

LAPLACE TRANSFORMS

Introduction:-

The mathematical tools such as Fourier series & Fourier transform are extremely useful in the analysis of signals & systems.

Fourier analysis techniques cannot be used for signals that are not absolutely integrable. For ex, signals $e^{at}u(t)$ (where $a > 0$), e^{-at} (where $-\infty < t < 0$), ramp & parabolic etc. such limitation of Fourier transform is overcome by introducing the convergent real exponential $e^{-\sigma t}$ in the transform integral. Thus, -ve exponential $e^{-j\omega t}$ in Fourier transform becomes e^{-st} where $s = \sigma + j\omega$. The introduction of the $e^{-\sigma t}$ will make the above functions to be absolutely integrable.

A linear time invariant (LTI) system is described by differential equations. Laplace transform is a powerful tool used to solve the differential equations.

Review of Laplace Transforms:-

The Laplace transform of a continuous time function $x(t)$ defined for all values of t is defined as

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt.$$

where s is a complex variable & is equal to $s = \sigma + j\omega$.
The operator L is called the Laplace transform operator which transforms the time-domain function $x(t)$ into frequency domain function $x(s)$.

The unilateral Laplace transform is defined as

$$L[x(t)] = x(s) = \int_0^{\infty} x(t) e^{-st} dt.$$

The unilateral Laplace transform is applicable for causal signals (i.e. $x(t) = 0$ for $t < 0$) only, whereas the bilateral Laplace transform is applicable for both causal & non-causal signals.

Inverse Laplace transform:-

The s -domain signal $x(s)$ can be transformed to time domain signal $x(t)$ by using inverse Laplace transform.

$$L^{-1}\{x(s)\} = x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\omega}^{s=\sigma+j\omega} x(s) e^{+st} ds.$$

Advantages of Laplace transform:-

1. Laplace transform exists for signals for which the Fourier transform does not exist.

Ex:- $x(t) = e^t u(t) \Rightarrow x(\omega) = \int_{-\infty}^{\infty} e^t u(t) e^{-j\omega t} dt$ does not exist because it is not absolutely integrable.
Multiply $x(t)$ with $e^{-\sigma t}$. The resultant signal $x(t) \cdot e^{-\sigma t} u(t) = e^{(1-\sigma)t} u(t)$ is absolutely integrable for $\sigma > 1$.

2. Convolution in time-domain can be obtained by multiplication in s-domain.
3. Differential equations can be easily solved.

Disadvantages :-

1. We cannot draw frequency domain plots using 's' domain in laplace transform. So frequency response of the system cannot be estimated. Instead we can only draw pole-zero diagram.
2. $s = j\omega$ is used only for sinusoidal steady state analysis.

Relation between L.T & F.T of a signal :-

The laplace transform of a function $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

where $s = \sigma + j\omega$

$$\begin{aligned} X(s) = X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{j\omega t} dt \end{aligned}$$

$$X(s) = \mathcal{F}[x(t) e^{-\sigma t}]$$

$$\boxed{\mathcal{L}\{x(t)\} = \mathcal{F}[x(t) e^{-\sigma t}]}$$

For $\sigma = 0$, $s = j\omega$ then $X(s) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt / \sigma = 0$

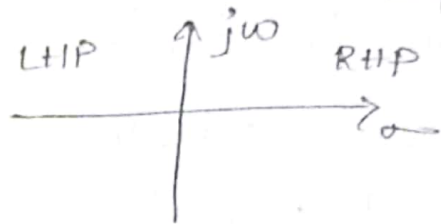
$$\boxed{\mathcal{L}[x(t)] = \mathcal{F}[x(t)]}$$

DSP

S-plane:-

The Laplace variable s is a complex variable defined by $s = \sigma + j\omega$, where σ is real part of s denoted as $\text{Re}(s)$ & ω is imaginary part of s denoted as $\text{Im}(s)$.

The most general form of the Laplace transform is a ratio of two polynomials



$$X(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

The zeros of $X(s)$ are the values of s for which the transform is zero.

The poles of $X(s)$ are the values of s for which the transform is infinity.

Example:-

$$X(s) = \frac{(s+2)(s+4)}{(s+6)(s-2)} \Rightarrow \begin{array}{l} \text{zeros} - s = -2, -4 \\ \text{poles} - s = -6, 2 \end{array}$$

Region of Convergence (ROC) for Laplace transform

The Laplace transform of a function $x(t)$ is defined by

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

where

$$s = \sigma + j\omega, \quad \omega = \text{Im}(s) \\ \sigma = \text{Re}(s)$$

The ROC is defined as the range of values of ' σ ' for which Laplace transform converges.

Find the Laplace transform & ROC of the following³ signals (i) $e^{-at} u(t)$ (ii) $-e^{-at} u(-t)$.

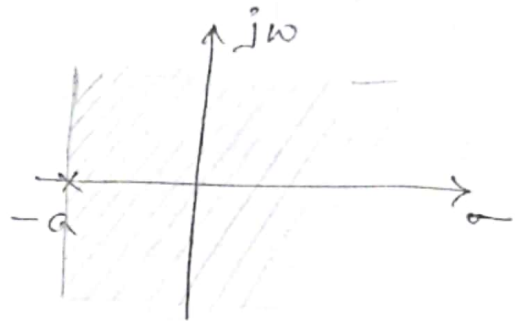
Sol:- (i) $e^{-at} u(t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-at} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s+a)t} dt \\ &= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^{\infty} \\ &= \frac{1}{s+a} \end{aligned}$$

ROC:-

$$\operatorname{Re}(s+a) > 0$$

$$\operatorname{Re}(s) > -a$$



$$\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a} \quad \text{ROC: } \boxed{\operatorname{Re}(s) > -a}$$

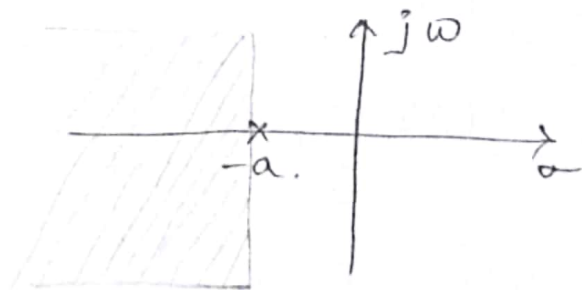
(ii) $-e^{-at} u(-t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} x(t) e^{-st} dt \\ &= - \int_{-\infty}^0 e^{-at} e^{-st} dt \\ &= - \int_{-\infty}^0 e^{-(s+a)t} dt \\ &= \int_0^{\infty} e^{(s+a)t} dt \end{aligned}$$

ROC:-

$$\operatorname{Re}(s+a) < 0$$

$$\operatorname{Re}(s) < -a$$



For the above integral to converge $\operatorname{Re}(s) < -a$.

$$= \frac{1}{s+a} \quad \mathcal{L}\{-e^{-at} u(-t)\} = \frac{1}{s+a} \quad \text{ROC: } \boxed{\operatorname{Re}(s) < -a}$$

→ Find the L.T of the signal $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$
 & find Roc. What are pole locations?

Sol:- $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$= \int_0^{\infty} e^{-at} e^{-st} dt + \int_{-\infty}^0 e^{-bt} e^{-st} dt$$

$$= \underbrace{\int_0^{\infty} e^{-(s+a)t} dt}_{\text{Converges if } \text{Re}[s] > -a} + \underbrace{\int_{-\infty}^0 e^{-(s+b)t} dt}_{\text{Converges if } \text{Re}[s] < -b}$$

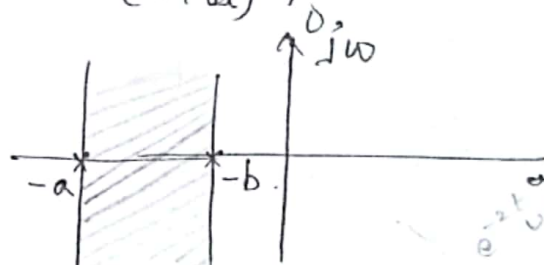
Converges if $\text{Re}[s] > -a$

Converges if $\text{Re}[s] < -b$

Converges if $-a < \text{Re}[s] < -b$

$$X(s) = \frac{-1}{s+b} e^{-(s+b)t} \Big|_{-\infty}^0 + \frac{1}{-(s+a)} e^{-(s+a)t} \Big|_0^{\infty}$$

$$= \frac{-1}{s+b} + \frac{1}{s+a}$$



Roc : $-a < \text{Re}[s] < -b$

Poles are at $s = -a$ & $s = -b$

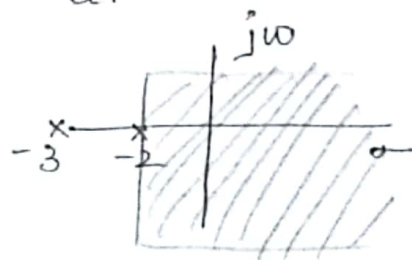
→ Find the L.T of the signal $x(t) = e^{-3t}u(t) + e^{-2t}u(-t)$

Sol:- $X(s) = \int_0^{\infty} e^{-3t} e^{-st} dt + \int_{-\infty}^0 e^{-2t} e^{-st} dt$

$$= \frac{1}{s+3} + \frac{1}{s+2}$$

$\text{Re}[s] > -3$

$\text{Re}[s] > -2$



Roc :- $\text{Re}[s] > -2$

→ Find the L.T of the signal $x(t) = e^{-b|t|}$. 4

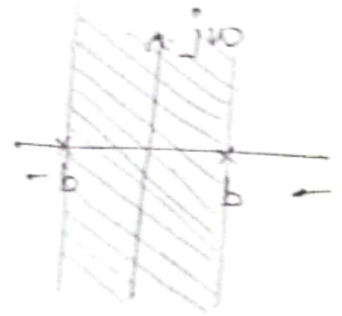
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^0 e^{bt} e^{-st} dt + \int_0^{\infty} e^{-bt} e^{-st} dt$$

$$= \underbrace{\int_{-\infty}^0 e^{-(s-b)t} dt}_{\text{Converges if } \operatorname{Re}[s] < b} + \underbrace{\int_0^{\infty} e^{-(s+b)t} dt}_{\text{Converges if } \operatorname{Re}[s] > -b}$$

$$= \left. \frac{e^{-(s-b)t}}{-(s-b)} \right|_{-\infty}^0 + \left. \frac{e^{-(s+b)t}}{-(s+b)} \right|_0^{\infty}$$

$$= \frac{-1}{s-b} + \frac{1}{s+b} \quad \text{ROC: } -b < \operatorname{Re}[s] < b$$



