

CHAPTER 8

State Space Analysis of Control System

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8.1. ANALYSIS OF SYSTEMS

The procedure for determining the state of a system is called state variable analysis. The state of a dynamic system is the smallest set of variables such that the knowledge of these variables at $t = t_0$ with the knowledge of the input for $t \geq t_0$ completely determines the behaviour of the system for any time $t \geq t_0$. This set of variables is called state variables.

In earlier chapters we studied the linear system by transfer function, block diagram etc. The transfer function has some drawbacks e.g. transfer function is only defined under zero initial conditions and also it is applicable to linear time invariant systems.* Therefore due to these limitation state variable approach is developed. This technique can be used for analysis and design of linear and non-linear, time invariant or time variant and multi input multi-output systems.** The state space analysis involves the description of the system in terms of 1 order differential equations by selecting suitable state variables, the first order derivatives are arranged on left hand side and on right hand side the terms are free from derivatives. The state space techniques have many advantages (Given in next article i.e., 8.2).

8.2. ADVANTAGES OF STATE SPACE TECHNIQUES

This technique has the following advantages.

* If the characteristic of a system does not change with time, then the system is said to be time invariant.

** A system is said to be a single variable system if and only if it has only one input terminal and only one output terminal. A system is said to be multivariable system if and only if it has more than one input terminal or more than one output terminal.

1. This approach can be applied to linear or nonlinear, time variant or time invariant systems.
2. It is easier to apply where the Laplace transform cannot be applied
3. n^{th} order differential equations can be expressed as ' n ' equation of first order whose solutions are easier.
4. It is a time domain approach.
5. This method is suitable for digital computer computation because this is a time domain approach.
6. The system can be designed for optimal conditions with respect to given performance indices.

8.3. SOME IMPORTANT DEFINITIONS

State : The state of a system at any time ' t_0 ' is the minimum set of numbers $x_1, x_2, \dots, x_n$ which along with the input to the system for time $t \geq t_0$ is sufficient to determine the behaviour of the system for all $t \geq t_0$. In other words, the state of a system represents the minimum amount of information that we need to know about a system at " t_0 " such that its future behaviour can be determined without reference to the input before ' t_0 '. The state can also be defined as the state of a system at time t_0 is the amount of information at t_0 that, together with input $u(t_0, \infty)$ determines the unique behaviour of the system for all $t \geq t_0$. By the behaviour of the system, we mean all responses, including the state of the system. If the system is a network we mean the voltage and current of every branch of the network.

Consider the network shown in Fig. 8.1 if the initial current through the inductor and initial voltage across the capacitor are known, then for any driving voltage the behaviour of the network can be determined. Hence, the inductor current and capacitor voltage can be considered as the state of the network.

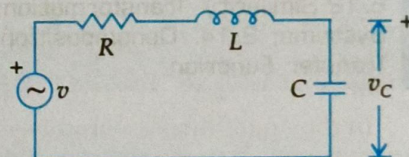


Fig. 8.1.

State Variables : The definition is given in Article 8.1

State Vector : If we need n variables to completely describe the behaviour of a given system, then these n state variables may be considered as n component of a vector x . Such a vector is called state vector. A state vector is thus a vector which determines uniquely the system state $x(t)$ for any time $t \geq t_0$ once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State space : The n -dimensional space whose coordinate axes consists of the x_1 axis, x_2 axis, x_n axis is called state space. Any state can be represented by a point in the state space.

8.4. STATE SPACE REPRESENTATION

8.4.1. State Space Representation For Electrical Network (Physical Variable Form)

Consider an RLC network shown in Fig. 8.2. Let, the current at time $t = 0$ be $i_L(0)$ and capacitor voltage at time $t = 0$ be $V_c(0)$. Thus, the state of the network at time $t = 0$ is specified by the inductor current and capacitor voltage. Hence, the pair $i_L(0), V_c(0)$ is called the initial state of the network. Similarly at time ' t ', the pair $i_L(t), V_c(t)$ is called the state of the network at ' t '. The variable i_L and V_c are called state variables of the network.

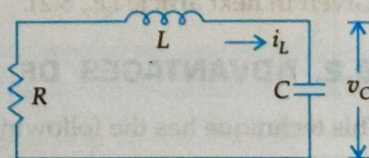


Fig. 8.2.

Apply KVL

$$Ri_L + L \frac{di_L}{dt} + V_c = 0 \quad \dots(8.1)$$

Also,
$$i_c = i_L = C \frac{dV_c}{dt} \quad \dots(8.2)$$

From equation (8.1)
$$\frac{di_L}{dt} = -\frac{R}{L} i_L - \frac{1}{L} V_c \quad \dots(8.3)$$

$$\frac{dV_c}{dt} = \frac{1}{C} i_L \quad \dots(8.4)$$

Equations of this form are called state equations. In such equations all the variables present are state variables.

