

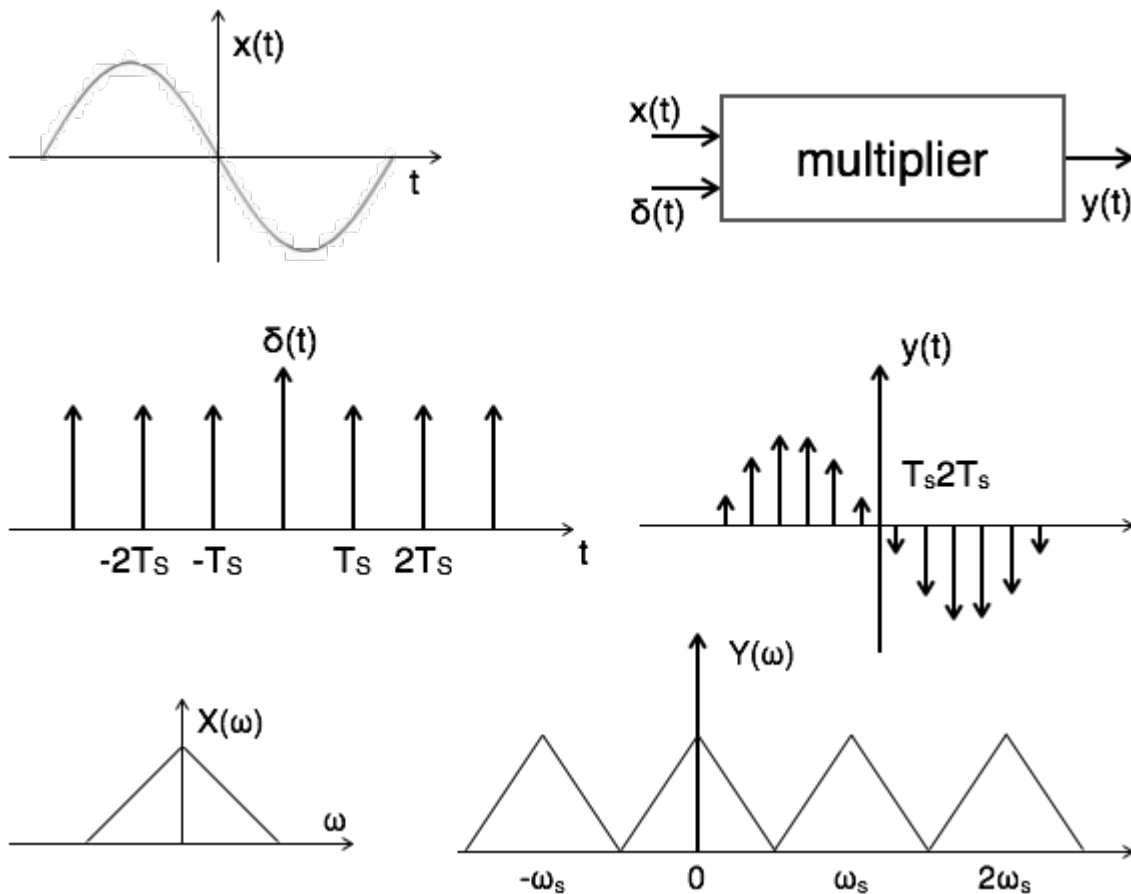
SIGNALS SAMPLING THEOREM

Statement: A continuous time signal can be represented in its samples and can be recovered back when sampling frequency f_s is greater than or equal to the twice the highest frequency component of message signal. i. e.

$$f_s \leq 2f_m.$$

Proof: Consider a continuous time signal $x(t)$. The spectrum of $x(t)$ is a band limited to f_m Hz i.e. the spectrum of $x(t)$ is zero for $|\omega| > \omega_m$.