Laplace transform of some standard signals:-

1. Impulse function:-

$$\delta(t) = \begin{cases} 1 & t=0 \\ 0 & t \neq 0 \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

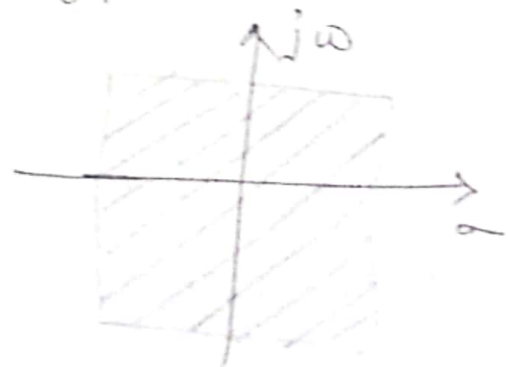
$\delta(t)$ is defined only at $t=0$.

$$X(s) = \delta(0) e^{-s(0)}$$

$$= 1$$

ROC:- Entire s-plane

$$\boxed{L\{\delta(t)\} \rightarrow 1}$$



2. Unit step function:-

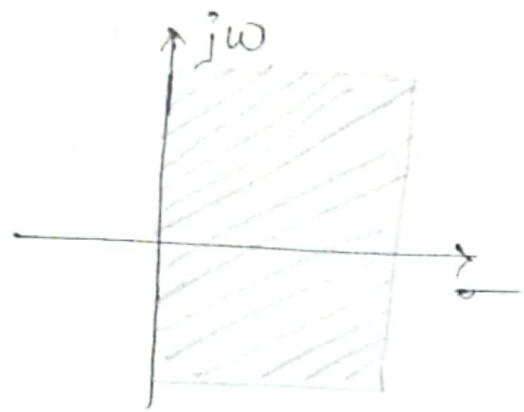
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\begin{aligned} \mathcal{L}\{u(t)\} &= \int_0^{\infty} e^{-st} dt \\ &= \left. \frac{e^{-st}}{-s} \right|_0^{\infty} = \frac{1}{s} \end{aligned}$$

$$u(t) \xleftrightarrow{\mathcal{L}\mathcal{T}} \frac{1}{s}$$

ROC: $\text{Re}[s] > 0$

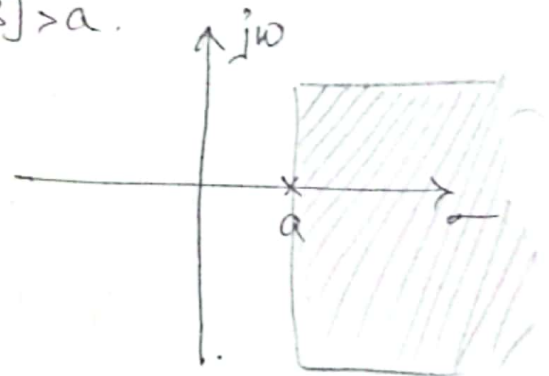


3. One-sided Exponential Rise function

$$x(t) = \begin{cases} e^{at} u(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{x(t)\} &= \int_0^{\infty} e^{at} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt \\ &= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^{\infty} = \frac{1}{s-a} \end{aligned}$$

ROC
 $\text{Re}[s-a] > 0$
 $\text{Re}[s] > a$



$$\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a}$$

ROC: $\text{Re}[s] > a$

4. One-sided Exponential Decay function

$$x(t) = \begin{cases} e^{-at} u(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\{x(t)\} = \int_0^{\infty} e^{-at} e^{-st} dt$$

$$\mathcal{L}\{e^{-at} u(t)\} = \int_0^{\infty} e^{-(s+a)t} dt$$

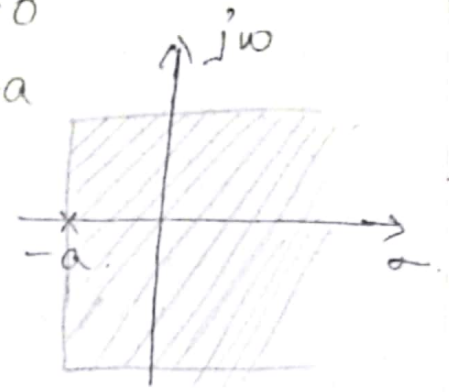
$$= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^{\infty}$$

$$\text{ROC: } \text{Re}[s+a] > 0$$

$$\text{Re}[s] > -a$$

$$\mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a}$$

$$\text{ROC: } \text{Re}[s] > -a$$



5. Sinusoidal Function:-

$$x(t) = \sin(\omega_0 t) u(t)$$

$$= \frac{1}{2j} \left[e^{j\omega_0 t} u(t) - e^{-j\omega_0 t} u(t) \right]$$

We know that

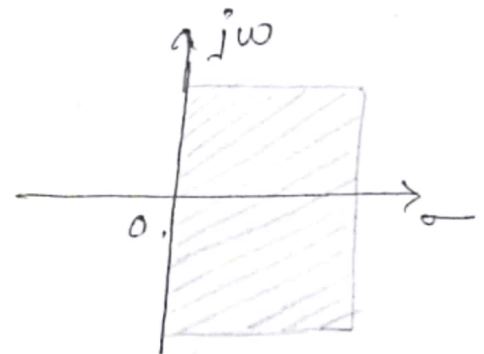
$$e^{at} u(t) \xleftrightarrow{\mathcal{L.T.}} \frac{1}{s-a} \quad \sigma > a$$

$$e^{-at} u(t) \xleftrightarrow{\mathcal{L.T.}} \frac{1}{s+a} \quad \sigma > -a$$

Therefore,

$$e^{j\omega_0 t} u(t) \xleftrightarrow{\mathcal{L.T.}} \frac{1}{s-j\omega_0} \quad \sigma > 0$$

$$e^{-j\omega_0 t} u(t) \xleftrightarrow{\mathcal{L.T.}} \frac{1}{s+j\omega_0} \quad \sigma > 0$$



$$X(s) = \frac{1}{2j} \left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right]$$

$$= \frac{1}{2j} \left[\frac{s+j\omega_0 - s+j\omega_0}{s^2 + \omega_0^2} \right]$$

$$= \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\sin(\omega_0 t) u(t) \xleftrightarrow{\mathcal{L.T.}} \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\text{ROC: } \text{Re}[s] > 0$$

6. Cosine function :-

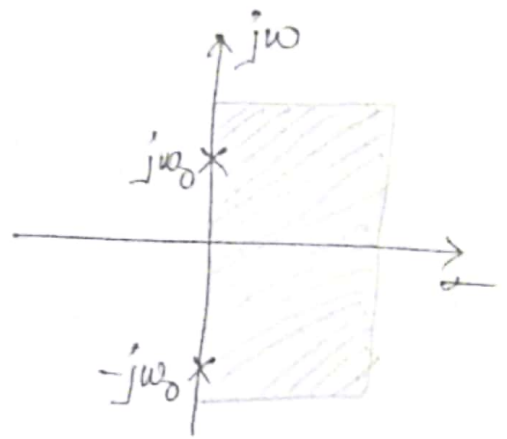
$$x(t) = \cos(\omega_0 t) u(t)$$

$$= \frac{1}{2} \left[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t) \right]$$

$$X(s) = \frac{1}{2} \left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0} \right]$$

$$= \frac{1}{2} \left[\frac{s + j\omega_0 + s - j\omega_0}{s^2 + \omega_0^2} \right]$$

$$= \frac{s}{s^2 + \omega_0^2}$$



$$\cos(\omega_0 t) u(t) \xleftrightarrow{\text{L.T.}} \frac{s}{s^2 + \omega_0^2} \quad \text{ROC: } \text{Re}[s] > 0$$

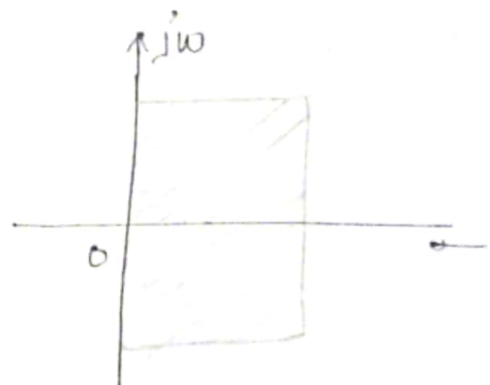
7. Ramp function :-

$$x(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\{x(t)\} = \int_0^{\infty} t \cdot e^{-st} dt$$

$$= \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty}$$

$$= \frac{1}{s^2}$$



$$t u(t) \xleftrightarrow{\text{L.T.}} \frac{1}{s^2} \quad \text{ROC: } \text{Re}[s] > 0$$

8. Damped sine wave :-

$$x(t) = e^{-at} \sin \omega t$$

$$L(x(t)) = L\{e^{-at} \sin \omega t\}$$

$$= L\left\{e^{-at} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right)\right\}$$

$$= \frac{1}{2j} L\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2j} \left[\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)} \right]$$

$$= \frac{1}{2j} \left[\frac{s+a+j\omega - s-a+j\omega}{(s+a)^2 + \omega^2} \right]$$

$$= \frac{1}{2j} \left[\frac{2j\omega}{(s+a)^2 + \omega^2} \right] = \frac{\omega}{(s+a)^2 + \omega^2}$$

Roc :-

$$\text{Re}[s] > -a$$

$$\text{Re}[s] > -a$$

$$\sigma > -a$$

9. Damped cosine wave :-

$$x(t) = e^{-at} \cos \omega t$$

$$L(x(t)) = L\left(e^{-at} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right)\right)$$

$$= \frac{1}{2} L\left(e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right)$$

$$= \frac{1}{2} \left[\frac{1}{s+(a-j\omega)} + \frac{1}{s+(a+j\omega)} \right]$$

$$= \frac{1}{2} \left[\frac{s+a+j\omega + s+a-j\omega}{(s+a)^2 + \omega^2} \right] = \frac{s+a}{(s+a)^2 + \omega^2}$$

