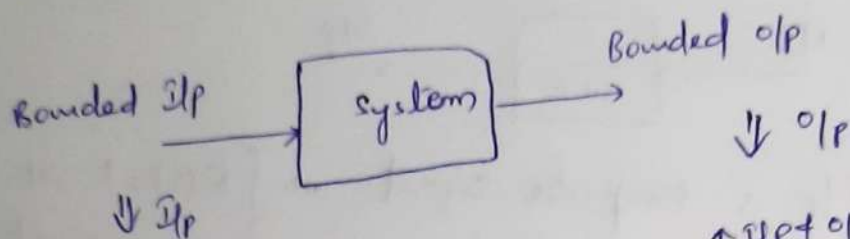


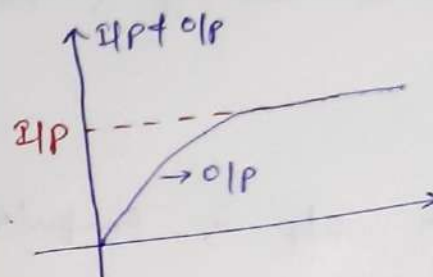
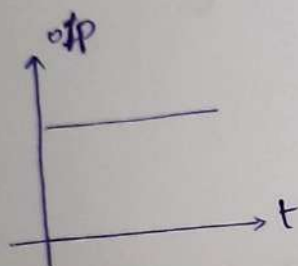
IV Stability analysis of control systems

(119)

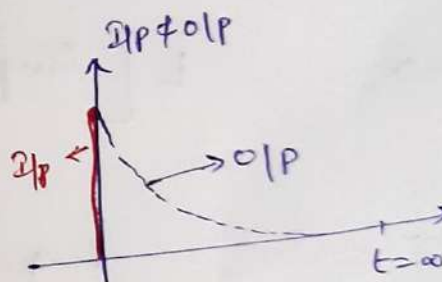
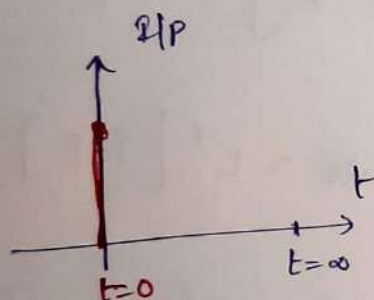
⇒ Stability of the system is defined as "when the system is subjected to bounded I/P, the output should be bounded".



Ex ①



Ex ②

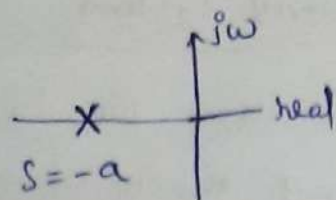


⇒ BIBO [Bounded Input Bounded output] stability means generally the impulse response of the system should tend to zero as the time approaches to infinity.

⇒ Stability analysis using location of the poles in s-plane.

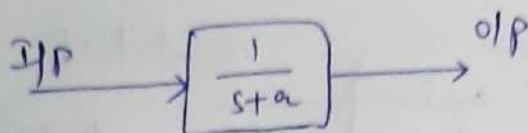
⇒ In this criterion, the stability of the system depends on roots of characteristic equation $[1 + G(s)H(s) = 0]$.
⇒ i.e. it depends on the poles location [closed loop poles].

Ex



A diagram of the complex s-plane. The horizontal axis is labeled 'real' and the vertical axis is labeled 'jw'. A pole, represented by an 'x', is located on the negative real axis at a point labeled 's = -a'.

$$\Rightarrow T.F = T(s) = \frac{1}{s+a}$$



$\Rightarrow$ let the I/P is impulse signal. $\Rightarrow \begin{cases} R(s) = 1 & \text{at } t=0 \\ = 0 & \text{at } t \neq 0 \end{cases}$

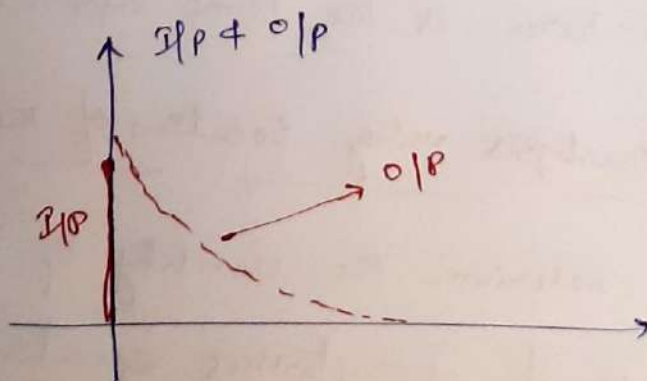
$\Rightarrow$ O/P is impulse response.

$$\Rightarrow L[\text{impulse response}] = T.F$$

$$\text{impulse response} = L^{-1}[T.F]$$

$$\text{impulse response} = L^{-1}\left[\frac{1}{s+a}\right]$$

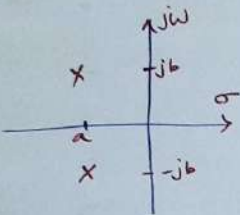
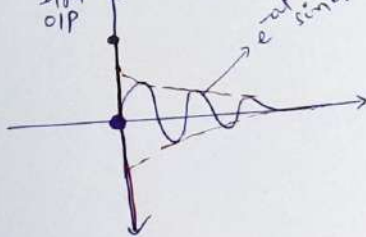
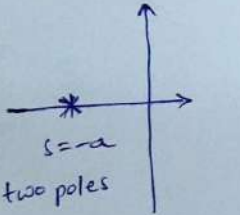
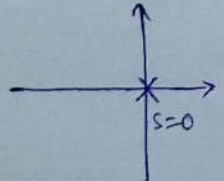
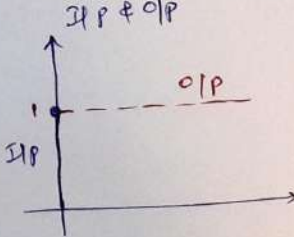
$$\text{O/P} = \text{impulse response} = e^{-at}$$

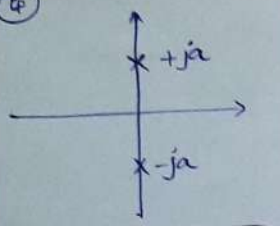
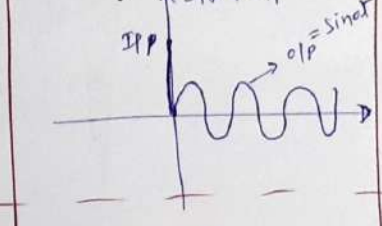
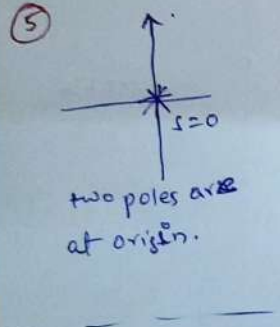
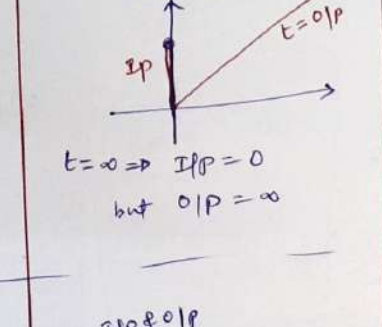
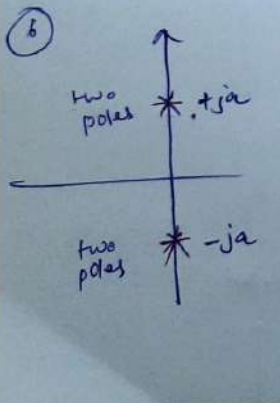
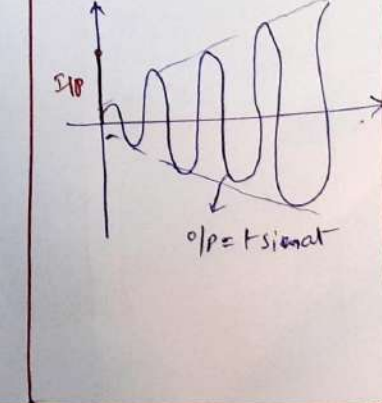


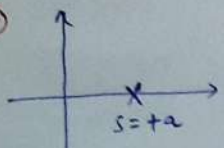
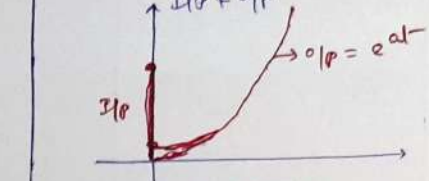
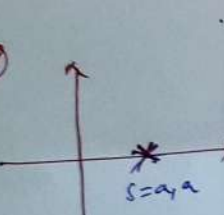
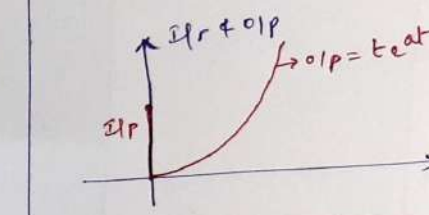
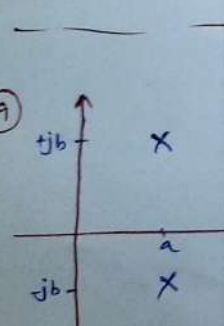
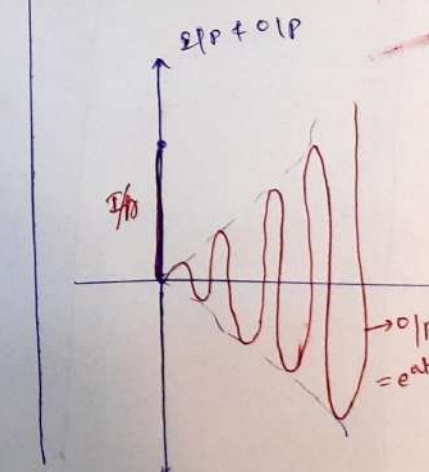
$$\text{at } t \rightarrow \infty \Rightarrow I/P = 0 \neq O/P = 0$$

$\Rightarrow$ i.e. for Bounded I/P the system exhibiting bounded output $\Rightarrow$ Stable system

If the pole lies in the left half of the s -plane
then the system is stable.

122 Location of closed loop poles	Transfer function	Impulse response expression	Imp and o/p	Comment on the stability
① 	$T(s) = \frac{1}{(s+a)^2 + b^2}$	$\begin{aligned} \text{o/p} &= \mathcal{L}^{-1}[T(s)] \\ &= e^{-at} \sin bt \end{aligned}$		stable system (or) absolute stable system
② 	$T(s) = \frac{1}{(s+a)^2}$	$\begin{aligned} \text{o/p} &= \mathcal{L}^{-1}[T(s)] \\ &= t e^{-at} \end{aligned}$		Absolute stable system
③ 	$T(s) = \frac{1}{s}$	$\begin{aligned} \text{o/p} &= \mathcal{L}^{-1}[T(s)] \\ &= 1 \end{aligned}$		marginally stable (or) critically stable.