Sampling of input signal $x(t)$ can be obtained by multiplying $x(t)$ with an impulse train $\delta(t)$ of period T_s . The output of multiplier is a discrete signal called sampled signal which is represented with $y(t)$ in the following diagrams:



Here, you can observe that the sampled signal takes the period of impulse. The process of sampling can be explained by the following mathematical expression:

$$\text{Sampled signal } y(t) = x(t) \cdot \delta(t) \dots \dots (1)$$

The trigonometric Fourier series representation of $\delta(t)$ is given by

$$\delta(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_s t + b_n \sin n\omega_s t) \dots \dots (2)$$

$$\text{Where } a_0 = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) dt = \frac{1}{T_s} \delta(0) = \frac{1}{T_s}$$

$$a_n = \frac{2}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) \cos n\omega_s t dt = \frac{2}{T_s} \delta(0) \cos n\omega_s 0 = \frac{2}{T_s}$$

$$b_n = \frac{2}{T_s} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) \sin n\omega_s t dt = \frac{2}{T_s} \delta(0) \sin n\omega_s 0 = 0$$

Substitute above values in equation 2.

$$\therefore \delta(t) = \frac{1}{T_s} + \sum_{n=1}^{\infty} \left(\frac{2}{T_s} \cos n\omega_s t + 0 \right)$$

Substitute δt in equation 1.

$$\rightarrow y(t) = x(t) \cdot \delta(t)$$

$$= x(t) \left[\frac{1}{T_s} + \sum_{n=1}^{\infty} \left(\frac{2}{T_s} \cos n\omega_s t \right) \right]$$

$$= \frac{1}{T_s} [x(t) + 2 \sum_{n=1}^{\infty} (\cos n\omega_s t) x(t)]$$

$$y(t) = \frac{1}{T_s} [x(t) + 2 \cos \omega_s t \cdot x(t) + 2 \cos 2\omega_s t \cdot x(t) + 2 \cos 3\omega_s t \cdot x(t) \dots \dots]$$

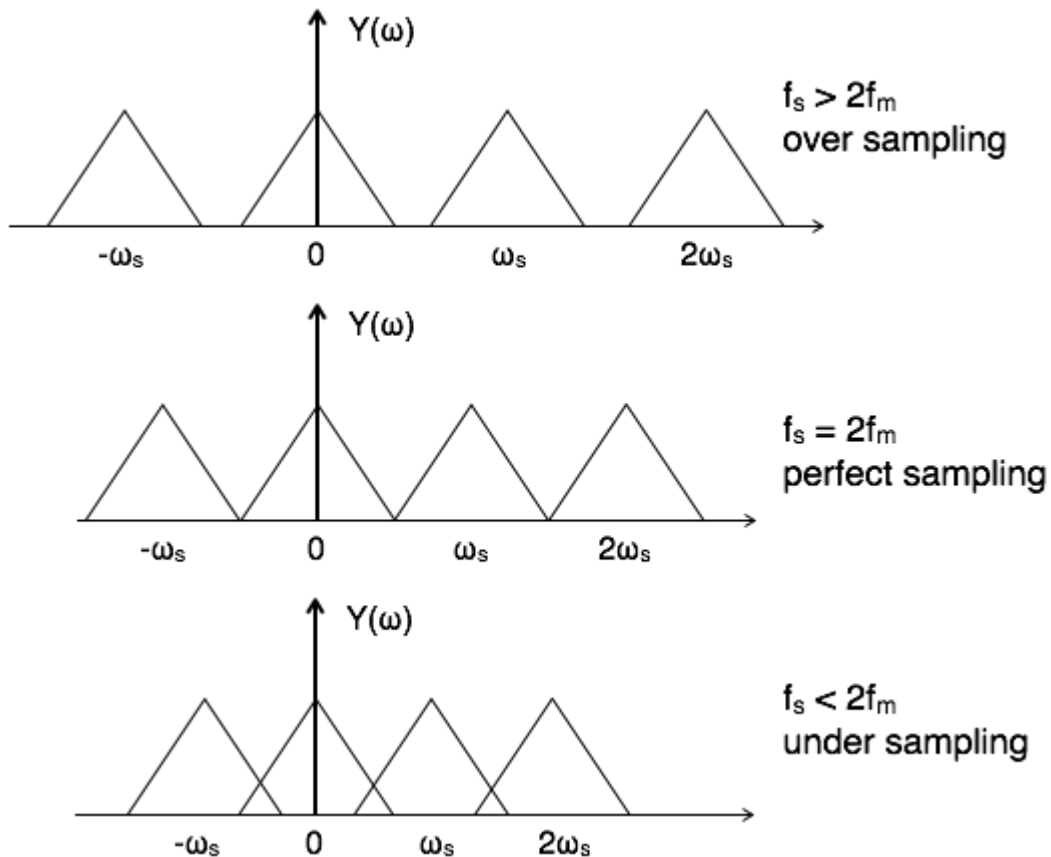
Take Fourier transform on both sides.