Roc :-

$$\text{Re}[s] > -a$$

$$\sigma > -a$$

$$\boxed{L(e^{-at} \cos \omega t) = \frac{s+a}{(s+a)^2 + \omega^2}}$$

Roc :

$$\text{Re}[s] > -a$$

10. Hyperbolic sin and cosine function:-

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

$$L(\sinh(at)) = \frac{1}{2} L(e^{at} - e^{-at}) \quad L(\cosh(at)) = \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right)$$

$$L(\sinh(at)) = \frac{a}{s^2 - a^2}$$

$$L(\cosh(at)) = \frac{s}{s^2 - a^2}$$

ROC: $\text{Re}(s) > a$

ROC: $\text{Re}(s) > a$

Properties of ROC:-

1. The ROC consists of strips of area parallel to jw axis in s plane
2. ROC does not contain any poles
3. If $x(t)$ is of finite duration & is absolutely integrable then the ROC is the entire s plane
4. If $x(t)$ is right-sided signal & the line $\text{Re}(s) = \sigma_0$ is in ROC then all values of s for which $\text{Re}(s) > \sigma_0$ will also be in ROC
5. If $x(t)$ is left-sided signal & the line $\text{Re}(s) = \sigma_0$ is in ROC then all values of s for which $\text{Re}(s) < \sigma_0$ will also be in ROC
6. If $x(t)$ is two-sided, & if the line passing through $\text{Re}(s) = \sigma_0$ is in the ROC, then the ROC will consist of a strip in s plane that includes the line passing through $\text{Re}(s) = \sigma_0$
7. The ROC is bounded by poles or extends to infinity

8. If $x(t)$ is right sided, the ROC is region to right side of rightmost pole.

9. If $x(t)$ is left sided, then the ROC is region in s -plane to left side of leftmost pole.

Properties of Laplace Transforms:-

① Linearity property:-

$$\begin{aligned} \text{If } x_1(t) &\xleftrightarrow{\text{LT}} X_1(s) && \text{with ROC} = R_1 \\ x_2(t) &\xleftrightarrow{\text{LT}} X_2(s) && \text{with ROC} = R_2 \end{aligned}$$

Then $\boxed{ax_1(t) + bx_2(t) \xleftrightarrow{\text{LT}} aX_1(s) + bX_2(s)}$ with $\text{ROC} = R_1 \cap R_2$.

Proof:-

$$\begin{aligned} \mathcal{L}\{x(t)\} &= X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ \mathcal{L}\{ax_1(t) + bx_2(t)\} &= \int_{-\infty}^{\infty} [ax_1(t) + bx_2(t)] e^{-st} dt \\ &= a \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-st} dt \\ &= aX_1(s) + bX_2(s) \end{aligned}$$

② Time shifting property:-

$$\text{If } x(t) \xleftrightarrow{\text{LT}} X(s) \quad \text{with ROC} = R$$

then $\boxed{x(t-t_0) \xleftrightarrow{\text{LT}} X(s) e^{-st_0}}$ with $\text{ROC} = R$

Proof:-

$$\begin{aligned} \mathcal{L}\{x(t)\} &= \int_{-\infty}^{\infty} x(t) e^{-st} dt \\ \mathcal{L}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau+t_0)} d\tau \\ &= e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau = e^{-st_0} X(s) \end{aligned} \quad \begin{aligned} \text{Let } t-t_0 &= \tau \\ dt &= d\tau \end{aligned}$$

③ Shifting in s-domain:-

$$\mathcal{L}\{x(t)\} \xleftrightarrow{\mathcal{L}\&T} X(s) \quad \text{with ROC} = R$$

then $\boxed{e^{+s_0 t} x(t) = X(s-s_0)}$ with ROC: $R + \text{Re}(s_0)$

Proof:-

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\mathcal{L}\{e^{s_0 t} x(t)\} = \int_{-\infty}^{\infty} x(t) e^{s_0 t} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t} dt$$

$$= X(s-s_0)$$

④ Time Scaling property:-

$$\mathcal{L}\{x(t)\} \xleftrightarrow{\mathcal{L}\&T} X(s)$$

with ROC = R

then $x(at) \xleftrightarrow{\mathcal{L}\&T} \frac{1}{|a|} X\left(\frac{s}{a}\right)$

with ROC = $\frac{R}{a}$

Proof:-

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

Let $at = \tau \Rightarrow$

$$dt = \frac{d\tau}{a} \quad t = \tau/a$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau/a)} d\tau/a$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau) e^{-(s/a)\tau} d\tau$$

$$= \frac{1}{|a|} X(s/a)$$

⑤ Time Reversal property:-

$$\text{If } x(t) \xleftrightarrow{L.T} X(s) \text{ with ROC} = R$$

$$\text{then } x(-t) \xleftrightarrow{L.T} X(-s) \text{ with ROC} = R.$$

Proof:-

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\mathcal{L}\{x(-t)\} = \int_{-\infty}^{\infty} x(-t) e^{-st} dt$$

$$= \int_{\infty}^{-\infty} x(\tau) e^{s\tau} (-d\tau)$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{s\tau} d\tau$$

$$= X(-s)$$

Let

$$-t = \tau$$

$$dt = -d\tau$$

⑥ Differentiation in Time Domain:-

$$\text{If } x(t) \xleftrightarrow{L.T} X(s), \text{ ROC} = R$$

$$\text{then } \frac{d}{dt} x(t) \xleftrightarrow{L.T} sX(s), \text{ ROC} = R$$

Proof:-

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X(s) e^{st} ds$$

$$\frac{d}{dt} [x(t)] = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} s X(s) e^{st} ds$$

$$= \mathcal{L}\{sX(s)\}$$

⑦ Differentiation in s-domain property : —

$$\mathcal{F}\{x(t)\} \xleftrightarrow{\text{LT}} X(s) \quad \text{ROC} = R$$

$$\text{then } \mathcal{F}\{t x(t)\} \xleftrightarrow{\text{LT}} -\frac{d}{ds} X(s) \quad \text{ROC} = R.$$

Proof:-

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt.$$

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} (-t) dt.$$

$$= -\int_{-\infty}^{\infty} [t x(t)] e^{-st} dt$$

$$-\frac{d}{ds} X(s) = \mathcal{L}\{t x(t)\}$$

⑧ Time Convolution property : —

$$\mathcal{F}\{x_1(t)\} \xleftrightarrow{\text{LT}} X_1(s) \quad \text{with ROC} = R_1$$

$$\mathcal{F}\{x_2(t)\} \xleftrightarrow{\text{LT}} X_2(s) \quad \text{with ROC} = R_2$$

$$\text{then } x_1(t) * x_2(t) \xleftrightarrow{\text{LT}} X_1(s) X_2(s) \quad \text{with ROC} = R_1 \cap R_2.$$

Proof:-

By definition of Convolution

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau.$$

Taking LT on both sides.

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-st} dt.$$

$$= \int_{-\infty}^{\infty} x_1(\tau) d\tau \int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt \rightarrow (1)$$

Using time shifting property.

$$\int_{-\infty}^{\infty} x_2(t-\tau) e^{-st} dt = e^{-s\tau} X_2(s) \rightarrow (2)$$

Substituting (2) in (1).

$$\begin{aligned} \mathcal{L}\{x_1(t) * x_2(t)\} &= \int_{-\infty}^{\infty} x_1(\tau) d\tau \cdot e^{-s\tau} x_2(s) \\ &= \int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau \cdot x_2(s) \\ &= X_1(s) \cdot x_2(s). \end{aligned}$$

⑨ Multiplication or Modulation or Convolution in s-domain properly:-

$$\begin{aligned} \mathcal{L}\{x_1(t)\} &\xleftrightarrow{\mathcal{L}\mathcal{T}} X_1(s) \\ \mathcal{L}\{x_2(t)\} &\xleftrightarrow{\mathcal{L}\mathcal{T}} X_2(s) \end{aligned}$$

Then $x_1(t) x_2(t) \xleftrightarrow{\mathcal{L}\mathcal{T}} \frac{1}{2\pi j} [X_1(s) * X_2(s)]$.

Proof:-

$$\begin{aligned} \mathcal{L}\{x_1(t) x_2(t)\} &= \int_{-\infty}^{\infty} x_1(t) x_2(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} x_1(u) e^{ut} du \right] x_2(t) e^{-st} dt \\ &= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} x_1(u) \left[\int_{-\infty}^{\infty} x_2(t) e^{-(s-u)t} dt \right] du \\ &= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} x_1(u) X_2(s-u) du \\ &= \frac{1}{2\pi j} [X_1(s) * X_2(s)] \end{aligned}$$

⑩ Parseval's Theorem:-

$$\begin{aligned} \mathcal{L}\{x_1(t)\} &\xleftrightarrow{\mathcal{L}\mathcal{T}} X_1(s) \\ \mathcal{L}\{x_2(t)\} &\xleftrightarrow{\mathcal{L}\mathcal{T}} X_2(s) \end{aligned}$$

Then $\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt \longleftrightarrow \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) X_2^*(-s^*) ds$

Proof:-

$$x_1(t) = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) e^{st} ds$$

Consider $\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) e^{st} ds \right] x_2^*(t) dt$

$$= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) \left\{ \int_{-\infty}^{\infty} x_2^*(t) e^{st} dt \right\} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) \left[\int_{-\infty}^{\infty} x_2(t) e^{-(-s^*)t} dt \right]^* ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) (X_2(-s^*))^* ds$$

$$= \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} X_1(s) X_2^*(-s^*) ds$$

(II) Initial Value Theorem:-

If $x(t) \xleftrightarrow{\mathcal{L}T} X(s)$

Then $\lim_{t \rightarrow 0} x(t) = x(0) = \lim_{s \rightarrow \infty} s X(s)$

Proof:-

$$\mathcal{L}\{x'(t)\} = s \mathcal{L}\{x(t)\} - x(0)$$

Taking $s \rightarrow \infty$ on both sides

$$\lim_{s \rightarrow \infty} \mathcal{L}\{x'(t)\} = \lim_{s \rightarrow \infty} [s X(s) - x(0)]$$

$$\lim_{s \rightarrow \infty} \int_0^{\infty} x'(t) e^{-st} dt = \lim_{s \rightarrow \infty} s x(s) - x(0).$$

As $s \rightarrow \infty$ the integration of LHS becomes zero.