Equations (8.3) and (8.4) can be written in matrix form as

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dV_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i_L \\ V_c \end{bmatrix} \quad \dots(8.5)$$

let $x(t) \triangleq \begin{bmatrix} i_L \\ V_c \end{bmatrix}$ and $A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$

then equation (8.5) can be written as

$$\frac{d}{dt} x(t) = A x(t)$$

or $\dot{x}(t) = A x(t)$

In the linear time-invariant systems, the general form of state equations are

$$\dot{x}(t) = A x(t) + B u(t) \quad \dots(8.6)$$

$$y(t) = C x(t) + D u(t) \quad \dots(8.7)$$

These equations are vector differential equations where 'x' is the n -dimensional state vector

y = n -dimensional output vector

u = r -dimensional control vector or input vector

A = $n \times n$ system matrix

B = $n \times r$ control matrix

C = $n \times n$ output matrix

In some cases there is no direct connection between input and output so $D u(t)$ will not be there.

$$y(t) = C x(t) \quad \dots(8.8)$$

Equations (8.6) and (8.8) can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \dots(8.9)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \dots & C_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \dots(8.10)$$

8.4.2. State Space Representation of n th Order Differential Equations

Consider the following examples.

(a) For n th Order Differential Equation

EXAMPLE 8.1. A system is described by the differential equation

$$\frac{d^3 y}{dt^3} + 6 \frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 10y = 8u(t)$$

where y is the output and u is the input to the system. Obtain state space representation of the system. (R.M.L., University, Faizabad, 2003)

Solution : Select the state variables as

$$x_1 = y, x_2 = \dot{y} \text{ and } x_3 = \ddot{y} \text{ then}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 8u(t) - 10x_1 - 11x_2 - 6x_3$$

The last equation is obtained from the given equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} u(t) \quad \dots(8.11)$$

Compare equation (8.11) with equation (8.6) we get

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}, x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) State Space Representation of n th Order Linear System with r Forcing Function

Consider the following example

EXAMPLE 8.2. A system is described by the following differential equation. Represent the system in state space.

$$\frac{d^3 x}{dt^3} + 3 \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)$$

and outputs are

$$y_1 = 4 \frac{dx}{dt} + 3u_1$$

$$y_2 = \frac{d^2 x}{dt^2} + 4u_2 + u_3$$

Solution : Select the state variables as

$$x_1 = x$$

$$\dot{x}_1 = \dot{x} = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u_1(t) + 3u_2(t) + 4u_3(t) - 3x_3 - 4x_2 - 4x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Outputs :

$$\begin{aligned} y_1 &= 4x_2 + 3u_1 \\ y_2 &= x_3 + 4u_2 + u_3 \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned}$$

8.4.3. State Space Representation for Transfer Function

Consider the following example

EXAMPLE 8.3. For the given transfer function, obtain the state model.

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{s^3 + a_3s^2 + a_2s + a_1}$$

Solution : This transfer function has no zeros.

$$(s^3 + a_3s^2 + a_2s + a_1) y(s) = Ku(s)$$

$$\text{or } s^3y(s) + a_3s^2y(s) + a_2sy(s) + a_1y(s) = Ku(s)$$

Taking inverse Laplace

$$\ddot{y}(t) + a_3 \ddot{y}(t) + a_2 \dot{y}(t) + a_1 y(t) = K u(t)$$

$$\text{or } \ddot{y}(t) = Ku(t) - a_3 \ddot{y}(t) - a_2 \dot{y}(t) - a_1 y(t)$$

Select the state variables as, first state variable as output

$$y(t) = x_1$$

$$\dot{y}(t) = \dot{x}_1 = x_2$$

$$\ddot{y}(t) = \dot{x}_2 = x_3$$

$$\ddot{\ddot{y}}(t) = \dot{x}_3$$

$$\therefore \ddot{\ddot{y}}(t) = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

Rewriting the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0 \ 0] x_1(t)$$

BLOCK DIAGRAM :

The block diagram of the given transfer function is shown in Fig. 8.3.

Now consider another case when the transfer function has zeros.

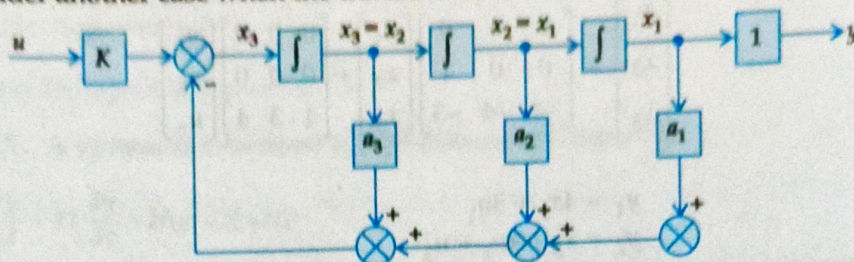


Fig. 8.3.

EXAMPLE 8.4. Obtain the state model for the given transfer function.

$$G(s) = \frac{y(s)}{u(s)} = \frac{K(C_2s + C_1)}{s^3 + a_3s^2 + a_2s + a_1}$$

Solution : Break the transfer function in two parts

$$\frac{y(s)}{u(s)} = \frac{x_1(s)}{u(s)} \cdot \frac{y(s)}{x_1(s)} = \frac{K}{s^3 + a_3s^2 + a_2s + a_1} (C_2s + C_1)$$