Location of closed loop poles	Transfer function	Impulse response expression	I/p & o/p	Comment on stability (123)
4 	$T(s) = \frac{1}{s^2 + a^2}$	$\begin{aligned} o/p &= \mathcal{L}^{-1}[T(s)] \\ &= \sin at \end{aligned}$		Marginally stable system
5  two poles are at origin.	$T(s) = \frac{1}{s^2}$	$\begin{aligned} o/p &= \mathcal{L}^{-1}[T.F.] \\ &= t \end{aligned}$	 $t = \infty \Rightarrow I/p = 0$ but $o/p = \infty$	Unstable
6  two poles $+ja$ two poles $-ja$	$T(s) = \frac{1}{(s^2 + a^2)^2}$	$\begin{aligned} o/p &= \mathcal{L}^{-1}[T.F.] \\ &= t \sin at \end{aligned}$	 $o/p = t \sin at$	Unstable.

Location of closed loop poles	Transfer function	Impulse response Expression	I/p & o/p	Comment on Stability (124)
⑦ 	$T(s) = \frac{1}{s-a}$	$o/p = \mathcal{L}^{-1}[T(s)] = e^{at}$	 at $t=0 \Rightarrow I/p=0$ o/p $= \infty$	Unstable
⑧ 	$T(s) = \frac{1}{(s-a)^2}$	$o/p = \mathcal{L}^{-1}[T(s)] = te^{at}$		Unstable
⑨ 	$T(s) = \frac{1}{(s-a)^2 + b^2}$	$o/p = \mathcal{L}^{-1}[T(s)] = e^{at} \sin bt$		Stable.

Stability using Routh - Hurwitz Criterion

⇒ Routh - Hurwitz Criterion is also known as R-H criterion.

⇒ To comment on the stability of the system using R-H criterion, we need to form Routh array.

Ex let closed loop transfer function = $\frac{4(s)}{1 + 4(s)H(s)}$ = $\frac{1}{s^4 + 8s^3 + 18s^2 + 16s + 5}$

characteristic equation (CE) = $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Routh array

s^4	1	18	5
s^3	8	16	0
s^2	b_1	b_2	b_3
s^1	c_1	c_2	c_3
s^0	d_1	d_2	d_3

↓ 1st column

s^4	1	18	5
s^3	8	16	0
s^2	16	5	0
s^1	13.5	0	0
s^0	5	0	0

$$b_1 = \frac{8 \times 18 - 16 \times 1}{8} = 16$$

$$b_2 = \frac{8 \times 5 - 0 \times 1}{8} = 5$$

$$b_3 = 0$$

$$c_1 = \frac{b_1 \times 16 - b_2 \times 8}{b_1} = 13.5$$

$$c_2 = \frac{b_1 \times 0 - b_3 \times 8}{b_1} = 0$$

$$c_3 = 0$$

$$d_1 = \frac{c_1 \times b_2 - c_2 \times b_1}{c_1} = \frac{13.5 \times 5 - 0 \times 16}{13.5} = 5$$

$$d_2 = 0$$

$$d_3 = 0$$

Note If there are no sign changes in the 1st column of Routh array then the system is stable. (126)

Ex: A closed loop system has the following characteristics equation. Comment on the stability of the system using R-H criterion.

$$CE = s^4 + 3s^3 + 3s^2 + 2s + 1 = 0.$$

Sol

Routh array

s^4	1	3	1
s^3	3	2	0
s^2	$\frac{3 \times 3 - 2 \times 1}{3}$ $= 7/3$	$\frac{3 \times 1 - 0 \times 1}{3}$ $= 1$	0
s^1	$\frac{7/3 \times 2 - 1 \times 3}{7/3}$ $= 5/7$	$\frac{7/3 \times 0 - 0 \times 3}{7/3}$ $= 0$	0
s^0	$\frac{5/7 \times 1 - 0 \times 7/3}{5/7}$ $= 1$	0	0

s^3	1	3	1
s^2	3	2	0
s^1	$7/3$	1	
s^0	$5/7$	0	
s^0	1		

↑ 1st Column of Routh array.

$\Rightarrow$ NO sign changes $\Rightarrow$ stable.

Q Comment on the stability of the system using R-H criterion
whose CE = $s^3 + s^2 + 2s + 24 = 0$.

Sol

Routh array

s^3	1	2	
s^2	1	24	
s^1	$\frac{1 \times 2 - 1 \times 24}{1}$ $= -22$	0	
s^0	$\frac{(-22 \times 24) - (0 \times 1)}{-22}$ $= 24$	0	

Routh array

s^3	1	2	
s^2	1	24	
s^1	-22	0	
s^0	24	0	

$\Rightarrow$ 1st sign change $\rightarrow$ s^1 row
2nd sign change $\rightarrow$ s^0 row

Note

No. of sign changes in the 1st column of

(128)

Routh array = No. of poles in the Right half of the s-plane.

ie in the above question

No. of RHS poles = 2

∴ The system is unstable.

Q The characteristic equation of the system is given below.

Determine the stability of the system?

$$4s^4 + 8s^3 + 2s^2 + 10s + 3 = 0$$

Sol

Routh array

s^4	4	2	3
s^3	8	10	0
s^2	$\frac{8 \times 2 - 10 \times 4}{8} = -3$	$\frac{8 \times 3 - 0 \times 4}{8} = 3$	0
s^1	$\frac{-3 \times 10 - 3 \times 8}{-3} = 18$	0	
s^0	$\frac{18 \times 3 - 0 \times (-3)}{18} = 3$		

$\Rightarrow$

s^4	4	2	3
s^3	8	10	0
s^2	-3	3	
s^1	18		
s^0	3		

no. of sign changes = 2

no. of poles in RHS = 2

unstable system.

Difficulty - 1 when the first element of any row is zero, while the rest of the row has at least one non-zero element then in such cases it will be difficult to form complete Routh array. In this case substitute a small +ve number " ϵ " in place of ~~zero~~ zero and evaluate the rest of Routh array in terms of " ϵ ". Check for sign changes in the 1st column by taking limit " ϵ " approaches to zero (i.e. $\epsilon \rightarrow 0$).

Q Comment on the stability of the system using R-H criterion whose C.E is

(130)

$$CE = s^5 + 1.5s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$$

Sol

Routh array $\Rightarrow$ let us observe the difficulty

s^5	1	2	3
s^4	1	2	15
s^3	0	-12	0
s^2	$\frac{0 \times 2 - 1 \times (-12)}{0} = \infty$	$\frac{0 \times 15 - 0 \times 1}{0} = \frac{0}{0}$	
s^1	$\downarrow$ further Routh array can't be evaluated.		
s^0			

$\Rightarrow$ in 3rd row $0 \rightarrow \epsilon$

Routh array

s^5	1	2	3
s^4	1	2	15
s^3	ϵ	-12	0
s^2	$\frac{2\epsilon - 1 \times (-12)}{\epsilon} = \frac{12 + 2\epsilon}{\epsilon}$	$\frac{15\epsilon - 0 \times 1}{\epsilon} = 15$	
s^1	$\frac{-24\epsilon - 144 - 15\epsilon}{2\epsilon + 12}$	0	
s^0	15		

By substituting $s \rightarrow 0$

(131)

$$\begin{array}{c|cccc}
 s^5 & 1 & 2 & 3 & \\
 s^4 & 1 & 2 & 15 & \\
 s^3 & \text{let } E=0 & -12 & 0 & = 0 \\
 s^2 & \text{let } \frac{2E+12}{E} = 0 & 15 & & \\
 s^1 & \text{let } \frac{-24E-144-15E}{2E+12} & & & \\
 s^0 & 15 & & &
 \end{array}$$

$E \rightarrow 0$

$= -12$

$$\begin{array}{c|cccc}
 s^5 & 1 & 2 & 3 & \\
 s^4 & 1 & 2 & 15 & \\
 s^3 & 0 & -12 & 0 & \\
 s^2 & +\infty & 15 & & \\
 s^1 & -12 & & & \\
 s^0 & 15 & & &
 \end{array}$$

1st sign change $\rightarrow$

2nd sign change $\rightarrow$

$\Rightarrow$ No. of sign changes = 2

$\Rightarrow$ No. of R.H.S poles = 2

$\Rightarrow$ Unstable.

Q $CE = s^5 + 2s^4 + 2s^3 + 4s^2 + 15s + 10 = 0$. Comment on the stability of the system.

Sol

Routh array

$$\begin{array}{c|cccc}
 s^5 & 1 & 2 & 11 & \\
 s^4 & 2 & 4 & 10 & \\
 s^3 & 0 \rightarrow E & 6 & & \\
 s^2 & \frac{4E-12}{E} & 10 & & \\
 s^1 & 6 & & & \\
 s^0 & 10 & & &
 \end{array}$$

$$\begin{array}{c|cccc}
 s^5 & 1 & 2 & 11 & \\
 s^4 & 2 & 4 & 10 & \\
 s^3 & 0 & & & \\
 s^2 & -12 & & & \\
 s^1 & 6 & & & \\
 s^0 & 10 & & &
 \end{array}$$

let $E \rightarrow 0$

$\Rightarrow$

1st sign change $\rightarrow$

2nd sign change $\rightarrow$

$\Rightarrow$ unstable.

Difficulty - 2

(132)

⇒ When a complete row of Routh array become zero then in such cases it will be difficult to evaluate complete Routh array.

⇒ In the above difficulty cases, construct an Auxiliary equation (AE) and differentiate it to get new co-efficients to evaluate the rest of Routh array.

⇒ Check the roots of auxiliary equation to comment on the stability of the system.

⇒ Q. CE = $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$.
Comment on the stability of the system.

Sol

s^6	1	8	20	16
s^5	2	12	16	0
s^4	$\frac{2 \times 8 - 1 \times 12}{2} = 2$	$\frac{2 \times 20 - 16 \times 1}{12} = 12$	$\frac{2 \times 16 - 0 \times 1}{2} = 16$	0
s^3	$\frac{2 \times 12 - 2 \times 2}{2} = 10$	$\frac{2 \times 16 - 16 \times 2}{2} = 0$	$\frac{2 \times 0 - 0 \times 2}{2} = 0$	0
s^2				
s				
s^0				

⇒ complete row has zero elements

↓
difficult to evaluate since the on defined elements.

⇒ Construct Auxiliary equation (AE) for s^4 row.