$$\lim_{s \rightarrow \infty} s x(s) - x(0) = 0.$$

$$\lim_{s \rightarrow \infty} s x(s) = x(0) = \lim_{t \rightarrow 0} x(t).$$

⑫ Final Value Theorem:-

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s x(s).$$

Proof:-

$$\mathcal{L}\{x'(t)\} = s x(s) - x(0).$$

Taking limit $s \rightarrow 0$ on both sides

$$\lim_{s \rightarrow 0} \mathcal{L}\{x'(t)\} = \lim_{s \rightarrow 0} [s x(s) - x(0)].$$

$$\lim_{s \rightarrow 0} \int_0^{\infty} x'(t) e^{-st} dt = \lim_{s \rightarrow 0} [s x(s) - x(0)]$$

$$\int_0^{\infty} x'(t) dt = \lim_{s \rightarrow 0} [s x(s) - x(0)].$$

$$x(t) \Big|_0^{\infty} = \lim_{s \rightarrow 0} s x(s) - x(0).$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s x(s).$$

Q. Determine the initial & final values of the function whose Laplace transform is given as

(a) $X(s) = \frac{5s+50}{s(s+5)}$

Sol:-

(a) $X(s) = \frac{5s+50}{s(s+5)}$

Initial value

$$x(0^+) = \lim_{s \rightarrow \infty} s X(s)$$

$$= \lim_{s \rightarrow \infty} s \frac{5s+50}{s(s+5)}$$

$$= \lim_{s \rightarrow \infty} \frac{5s+50}{1+s/s}$$

$$= 5$$

(b) $X(s) = \frac{s+4}{s^2+3s+5}$

Initial value

$$x(0^+) = \lim_{s \rightarrow \infty} s X(s)$$

$$= \lim_{s \rightarrow \infty} \frac{s(s+4)}{s^2+3s+5} \quad s = \frac{1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{1/x^2 + 4/x}{(\frac{1}{x})^2 + 3(\frac{1}{x}) + 5}$$

$$= \lim_{x \rightarrow 0} \frac{1+4x}{1+3x+5x^2}$$

$$= 1$$

Final value

$$x(\infty) = \lim_{s \rightarrow 0} s X(s) = \lim_{s \rightarrow 0} \frac{s(s+4)}{s^2+3s+5}$$

$$= 0$$

(b) $X(s) = \frac{s+4}{s^2+3s+5}$

(c) $X(s) = \frac{s^2+3s+6}{s^2+2s+3}$

Final Value.

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s X(s)$$

$$= \lim_{s \rightarrow 0} s \left(\frac{s^2+3s+6}{s^2+2s+3} \right)$$

$$= \frac{6}{3}$$

$$= 2$$

(c) $X(s) = \frac{s^2+3s+6}{s^2+2s+3}$

Initial value

$$x(0^+) = \lim_{s \rightarrow \infty} s X(s)$$

$$= \infty$$

Final Value

$$x(\infty) = \lim_{s \rightarrow 0} s X(s)$$

$$= \lim_{s \rightarrow 0} \frac{s(s^2+3s+6)}{s^2+2s+3}$$

$$= 0$$

Inversion of Bilateral Laplace Transforms:-

If ROC is $\text{Re}[s] > a$, then signal $x(t)$ is a causal signal.
If ROC is $\text{Re}[s] < a$, then signal $x(t)$ is a non-causal signal.

If a pole of $x(s)$ lies to the left of the ROC, then this pole gives rise to a causal signal.

If a pole of $x(s)$ lies to the right of ROC, then this pole gives a non-causal signal.

Problems:-

- (1) Find the inverse L.T of $x(s) = \frac{2}{(s+4)(s+1)}$ if the region of convergence is
(a) $-4 < \text{Re}(s) < 1$ (b) $\text{Re}(s) > 1$ (c) $\text{Re}(s) < -4$

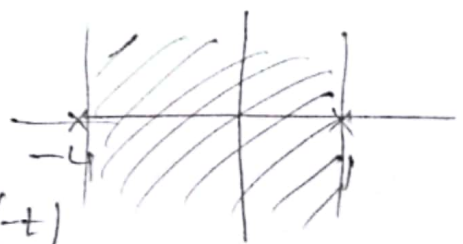
Sol:- $x(s) = \frac{2}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$

On solving $A = \frac{2}{-4+1} = -\frac{2}{3}$, $B = \frac{2}{+1+4} = \frac{2}{5}$.

$$x(s) = \frac{-2}{3} \frac{1}{s+4} + \frac{2}{5} \frac{1}{s+1}$$

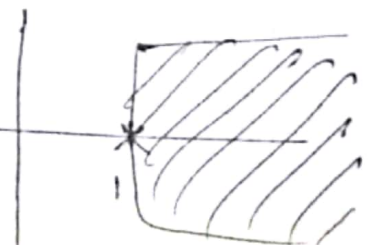
(a) $-4 < \text{Re}(s) < 1$.

$$x(t) = \frac{-2}{3} e^{-4t} u(t) + \frac{2}{5} (-e^t) u(-t)$$



(b) $\text{Re}(s) > 1$

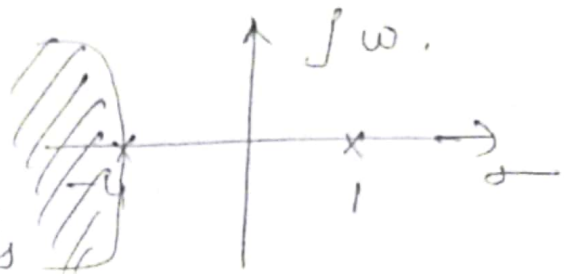
Both poles lie on the left of ROC, so poles correspond to causal signal.



$$x(t) = \frac{-2}{5} e^{-4t} u(t) + \frac{2}{5} e^t u(t).$$

(c) ROC $\leftarrow -4$

Poles lie to the right of ROC. So poles corresponds to non-causal signal.



$$x(t) = \frac{2}{5} e^{-4t} u(-t) - \frac{2}{5} e^t u(-t)$$

(2) Find the signal whose bilateral transform is

$$X(s) = \frac{1}{(s+5)(s+1)} \quad -5 < \text{Re}(s) < -1.$$

Sol:-

$$X(s) = \frac{1}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}$$

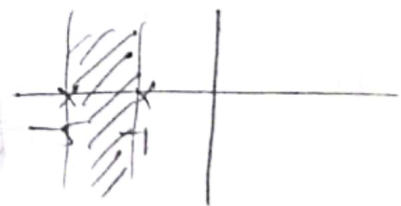
On solving $A = \frac{1}{-5+1} = -\frac{1}{4}$, $B = \frac{1}{-1+5} = \frac{1}{4}$.

$$X(s) = -\frac{1}{4} \frac{1}{s+5} + \frac{1}{4} \frac{1}{s+1}$$

$$x(t) = -\frac{1}{4} e^{-5t} u(t) + \frac{1}{4} e^{-t} u(-t).$$

$$\text{Re}(s) < -5$$

$$\text{Re}(s) < -1$$



2) find inverse laplace transform of the following

(i) $\frac{s^2 + 6s + 7}{s^2 + 3s + 2} \quad \text{Re}[s] > -1$

(ii) $\frac{s^3 + 2s^2 + 6}{s^2 + 3s} \quad \text{Re}[s] > 0$

Sol (i) $\frac{s^2 + 6s + 7}{s^2 + 3s + 2}$

$$\frac{s^2 + 3s + 2}{s^2 + 3s + 2} \cdot \frac{s^2 + 6s + 7}{s^2 + 3s + 2} = \frac{1}{s^2 + 3s + 2}$$

$$X(s) = 1 + \frac{3s + 5}{s^2 + 3s + 2}$$

$$\frac{3s + 5}{s^2 + 3s + 2} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow \frac{2}{s+1} + \frac{1}{s+2}$$

$$\begin{aligned} L^{-1}[X(s)] &= L^{-1}\left[1 + \frac{2}{s+1} + \frac{1}{s+2}\right] \\ &= L^{-1}[1] + L^{-1}\left[\frac{2}{s+1}\right] + L^{-1}\left[\frac{1}{s+2}\right] \end{aligned}$$

$$x(t) = \delta(t) + 2e^{-t}u(t) + e^{-2t}u(t)$$

(ii) $\frac{s^3 + 2s^2 + 6}{s^2 + 3s} \Rightarrow s - 1 + \frac{3s + 6}{s^2 + 3s}$

$$\frac{3s + 6}{s^2 + 3s} = \frac{A}{s} + \frac{B}{s+3} = \frac{2}{s} + \frac{1}{s+3}$$

$$X(s) = s - 1 + \frac{2}{s} + \frac{1}{s+3}$$

$$x(t) = \delta'(t) - \delta(t) + (2 + e^{-3t})u(t)$$

$$\begin{aligned} \frac{s^3 + 2s^2 + 6}{s^2 + 3s} &= \frac{s^2 + 3s}{s^2 + 3s} + \frac{s^2 + 2s^2 + 6}{s^2 + 3s} \\ &= 1 + \frac{-s^2 + 6}{s^2 + 3s} \\ &= 1 + \frac{-s^2 - 3s + 3s + 6}{s^2 + 3s} \\ &= 1 + \frac{3s + 6}{s^2 + 3s} \end{aligned}$$

$$\frac{d}{dt}[\delta(t)]$$

$$\frac{d^2}{dt^2}[\delta(t)] = s^2 F(s) - s f(0) - f'(0)$$

$$s F(s) - f(0)$$

$$= s - 1$$

⑤ Find the inverse laplace transform of

$$(i) X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2 + 2s + 1)}$$

$$(ii) X(s) = \frac{2s+1}{(s+1)(s^2 + 2s + 2)}$$

Sol:-

$$X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2 + 2s + 1)} = \frac{3s^2 + 8s + 6}{(s+2)(s+1)^2}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