Now consider
$$\frac{x_1(s)}{u(s)} = \frac{K}{s^3 + a_3s^2 + a_2s + a_1}$$

$$[s^3 + a_3s^2 + a_2s + a_1] x_1(s) = Ku(s)$$

Taking inverse Laplace

$$\ddot{x}_1(t) + a_3 \dot{x}_1(t) + a_2 \dot{x}_1(t) + a_1 x_1(t) = Ku(t)$$

$$\ddot{x}_1(t) = -a_3 \dot{x}_1(t) - a_2 x_1(t) - a_1 x_1(t) + Ku(t)$$

Select the state variables as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \dot{x}_3 = x_3$$

$$\dot{x}_3 = \ddot{x}_1$$

Rewrite

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_3 x_3 - a_2 x_2 - a_1 x_1 + Ku$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u(t)$$

Consider,
$$\frac{y(s)}{x_1(s)} = C_2s + C_1$$

$$y(s) = x_1(s) [C_2s + C_1]$$

Take inverse Laplace

$$y(t) = C_1 x_1 + C_2 \dot{x}_1$$

$$y(t) = [C_1 \ C_2 \ 0] x(t)$$

$$= [C_1 \ C_2 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The block diagram is shown in Fig. 8.4.

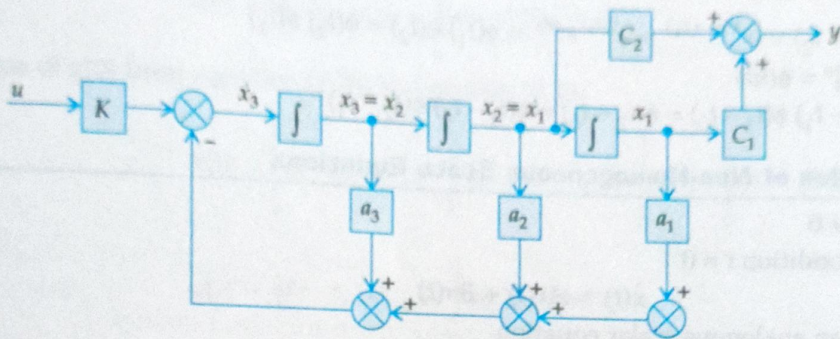


Fig. 8.4.

8.5. SOLUTION OF THE TIME-INVARIANT STATE EQUATION

8.5.1. Solution of Homogeneous State Equation : Laplace Transform Method

We know that

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \dots(8.12)$$

$u(t) = 0$ for unforced response

Then $\dot{x}(t) = Ax(t)$... (8.13)

Let us consider the analogous scalar equation

$$\dot{x}(t) = ax(t) \quad \dots(8.14)$$

Take the Laplace transform of equation (8.14)

$$sX(s) - x(0) = aX(s)$$

$$(s - a) X(s) = x(0)$$

OR

$$X(s) = (s - a)^{-1} x(0) \quad \dots(8.15)$$

Take the inverse laplace of equation (8.15)

$$x(t) = e^{at} x(0) \quad \dots(8.16)$$

If equation (8.16) is the solution of equation (8.14) then the solution of equation (8.13)

$$x(t) = e^{At} x(0)$$

$$e^{At} = \phi(t) = \text{State Transition Matrix (STM)}$$

$$= I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots = \sum_{i=0}^n \frac{A^i t^i}{i!} \quad \dots(8.17)$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \mathcal{L}^{-1}[sI - A]^{-1}$$

where $\phi(s)$ = Resolvent matrix

8.5.2. Properties of State Transition Matrices

For time invariant system $\dot{x} = Ax$ and

$$\phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

$$(i) \quad \phi(0) = e^{A(0)} = I$$

$$(ii) \quad \phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$$

$$(iii) \quad \phi^{-1}(t) = \phi(-t)$$

$$(iv) \quad \phi(t_1 + t_2) = e^{A(t_1 + t_2)} = e^{At_1} e^{At_2} = \phi(t_1) \phi(t_2) = \phi(t_2) \phi(t_1)$$

$$(v) \quad [\phi(t)]^n = \phi(nt)$$

$$(vi) \quad \phi(t_2 - t_1) \phi(t_1 - t_0) = \phi(t_2 - t_0) = \phi(t_1 - t_0) \phi(t_2 - t_1)$$

8.5.3. Solution of Non-Homogeneous State Equations

Here $u(t) \neq 0$

a. Initial condition $t = 0$

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \dots(8.18)$$

Consider an analogous scalar equation

$$\dot{x}(t) = ax(t) + bu(t) \quad \dots(8.19)$$

where 'a' and 'b' are the scalar quantities. Take Laplace transform of equation (8.19)

$$sX(s) - x(0) = aX(s) + bu(s)$$

or

$$(s - a) X(s) = x(0) + bu(s)$$

Multiplying both the sides by $(s - a)^{-1}$ we get

$$X(s) = (s - a)^{-1} x(0) + (s - a)^{-1} bu(s) \quad \dots(8.20)$$

Take inverse laplace of equation (8.20)

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \quad \dots(8.21)$$

Equation (8.21) is the solution of equation (8.19), then the solution of equation (8.18) will be

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} bu(\tau) d\tau \quad \dots(8.22)$$

or

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t - \tau) bu(\tau) d\tau \quad \dots(8.23)$$

b. When initial time is given by t_0 i.e., $t = t_0$

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t - \tau) Bu(\tau) d\tau$$

At

$$t = t_0$$

$$x(t_0) = \phi(t_0) x(0) + \int_0^{t_0} \phi(t_0 - \tau) Bu(\tau) d\tau \quad \dots(8.24)$$