133

$$\Rightarrow \textcircled{1} \text{ AE} = 2s^4 + 12s^2 + 16$$

$$\Rightarrow \textcircled{2} \frac{d \text{AE}}{ds} = 8s^3 + 24s + 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	0
s^1	2 + 6	0	0	0
s^0	16	0	0	0

To comment on the stability, roots of ~~AE~~ AE needed to be known.

$$2s^4 + 12s^2 + 16 = 0$$

$$\text{let } s^2 = x$$

$$2x^2 + 12x + 16 = 0$$

$$\Rightarrow x = -2, -4$$

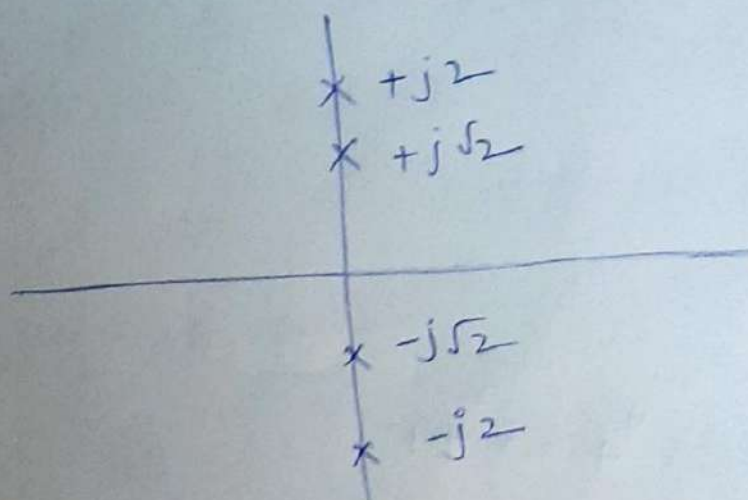
$$s^2 = x$$

$$s = \sqrt{-2},$$

$$s = \sqrt{-4}$$

$$s = \pm j\sqrt{2}$$

$$s = \pm j2$$



$\Rightarrow$ No. of poles on $j\omega$ axis = 4

$\Rightarrow$ No. sign changes in the Routh array
therefore no. of poles in LHS = 2.

$\Rightarrow$ marginally stable.

Q CE = $2s^5 + s^4 + 4s^3 + 2s^2 + 2s + 1 = 0$.
comment on the stability.

s^5	2	4	2	
s^4	1	2	1	
s^3	$\frac{1 \times 4 - 2 \times 2}{1} = 0$	$\frac{1 \times 2 - 1 \times 2}{1} = 0$	0	$\Rightarrow$ complete row is zero.
s^2				
s				
s^0				

$$\Rightarrow AE = s^4 + 2s^2 + 1$$

$$\frac{dAE}{ds} = 4s^3 + 4s$$

s^5	2	4	2	
s^4	1	2	1	
s^3	4	4	0	
s^2	1	1	0	
s^1	$\frac{1 \times 4 - 1 \times 4}{1} = 0$	$\frac{1 \times 0 - 0 \times 4}{1} = 0$	0	$\Rightarrow$ complete row is zero.
s^0				

$$AE = s^2 + 1$$

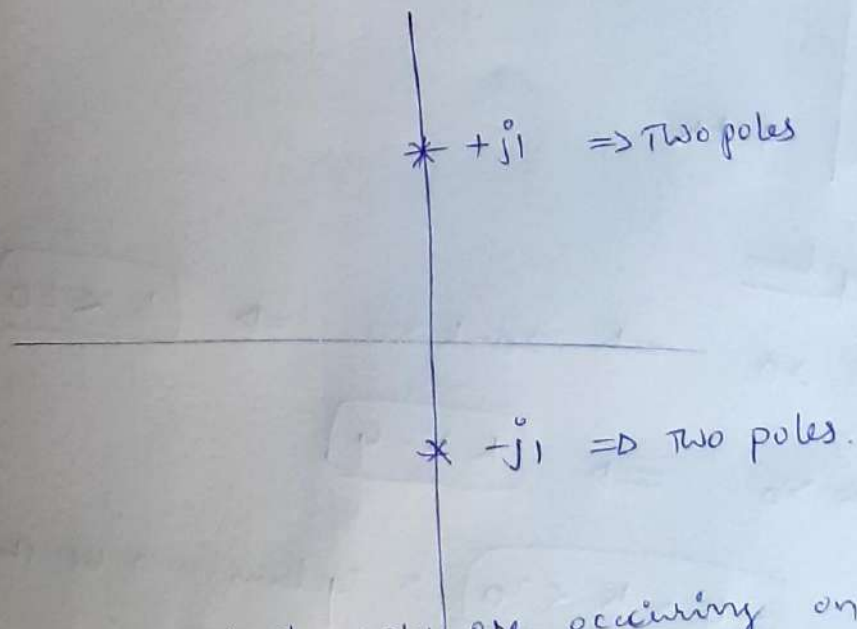
$$\frac{dAE}{ds} = 2s$$

s^5	2	4	2
s^4	1	2	1
s^3	4	1	0
s^2	1	1	0
s^1	2	0	0
s^0	1	0	0

Roots of characteristic equation

$$AE \Rightarrow s^4 + 2s^2 + 1 = 0$$

$$s_{1,2} = \pm j1, \quad s_{3,4} = \pm j1$$



$\Rightarrow$ Repeated pair of poles are occurring on $j\omega$ axis.

$\Rightarrow$ System is marginally stable.

Q. CE = $s^4 + 2s^3 + 3s^2 + 2s + K = 0$, comment on the stability of the system.

137

Sol

s^4	1	3	K	
s^3	2	2	0	
s^2	$\frac{2 \times 3 - 1 \times 2}{2}$ $= 2$	$\frac{2K - 0 \times 1}{2}$ $= K$	0	$\Rightarrow$
s	$\frac{4 - 2K}{2}$	0	0	
s^0	K	0	0	

s^4	1	3	K	
s^3	2	2	0	
s^2	2	K	0	
s	$\frac{2-K}{2}$	0	0	
s^0	K	0	0	

① For stability

① $\frac{2-K}{2} > 0 \Rightarrow 2-K > 0 \Rightarrow 2 > K \Rightarrow K < 2$

② $K > 0 \Rightarrow$

$\Rightarrow 0 < K < 2$

for stable system.

② Marginally Stable

$\frac{2-K}{2} = 0 \Rightarrow K = 2$

$\therefore$ for $K = 2$, s^1 row is completely zero.

$$AE \Rightarrow 2s^2 + 2 = 0$$

$$s^2 + 1 = 0$$

$$s = \pm j1 \Rightarrow \text{system is marginally stable.}$$

$$s = \pm j\omega$$

$$\omega = \omega_{\text{nat}} = 1 \text{ rad/sec.}$$

Unstable system

$\Rightarrow$ Other than $0 < K < 2$, $K = 2$ system will be unstable.

conditionally stable systems

A system is said to be conditionally stable if its stability depends on one (or) more parameters.

Q

A closed loop system with unity feedback has the following open loop transfer function $G(s)H(s) =$

139

$$\frac{K}{(s^2+2s+2)(s+2)}$$

comment on the stability of system.

Sol

$$C \cdot E = 1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K}{(s^2+2s+2)(s+2)} = 0$$

$$\Rightarrow s^3 + 4s^2 + 6s + (4+K) = 0$$

s^3	1	6
s^2	4	$4+K$
s	$\frac{20-K}{4}$	0
s^0	$4+K$	

For stable system

$$(i) \quad \frac{20-K}{4} > 0 \Rightarrow 20-K > 0 \Rightarrow \boxed{K < 20}$$

$$(ii) \quad 4+K > 0 \Rightarrow \boxed{K > -4}$$

$$\Rightarrow \boxed{-4 < K < 20} \Rightarrow \text{System will be stable.}$$

For marginally stable system

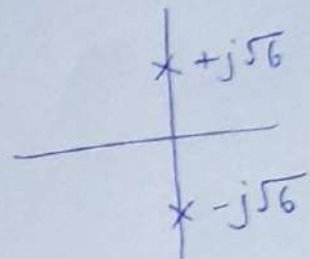
$\Rightarrow$ when $K=20$, s^1 zero completely zero.

$$AE \Rightarrow 4s^2 + (4+K) = 0$$

$$4s^2 + (4+20) = 0$$

$$s^2 + 6 = 0$$

$$s = \pm j\sqrt{6}$$



$\Rightarrow$ system is marginally stable.

for unstable system

Other than

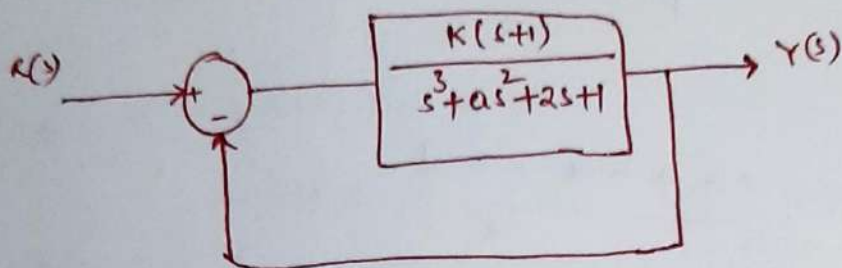
unstable.

$-4 < K < 20$, $K = 20$, System will be

Q

The feedback system shown below oscillates at 2 rad/sec for the values of K and a ?

(140)



sol

$$H(s) = 1$$

$$G(s) = \frac{K(s+1)}{s^3 + as^2 + 2s + 1}$$

$$\Rightarrow 1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K(s+1)}{s^3 + as^2 + 2s + 1} = 0$$

$$\Rightarrow s^3 + as^2 + 2s + 1 + Ks + K = 0$$

$$\Rightarrow s^3 + as^2 + (K+2)s + (K+1) = 0$$

s^3		1	$K+2$
s^2		a	$K+1$
s		$\frac{2a+Ka-K-1}{a}$	0
s^0		$K+1$	

when $\frac{2a + ka - k - 1}{a} = 0$, s' is completely zero. Then the system will oscillates. (141)

$$\Rightarrow \frac{2a + ka - k - 1}{a} = 0$$

$$\Rightarrow \boxed{a(k+2) = k+1} \rightarrow (1)$$

$\Rightarrow$ Auxiliary equation (AE) $\Rightarrow as^2 + k+1 = 0$

$$as^2 = -(k+1)$$

$$s^2 = \frac{-(k+1)}{a}$$

$$\boxed{s = \pm j \sqrt{\frac{k+1}{a}}} \rightarrow (2)$$

but for eqⁿ $\Rightarrow s = \pm j 2$

$$\therefore \sqrt{\frac{k+1}{a}} = 2$$

$$k+1 = 4a$$

$$\boxed{k = 4a - 1} \rightarrow (3)$$

from (1) & (3)