On simplifying

$$A = 2 \quad B = 1 \quad C = 1$$

$$X(s) = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$X(t) = L^{-1}[X(s)]$$

$$= 2e^{-2t}u(t) + e^{-t}u(t) + te^{-t}u(t)$$

$$(ii) X(s) = \frac{2s+1}{(s+1)(s^2 + 2s + 2)}$$

$$s^2 + 2s + 2$$

$$s_1, s_2 = -1 \pm j1$$

$$= \frac{A}{s+1} + \frac{B}{s - (-1+j)} + \frac{B^*}{s - (-1-j)}$$

On simplifying

$$A = -1 \quad B = 0.5 - j$$

$$X(s) = \frac{-1}{s+1} + \frac{0.5-j}{s - (-1+j)} + \frac{0.5+j}{s - (-1-j)}$$

$$= -e^{-t}u(t) + (0.5-j)e^{(-1+j)t}u(t) + (0.5+j)e^{(-1-j)t}u(t)$$

$$= -e^{-t}u(t) + 0.5e^{-t}(e^{jt} + e^{-jt})u(t) - je^{-t}(e^{jt} - e^{-jt})u(t)$$

$$= -e^{-t}u(t) + e^{-t}\cos t u(t) + 2e^{-t}\sin t u(t)$$

$$= -e^{-t}u(t) + e^{-t}(\cos t + 2\sin t)u(t)$$

6) Find the inverse Laplace transform of

(i) $\frac{s^3+1}{s(s+1)(s+2)}$ (ii) $\frac{s-1}{(s+1)(s^2+2s+5)}$

Sol:-

$$\frac{s^3+1}{s(s+1)(s+2)} = 1 - \frac{8s^2+2s-1}{s(s+1)(s+2)}$$

$$= 1 - \left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \right] \quad \begin{matrix} A = -1/2, B = 0 \\ C = 7/2 \end{matrix}$$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= \mathcal{L}^{-1} \left[1 + \frac{1/2}{s} - \frac{7}{2(s+2)} \right]$$

$$= \delta(t) + \frac{1}{2}u(t) - \frac{7}{2}e^{-2t}u(t)$$

(ii) $\frac{s-1}{(s+1)(s^2+2s+5)}$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+5}$$

$$A = -1/2$$

$$B = 1/2$$

$$C = 3/2$$

$$f(s) = \frac{-1/2}{s+1} + \frac{s+3}{2(s^2+2s+5)}$$

$$= \frac{-1/2}{s+1} + \frac{1}{2} \left[\frac{s+1}{(s^2+2s+1)+4} + \frac{2}{(s^2+2s+1)+4} \right]$$

$$= \frac{-1}{2} \frac{1}{s+1} + \frac{1}{2} \left[\frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4} \right]$$

$$f(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-t}\cos 2t u(t) + \frac{1}{2}e^{-t}\sin 2t u(t)$$

(7) Find the initial & final values of the function

$$F(s) = \frac{17s^3 + 7s^2 + s + 6}{s^5 + 3s^4 + 5s^3 + 4s^2 + 2s}$$

Sol:- initial value

$$f(0) = \lim_{s \rightarrow \infty} s F(s)$$

$$= \lim_{s \rightarrow \infty} s \cdot \frac{17s^3 + 7s^2 + s + 6}{s^5 + 3s^4 + 5s^3 + 4s^2 + 2s}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{17}{x^3} + 7/x^2 + 1/x + 6}{\frac{1}{x^5} + \frac{3}{x^4} + \frac{5}{x^3} + \frac{4}{x^2} + \frac{2}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{17 + 7x + x^2 + 6x^3}{1 + 3x + 5x^2 + 4x^3 + 2x^4} \cdot \frac{x^5}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{17x + 7x^2 + x^3 + 6x^4}{1 + 3x + 5x^2 + 4x^3 + 2x^4}$$

$$= 0$$

Final value

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$$

$$= \lim_{s \rightarrow 0} s \cdot \frac{17s^3 + 7s^2 + s + 6}{s^5 + 3s^4 + 5s^3 + 4s^2 + 2s}$$

$$= \lim_{s \rightarrow 0} \frac{17s^3 + 7s^2 + s + 6}{s^4 + 3s^3 + 5s^2 + 4s + 2} = \frac{6}{2} = 3$$

Find the Laplace transform of $x(s) = \frac{1}{(s+4)(s-2)}$, if the (15)
region of convergence is

(a) $-4 < \text{Re}[s] < 2$

(b) $\text{Re}[s] > 2$

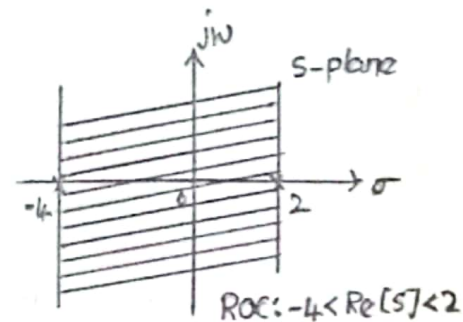
(c) $\text{Re}[s] < -4$

(d) $2 < \text{Re}[s] < -4$

3d: Given $x(s) = \frac{1}{(s+4)(s-2)} = \frac{A}{s-2} + \frac{B}{s+4} = \left[\frac{1}{6} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+4} \right]$

The given $x(s)$ has poles at $s = -4$ and $s = 2$

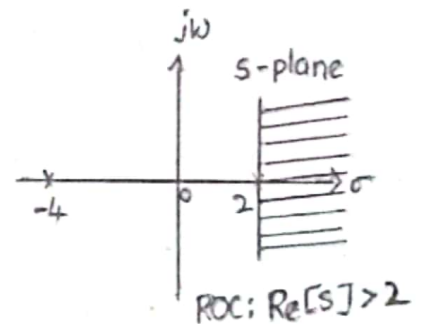
(a) ROC: $-4 < \text{Re}[s] < 2$, for this the pole at $s = -4$ correspond to causal signal and the pole at $s = 2$ correspond to non-causal signal



$$x(t) = L^{-1} \left[\frac{1}{6} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+4} \right]$$

$$= -\frac{1}{6} e^{2t} u(-t) - \frac{1}{6} e^{-4t} u(t)$$

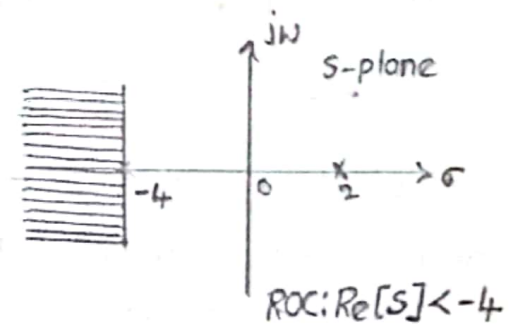
(b) ROC: $\text{Re}[s] > 2$, Both the poles lie left of ROC, so both the poles corresponds to causal signal



$$\therefore x(t) = L^{-1} \left[\frac{1}{6} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+4} \right]$$

$$= \frac{1}{6} e^{2t} u(t) - \frac{1}{6} e^{-4t} u(t)$$

(c) ROC: $\text{Re}[s] < -4$; Both the poles lie right of ROC, so both the poles corresponds to non-causal signal



$$\therefore x(t) = L^{-1} \left[\frac{1}{6} \frac{1}{s-2} - \frac{1}{6} \frac{1}{s+4} \right]$$

$$= -\frac{1}{6} e^{2t} u(-t) + \frac{1}{6} e^{-4t} u(-t)$$

(d) Since, ROC of the sum of the signals is equal to the intersection of their ROC's, it is not possible for to have ROC: $2 < \text{Re}[s] < -4$. So the inverse Laplace transform of the given $x(s)$ for this ROC does not exist.

Find all possible inverse Laplace transforms of

$$X(s) = \frac{1}{s(s+1)(s+2)(s+3)}$$

Sol: Given $X(s) = \frac{1}{s(s+1)(s+2)(s+3)}$

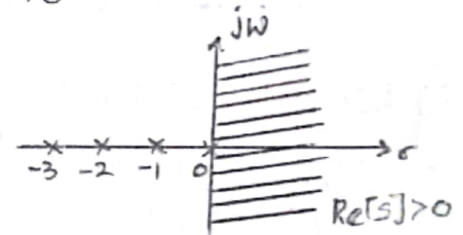
$$= \frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right)$$

The given $X(s)$ has poles at $s=0$, $s=-1$, $s=-2$ and $s=-3$

Case (i): ROC; $\text{Re}[s] > 0$; for this, all the terms must correspond to causal signals

$$x(t) = L^{-1} \left[\frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right) \right]$$

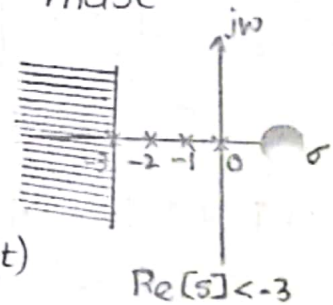
$$= \frac{1}{6} u(t) - \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-2t} u(t) - \frac{1}{6} e^{-3t} u(t)$$



Case (ii) ROC; $\text{Re}[s] < -3$; for this, all the terms must correspond to non-causal signals

$$x(t) = L^{-1} \left[\frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right) \right]$$

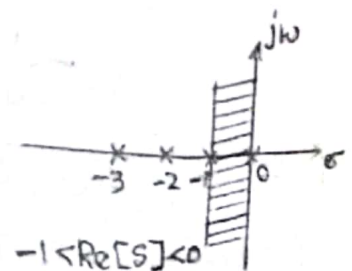
$$= -\frac{1}{6} u(-t) + \frac{1}{2} e^{-t} u(-t) - \frac{1}{2} e^{-2t} u(-t) + \frac{1}{6} e^{-3t} u(-t)$$



Case (iii): ROC; $-1 < \text{Re}[s] < 0$; for this, the poles at $s=-3$, $s=-2$ and $s=-1$ correspond to causal signals and the pole at $s=0$ correspond to non-causal signal.

$$x(t) = L^{-1} \left[\frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right) \right]$$

$$= -\frac{1}{6} u(-t) - \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-2t} u(t) - \frac{1}{6} e^{-3t} u(t)$$



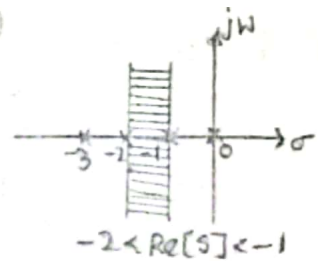
Case (iv): ROC; $-2 < \text{Re}[s] < -1$; for this, the poles at $s=-3$, $s=-2$ corresponds to causal signals, and the poles at $s=-1$ and

$S=0$ correspond to non-causal signals.

