverse F.T on both sides

$$y(t) = \mathcal{F}^{-1}[Y(\omega)]$$

$$= \frac{1}{2}e^{-2t}u(t) - e^{-3t}u(t) + \frac{1}{2}e^{-4t}u(t)$$

- (18) The i/p & o/p of a causal LTI system are described by the differential equation.

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t).$$

- (a) Find the frequency response of the system.
(b) Find the impulse response of the system.
(c) What is the response of the system if $x(t) = te^{-t} u(t)$?

Sol: (a) $\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t).$

Taking F.T on both sides

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = X(\omega).$$

Frequency response $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3j\omega + 2}.$

(b). $H(\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2} = \frac{1}{(j\omega + 2)(j\omega + 1)}.$

Taking inverse F.T on both sides

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}\left[\frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}\right]$$
$$= e^{-t} u(t) - e^{-2t} u(t).$$

(c). $x(t) = te^{-t} u(t)$ $X(\omega) = \frac{1}{(1 + j\omega)^2}.$

$$Y(\omega) = X(\omega) H(\omega) = \frac{1}{(1 + j\omega)^2} \cdot \frac{1}{(1 + j\omega)(2 + j\omega)}$$

$$Y(\omega) = \frac{1}{(1+j\omega)^3(j\omega+2)}$$

$$= \frac{A}{j\omega+1} + \frac{B}{(j\omega+1)^2} + \frac{C}{(j\omega+1)^3} + \frac{D}{j\omega+2}$$

$$= \frac{1}{j\omega+1} - \frac{1}{(j\omega+1)^2} + \frac{1}{(j\omega+1)^3} - \frac{1}{j\omega+2}$$

Taking inverse F.T on both sides,

$$y(t) = e^{-t} u(t) - t e^{-t} u(t) + \frac{t^2}{2} e^{-t} u(t) - e^{-2t} u(t)$$

Properties of Fourier transform:-

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

① Linearity property:-

$$\text{If } \mathcal{F}[f_1(t)] = F_1(\omega) \text{ \& } \mathcal{F}[f_2(t)] = F_2(\omega)$$

$$\text{then } \mathcal{F}[a f_1(t) + b f_2(t)] = a F_1(\omega) + b F_2(\omega)$$

Proof:-

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\begin{aligned} \mathcal{F}[a f_1(t) + b f_2(t)] &= \int_{-\infty}^{\infty} [a f_1(t) + b f_2(t)] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} a f_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} b f_2(t) e^{-j\omega t} dt \\ &= a \int_{-\infty}^{\infty} f_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} f_2(t) e^{-j\omega t} dt \\ &= a F_1(\omega) + b F_2(\omega) \end{aligned}$$

② Scaling property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Proof:-

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

$$\text{Let } at = x$$

$$t = x/a, \quad dt = \frac{dx}{a}$$

$$\begin{aligned}
 \mathcal{F}[\mathcal{F}(at)] &= \int_{-\infty}^{\infty} \mathcal{F}(x) e^{-j\omega\left(\frac{x}{a}\right)} \frac{dx}{a} \\
 &= \frac{1}{a} \int_{-\infty}^{\infty} \mathcal{F}(x) e^{-j\left(\frac{\omega}{a}\right)x} dx \\
 &= \frac{1}{a} f\left(\frac{\omega}{a}\right)
 \end{aligned}$$

Hence

$$\mathcal{F}(at) \leftrightarrow \frac{1}{a} f\left(\frac{\omega}{a}\right)$$

Similarly, it can be shown that if $a < 0$

$$\mathcal{F}(at) \leftrightarrow \frac{1}{a} f\left(\frac{\omega}{a}\right)$$

$$\therefore \mathcal{F}(at) \leftrightarrow \frac{1}{|a|} f\left(\frac{\omega}{a}\right)$$

Significance:—

The func. $\mathcal{F}(at)$ represents func. $\mathcal{F}(t)$ compressed in the time scale by a factor of a . Similarly the func. $F\left(\frac{\omega}{a}\right)$ represents a func. $F(\omega)$ expanded in the freq. scale by the same factor a .

The scaling property therefore states that compression in the time domain is equivalent to expansion in the freq. domain & vice versa.

③ Symmetry property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(t) \longleftrightarrow 2\pi f(-\omega)$$

Proof:-

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{jxt} dx.$$

$$2\pi f(t) = \int_{-\infty}^{\infty} f(x) e^{jxt} dx.$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} f(x) e^{-jxt} dx.$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} f(x) e^{-jx\omega} dx.$$

Replacing the dummy variable x by another variable t , we get.