$$Y(\omega) = \frac{1}{T_s} [X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega + 2\omega_s) + \dots]$$

$$\therefore Y(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \quad \text{where } n = 0, \pm 1, \pm 2, \dots$$

To reconstruct $x(t)$, you must recover input signal spectrum $X(\omega)$ from sampled signal spectrum $Y(\omega)$, which is possible when there is no overlapping between the cycles of $Y(\omega)$.

Possibility of sampled frequency spectrum with different conditions is given by the following diagrams:



Aliasing Effect

The overlapped region in case of under sampling represents aliasing effect, which can be

removed by

- considering $f_s > 2f_m$
- By using anti aliasing filters.

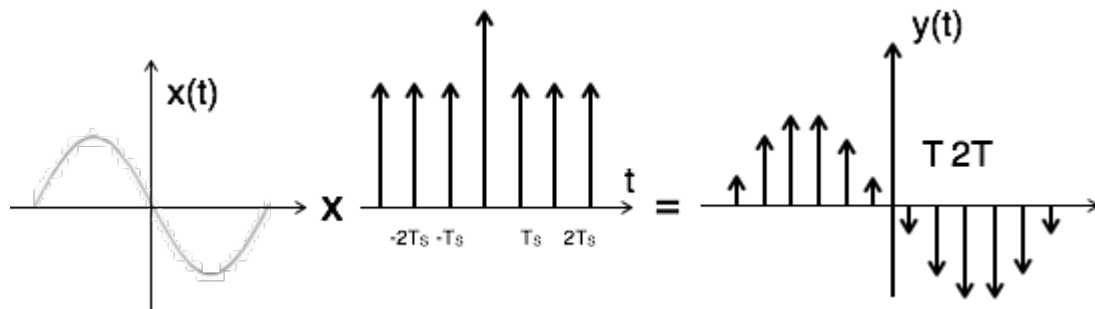
SIGNALS SAMPLING TECHNIQUES

There are three types of sampling techniques:

- Impulse sampling.
- Natural sampling.
- Flat Top sampling.

Impulse Sampling

Impulse sampling can be performed by multiplying input signal $x(t)$ with impulse train $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ of period 'T'. Here, the amplitude of impulse changes with respect to amplitude of input signal $x(t)$. The output of sampler is given by



$$y(t) = x(t) \times \text{impulse train}$$

$$= x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$y(t) = y_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \dots \dots 1$$

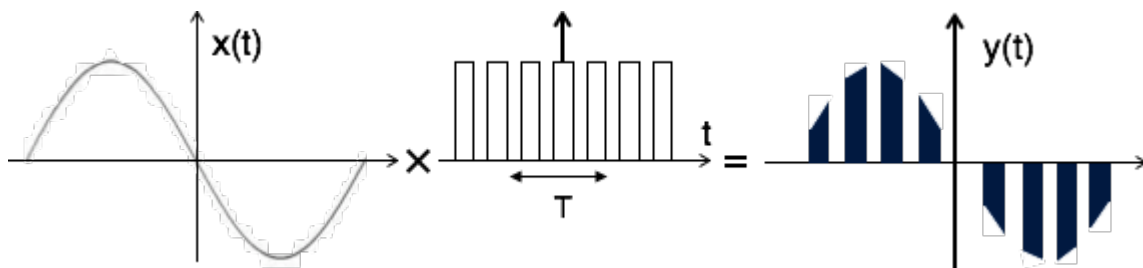
To get the spectrum of sampled signal, consider Fourier transform of equation 1 on both sides

$$Y(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

This is called ideal sampling or impulse sampling. You cannot use this practically because pulse width cannot be zero and the generation of impulse train is not possible practically.