Multiply both the sides by $\phi^{-1}(t_0)$

$$\begin{aligned} \phi^{-1}(t_0) x(t_0) &= x(0) + \int_0^{t_0} \phi(-t_0) \phi(t_0 - \tau) Bu(\tau) d\tau \\ &= x(0) + \int_0^{t_0} \phi(-t_0 + t_0 - \tau) Bu(\tau) d\tau \\ \phi^{-1}(t_0) x(t_0) &= x(0) + \int_0^{t_0} \phi(-\tau) Bu(\tau) d\tau \end{aligned}$$

or

$$x(0) = \phi(-t_0) x(t_0) - \int_0^{t_0} \phi(-\tau) Bu(\tau) d\tau \quad \dots(8.25)$$

Put the value of $x(0)$ from equation (8.25) in equation (8.23)

$$x(t) = \left[\phi(-t_0) x(t_0) - \int_0^{t_0} \phi(-\tau) Bu(\tau) d\tau \right] \phi(t) + \int_0^t \phi(t-\tau) Bu(\tau) d\tau$$

or

$$x(t) = \phi(t-t_0)x(t_0) + \int_{t_0}^t \phi(t-\tau) Bu(\tau) d\tau \quad \dots(8.26)$$

Equation (8.26) can also be written as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \quad \dots(8.27)$$

EXAMPLE 8.5. Find the time response of the system described by the equation

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, u(t) = 1, t > 0$$

Solution :

$$x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) Bu(\tau) d\tau \quad \dots(8.28)$$

Step 1 : Calculation of $\phi(t)$:

$$\phi(t) = \mathcal{L}^{-1} \phi(s) = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$\phi(s) = [sI - A]^{-1}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{s+2}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ 0 & \frac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1} \phi(s) = \begin{bmatrix} \mathcal{L}^{-1} \frac{s+2}{s^2+3s+2} & \mathcal{L}^{-1} \frac{1}{s^2+3s+2} \\ 0 & \mathcal{L}^{-1} \frac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

Put all the values in equation (8.28)

$$\begin{aligned} x(t) &= \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 d\tau \\ &= \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ e^{-2(t-\tau)} \end{bmatrix} d\tau \\ x_1(t) &= -e^{-t} + \int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)} d\tau \\ x_2(t) &= \int_0^t e^{-2(t-\tau)} d\tau \end{aligned}$$

or,

$$x_1(t) = \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \quad \text{Ans.} \quad \dots(8.29)$$

$$x_2(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

if

$$y = Cx \quad C = [1 \ 0]$$

$$y = [1 \ 0] \begin{bmatrix} \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} = \frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \quad \dots(8.30)$$

Equations 8.29 and 8.30 are the required answers.

8.6. TRANSFER MATRIX

We know that

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \dots(8.31)$$

$$y(t) = Cx(t) \quad \dots(8.32)$$

Take the Laplace transform, assuming initial conditions zero*

$$sX(s) = AX(s) + Bu(s) \quad \dots(8.33)$$

$$Y(s) = CX(s) \quad \dots(8.34)$$

Consider the equation (8.33), I is the identity matrix

$$sIX(s) = AX(s) + Bu(s)$$

or

$$X(s)[sI - A] = Bu(s)$$

Multiplying both the side by $[sI - A]^{-1}$ we get

$$X(s) = u(s)[sI - A]^{-1}B \quad \dots(8.35)$$

Put the values of $X(s)$ from equation (8.35) in equation (8.34)

$$Y(s) = C[sI - A]^{-1}Bu(s) \quad \dots(8.36)$$

We know that

$$Y(s) = G(s)u(s) \quad \dots(8.37)$$

Compare the equation (8.36) and equation (8.37) we get

$$G(s) = C[sI - A]^{-1}B \quad \dots(8.38)$$

where $G(s)$ = Transfer matrix

The transfer matrix given by the equation (8.38) is for MIMO system. For single input single output system, the transfer matrix is given by

$$G(s) = C^T[sI - A]^{-1}b \quad \dots(8.39)$$

* By the definition of transfer function.

Also, $[sI - A]^{-1} = \phi(s)$ = Resolvent matrix, Then equation (8.39) becomes
 For MIMO system
 $G(s) = C\phi(s)B$... (8.40)
 For SISO system
 $G(s) = C^T \phi(s) b$... (8.41)

EXAMPLE 8.6. The system equations are given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0] x(t)$$

Find the transfer function of the system.

Solution : Given that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C^T = [1 \ 0]$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \quad \therefore [sI - A]^{-1} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ -\frac{2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

$$C^T [sI - A]^{-1} \cdot b = [1 \ 0] \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ -\frac{2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+3s+2}$$

Hence, required transfer function = $\frac{1}{s^2+3s+2}$ **Ans.**

8.7. COMPUTATION OF STATE TRANSITION MATRIX : e^{At}

8.7.1. Laplace Transform Method

We know that

$$\dot{x}(t) = Ax + Bu(t)$$

For unforced system $u(t) = 0$

$$\dot{x}(t) = Ax$$

Take the Laplace transform

$$sX(s) - x(0) = AX(s)$$

or

$$[sI - A]X(s) = x(0)$$

Multiply both the sides by $[sI - A]^{-1}$

$$X(s) = [sI - A]^{-1} x(0)$$

$[sI - A]^{-1} = \phi(s)$ = Resolvent matrix

$$\phi(t) = e^{At} = \mathcal{L}^{-1} \phi(s) = \mathcal{L}^{-1} [sI - A]^{-1}$$

where $\phi(t) = e^{At}$ = state transition matrix

EXAMPLE 8.7. Compute the STM when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution :

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s-2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{s-2}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ 0 & \frac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}[sI - A]^{-1} = \begin{bmatrix} \mathcal{L}^{-1} \frac{s-2}{s^2+3s+2} & \mathcal{L}^{-1} \frac{1}{s^2+3s+2} \\ 0 & \mathcal{L}^{-1} \frac{s+1}{s^2+3s+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

8.7.2. Series Summation Method

8.7.3. Eigen Values

The roots of the characteristic equation are known as eigen values of matrix.

8.7.4. Eigen Vectors

Any non zero vector P_i that satisfies the matrix equation $(\lambda_i I - A) P_i = 0$ where λ_i is i^{th} eigenvalue of A is known as eigenvector of A associated with the eigenvalue λ_i .

8.7.5. Use of Diagonal

Let

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} + \dots = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

Let λ_i is the eigenvalue of A and P_i is the eigenvector, then for each of the n eigenvalues of A

$$AP_1 = \lambda_1 P_1$$

$$AP_2 = \lambda_2 P_2$$

$$\vdots$$

$$AP_n = \lambda_n P_n$$

or

$$AM = M \Lambda$$

...(8.42)

where Λ = diagonal matrix with eigenvalues on its main diagonal

M = modal matrix

$$\Lambda = M^{-1}AM$$

or

$$e^{\Lambda t} = M^{-1}e^{At}M$$

or

$$e^{At} = M e^{\Lambda t} M^{-1}$$

EXAMPLE 8.9. Compute e^{At} when

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

Solution :

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 3 & \lambda + 4 \end{bmatrix} \quad \dots(8.43)$$

$$|\lambda I - A| = \lambda^2 + 4\lambda + 3 = 0$$

or

$$\lambda_1 = -1$$

$$\lambda_2 = -3$$

Let the eigenvector

$$P_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

Put $\lambda = -1$ in equation (8.43)

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-p_{11} - p_{21} = 0$$

$$3p_{11} + 3p_{21} = 0$$

Let

$$p_{11} = 1 \quad \text{then} \quad p_{21} = -1$$

$\therefore$

$$P_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Put

$$\lambda = -3$$

$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3p_{12} - p_{22} = 0$$

$$3p_{12} + p_{22} = 0$$

Let $p_{12} = 1$ then $p_{22} = -3$

$$p_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\therefore$

$$M = [P_1 \ P_2] = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\therefore e^{At} = M e^{\Lambda t} M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0.5e^{-t} - 0.5e^{-3t} \\ 1.5e^{-3t} - 1.5e^{-t} & 1.5e^{-3t} - 0.5e^{-t} \end{bmatrix} \quad \text{Ans.}$$

8.8. BLOCK DIAGRAM OF A LINEAR SYSTEM IN STATE VARIABLE FORM

8.8.1. MIMO System

8.9. CONTROLLABILITY AND OBSERVABILITY

A system is said to be controllable if any initial state $x(t_0)$ or x_0 can be transferred to any final state $x(t_f)$ in a finite time interval $(t_f - t_0)$, $t \geq 0$ by some control u .

A system is said to be observable if every state x_0 can be exactly determined from the measurement of the output 'y' over a finite interval of time $0 \leq t \leq t_f$.

The concept of controllability and observability were introduced by Kalman and they play an important role in the design of control system in state space.

There are four possible states for the system.

- (i) The system is controllable and observable
- (ii) The system is controllable and unobservable
- (iii) The system is uncontrollable and observable
- (iv) The system is uncontrollable and unobservable

Consider the following example to determine the controllability and observability.

8.9.1. With the Help of Block Diagram

EXAMPLE 8.10. Determine the controllability and observability of the system described by the state equation. Find out the transfer function and draw the block diagram.

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0 \ +2] x(t)$$

Solution :

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + u \\ \dot{x}_3 &= -3x_3 \\ y &= x_1 + 2x_3 \end{aligned}$$

The block diagram is shown in Fig. 8.7.

From the block diagram, the first state is controllable and observable, second state is controllable and unobservable (*scu*), the third state is uncontrollable and observable (*suo*). Since, all the states are not controllable and observable the whole system is uncontrollable and unobservable.

The transfer function is available only for the state which is both controllable and observable. Hence, the required transfer function is

$$G(s) = \frac{1/s}{1+1/s} = \frac{1}{s+1}$$

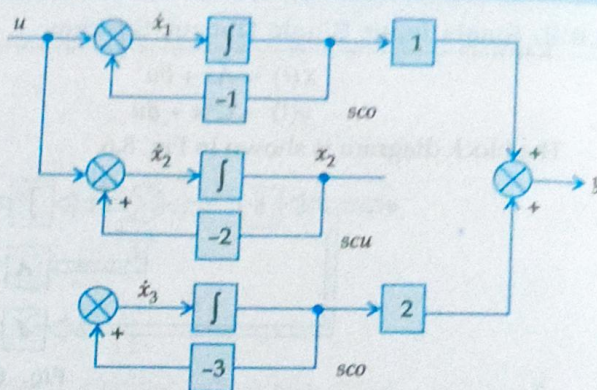


Fig. 8.7.