(16)

$$x(t) = L^{-1} \left[\frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right) \right]$$

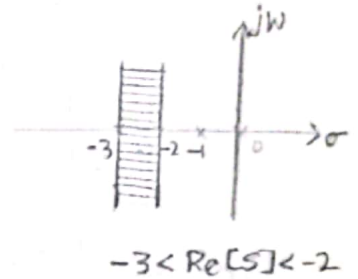
$$= -\frac{1}{6} u(-t) + \frac{1}{2} e^{-t} u(-t) + \frac{1}{2} e^{-2t} u(t) - \frac{1}{6} e^{-3t} u(t)$$



Case 5: ROC; $-3 < \text{Re}[s] < -2$; for this, the pole at $s=-3$ corresponds to causal term and the poles at $s=0$, $s=-1$ and $s=-2$ correspond to non-causal signals

$$x(t) = L^{-1} \left[\frac{1}{6} \cdot \frac{1}{s} - \frac{1}{2} \left(\frac{1}{s+1} \right) + \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{6} \left(\frac{1}{s+3} \right) \right]$$

$$= -\frac{1}{6} u(-t) + \frac{1}{2} e^{-t} u(t) - \frac{1}{2} e^{-2t} u(-t) - \frac{1}{6} e^{-3t} u(t)$$



Solution of differential equations using Laplace Transform:

The ratio of the Laplace transform of the output to the Laplace transform of the input with the initial conditions neglected is called the transfer function of the system.

The transfer function of a system is also defined as the Laplace transform of the impulse response of the system.

$$\text{Thus, } H(s) = \frac{Y(s)}{X(s)} = \frac{\text{L.T of output}}{\text{L.T of Input}} \bigg|_{\text{Initial conditions} = 0}$$

$$\Rightarrow Y(s) = X(s) H(s)$$

$$y(t) = L^{-1}[X(s) H(s)]$$

is known as the zero state response

If the input $x(t) = \delta(t)$, then $X(s) = 1$. Thus,

$$y(t) = L^{-1}[H(s)] = h(t)$$

is the impulse response of the system.

(2) Find the system transfer function for each of the systems described by the following differential equations:

$$(a) \frac{d^2 y(t)}{dt^2} + 8 \frac{dy(t)}{dt} + 14 y(t) = 4 \frac{dx(t)}{dt} + 7 x(t)$$

$$(b) \frac{d^3 y(t)}{dt^3} + 5 \frac{d^2 y(t)}{dt^2} + 12 \frac{dy(t)}{dt} + 6 y(t) = 2 \frac{d^2 x(t)}{dt^2} + 5 \frac{dx(t)}{dt} + 7 x(t)$$

sol: The given differential is

$$\frac{d^2 y(t)}{dt^2} + 8 \frac{dy(t)}{dt} + 14 y(t) = 4 \frac{dx(t)}{dt} + 7 x(t)$$

Apply the Laplace transforms on both sides and assuming zero initial conditions, we have

$$s^2 Y(s) + 8s Y(s) + 14 Y(s) = 4s X(s) + 7 X(s)$$

$$[s^2 + 8s + 14] Y(s) = (4s + 7) X(s)$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{4s + 7}{s^2 + 8s + 14}$$

(b) The given differential equation is

$$\frac{d^3 y(t)}{dt^3} + 5 \frac{d^2 y(t)}{dt^2} + 12 \frac{dy(t)}{dt} + 6 y(t) = 2 \frac{d^2 x(t)}{dt^2} + 5 \frac{dx(t)}{dt} + 7 x(t)$$

Apply Laplace transforms on both sides and assuming zero initial conditions, we have

$$s^3 Y(s) + 5s^2 Y(s) + 12s Y(s) + 6 Y(s) = 2s^2 X(s) + 5s X(s) + 7 X(s)$$

$$Y(s) [s^3 + 5s^2 + 12s + 6] = [2s^2 + 5s + 7] X(s)$$

T

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s^2 + 5s + 7}{s^3 + 5s^2 + 12s + 6}$$

Find the impulse and step response of the following systems.

$$(a) H(s) = \frac{5}{s^2 + 4s + 5}$$

$$(b) H(s) = \frac{s+3}{s^2 + 6s + 8}$$

Sol. The given transfer function is

$$H(s) = \frac{5}{s^2 + 4s + 5} = \frac{Y(s)}{X(s)}$$

For an impulse input $x(t) = \delta(t)$

$$x(t) = \delta(t)$$

$$\therefore X(s) = 1$$

$$\therefore \text{Response } Y(s) = H(s)X(s) = \frac{5}{s^2 + 4s + 5} (1) = \frac{5}{(s+2)^2 + 1^2}$$

Apply Inverse Laplace transform on both sides we get

$$y(t) = 5e^{-2t} \sin t u(t)$$

for a step input:

$$x(t) = u(t)$$

$$\therefore X(s) = 1/s$$

$$\therefore \text{Response } Y(s) = H(s)X(s) = \frac{5}{s^2 + 4s + 5} (1/s) = \frac{5}{s(s^2 + 4s + 5)}$$

Taking partial fractions, we have

$$\begin{aligned} \frac{5}{s(s^2 + 4s + 5)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5} \\ &= \frac{A[(s^2 + 4s + 5) + s(Bs + C)]}{s(s^2 + 4s + 5)} \end{aligned}$$

$$5 = (A+B)s^2 + (4A+C)s + 5A$$

Comparing the coefficient of

$$s^2 \Rightarrow A+B=0 \Rightarrow B=-A$$

$$s \Rightarrow 4A+C=0 \Rightarrow C=-4A$$

$$\text{Constants} \Rightarrow 5A=5 \Rightarrow A=1, \text{ so } B=-1 \\ C=-4$$

$$\therefore Y(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+5}$$

$$= \frac{1}{s} + -\frac{s+2}{(s+2)^2+1^2} - 2 \left[\frac{1}{(s+2)^2+1^2} \right]$$

☞ Taking Inverse Laplace transform on both sides, we get

$$y(t) = u(t) - e^{-2t} \cos t u(t) - 2e^{-2t} \sin t u(t)$$

(b) The given transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+6s+8}$$

For an impulse input

$$x(t) = \delta(t)$$

$$\therefore X(s) = 1$$

$$\therefore \text{Response } Y(s) = H(s)X(s) = \frac{s+3}{s^2+6s+8} \quad (1)$$

$$= \frac{s+3}{s^2+6s+8}$$

Taking partial fractions, we have

$$Y(s) = \frac{s+3}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

$$= \frac{1}{2} \left(\frac{1}{s+2} \right) + \frac{1}{2} \left(\frac{1}{s+4} \right)$$

Taking Inverse Laplace transform on both sides, we get the impulse response

$$y(t) = \frac{1}{2} e^{-2t} u(t) + \frac{1}{2} e^{-4t} u(t)$$

for a step input

$$x(t) = u(t)$$

$$\therefore x(s) = \frac{1}{s}$$

$$\therefore \text{Response } y(s) = H(s)x(s) = \frac{s+3}{(s+2)(s+4)} \cdot \frac{1}{s} = \frac{s+3}{s(s+2)(s+4)}$$

Taking partial fractions, we have

$$y(s) = \frac{s+3}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$= \frac{3/8}{s} - \frac{1/4}{s+2} - \frac{1/8}{s+4}$$

Taking inverse Laplace transforms on both sides, we get the step response

$$y(t) = \frac{3}{8} u(t) - \frac{1}{4} e^{-2t} u(t) - \frac{1}{8} e^{-4t} u(t)$$

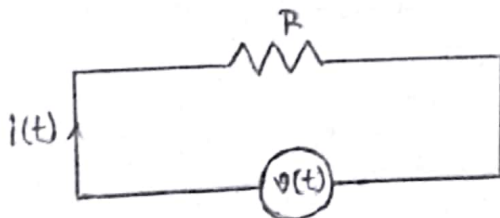
Circuit Analysis using Laplace transforms:-

Pure resistive element:-

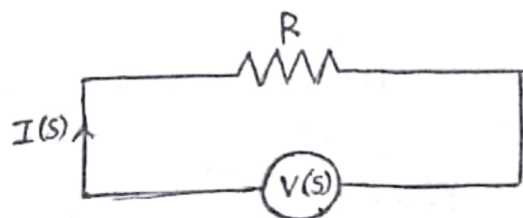
$$v(t) = Ri(t)$$

$$\therefore V(s) = RI(s)$$

i.e. Resistance R in time-domain remains as R in s -domain also
It has no initial conditions.



Pure resistive



Laplace transformed version

Pure Inductive element:-

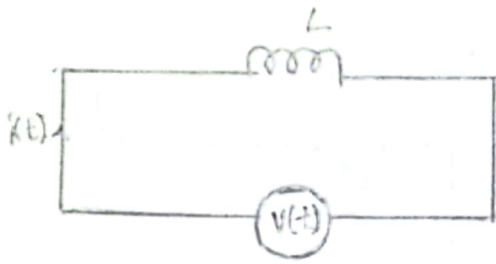
$$v(t) = L \frac{di(t)}{dt}$$

$$V(s) = L[sI(s) - i(0^-)]$$

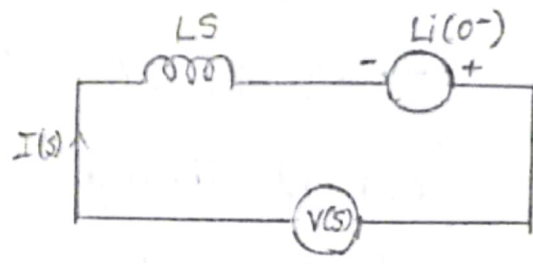
(19)

$$\therefore V(s) = LsI(s) - Li(0^-)$$

Where $i(0^-)$ is the initial current through the inductor



Pure Inductive element



Its Laplace transform version

→ Inductor L in time-domain becomes Ls in s -domain

Pure Capacitive element:-

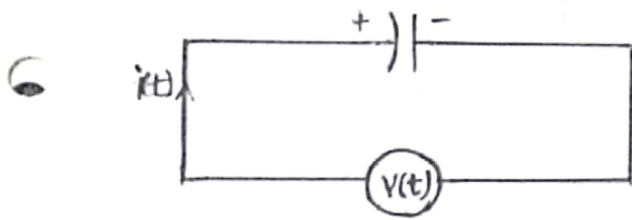
$$i(t) = C \frac{dv(t)}{dt}$$

$$\therefore I(s) = C[sV(s) - v(0^-)] = CsV(s) - Cv(0^-) = CsV(s) - q(0^-)$$

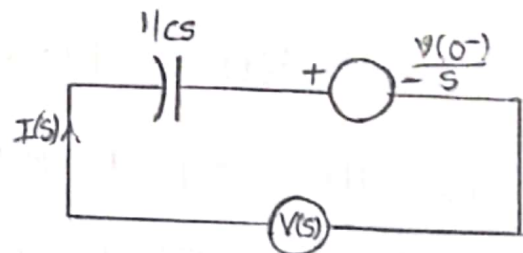
$$V(s) = \frac{1}{Cs} I(s) + \frac{q(0^-)}{Cs} = \frac{1}{Cs} I(s) + \frac{v(0^-)}{s}$$

Where $v(0^-)$ is the initial voltage across the capacitor.