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$= \mathcal{F}[f(t)]$$

$$\therefore \boxed{f(t) \longleftrightarrow 2\pi f(-\omega)}$$

④ Time-shifting property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(t-t_0) \longleftrightarrow F(\omega) e^{-j\omega t_0}.$$

Proof:-

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0) e^{-j\omega t} dt.$$

$$\text{let } t - t_0 = x.$$

$$\begin{aligned}\mathcal{F}[f(t-t_0)] &= \int_{-\infty}^{\infty} f(x) e^{-j\omega(x+t_0)} dx \\&= \int_{-\infty}^{\infty} f(x) e^{-j\omega x} e^{-j\omega t_0} dx \\&= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \\&= e^{-j\omega t_0} F(\omega).\end{aligned}$$

This theorem states that if a function is shifted in the time domain by t_0 sec's then its magnitude spectrum $F(\omega)$ remains unchanged, but the phase spectrum is changed by an amount $-\omega t_0$.

Ex:-

$$\mathcal{F}[\text{rect}(t - \tau/2)] = ?$$

$$\mathcal{F}[\text{rect}(\frac{t}{\tau})] = \tau \text{sinc}(\frac{\omega\tau}{2}).$$

$$\mathcal{F}[\text{rect}(t - \tau/2)] = \tau \text{sinc}(\frac{\omega\tau}{2}) e^{-j\omega\tau/2}$$

⑤ Frequency-shifting property:- (or Frequency-translation (or) modulation theorem)

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } f(t) e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$$

Proof:-

$$\begin{aligned}\mathcal{F}[f(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} f(t) e^{j\omega_0 t} e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-j(\omega - \omega_0)t} dt \\ &= F(\omega - \omega_0)\end{aligned}$$

The theorem states that a shift of ω_0 in the freq. domain is equivalent to multiplication by $e^{j\omega_0 t}$ in the time domain.

This is known as freq. translation theorem since the signal multiplied by a factor $e^{j\omega_0 t}$ translates the whole freq. spectrum $F(\omega)$ by an amount ω_0 .

Ex:-

If a signal of freq. ω_0 ~~can~~ is expressed as the sum of exponentials, then the multiplication of the sig. $f(t)$ by a sinusoidal sig. (modulation) will translate the whole freq. spectrum. That is

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2} [f(t)e^{j\omega_0 t} + f(t)e^{-j\omega_0 t}]$$

Using freq. shift theorem

$$f(t) \longleftrightarrow F(\omega)$$

then

$$f(t) \cos \omega_0 t \longleftrightarrow \frac{1}{2} [F(\omega + \omega_0) + F(\omega - \omega_0)]$$

similarly

$$f(t) \sin \omega_0 t \longleftrightarrow \frac{j}{2} [F(\omega + \omega_0) - F(\omega - \omega_0)]$$

⑥ Time Differentiation property:-

$$f(t) \longleftrightarrow F(\omega)$$

$$\frac{df}{dt} \longleftrightarrow j\omega F(\omega)$$

Proof:-

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\frac{df}{dt} = \frac{1}{2\pi} \frac{d}{dt} \left[\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} F(\omega) j\omega e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} (j\omega F(\omega)) e^{j\omega t} d\omega \right]$$

$$= j\omega F(\omega)$$

$$\Rightarrow \frac{df}{dt} \longleftrightarrow j\omega F(\omega)$$

Similarly

$$\frac{d^n f}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$$

④ Frequency Differentiation Property:-

$$\mathcal{F}\{f(t)\} \longleftrightarrow F(\omega)$$

then

$$-j\omega f(t) \longleftrightarrow \frac{dF}{d\omega}$$

Proof:-

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\frac{dF}{d\omega} = \frac{d}{d\omega} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right]$$

$$= \int_{-\infty}^{\infty} f(t) (-j) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} (-j) f(t) e^{-j\omega t} dt$$

$$-j\omega f(t) \longleftrightarrow \frac{dF}{d\omega}$$

Similarly

$$(-j\omega)^n f(t) = \frac{d^n F}{d\omega^n}$$

⑧ Time - Convolution property:-

$$\mathcal{F}\{f_1(t)\} \longleftrightarrow F_1(\omega)$$

$$\mathcal{F}\{f_2(t)\} \longleftrightarrow F_2(\omega)$$

then

$$f_1(t) * f_2(t) \longrightarrow F_1(\omega) \cdot F_2(\omega)$$

Proof:-

According to the Convolution property of signals

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau.$$

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

$$\mathcal{F}[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [f_1(t) * f_2(t)] e^{-j\omega t} dt.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_1(\tau) f_2(t-\tau) d\tau] e^{-j\omega t} dt.$$