Natural Sampling

Natural sampling is similar to impulse sampling, except the impulse train is replaced by pulse train of period T. i.e. you multiply input signal $x(t)$ to pulse train $\sum_{n=-\infty}^{\infty} P(t - nT)$ as shown below



The output of sampler is

$$y(t) = x(t) \times \text{pulse train}$$

$$= x(t) \times p(t)$$

$$= x(t) \times \sum_{n=-\infty}^{\infty} P(t - nT) \dots \dots (1)$$

The exponential Fourier series representation of pt can be given as

$$p(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \dots \dots (2)$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t}$$

Where $F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} p(t) e^{-jn\omega_s t} dt$

$$= \frac{1}{TP}(n\omega_s)$$

Substitute F_n value in equation 2

$$\therefore p(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} P(n\omega_s) e^{jn\omega_s t}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

Substitute pt in equation 1

$$y(t) = x(t) \times p(t)$$

$$= x(t) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) e^{jn\omega_s t}$$

$$y(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) x(t) e^{jn\omega_s t}$$

To get the spectrum of sampled signal, consider the Fourier transform on both sides.

$$F.T[y(t)] = F.T\left[\frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) x(t) e^{jn\omega_s t}\right]$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) F.T[x(t) e^{jn\omega_s t}]$$

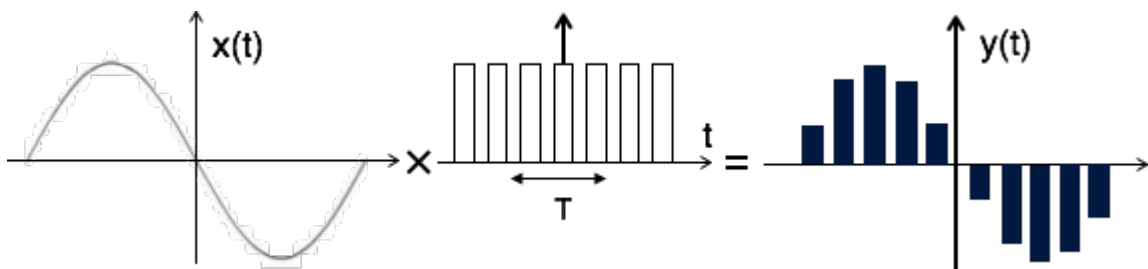
According to frequency shifting property

$$F.T[x(t) e^{jn\omega_s t}] = X[\omega - n\omega_s]$$

$$\therefore Y[\omega] = \frac{1}{T} \sum_{n=-\infty}^{\infty} P(n\omega_s) X[\omega - n\omega_s]$$

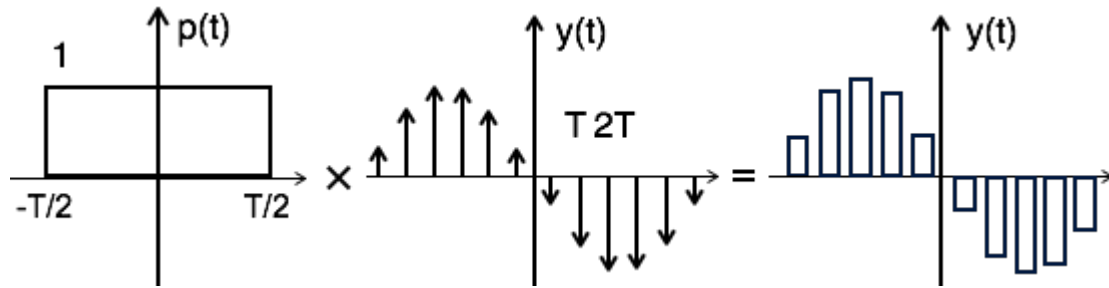
Flat Top Sampling

During transmission, noise is introduced at top of the transmission pulse which can be easily removed if the pulse is in the form of flat top. Here, the top of the samples are flat i.e. they have constant amplitude. Hence, it is called as flat top sampling or practical sampling. Flat top sampling makes use of sample and hold circuit.