→ The capacitor C in time-domain becomes $1/Cs$ in s -domains



Pure capacitive element



Its Laplace transformed version

Step and Impulse Responses of series R-L circuit:-

Evaluation of step response.

In the series R-L circuit shown fig, Let us consider that the switch S is closed at time $t=0$. For the step response, the input excitation is $x(t) = Vu(t)$. Applying Kirchhoff's voltage law to the circuit, we get the

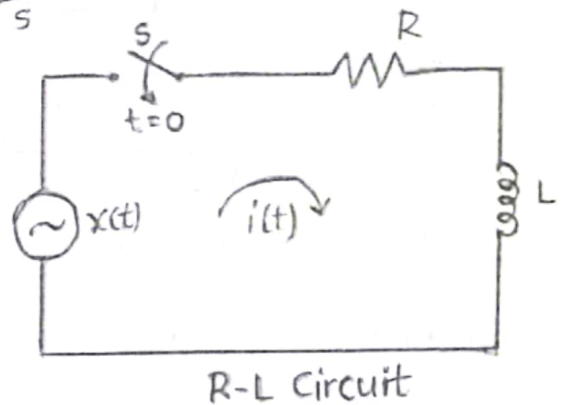
following differential equation.

$$Ri(t) + L \frac{di(t)}{dt} = Vu(t)$$

Taking Laplace transform on both sides, we have

$$RI(s) + L[sI(s) - i(0^+)] = \frac{V}{s}$$

Since the current through an inductor cannot change instantaneously and since the current before $t=0$ is zero, $i(0^+) = 0$



$$\therefore [Ls + R]I(s) = \frac{V}{s}$$

$$\begin{aligned} I(s) &= \frac{V}{s} \cdot \frac{1}{(Ls + R)} = \frac{V}{L} \cdot \frac{1}{s(s + R/L)} \\ &= \frac{V}{L} \cdot \frac{L}{R} \left[\frac{1}{s} - \frac{1}{s + R/L} \right] \\ &= \frac{V}{R} \left[\frac{1}{s} - \frac{1}{s + R/L} \right] \end{aligned}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{V}{R} [1 - e^{-(R/L)t}]$$

which is the step response.

Evaluation of Impulse response:

We know that for the impulse response, the input excitation is $x(t) = \delta(t)$. Therefore, the differential equation describing the R-L network would become

$$Ri(t) + L \frac{di(t)}{dt} = \delta(t)$$

Taking Laplace transform on both sides, we have

$$RI(s) + L[sI(s) - i(0^-)] = 1$$

Now, $i(0^-) = 0$, $\therefore (R + Ls)I(s) = 1$

$$\therefore I(s) = \frac{1}{R+Ls} = \frac{1}{L} \cdot \frac{1}{s+R/L}$$

20

Taking Inverse Laplace transform, we obtain

$$i(t) = \frac{1}{L} e^{-(R/L)t} u(t)$$

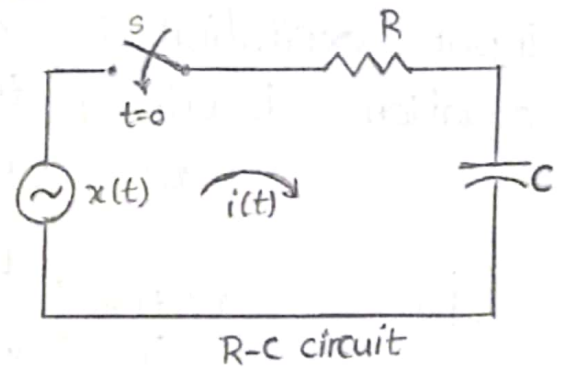
which is the impulse response

Step and impulse Responses of series R-C circuit

Step response:-

In the series R-C circuit shown in figure; let us consider that the switch s is closed at $t=0$

We know that for the step response, the input excitation is $x(t) = V u(t)$. The integro-differential equation describing the series R-C circuit is



$$Ri(t) + \frac{1}{C} \int_{-\infty}^t i(t) dt = V u(t)$$

● This can be written as

$$Ri(t) + \frac{1}{C} \int_0^t i(t) dt + \frac{1}{C} \int_{-\infty}^0 i(t) dt = V u(t)$$

Taking Laplace transform on both sides, we have

$$RI(s) + \frac{1}{C} \left[\frac{I(s)}{s} \right] + \frac{1}{C} \{ L[q(0^+)] \} = \frac{V}{s}$$

$$(or) \quad RI(s) + \frac{1}{C} \left[\frac{I(s)}{s} + \frac{q(0^+)}{s} \right] = \frac{V}{s}$$

where $q(0^+)$ is the charge on the capacitor at $t=0^+$. If the capacitor is initially relaxed, $q(0^+) = 0$

$$\text{Hence,} \quad I(s) \left(R + \frac{1}{Cs} \right) = \frac{V}{s}$$

$$\text{i.e. } I(s) = \frac{V}{s} \frac{1}{R + (1/cs)} = \frac{V}{s} \frac{cs}{Rcs + 1} = \frac{Vc}{RC[s + (1/RC)]}$$

$$(\text{or}) \quad I(s) = \frac{V/R}{s + (1/RC)}$$

Taking inverse Laplace transform on both sides, we get

$$i(t) = \frac{V}{R} e^{-t/RC}$$

Which is the step response.

Impulse response:

We know that for the impulse response, the input excitation is $x(t) = \delta(t)$. Hence, the differential equation describing the circuit is:

$$Ri(t) + \frac{1}{C} \int_{-\infty}^t i(t) dt = \delta(t)$$

$$\text{i.e. } Ri(t) + \frac{1}{C} \int_0^t i(t) dt + \frac{1}{C} \int_{-\infty}^0 i(t) dt = \delta(t)$$

Taking Laplace transform on both sides, we have

$$RI(s) + \frac{1}{C} \left[\frac{I(s)}{s} \right] + \frac{1}{C} \{ L[q(0^+)] \} = 1$$

$$RI(s) + \frac{1}{C} \left[\frac{I(s)}{s} + \frac{q(0^+)}{s} \right] = 1$$

Now, since $q(0^+) = 0$ we have

$$I(s) \left(R + \frac{1}{Cs} \right) = 1$$

$$\text{Hence, } I(s) = \frac{1}{R[1 + (1/Rcs)]} = \frac{Rcs}{R(Rcs + 1)} = \frac{cs}{RC[s + (1/RC)]}$$

$$= \frac{s}{R[s + (1/RC)]}$$

$$I(s) = \frac{1}{R} \left[\frac{[s + (1/RC)] - (1/RC)}{s + (1/RC)} \right]$$

$$I(s) = \frac{1}{R} \left\{ 1 - \frac{1}{RC} \frac{1}{[s + (1/RC)]} \right\}$$

Taking inverse Laplace transform on both sides, we have (2)

$$i(t) = \frac{1}{R} \left[\delta(t) - \frac{1}{RC} e^{-t/RC} u(t) \right]$$

Step Response of series R-L-C circuit:

Figure shows the series R-L-C circuit. The integro-differential equation describing it is

$$Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int_{-\infty}^t i(t) dt = v u(t)$$

$$Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int_{-\infty}^0 i(t) dt + \frac{1}{C} \int_0^t i(t) dt = v u(t)$$

Taking Laplace transform on both sides, we have

$$RI(s) + L[SI(s) - i(0^+)] + \frac{1}{C} L[q(0^+)] + \frac{1}{C} \frac{I(s)}{s} = \frac{V}{s}$$

$$\text{or } RI(s) + L[SI(s) - i(0^+)] + \frac{1}{C} \frac{q(0^+)}{s} + \frac{1}{C} \frac{I(s)}{s} = \frac{V}{s}$$

Where $i(0^+)$ is the initial current through the inductor and $q(0^+)$ is the initial charge on the capacitor. Neglecting the initial conditions, we get.

$$RI(s) + LSI(s) + \frac{1}{Cs} I(s) = \frac{V}{s}$$

$$I(s) \left(Ls + R + \frac{1}{Cs} \right) = \frac{V}{s}$$

$$I(s) = \frac{V}{Ls^2 + Rs + (1/LC)}$$

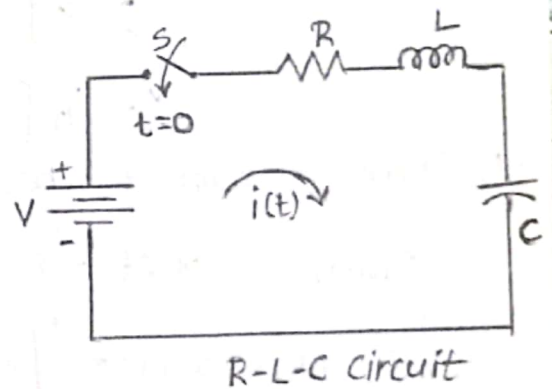
$$= \frac{V}{L \left[s^2 + (R/L)s + (1/LC) \right]}$$

$$= \frac{V}{L (s - P_1)(s - P_2)}$$

Where

$$P_1, P_2 = -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{R^2 - 4\frac{L}{C}}$$

Hence,
$$i(t) = \frac{(V/L)}{(P_1 - P_2)} [e^{P_1 t} - e^{P_2 t}]$$



Miscellaneous problems:-

(1) Find the convolution of the signals $x_1(t) = e^{-2t} u(t)$ and $x_2(t) = e^{-4t} u(t)$ using the convolution property of Laplace transforms. (b) verify the result by time domain method.