According to the time shifting property

$$\mathcal{F}[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau) F_2(\omega) e^{-j\omega \tau} d\tau.$$

$$= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega \tau} d\tau.$$

$$= F_2(\omega) \cdot F_1(\omega).$$

$$= F_1(\omega) \cdot F_2(\omega).$$

⑨ Frequency Convolution Property:-

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$$F_1(\omega) \xrightarrow{\mathcal{F}^{-1}} f_1(t)$$

$$F_2(\omega) \xrightarrow{\mathcal{F}^{-1}} f_2(t)$$

then

$$F_1(\omega) * F_2(\omega) \xrightarrow{\mathcal{F}^{-1}} 2\pi f_1(t) f_2(t)$$

Proof:-

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} [F_1(\omega) * F_2(\omega)] e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(\omega_0) F_2(\omega - \omega_0) e^{j\omega t} d\omega d\omega_0$$

$$= \int_{-\infty}^{\infty} F_1(\omega_0) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(\omega - \omega_0) e^{j\omega t} d\omega \right] d\omega_0$$

According to the frequency shifting property

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = \int_{-\infty}^{\infty} F_1(\omega_0) f_2(t) e^{j\omega_0 t} d\omega_0$$

On Multiplying & dividing by 2π .

$$\mathcal{F}^{-1}[F_1(\omega) * F_2(\omega)] = 2\pi f_2(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega_0) e^{j\omega_0 t} d\omega_0 \right]$$

$$= 2\pi f_2(t) f_1(t)$$

$$= 2\pi f_1(t) f_2(t)$$

⑩ Time-Integration Property:-

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } \int_{-\infty}^t f(t) dt \longrightarrow \frac{F(\omega)}{j\omega}$$

Proof:-

$$\text{Let } \int_{-\infty}^t f(t) dt = \phi(t) \rightarrow (1)$$

Differentiating on both sides

$$\frac{d}{dt} [f(t)] = \frac{d}{dt} [\phi(t)]$$

According to time differentiation property

$$\frac{d}{dt} [f(t)] \rightarrow j\omega f(\omega)$$

$$\text{i.e. } \frac{d}{dt} [\phi(t)] \rightarrow j\omega \phi(\omega) \rightarrow (2)$$

On substituting Eq. (1) in Eq. (2)

$$\frac{d}{dt} \left[\int_{-\infty}^t f(t) dt \right] \rightarrow j\omega \phi(\omega) \rightarrow (3)$$

$$\text{But as } \mathcal{F}[f(t)] \rightarrow f(\omega) \rightarrow (4)$$

Equating Eq. (3) & Eq. (4)

$$f(\omega) = j\omega \phi(\omega)$$

$$\phi(\omega) = \frac{f(\omega)}{j\omega}$$

$$\mathcal{F}[\phi(t)] = \frac{f(\omega)}{j\omega}$$

$$\mathcal{F} \left[\int_{-\infty}^t f(t) dt \right] \rightarrow \frac{f(\omega)}{j\omega}$$

II) Frequency - Integration Property:-

$$\mathcal{F}[f(t)] \leftrightarrow f(\omega)$$

$$\text{Then } \int_{-\infty}^{\infty} f(\omega) d\omega \rightarrow \frac{f(t)}{-jt}$$

Proof:—

$$\text{Let } \int_{-\infty}^{\infty} F(\omega) d\omega = \psi(\omega) \longrightarrow (1)$$

According to freq. differentiation property

$$\frac{d}{d\omega} [F(\omega)] \longrightarrow -j\omega f(t)$$

$$\text{i.e. } \frac{d}{d\omega} [\psi(\omega)] \longrightarrow -j\omega \psi(t) \longrightarrow (2)$$

Substituting Eq (1) in Eq (2)

$$\frac{d}{d\omega} \left[\int_{-\infty}^{\infty} F(\omega) d\omega \right] \longrightarrow -j\omega \psi(t)$$

$$F(\omega) d\omega \longrightarrow -j\omega \psi(t) \longrightarrow (3)$$

$$\text{But } \mathcal{F}^{-1}[F(\omega)] = f(t) \longrightarrow (4)$$

Equating Eq. (3) & (4)

$$f(t) = -j\omega \psi(t)$$

$$\psi(t) = \frac{f(t)}{-j\omega}$$

$$\int_{-\infty}^{\infty} F(\omega) d\omega \longrightarrow \frac{f(t)}{-j\omega}$$

(12) Parseval's Theorem:— (Energy density spectrum)

$$\mathcal{F}^{-1}[\mathcal{F}(t)] = f(\omega)$$

then According to Parseval's Theorem

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Proof:-

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} f(t) f^*(t) dt \longrightarrow (1)$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

$$\mathcal{F}^{-1}[F^*(\omega)] = f^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{j\omega t} d\omega \longrightarrow (2)$$

On substituting Eq. (2) in Eq. (1).

$$E = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$