Theoretically, the sampled signal can be obtained by convolution of rectangular pulse pt with ideally sampled signal say $y_\delta t$ as shown in the diagram:

$$\text{i.e. } y(t) = p(t) \times y_{\delta}(t) \dots \dots (1)$$



To get the sampled spectrum, consider Fourier transform on both sides for equation 1

$$Y[\omega] = F.T [P(t) \times y_{\delta}(t)]$$

By the knowledge of convolution property,

$$Y[\omega] = P(\omega) Y_{\delta}(\omega)$$

$$\text{Here } P(\omega) = T \text{Sa}\left(\frac{\omega T}{2}\right) = 2 \sin \omega T / \omega$$

Nyquist Rate

It is the minimum sampling rate at which signal can be converted into samples and can be recovered back without distortion.

$$\text{Nyquist rate } f_N = 2f_m \text{ hz}$$

$$\text{Nyquist interval} = \frac{1}{f_N} = \frac{1}{2f_m} \text{ seconds.}$$

Samplings of Band Pass Signals

In case of band pass signals, the spectrum of band pass signal $X[\omega] = 0$ for the frequencies outside the range $f_1 \leq f \leq f_2$. The frequency f_1 is always greater than zero. Plus, there is no aliasing effect when $f_s > 2f_2$. But it has two disadvantages:

- The sampling rate is large in proportion with f_2 . This has practical limitations.
- The sampled signal spectrum has spectral gaps.

To overcome this, the band pass theorem states that the input signal $x(t)$ can be converted into its samples and can be recovered back without distortion when sampling frequency $f_s < 2f_2$.

Also,

$$f_s = \frac{1}{T} = \frac{2f_2}{m}$$

Where m is the largest integer $< \frac{f_2}{B}$

and B is the bandwidth of the signal. If $f_2 = KB$, then

$$f_s = \frac{1}{T} = \frac{2KB}{m}$$

For band pass signals of bandwidth $2f_m$ and the minimum sampling rate $f_s = 2B = 4f_m$,

the spectrum of sampled signal is given by $Y[\omega] = \frac{1}{T} \sum_{n=-\infty}^{\infty} X[\omega - 2nB]$

Nyquist Rate of Sampling:-

Nyquist rate of sampling is the minimum sampling rate at which the signal can be sampled & still be reconstructed from its samples without any distortion.

A signal sampled at greater than Nyquist rate is said to be oversampled.

A signal sampled at less than Nyquist rate is said to be undersampled.

Nyquist Interval:-

Nyquist interval is the time interval b/w any two adjacent samples when sampling rate is nyquist rate.

$$\text{Nyquist interval} = \frac{1}{\text{Nyquist rate}} = \frac{1}{2f_m}$$

Problems:-

- ① A signal $x(t) = \text{sinc}(150\pi t)$ is sampled at a rate of (a) 100 Hz (b) 200 Hz (c) 300 Hz. For each of these cases, explain if you can recover the signal $x(t)$ from the sampled signal.

Sol:- $x(t) = \text{sinc}(150\pi t) =$

$$2\pi f_m = 150\pi$$

$$f_m = 75 \text{ Hz.}$$

$$\text{Nyquist rate} = 2f_m = 150 \text{ Hz.}$$

(a) $f_s = 100 \text{ Hz}$ $f_s < 2f_m \Rightarrow x(t)$ cannot be recovered from samples

(b) $f_s = 200 \text{ Hz}$ $f_s > 2f_m$
 (c) $f_s = 300 \text{ Hz}$ $f_s > 2f_m$ $\Rightarrow x(t)$ can be recovered from samples.