$$a(4a-1+2) = (4a-1)+1$$

$$4a^2 + a = 4a$$

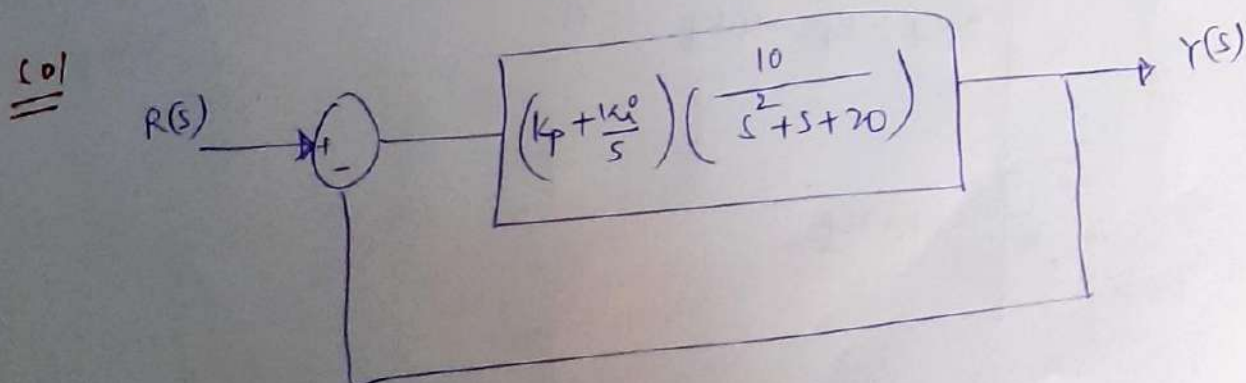
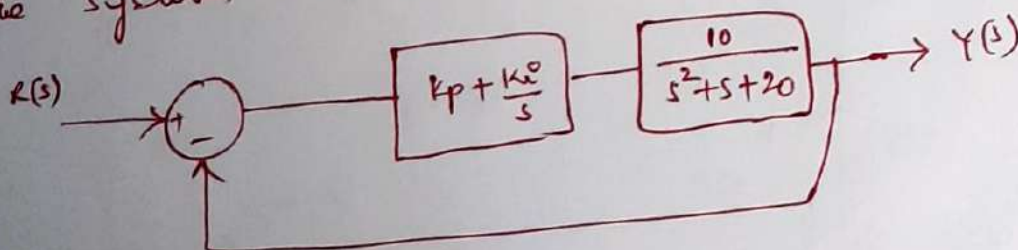
$$\Rightarrow \cancel{4}(4a+1) = 4\cancel{a}$$

$$\Rightarrow \boxed{a = 3/4}$$

From (3) $K = 4 \left(\frac{3}{4} \right) - 1$

$$\boxed{K = 2}$$

Q For the given systems find the range of K_p and K_I , the systems to be stable.



$$H(s) = 1$$

$$G(s) = \left(K_p + \frac{K_I}{s} \right) \left(\frac{10}{s^2 + s + 20} \right)$$

$$CE \Rightarrow (1 + K(s))H(s) = 0$$

$$\Rightarrow 1 + \left(K_p + \frac{K_I}{s} \right) \left(\frac{10}{s^2 + s + 20} \right) = 0$$

$$\Rightarrow 1 + \frac{K_p s + K_I}{s} \cdot \frac{10}{s^2 + s + 20} = 0$$

$$\Rightarrow 1 + \frac{10(K_p s + K_I)}{s(s^2 + s + 20)} = 0$$

$$\Rightarrow s^3 + s^2 + (20 + 10K_p)s + 10K_I = 0$$

s^3	1	$20 + 10K_p$
s^2	1	$10K_I$
s	$20 + 10K_p - 10K_I$	0
s^0	$10K_I$	

The system to be stable

$$\textcircled{1} \quad 20 + 10K_p - 10K_I > 0$$

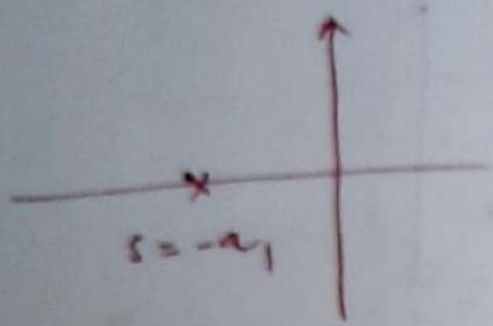
$$2 + K_p - K_I > 0 \Rightarrow$$

$$\boxed{K_p > 2 - K_I}$$

② $10k_2 > 0$

$k_2 > 0$

Relative stability analysis



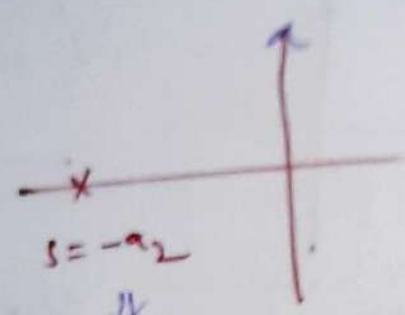
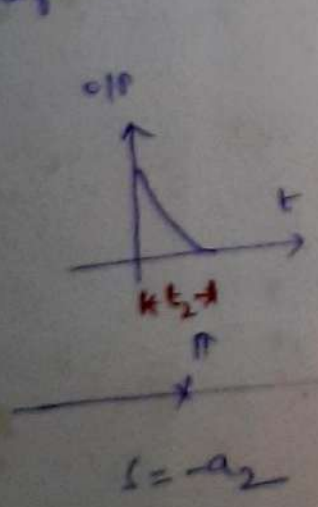
↓
system-1
↓

$T(s) = \frac{1}{s+a_1}$

$\Rightarrow$ let $a_2 > a_1$

↓

Impulse response $\Rightarrow$ o/p = $e^{-a_1 t}$

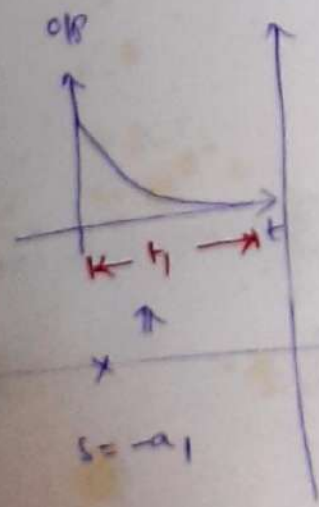


↓
system-2
↓

$T(s) = \frac{1}{s+a_2}$

↓

o/p = $e^{-a_2 t}$



⇒ From the above analysis,

Both system-1 and system-2 are absolutely stable.

⇒ System-2 is said to be relatively more stable than system-1 since $t_2 \ll t_1$.

⇒ System-2 can recover quickly from the impulse inputs because its response time is less when compare to system-1.

⇒ ∴ System-2 is more stable than system-1.

⇒ Root Locus

⇒ Root Locus is designed as locus of closed loop poles when the system Gain (K) varies from 0 to ∞ .

⇒ Root Locus determines the relative stability of the systems.

Angle and magnitude condition

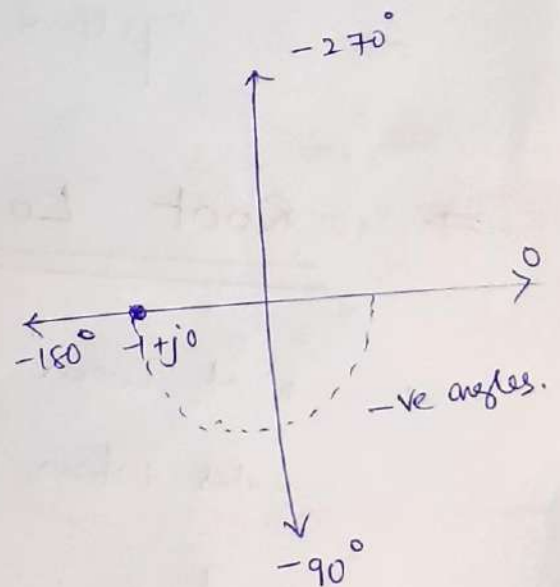
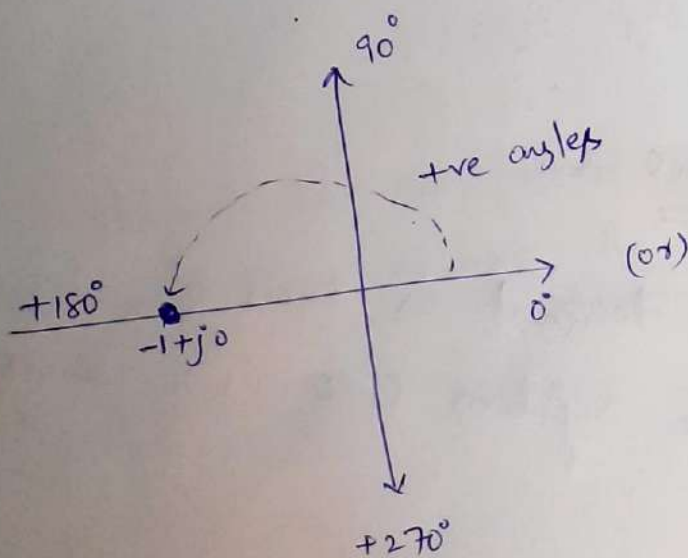
$\Rightarrow$ The angle condition is used for checking whether any point lies on root locus or not.

$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1$$

$$\Rightarrow G(s)H(s) = -1 + j0$$

$$\angle G(s)H(s) = 180^\circ - \tan^{-1}(0/1) = 180^\circ$$



$$\therefore \angle G(s)H(s) = \pm 180^\circ$$

(or)

$$\angle G(s)H(s) = \pm (2q + 1) 180^\circ$$

Ex

$$y = 2x + 5$$

$\Rightarrow$ let $x = 0$

$$y = 2 \times 0 + 5$$

$$y = 5$$

$(x, y) = (0, 5)$
will be on $y = 2x + 5$

⇒ For a point, the angle condition may be stated as it lies on the root locus if the angle made by it is $\pm 180^\circ$ or odd multiples of $\pm 180^\circ$.

⇒ Magnitude condition is used for finding the value of system gain 'K' at any point on the root locus.

$$1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1 + j0$$

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

$$|G(s)H(s)| = 1$$

Q check whether the following points are on the root locus or not. The system is $G(s)H(s) = \frac{K}{(s+1)^4}$.