8.9.2. Kalman's Test

A linear time invariant continuous system described by the state equation

$$\dot{x} = Ax + Bu \quad \dots(8.44)$$

$$y = Cx$$

is completely controllable if and only if the rank of the controllability matrix is defined as

$$Q = [B : AB : A^2B : \dots : A^{n-1}B]$$

is equal to rank ' n '.

The system represented by the equation (8.44) is observable if and only if, the $n \times n_m$ composite matrix is obtained as

$$Q = [C^T : A^T C^T : (A^T)^2 C^T : \dots : A^{T(n-1)} C^T]$$

EXAMPLE 8.11. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Test for controllability and observability.

Solution :

$$Q = [B : AB]$$

Given that

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

Rank of 'Q' is one. Hence system is uncontrollable.*

$$C = [0 \ 1] \therefore C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} A^T C^T = \begin{bmatrix} -0.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

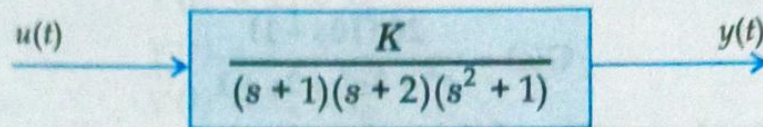
$$Q' = [C^T : A^T C^T]$$

$$Q' = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

Rank of 'Q' is one. Hence the system is unobservable. For controllability and observability the rank should be two.

The given system is uncontrollable and unobservable.

EXAMPLE 8.14. A system is given below, find the state and output equation of the system.
(R.M.L. University Faizabad, 2001)



Solution :

$$\frac{y(s)}{u(s)} = G(s) = \frac{K}{(s+1)(s+2)(s^2+1)}$$

$$\frac{y(s)}{u(s)} = \frac{K}{s^4 + 3s^3 + 3s^2 + 3s + 2}$$

$$y(s)[s^4 + 3s^3 + 3s^2 + 3s + 2] = Ku(s)$$

$$s^4y(s) + 3s^3y(s) + 3s^2y(s) + 3sy(s) + 2y(s) = Ku(s)$$

Taking inverse Laplace

$$\ddot{\ddot{y}}(t) + 3\ddot{\ddot{y}}(t) + 3\ddot{\ddot{y}}(t) + 3\dot{y}(t) + 2y(t) = Ku(t)$$

$$\dot{y}(t) = x_1$$

$$\dot{\dot{y}}(t) = \dot{x}_1 = x_2$$

$$\dot{\ddot{y}}(t) = \dot{x}_2 = x_3$$

$$\dot{\ddot{\ddot{y}}}(t) = \dot{x}_3 = x_4$$

$$\dot{\ddot{\ddot{\ddot{y}}}}(t) = \dot{x}_4$$

$$\therefore \ddot{y}(t) = \dot{x}_4 = Ku(t) - 3x_4 - 3x_3 - 3x_2 - 2x_1$$

Rewriting

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = Ku(t) - 3x_4 - 3x_3 - 3x_2 - 2x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} u(t)$$

$$y(t) = x_1$$

$$y(t) = [1 \ 0 \ 0 \ 0]x(t)$$

Block diagram :

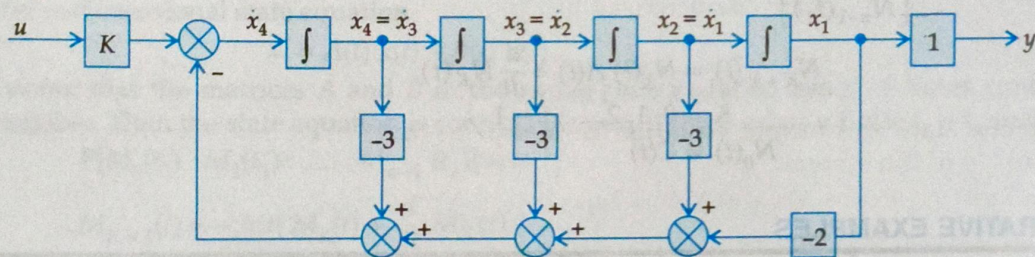


Fig. 8.8.

EXAMPLE 8.15. A system is described by the following transfer function

$$G(s) = \frac{20(10s+1)}{s^3+3s^2+2s+1}$$

Find the state and output equation of the system.

(A.M.U., Aligarh, 1989-90)

Solution :

$$G(s) = \frac{y(s)}{u(s)} = \frac{20(10s+1)}{s^3+3s^2+2s+1}$$

$$\frac{y(s)}{u(s)} = \frac{X_1(s)}{u(s)} \cdot \frac{y(s)}{X_1(s)} = \frac{20}{s^3+3s^2+2s+1} \cdot (10s+1)$$