- ② Determine the Nyquist sampling rate & Nyquist sampling interval for the following signals

(a) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$ (b) $x(t) = \frac{\sin 4000\pi t}{\pi t}$
 (c) $x(t) = \left[\sin \left(\frac{4000\pi t}{\pi t} \right) \right]^2$

Sol:- (a) $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$

$$\omega_{m1} = 2000\pi$$

$$\omega_{m2} = 4000\pi$$

max. freq. component of $x(t)$ is $\omega_m = \max(\omega_{m1}, \omega_{m2}) = 4000\pi$

$$2\pi f_m = 4000\pi$$

$$f_m = 2000 \text{ Hz.}$$

$$\text{Nyquist rate } f_N = 2f_m = 4000 \text{ Hz}$$

$$\text{Nyquist Interval} = \frac{1}{f_N} = 0.25 \text{ ms.}$$

$$(b) x(t) = \frac{\sin(4000\pi t)}{\pi t}$$

$$\omega_m = 4000\pi$$

$$2\pi f_m = 4000\pi \Rightarrow f_m = 2000 \text{ Hz}$$

$$\begin{aligned} \text{Nyquist rate } f_N &= 2f_m \\ &= 2 \times 2000 \\ &= 4000 \text{ Hz} \end{aligned}$$

$$\begin{aligned} \text{Nyquist interval} &= \frac{1}{f_N} \\ &= 0.25 \text{ ns} \end{aligned}$$

$$\begin{aligned} (c) x(t) &= \left[\frac{\sin(4000\pi t)}{\pi t} \right]^2 \\ &= \frac{\sin \frac{4000\pi t}{\pi t}}{\pi t} \sin \frac{4000\pi t}{\pi t} \\ &= \left[\frac{1 - \cos 8000\pi t}{(\pi t)^2} \right] \end{aligned}$$

$$\omega_m = 4000\pi + 4000\pi = 8000\pi$$

$$f_m = \frac{\omega_m}{2\pi} = \frac{8000\pi}{2\pi} = 4000 \text{ Hz}$$

$$\begin{aligned} \text{Nyquist rate } f_N &= 2f_m \\ &= 8000 \text{ Hz} \end{aligned}$$

$$\begin{aligned} \text{Nyquist interval } \frac{1}{f_N} &= \frac{1}{2f_m} \\ &= \frac{1}{8000} = 0.125 \text{ ns} \end{aligned}$$

③ Find the Nyquist rate & Nyquist interval for the following signals.

(a) $\text{rect}(300t)$

(b) $-10\sin 40\pi t + \cos 300\pi t$

Sol:-

$$(a) \text{rect}(300t) = \text{rect}\left(\frac{t}{T}\right)$$

$$\text{So } \frac{t}{T} = 300t \Rightarrow T = \frac{1}{300}$$

$$F\left[\text{rect}\left(\frac{t}{T}\right)\right] = T \text{sinc}\left(\frac{\omega T}{2}\right) =$$

$$x(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

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$$= \frac{1}{300} \operatorname{sinc}\left(\frac{2\pi f}{2} \times \frac{1}{300}\right)$$

$$= \frac{1}{300} \operatorname{sinc}\left(\frac{\pi f}{300}\right)$$

The F.T of a rectangular pulse is a sinc function & it stays upto infinity. Hence, the Nyquist rate is also infinite & Nyquist interval is zero.

$$(b) -10 \sin 40\pi t \cos 300\pi t$$

$$= -5 \left[2 \sin 40\pi t \cos 300\pi t \right]$$

$$= -5 \sin 340\pi t + 5 \sin 260\pi t$$

$$\omega_{m1} = 340\pi$$

$$\omega_{m2} = 260\pi$$

$$\omega_m = \max(\omega_{m1}, \omega_{m2})$$

$$= 340\pi$$

$$f_m = \frac{\omega_m}{2\pi} = \frac{340\pi}{2\pi} = 170 \text{ Hz}$$

$$\text{Nyquist rate} = 2f_m$$

$$= 340 \text{ Hz}$$

$$\text{Nyquist interval} = \frac{1}{340} \text{ sec.}$$

Determine the Nyquist sampling rate & Nyquist sampling interval for

(a) $x(t) = 2 \operatorname{sinc}(100\pi t)$

(b) $x(t) = \frac{1}{2} \operatorname{sinc}(100\pi t) + \frac{1}{3} \operatorname{sinc}(50\pi t)$