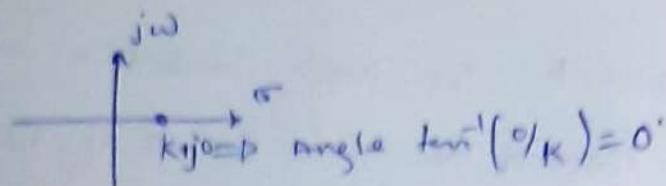
sol $G(s)H(s) = \frac{K}{(s+1)^4}$

$$s_1 = -3 + j4, \\ s_2 = -3 - j2$$

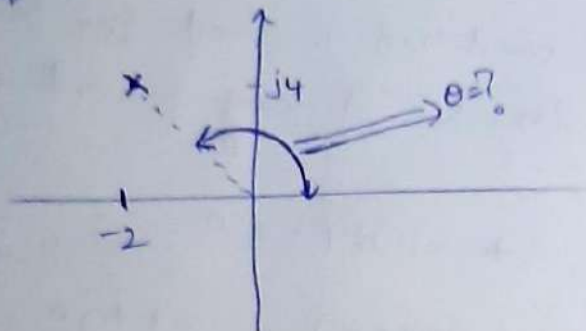
⇒ ① $\left| \frac{G(s)H(s)}{s_1 = -3 + j4} \right| = ?$

$$\left| \frac{G(s)H(s)}{s_1 = -3 + j4} \right| = \frac{K + j0}{[-2 + j4]^4}$$

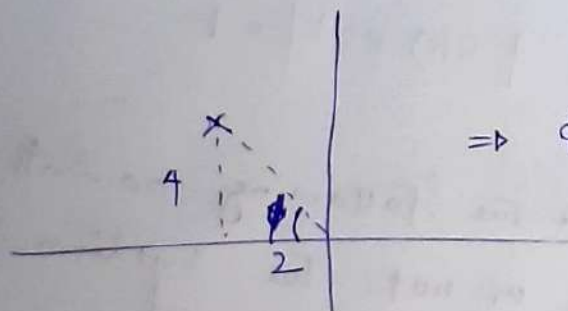
$$\Rightarrow K + j0 = 0$$



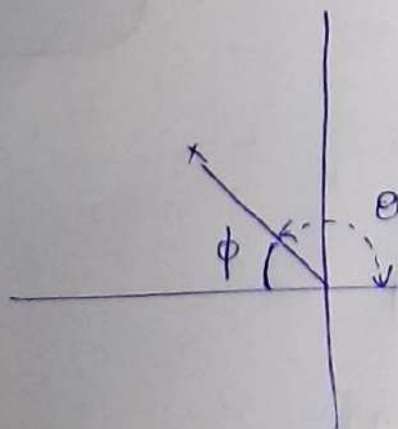
$$\Rightarrow -2 + j4 \Rightarrow$$



⇓



$$\Rightarrow \phi = \tan^{-1}(4/2) = 63.434$$



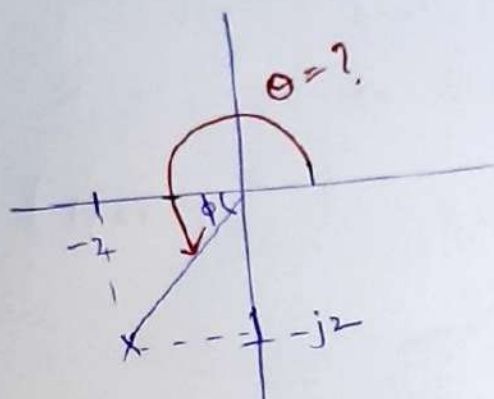
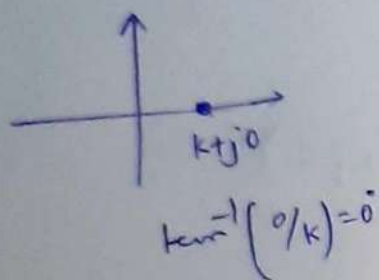
$$\theta = 180 - \phi = 180 - 63.434 = 116.5^\circ$$

$$\Rightarrow \left[\frac{K + j0}{-2 + j4} \right]^4 = \frac{0^\circ}{[116.5^\circ]^4} = \frac{0^\circ}{[116.5^\circ]^4} = \frac{0^\circ}{466^\circ}$$

$$\left| \frac{G(s)H(s)}{s_1 = -3+j4} \right| = -466^\circ$$

Therefore s_1 will not be on root locus.

$$\begin{aligned} \textcircled{2} \quad \left| \frac{G(s)H(s)}{s_2 = -3-j2} \right| &= \frac{K+j0}{[-3-j2+1]^4} \\ &= \frac{K+j0}{[-2-j2]^4} \end{aligned}$$



$$\Rightarrow \theta = 180 + \phi$$

$$\theta = 180 + \tan^{-1}\left(\frac{2}{2}\right)$$

$$\theta = 180 + 45^\circ$$

$$\theta = 225^\circ$$

$$\left| \frac{G(s)H(s)}{s_2 = -3-j2} \right| = \frac{0^\circ}{[225^\circ]^4} = \frac{0^\circ}{225 \times 4^\circ} = \frac{0^\circ}{900}$$

$$= -900^\circ$$

$$= -180^\circ \times 5 \quad \text{odd multiple.}$$

the point $s_2 = -3-j2$ will be on root locus.

(150)

⇒ Q at the point $s = -1+j1$, find the system gain 'K' for the system $G(s)H(s) = \frac{K}{s[s^2+7s+12]}$

sol

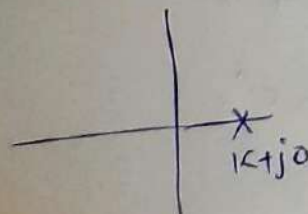
$$G(s)H(s) = \frac{K}{s[s^2+7s+12]}$$

at $s = -1+j1$

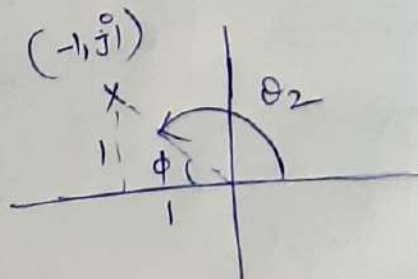
$$G(s)H(s) = \frac{K}{(-1+j1) \left[(-1+j1)^2 + 7(-1+j1) + 12 \right]}$$

$$= \frac{K}{(-1+j1)(5+j5)}$$

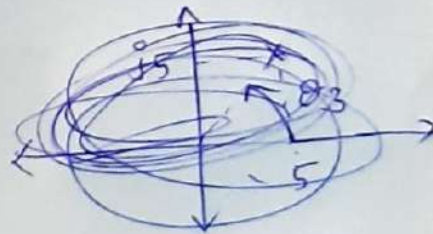
$$\left[G(s)H(s) \right]_{s=-1+j1} = \frac{K+j0}{(-1+j1)(5+j5)}$$

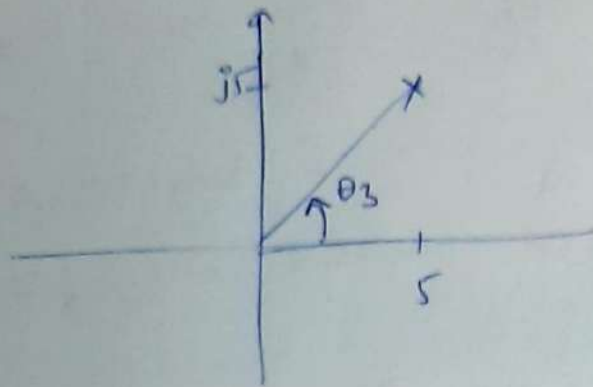


$$\theta_1 = \tan^{-1}\left(\frac{0}{K}\right) = 0^\circ$$



$$\begin{aligned} \theta_2 &= 180^\circ - \phi \\ &= 180^\circ - \tan^{-1}\left(\frac{1}{1}\right) \\ &= 135^\circ \end{aligned}$$





$$\theta_3 = \tan^{-1}(5/5) = 45^\circ$$

$$\therefore \left| G(s)H(s) \right|_{s=5+j5} = \frac{0^\circ}{(135^\circ)(45^\circ)}$$

$$= \frac{0^\circ}{180^\circ}$$

$$= -180^\circ$$

$\therefore$ The point $s = -1 + j1$ is on the root locus.

$$\Rightarrow \left| G(s)H(s) \right|_{s=-1+j1} = 1$$

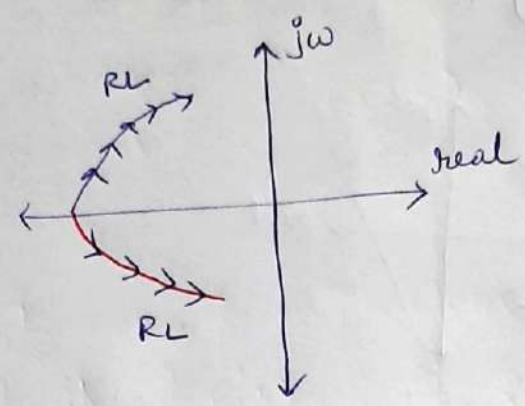
$$\left| \frac{K+j0}{(-1+j1)(5+j5)} \right| = 1$$

$$\begin{aligned} x+jy &= z \\ |z| &= \sqrt{x^2+y^2} \\ \angle z &= \tan^{-1}(y/x) \end{aligned}$$

$$\frac{\sqrt{K^2+0^2}}{\sqrt{(-1)^2+1^2} \cdot \sqrt{5^2+5^2}} = 1 \Rightarrow \boxed{K=10}$$

Construction rules of Root Locus [RL]

Rule-1 : The Root Locus [RL] is symmetrical about real axis.



Rule-2 let No. of poles = P
No. of zeros = Z

If $P > Z$, No. of RL branches = P
 $\Rightarrow$ No. of RL branches terminating at zero = Z
 $\Rightarrow$ No. of RL branches terminating at infinity = $P - Z$

Rule-3 A point on real-axis is said to be on root locus if to the right side of that point the sum of open loop poles and zeros is odd number.