Sol: Given $x_1(t) = e^{-2t} u(t)$

$$x_1(s) = \frac{1}{s+2}$$

and $x_2(t) = e^{-4t} u(t) \Rightarrow x_2(s) = \frac{1}{s+4}$

According to convolution property of Laplace transforms,

$$L[x_1(t) * x_2(t)] = x_1(s) x_2(s)$$

$$\therefore x_1(s) x_2(s) = \frac{1}{s+2} \cdot \frac{1}{s+4} = \frac{1}{2} \left[\frac{1}{s+2} - \frac{1}{s+4} \right]$$
$$= L[x_1(t) * x_2(t)]$$

$$\therefore x_1(t) * x_2(t) = L^{-1}[x_1(s) x_2(s)] = L^{-1} \left[\frac{1}{2} \left(\frac{1}{s+2} - \frac{1}{s+4} \right) \right]$$
$$= \frac{1}{2} \left[L^{-1} \left(\frac{1}{s+2} \right) - L^{-1} \left(\frac{1}{s+4} \right) \right]$$
$$= \frac{1}{2} [e^{-2t} - e^{-4t}] u(t)$$

(b) Time domain method:-

Given $x_1(t) = e^{-2t} u(t)$ and $x_2(t) = e^{-4t} u(t)$

$$\therefore x_1(t) * x_2(t) = \int_0^t x_1(\tau) x_2(t-\tau) d\tau = \int_0^t e^{-2\tau} e^{-4(t-\tau)} d\tau$$
$$= e^{-4t} \int_0^t e^{-2\tau} e^{4\tau} d\tau = e^{-4t} \int_0^t e^{2\tau} d\tau$$
$$= e^{-4t} \left[\frac{e^{2\tau}}{2} \right]_0^t = \frac{1}{2} e^{-4t} [e^{2t} - 1] u(t)$$
$$= \frac{1}{2} [e^{-2t} - e^{-4t}] u(t)$$

Hence the result is verified.

(2) Find the Laplace transform of the following signals (22)
using properties of Laplace transform

(i) $x(t) = \delta(t)$ (ii) $x(t) = u(t-3)$ (iii) $x(t) = \sin at \cos bt u(t)$

(i) $x(t) = \delta(t)$

Sol: Given $x(t) = \delta(t)$

We know that $L[u(t)] = 1/s$ and $\frac{du(t)}{dt} = \delta(t)$

From the differentiation in time domain property, we have

$$L[\delta(t)] = L\left[\frac{du(t)}{dt}\right] = sL[u(t)] = s \times \frac{1}{s} = 1$$

(ii) $x(t) = u(t-3)$

Sol: Given $x(t) = u(t-3)$

We know that $L[u(t)] = 1/s$

using time shifting property, we have

$$L[x(t-t_0)] = e^{-st_0} L[x(t)]$$

$$\therefore L[u(t-3)] = e^{-3s} L[u(t)] = \frac{e^{-3s}}{s}$$

(iii) $x(t) = \sin at \cos bt u(t)$

Given $x(t) = \sin at \cos bt u(t)$

$$= \frac{1}{2} [\sin(a+b)t + \sin(a-b)t] u(t)$$

$$\therefore x(s) = L[x(t)] = L\left\{\frac{1}{2} [\sin(a+b)t + \sin(a-b)t] u(t)\right\}$$

$$= \frac{1}{2} \{ L[\sin(a+b)t u(t)] + L[\sin(a-b)t u(t)] \}$$

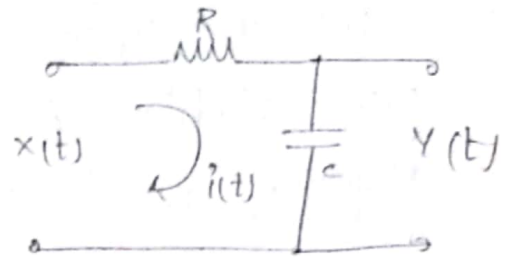
using superposition property, we have

$$\therefore x(s) = \frac{1}{2} \left[\frac{a+b}{s^2 + (a+b)^2} + \frac{a-b}{s^2 + (a-b)^2} \right]$$

→ Find the response of the circuit shown in figure 23 for the input $x(t) = r(t) - 2r(t-1) + r(t-2)$

Sol:- $x(t) = Ri(t) + \frac{1}{c} \int i(t) dt$

on applying laplace transform



$$X(s) = RI(s) + \frac{1}{cs} I(s) = \left(R + \frac{1}{cs}\right) I(s)$$

$$I(s) = X(s) \frac{cs}{1 + Rcs}$$

Output $Y(s) = \frac{1}{cs} I(s)$

$$= \frac{1}{cs} X(s) \frac{cs}{1 + Rcs}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{Rc} \frac{1}{\left(s + \frac{1}{Rc}\right)}$$

For input $x(t) = r(t) = tu(t)$, $X(s) = \frac{1}{s^2}$

$$Y(s) = \frac{1}{Rcs^2 \left(s + \frac{1}{Rc}\right)} = \frac{-Rc}{s} + \frac{1}{s^2} + \frac{Rc}{\left(s + \frac{1}{Rc}\right)}$$

Taking inverse laplace transform

$$Y(t) = \left[-Rc + t + Rc e^{-t/Rc}\right]$$

$$= t - Rc(1 - e^{-t/Rc})$$

using time shifting property

$$\mathcal{L}\{x(t - t_0)\} = e^{-st_0} x(s)$$

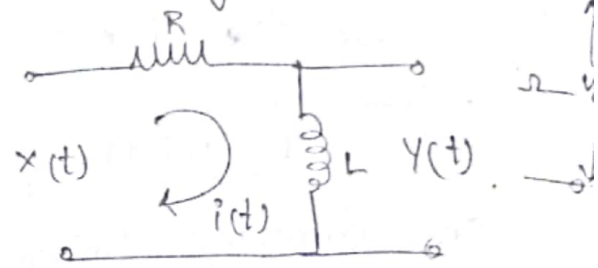
For input $x(t) = r(t) - 2r(t-1) + r(t-2)$

$$Y(t) = \left[t - Rc(1 - e^{-t/Rc})\right] - 2\left[(t-1) - Rc(1 - e^{-(t-1)/Rc})\right] + \left[t-2 - Rc(1 - e^{-(t-2)/Rc})\right]$$

Find whether the system shown in figure is

BIBO stable or not?

Sol:- For a system to be BIBO stable, its impulse response must be absolutely integrable.



$$X(s) = (R + Ls) I(s).$$

$$I(s) = \frac{X(s)}{R + Ls}.$$

$$\begin{aligned} \text{Output } Y(s) &= Ls I(s) \\ &= Ls \frac{X(s)}{R + Ls}. \end{aligned}$$

$$\frac{Y(s)}{X(s)} = \frac{Ls}{L \left[s + \frac{R}{L} \right]} = \frac{s + \frac{R}{L} - \frac{R}{L}}{s + \frac{R}{L}}.$$

$$= 1 - \frac{R/L}{s + R/L}.$$

Taking inverse Laplace transform on both sides

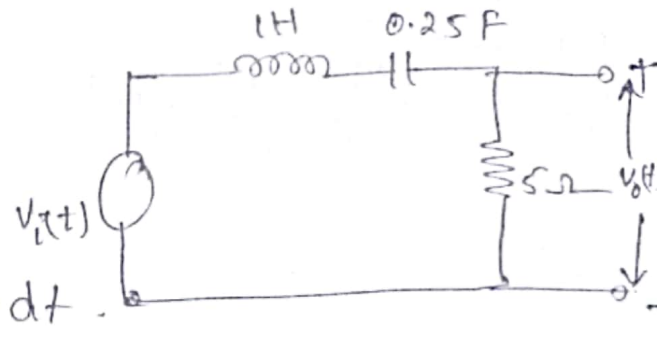
$$h(t) = \delta(t) - \frac{R}{L} e^{-(R/L)t} u(t).$$

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_0^{\infty} \left[\delta(t) - \frac{R}{L} e^{-(R/L)t} \right] dt. \\ &= \int_0^{\infty} \delta(t) dt - \frac{R}{L} \int_0^{\infty} e^{-(R/L)t} dt. \\ &= 1 - \frac{R}{L} \left[\frac{e^{-(R/L)t}}{-R/L} \right]_0^{\infty}. \\ &= 1 - (0 - 1) = 2 < \infty. \end{aligned}$$

The system is stable.

→ Obtain the voltage across the resistor in the circuit shown in figure as a function of time for $t > 0$. Assume that $i(0) = v_c(0) = 0$.

(a) $v_i(t) = \delta(t)$ (b) $v_i(t) = u(t)$.



Sol: - $v_i(t) = \frac{di(t)}{dt} + 5i(t) + \frac{1}{0.25} \int i(t) dt$

$$V_i(s) = sI(s) + 5I(s) + \frac{4}{s} I(s)$$

$$V_i(s) = \left[s + 5 + \frac{4}{s} \right] I(s) \Rightarrow I(s) = \frac{V_i(s) s}{s^2 + 5s + 4}$$

$$V_o(s) = 5I(s) = \frac{5s V_i(s)}{s^2 + 5s + 4}$$

Transfer function $\frac{V_o(s)}{V_i(s)} = \frac{5s}{(s+1)(s+4)}$

(a) $v_i(t) = \delta(t)$ [Impulse Response]

$$V_o(s) = \frac{5s}{(s+1)(s+4)} = \frac{-5/3}{s+1} + \frac{20/3}{s+4}$$

$$h(t) = v_o(t) = -\frac{5}{3} e^{-t} u(t) + \frac{20}{3} e^{-4t} u(t)$$

(b) $v_i(t) = u(t)$ [Step Response]

$$V_o(s) = \frac{1}{s} \cdot \frac{5s}{(s+1)(s+4)} = \frac{+5/3}{s+1} - \frac{5/3}{s+4}$$

$$s(t) = v_o(t) = \frac{5}{3} e^{-t} u(t) - \frac{5}{3} e^{-4t} u(t)$$

Comparison of Laplace & Fourier Transforms :-

Fourier transform	Laplace transform
1. The F.T of a function $x(t)$ can be represented by a continuous sum of exponential functions of the form $e^{j\omega t}$	1. The L.T of a function $x(t)$ can be represented by a continuous sum of complex exponential damped waves of the form e^{st} .
2. It does not have any convergence factor	2. It has a convergence factor & so is more general.
3. It is rarely used for solving differential equations because F.T does not exist for many signals as $ x(t) $ is not absolutely integrable	3. It is very widely used for solving differential equations because L.T exists for signals for which the F.T does not exist.
4. F.T cannot be used to analyse unstable systems.	4. L.T can be used to analyse even unstable systems.
5. F.T is the L.T evaluated along the imaginary axis of the s -plane. i.e. $x(\omega) = X(s) / s = j\omega$	5. L.T of $x(t)$ is the F.T of $x(t)e^{-\sigma t}$, i.e. $L\{x(t)\} = F\{x(t)e^{-\sigma t}\}$