Consider $\frac{X_1(s)}{u(s)} = \frac{20}{s^3+3s^2+2s+1}$

$$X_1(s) [s^3 + 3s^2 + 2s + 1] = 20 u(s)$$

Taking inverse Laplace

$$\ddot{X}_1(t) + 3\ddot{X}_1(t) + 2\dot{X}_1(t) + X_1(t) = 20 u(t)$$

Select

$$\dot{X}_1 = X_2$$

$$\ddot{X}_1 = \dot{X}_2 = X_3$$

Rewriting

$$\begin{aligned}\ddot{X}_1 &= \dot{X}_3 = X_4 \\ \dot{X}_1 &= 20u - 3\dot{X}_1 - 2\dot{X}_1 - X_1 = 20u - 3X_3 - 2X_2 - X_1 \\ \dot{X}_1 &= X_2 \\ \dot{X}_2 &= X_3 \\ \dot{X}_3 &= 20u - 3X_3 - 2X_2 - X_1\end{aligned}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} u$$

$$\frac{y(s)}{X_1(s)} = 10s + 1$$

or

$$y(s) = (10s + 1) X_1(s)$$

$$y(t) = X_1 + 10 \dot{X}_1$$

$$y(t) = [1 \ 10 \ 0] X(t)$$

EXAMPLE 8.16. A system characterised by the transfer function

$$\frac{Y(s)}{u(s)} = \frac{2}{s^3 + 6s^2 + 11s + 6}$$

Find the state and output equation in matrix form and also test the controllability and observability of the system.

Solution : $(s^3 + 6s^2 + 11s + 6) Y(s) = 2u(s)$.

Taking inverse Laplace

$$\ddot{Y}(t) + 6\dot{Y}(t) + 11Y(t) + 6Y(t) = 2u(t).$$

Let

$$Y(t) = x_1$$

$$\dot{Y}(t) = \dot{x}_1 = x_2$$

$$\ddot{Y}(t) = \dot{x}_2 = x_3$$

$$\ddot{Y}(t) = \dot{x}_3 = x_4$$

 $\therefore$

$$\ddot{Y}(t) = x_4 = \dot{x}_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1$$

Rewriting

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 2u(t) - 6x_3 - 11x_2 - 6x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0 \ 0] x(t)$$

For controllability

$$Q = [b : Ab : A^2 b]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix}$$

$$A^2b = \begin{bmatrix} 2 \\ -12 \\ 50 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -12 \\ 2 & -12 & 50 \end{bmatrix}$$

$$|Q| \neq 0$$

It is full of rank, hence the system is controllable

For observability :

$$Q' = [C^T : A^T C^T : A^{T^2} C^T]$$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A^{T^2} C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of the matrix = 3, Hence the system observable.

EXAMPLE 8.17. A system is described by the following equations

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} x(t)$$

Find the transfer function of the system.

(B.E., A.M.U., Aligarh, 89-90)

(R.M.L. University, Faizabad, 2003)

Solution : The given system is MIMO system.

$\therefore$

$$G(s) = C[sI - A]^{-1} B$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} \frac{1}{s^2 + 3s + 2}$$

$$C[sI - A]^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} \frac{1}{s^2 + 3s + 2} = \begin{bmatrix} s+2 & 2s+3 \\ s+2 & 1 \\ s+2 & s+2 \end{bmatrix} \frac{1}{s^2 + 3s + 2}$$

$$C[sI - A]^{-1} B = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+2 & 2s+3 \\ s+2 & 1 \\ s+2 & s+2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{s^3 - 3s^2 - 9s + 2}{s^2 + 3s + 2}$$

$$G(s) = \frac{s^3 - 3s^2 - 9s + 2}{s^2 + 3s + 2} \text{ is the required transfer function.}$$

EXAMPLE 8.18. A system is characterised by the equation

$$\frac{y(s)}{u(s)} = \frac{20(4s+2)}{s^3 + 5s^2 + 8s + 2}$$

Find its state and output equation and express in matrix form.

(B7.E., A.M.U., Aligarh, 1986-87)

Solution :

$$\frac{y(s)}{u(s)} = \frac{X_1(s)}{u(s)} \cdot \frac{y(s)}{X_1(s)} = \frac{20}{s^3 + 5s^2 + 8s + 2} \cdot (4s + 2)$$

$$\frac{X_1(s)}{u(s)} = \frac{20}{s^3 + 5s^2 + 8s + 2}$$

$$(s^3 + 5s^2 + 8s + 2) X_1(s) = 20 u(s)$$

Taking inverse laplace

$$\ddot{x}_1 + 5\dot{x}_1 + 8x_1 + 2x_1 = 20u$$

Select

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_1 = x_3 = -5x_3 - 8x_2 - 2x_1 + 20u$$

Rewriting

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -5x_3 - 8x_2 - 2x_1 + 20u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} u$$

$$\frac{y(s)}{X_1(s)} = 4s + 2$$

$$y(s) = (4s + 2) X_1(s)$$

$$y(t) = 4\dot{x}_1 + 2x_1 = 4x_2 + 2x_1$$

$$y(t) = \begin{bmatrix} 2 & 4 & 0 \end{bmatrix} x(t)$$

EXAMPLE 8.19. A system is described by the following transfer function

$$G(s) = \frac{s+2}{s^3+3s^2+2s+10}$$

Find the values of A , b and C^T .