Ex $G(s)H(s) = \frac{K(s+2)(s+6)}{s(s+4)(s+8)}$

Rule-2

$$p = 3, \quad z = 2$$

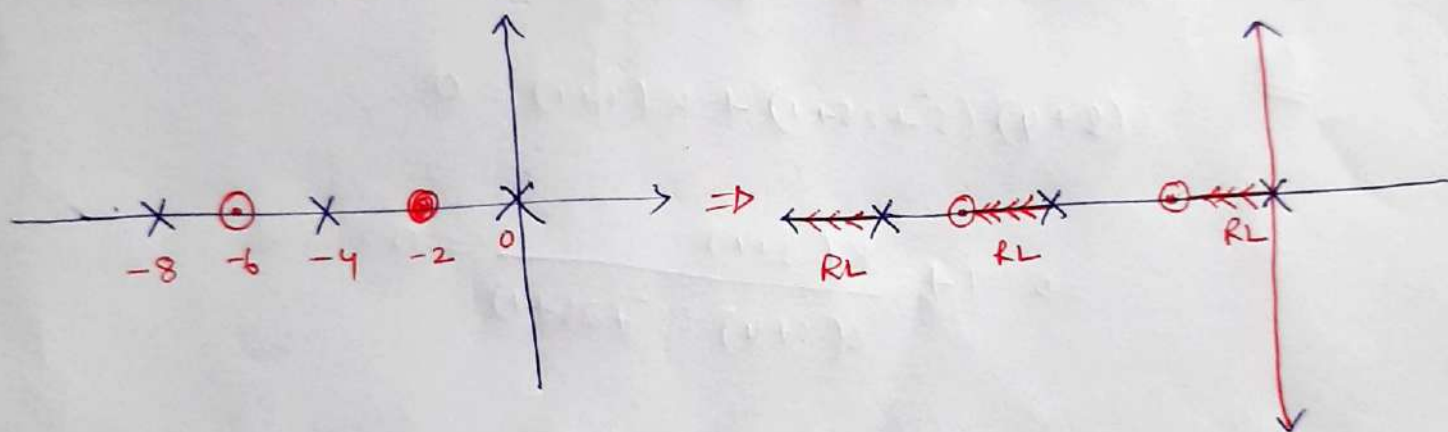
$$\Rightarrow \text{No. of RL branches} = p = 3$$

$$\Rightarrow \text{No. of RL branches ending at zero} = z = 2$$

$$\Rightarrow \text{No. of RL branches terminating at infinity} = p - z = 3 - 2 = 1$$

Rule-3

$$\text{poles} \Rightarrow s = 0, -4, -8, \quad \text{zeros} \Rightarrow -2, -6$$

Rule-4 Angle of asymptotes

$\Rightarrow$ If $p > z$, then $p - z$ RL branches terminate at infinity. They terminate at infinity along a certain straight lines known as Asymptotes.

$$\therefore \text{No. of Asymptotes} = p - z$$

$$\text{Angle of Asymptotes} \Rightarrow \theta = \frac{[2q+1]}{(p-z)} \times 180^\circ$$

where $q = 0, 1, 2, \dots$

Ex

let $p = 4, z = 2$

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$p > z$, No. of asymptotes $= p - z = \underline{2}$
 $\Rightarrow \sigma = \underline{0, 1}$

$$\theta_1 = \frac{2 \times 0 + 1}{2} \times 180^\circ = 90^\circ$$

$$\theta_2 = \frac{2 \times 1 + 1}{2} \times 180^\circ = 270^\circ$$

Q A system has the CE $s(s+4)(s^2+2s+1)+K(s+1)=0$.
Find angle of asymptotes?

Sol

$$s(s+4)(s^2+2s+1)+K(s+1)=0$$

$$\Rightarrow 1 + \frac{K(s+1)}{s(s+4)(s^2+2s+1)} = 0$$

$$\Rightarrow 1 + G(s)H(s) = 0$$

$$\therefore G(s)H(s) = \frac{K(s+1)}{s(s+4)(s^2+2s+1)}$$

$p = 4, z = 1$, No. of Asymptotes $\Rightarrow p - z = 4 - 1 = 3$.

$$\sigma = 0, 1, 2$$

$$\theta_1 = \frac{2 \times 0 + 1}{3} \times 180^\circ = 60^\circ$$

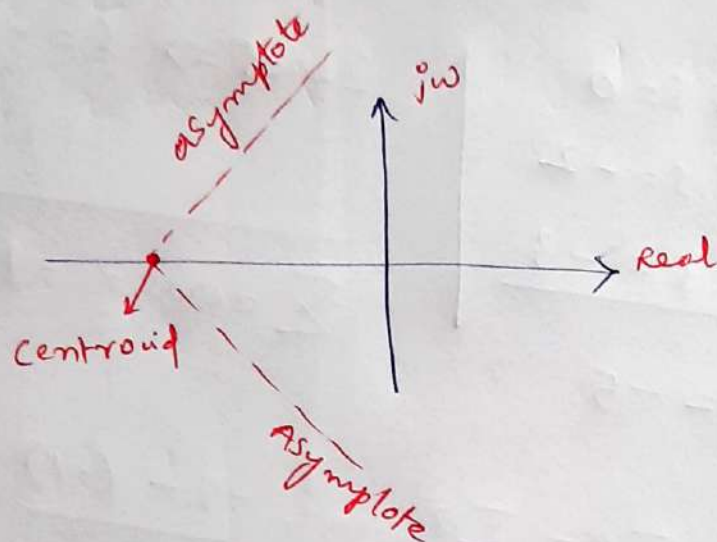
$$\theta_2 = \frac{2 \times 1 + 1}{3} \times 180^\circ = 180^\circ$$

$$\theta_3 = \frac{2 \times 2 + 1}{3} \times 180^\circ = 300^\circ$$

Rule-5

(155)

Centroid $\div$ It is the intersection point of the Asymptotes on real axis. It may or may not part of Root Locus.



$$\text{Centroid} = \sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{p - z}$$

$$\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{p - z}$$

~~Q~~ $s^3 + 5s^2 + (K+6)s + K = 0$

$$s^3 + 5s^2 + 6s + K(s+1) = 0$$

$$1 + \frac{K(s+1)}{s^3 + 5s^2 + 6s} = 0$$

$$\Rightarrow 1 + G(s)H(s) = 0 \Rightarrow \frac{K(s+1)}{s(s^2 + 5s + 6)} = G(s)H(s)$$

$$p = 3, z = 1, p - z = 3 - 1 = 2$$

$$\text{No. of Asymptotes} = p - z = 3 - 1 = 2$$

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

$$\begin{array}{l} \text{poles} \Rightarrow s = 0 \\ s = -2 \\ s = -3 \\ \hline \end{array}$$

$$\sum \text{poles} = -5$$

$$\text{zeros} \Rightarrow s = -1$$

$$\downarrow$$

$$\sum \text{zeros} = -1$$

$$\sigma = \frac{\sum p - \sum z}{p - z} = \frac{-5 - (-1)}{2} = -2$$

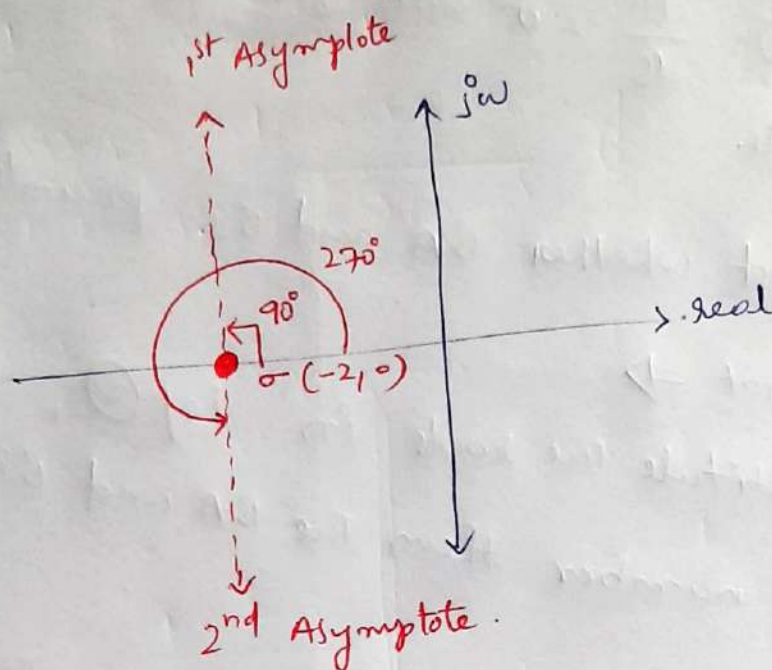
$$\therefore \sigma = (-2, 0)$$

$$\Rightarrow \text{No. of Asymptotes} = 2$$

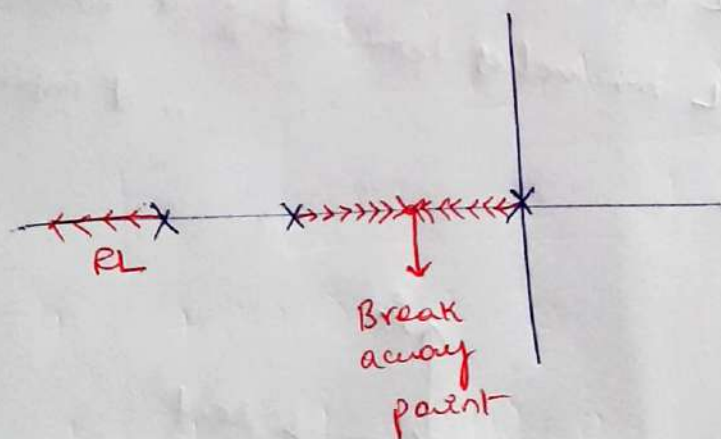
$$\alpha = 0, 1$$

$$\theta_1 = \frac{2 \times 0 + 1}{2} \times 180^\circ = 90^\circ$$

$$\theta_2 = \frac{2 \times 1 + 1}{2} \times 180^\circ = 270^\circ$$



Rule-6 $\Rightarrow$ Break away / Break in point
 They are those points where multiple roots of characteristic equation occur.



Steps to find Break away (BA) and Break in (BI) points.

- ① Construct $1 + G(s)H(s) = 0$
- ② write K in terms of 's'.
- ③ find $\frac{dK}{ds} = 0$

④ roots of $\frac{dK}{ds} = 0$ will give BA and BI points.

⑤ To test whether BA and BI points are valid or not $\Rightarrow$
Substitute the roots in step ②, if K is +ve number then the BA and BI are valid.

Q $G(s)H(s) = \frac{K}{s}$, plot RL for the system.

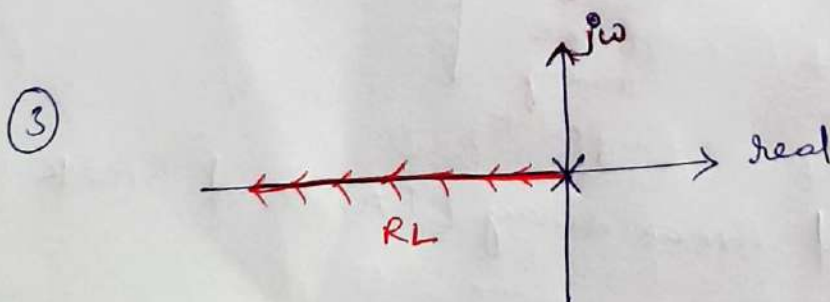
sol ② $P = 1, Z = 0$

No. of RL branches = $P = 1$

No. of RL branches terminating at zero = $Z = 0$.

" at infinity = $P - Z = 1$.

No. of "



④ No. of asymptotes $\Rightarrow P - Z = 1$
i.e. $q = 1$ we have to take

$$\theta_1 = \frac{2 \times 0 + 1 \times 180^\circ}{2} \Rightarrow 180^\circ = \theta_1$$

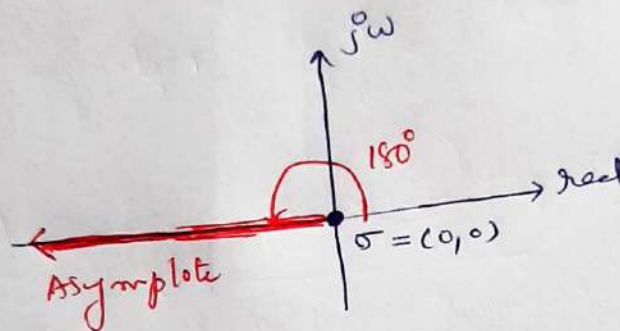
⑤

poles $\Rightarrow s=0$, 2 zeros $\Rightarrow$ no zeros.