(c) $x(t) = \operatorname{sinc}(100\pi t) + 3 \operatorname{sinc}^2(60\pi t)$ (d) $x(t) = \operatorname{sinc}(80\pi t) \operatorname{sinc}(120\pi t)$

Q.1 :-

(a) $x(t) = 2 \operatorname{sinc}(100\pi t)$

$$= 2 \frac{\sin(100\pi t)}{100\pi t} = \frac{2 \sin(\omega_m t)}{\omega_m t}$$

$$\omega_m = 100\pi \Rightarrow f_m = \frac{100\pi}{2\pi} = 50$$

Nyquist rate $= 2f_m$
 $= 2 \times 50$
 $= 100 \text{ Hz}$

Nyquist interval $= \frac{1}{2f_m}$
 $= \frac{1}{100 \text{ Hz}} = 10 \text{ ms}$

(b) $x(t) = \frac{1}{2} \operatorname{sinc}(100\pi t) + \frac{1}{3} \operatorname{sinc}(50\pi t)$

$$= \frac{1}{2} \frac{\sin(100\pi t)}{100\pi t} + \frac{1}{3} \frac{\sin(50\pi t)}{50\pi t}$$

$\omega_{m1} = 100\pi$ $\omega_{m2} = 50\pi$ $\omega_m = \max(100\pi, 50\pi)$
 $= 100\pi$

$$f_m = \frac{\omega_m}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$

Nyquist rate $= 2f_m = 100 \text{ Hz}$

Nyquist Interval $= \frac{1}{2f_m}$
 $= 10 \text{ ms}$

(c) $x(t) = \operatorname{sinc}(100\pi t) + 3 \operatorname{sinc}^2(60\pi t)$

$$= \frac{\sin(100\pi t)}{100\pi t} + 3 \left[\frac{\sin 60\pi t}{60\pi t} \right]^2$$

$$= \frac{\sin(100\pi t)}{100\pi t} + 3 \left[\frac{\sin 60\pi t}{60\pi t} \right] \left[\frac{\sin 60\pi t}{60\pi t} \right]$$

$$\omega_{m_1} = 100\pi \quad \omega_{m_2} = 60\pi + 60\pi = 120\pi$$

$$\begin{aligned} \omega_m &= \max(\omega_{m_1}, \omega_{m_2}) \\ &= \max(100\pi, 120\pi) = 120\pi \end{aligned}$$

$$f_m = \frac{\omega_m}{2\pi} = \frac{120\pi}{2\pi} = 60 \text{ Hz}$$

$$\begin{aligned} \text{Nyquist rate} &= 2f_m \\ &= 120 \text{ Hz} \end{aligned}$$

$$\begin{aligned} \text{Nyquist Interval} &= \frac{1}{2f_m} \\ &= \frac{1}{120} \text{ sec} \end{aligned}$$

$$\begin{aligned} \text{(d) } x(t) &= \text{sinc}(80\pi t) \sin(120\pi t) \\ &= \frac{\sin(80\pi t)}{80\pi t} + \frac{\sin(120\pi t)}{120\pi} \end{aligned}$$

$$\omega_{m_1} = 80\pi \quad \omega_{m_2} = 120\pi \quad \omega_m = \max(\omega_{m_1}, \omega_{m_2})$$

$$= 80\pi + 120\pi$$

$$= 200\pi$$

$$f_m = \frac{\omega_m}{2\pi} = 100 \text{ Hz}$$

$$\begin{aligned} \text{Nyquist rate} &= 2f_m \\ &= 200 \text{ Hz} \end{aligned}$$

$$\begin{aligned} \text{Nyquist Interval} &= \frac{1}{200} \\ &= 5 \text{ ms} \end{aligned}$$