Solution :

$$\frac{Y(s)}{u(s)} = \frac{s+2}{s^3+3s^2+2s+10}$$

$$\frac{Y(s)}{u(s)} = \frac{X_1(s)}{u(s)} \cdot \frac{y(s)}{X_1(s)} = \frac{1}{s^3+3s^2+2s+10} (s+2)$$

$$\frac{X_1(s)}{u(s)} = \frac{1}{s^3+3s^2+2s+10}$$

or $(s^3 + 3s^2 + 2s + 10) X_1(s) = u(s)$

Taking inverse Laplace

$$\ddot{\ddot{X}}_1 + 3\ddot{\ddot{X}}_1 + 2\dot{\ddot{X}}_1 + 10X_1 = u$$

$$\ddot{\ddot{X}}_1 = u - 3\ddot{\ddot{X}}_1 - 2\dot{\ddot{X}}_1 - 10X_1$$

Select

$$\dot{\ddot{X}}_1 = x_2$$

$$\ddot{\ddot{X}}_1 = \dot{\ddot{X}}_2 = x_3$$

$$\ddot{\ddot{X}}_1 = \dot{\ddot{X}}_3$$

Rewriting

$$\dot{\ddot{X}}_1 = x_2$$

$$\dot{\ddot{X}}_2 = x_3$$

$$\dot{\ddot{X}}_3 = -3x_3 - 2x_2 - 10x_1 + u$$

$$\begin{bmatrix} \dot{\ddot{X}}_1 \\ \dot{\ddot{X}}_2 \\ \dot{\ddot{X}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$\therefore$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{y(s)}{X_1(s)} = s + 2$$

$$y(s) = (s + 2) X_1(s) = \dot{\ddot{X}}_1 + 2X_1$$

$$y(t) = x_2 + 2x_1$$

$$y(t) = [2 \ 1 \ 0]x(t) \therefore C^T = [2 \ 1 \ 0]$$

EXAMPLE 8.20. A single input single output system is given as

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \ 0 \ 2]x(t)$$

Test for controllability and observability.

(R.M.L. University Faizabad, 2001)

Solution :

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = [1 \ 0 \ 2]$$

$$AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$Q = [B : AB : A^2B] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of the matrix = 2

Hence, the system is uncontrollable

For observability

$$Q' = [C^T : A^T C^T : A^{T^2} C^T]$$

$$A^T C^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -6 \end{bmatrix}$$

$$A^{T^2} C^T = \begin{bmatrix} 1 \\ 0 \\ 18 \end{bmatrix}$$

$$Q' = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & -6 & 18 \end{bmatrix}$$

Rank of matrix = 2

EXAMPLE 8.23. Obtain the transfer function of the system.

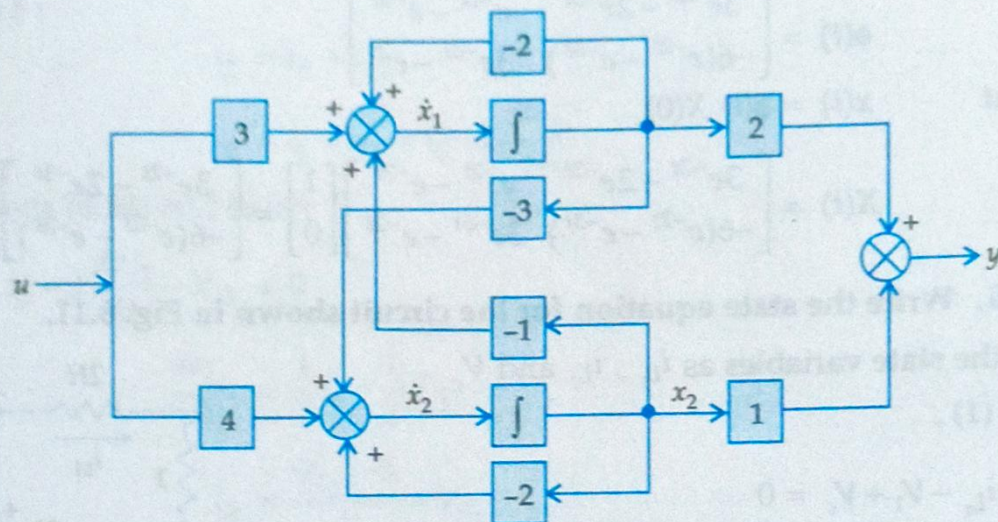


Fig. 8.10

Solution: From Fig. 8.10

$$\dot{x}_1 = -2x_1 - x_2 + 3u$$

$$\dot{x}_2 = -3x_1 - 2x_2 + 4u$$

Output equation:

$$y = 2x_1 + x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} u$$

$$y = [2 \ 1] x(t)$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 4s + 1} \begin{bmatrix} s+2 & -1 \\ -3 & s+2 \end{bmatrix}$$

$$C^T [sI - A]^{-1} b = \frac{7s + 3}{s^2 + 4s + 1}$$

Hence, required transfer function = $\frac{7s + 3}{s^2 + 4s + 1}$

EXAMPLE 8.24. Consider the following matrix

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the state transition matrix, also determine $x(t)$

Solution :

Given $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$\phi(s) = [sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\phi(s) = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6(e^{-2t} - e^{-3t}) & 3e^{-3t} - e^{-2t} \end{bmatrix}$$

We know that

$$x(t) = \phi(t) X(0)$$

$$X(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6(e^{-2t} - e^{-3t}) & 3e^{-3t} - e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} \\ -6(e^{-2t} - e^{-3t}) \end{bmatrix}$$

Ans.

EXAMPLE 8.25. Write the state equation for the circuit shown in Fig. 8.11