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$$\sigma = \frac{\sum p - \sum z}{p - z}$$

$$\sigma = \frac{0 - 0}{1} = 0.$$



⑥

BA & BI points

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s} = 0$$

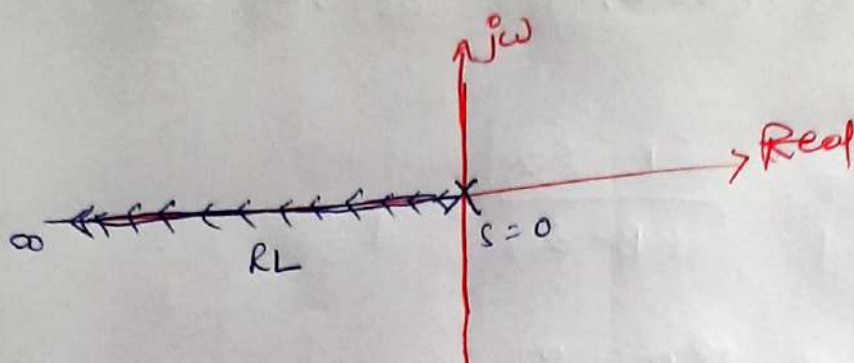
$$s + K = 0$$

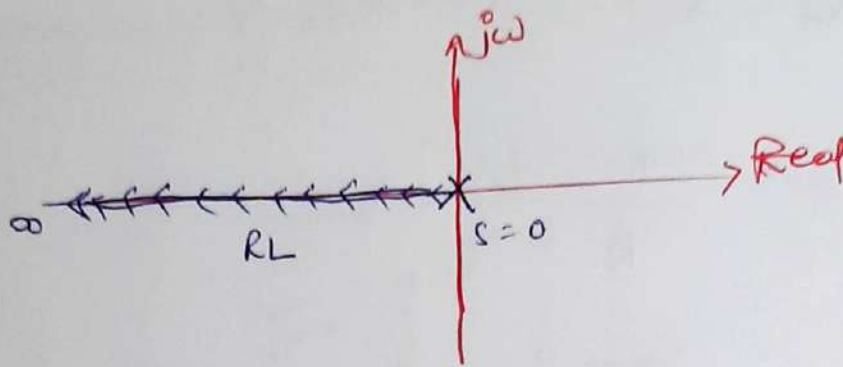
$$K = -s$$

$$\frac{dK}{ds} \Rightarrow \frac{dK}{ds} = \frac{d(-s)}{ds} = -1$$

$\Rightarrow$ there are no roots for $\frac{dK}{ds}$.

$\Rightarrow$ there are no BA and BI points.





Q The system dynamics are given as $G(s)H(s) = \frac{k}{s(s+2)}$
 Draw the Root Locus diagrams for the given system.

Sol

$$G(s)H(s) = \frac{k}{s(s+2)}$$

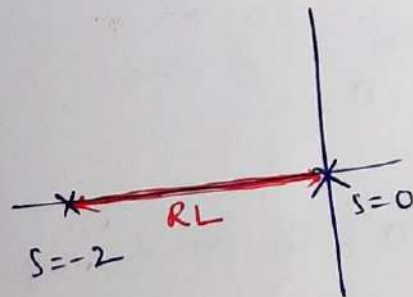
② $P = 2, \quad z = 0, \quad P - z = 2.$

$\therefore$ No. of RL branches terminate at infinity = 2

$\therefore$ No. of Asymptotes = 2

poles $\Rightarrow s = 0, s = -2$

③



④

$$\theta_1 = \frac{2 \times 0 + 1}{P - z} \times 180^\circ = \frac{1}{2} \times 180^\circ = 90^\circ$$

$$\theta_2 = \frac{2 \times 1 + 1}{P - z} \times 180^\circ = \frac{3}{2} \times 180^\circ = 270^\circ$$

(5)

$$\text{Centroid} = \sigma = \frac{\sum p - \sum z}{p - z}$$

$$= \frac{[0 + (-2)] - [0]}{2}$$

$$\sigma = -1$$

(161)

(6) BW | BA points

(a) $1 + k(s)H(s) = 0$

$$1 + \frac{k}{s(s+2)} = 0$$

$$s^2 + 2s + k = 0$$

(b) $k = -s^2 - 2s \rightarrow \textcircled{1}$

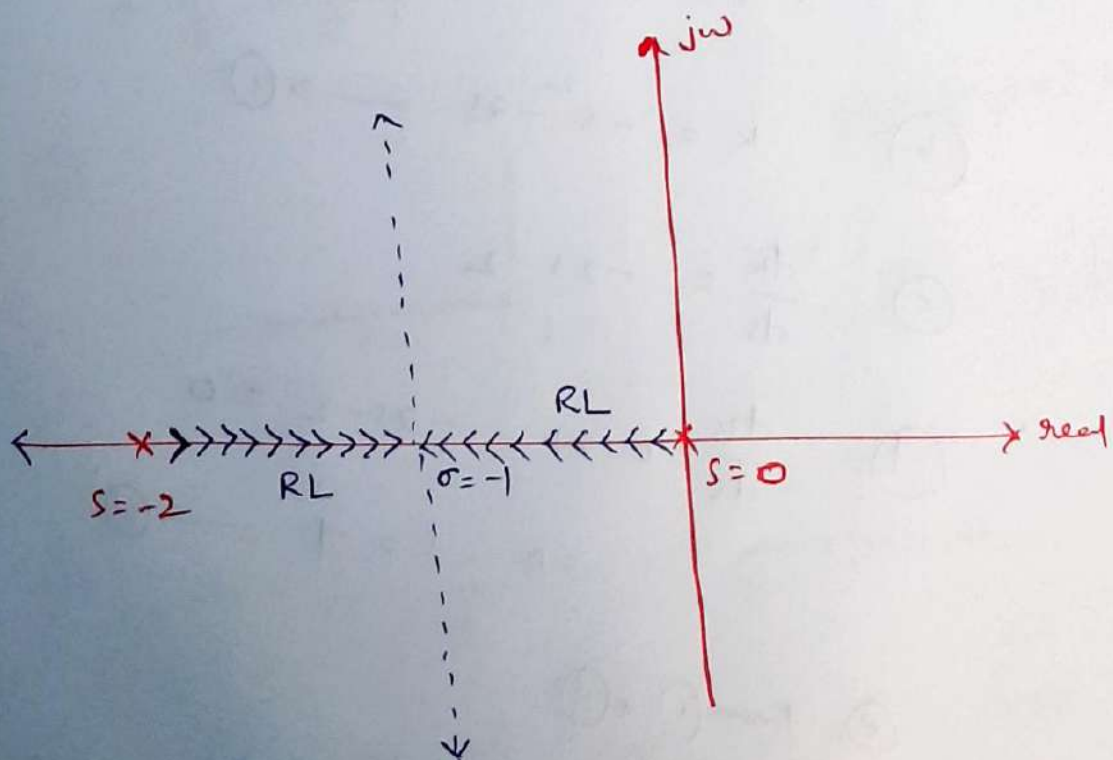
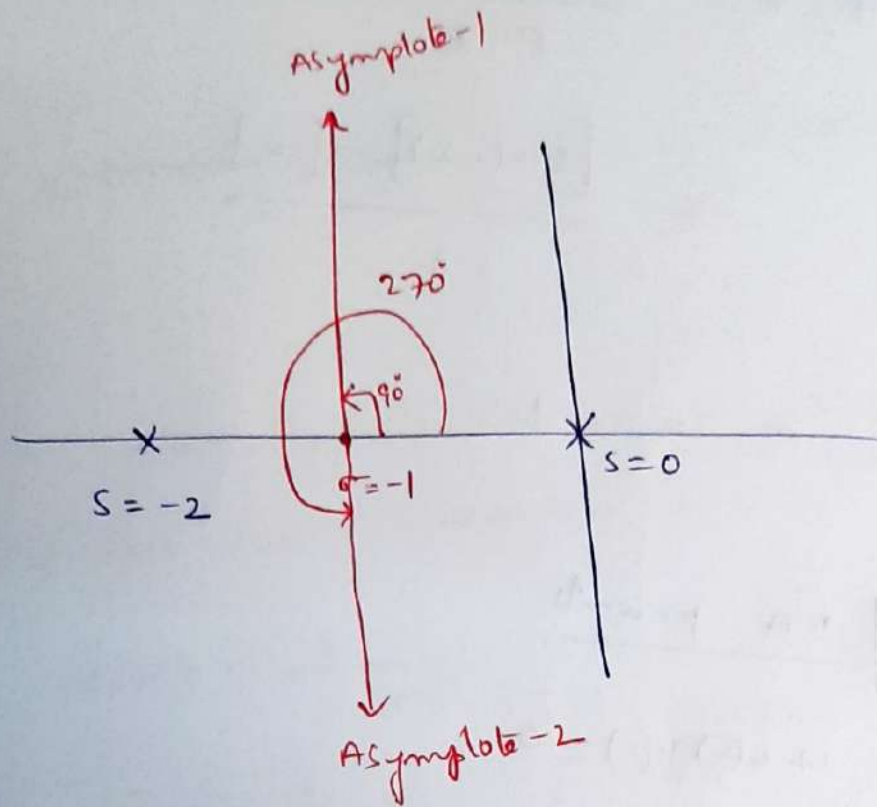
(c) $\frac{dk}{ds} = -2s - 2$

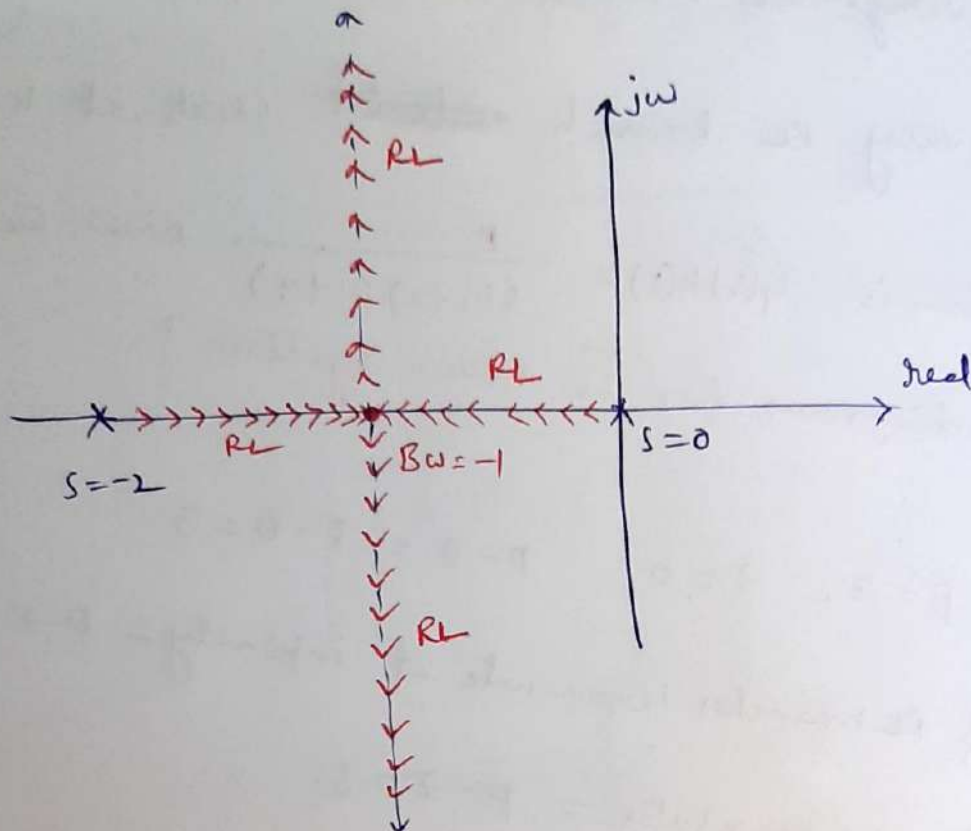
(d) $\frac{dk}{ds} = 0 \Rightarrow -2s - 2 = 0$
 $\Rightarrow s = -1 \rightarrow \textcircled{2}$

(e) from $\textcircled{1}$ & $\textcircled{2}$

$$k = -(-1)^2 - 2(-1) = +1 \Rightarrow \text{+ve value.}$$

the point is valid.





$\Rightarrow$ at $s = 0$

Magnitude condition $\Rightarrow \left| G(s) H(s) \right| = \left| \frac{K}{s(s+2)} \right|_{s=0} = 1$

$$\Rightarrow \frac{K}{1 \cdot 1} = 1$$

$$\boxed{K = 0}$$

$\Rightarrow$ at $s = -2 \Rightarrow s = -2 + j^0$

$$\left| G(s) H(s) \right| = 1 \Rightarrow \left| \frac{K}{(-2 + j^0)(-2 + j^0 + 2)} \right| = 1$$

$$\Rightarrow \boxed{K = 0}$$

⇒ Note ① Every RL branch starts at pole and (164)

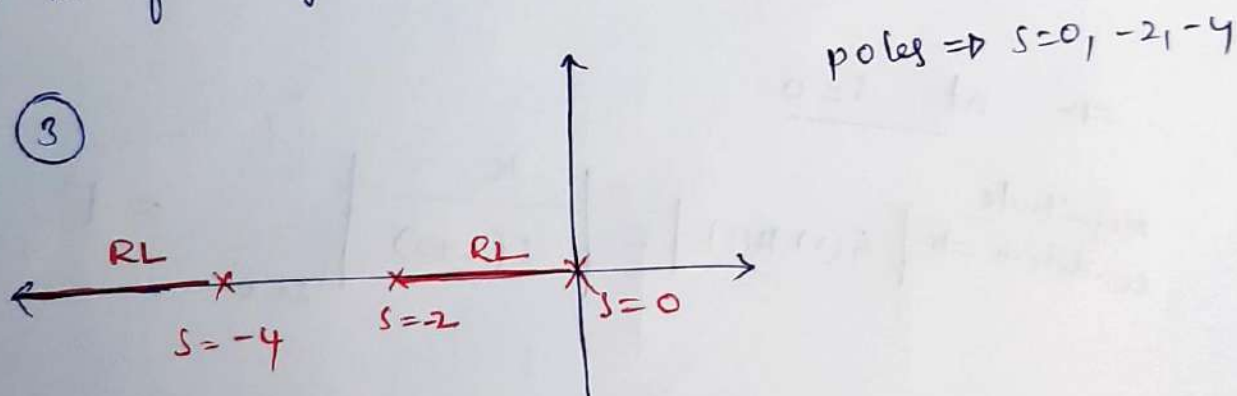
② Every RL branch ~~ends at~~ starts at $K=0$

Q The system is $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$. draw the root locus diagrams for the given system?

sol ② $P=3, Z=0, P-Z=3-0=3$.

∴ No. of RL branches terminate at infinity = $P-Z=3-0=3$.

∴ No. of Asymptotes = $P-Z=3$.



④ $\theta_1 = \frac{2 \times 0 + 1}{P-Z} \times 180^\circ = 60^\circ$

$$\theta_2 = \frac{2 \times 1 + 1}{P-Z} \times 180^\circ = 180^\circ$$

$$\theta_3 = \frac{2 \times 2 + 1}{P-Z} \times 180^\circ = 300^\circ$$

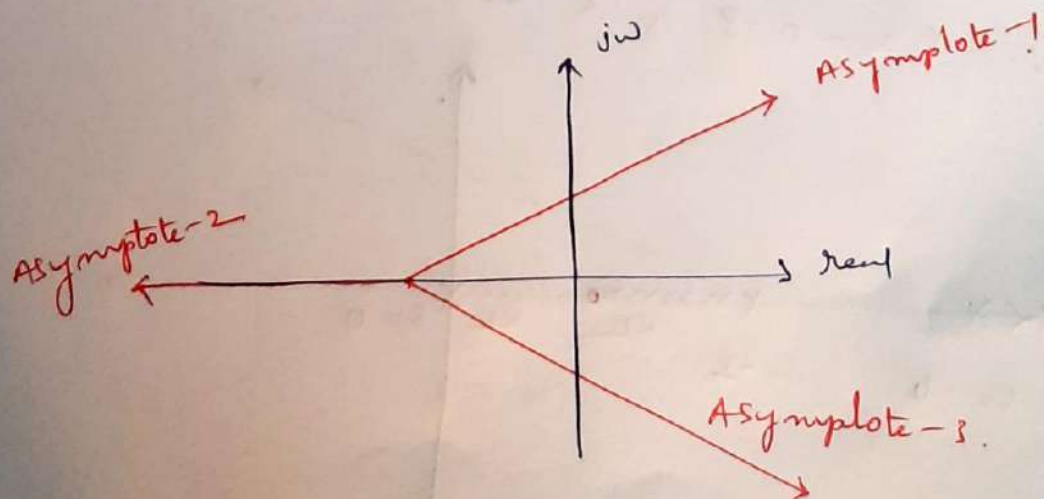
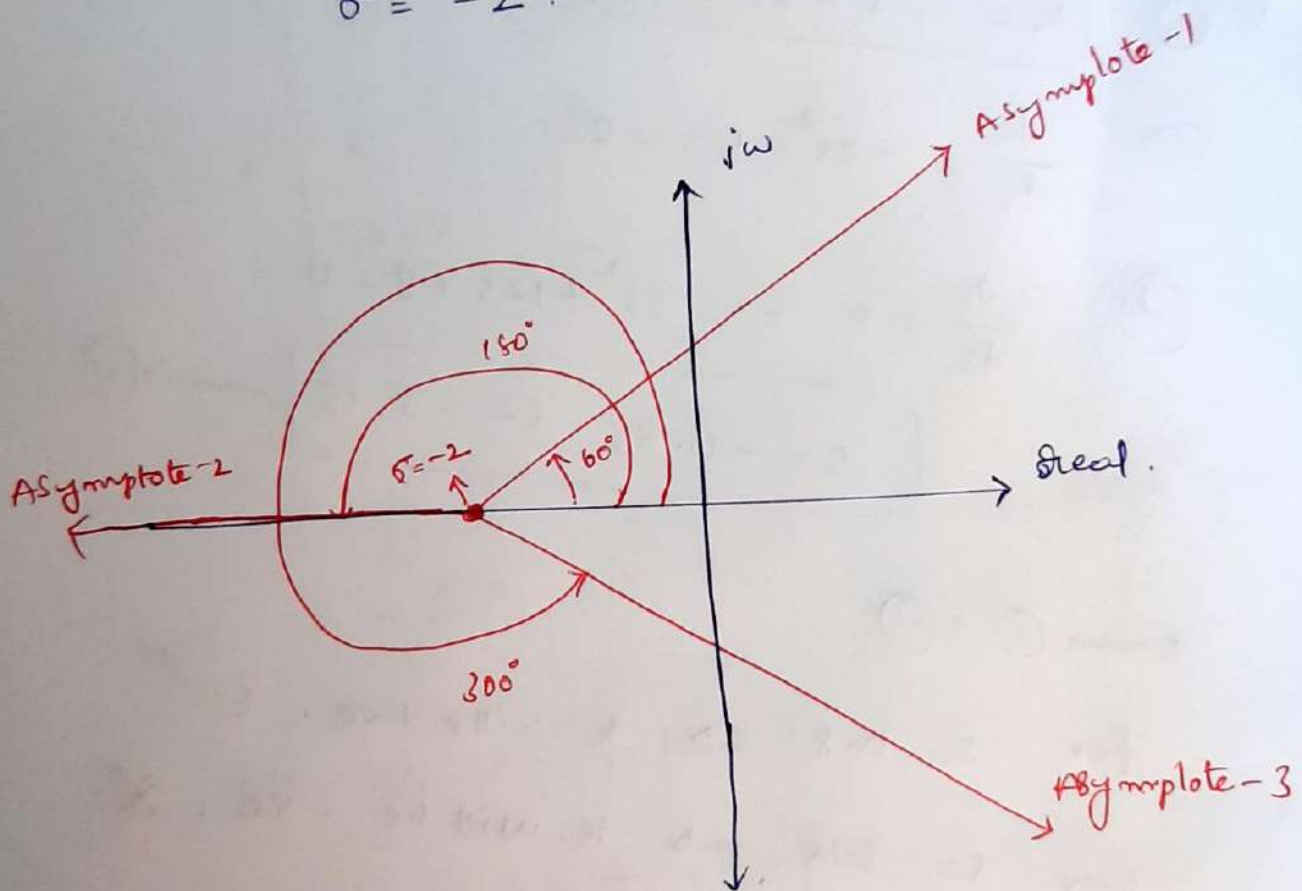
⑤

$$\text{centroid} = \frac{\sum P - \sum Z}{p - z}$$

$$= \frac{[0 + (-2) + (-4)] - [0]}{3}$$

$$= -6/3 = -2$$

$$\sigma = -2$$



6

BW/BZ point

166

a) $1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$s(s+2)(s+4) + K = 0$$

b) $K = -s^3 - 6s^2 - 8s \rightarrow (1)$

c) $\frac{dK}{ds} = -3s^2 - 12s - 8$

d) $\frac{dK}{ds} = 0 \Rightarrow 3s^2 + 12s + 8 = 0$

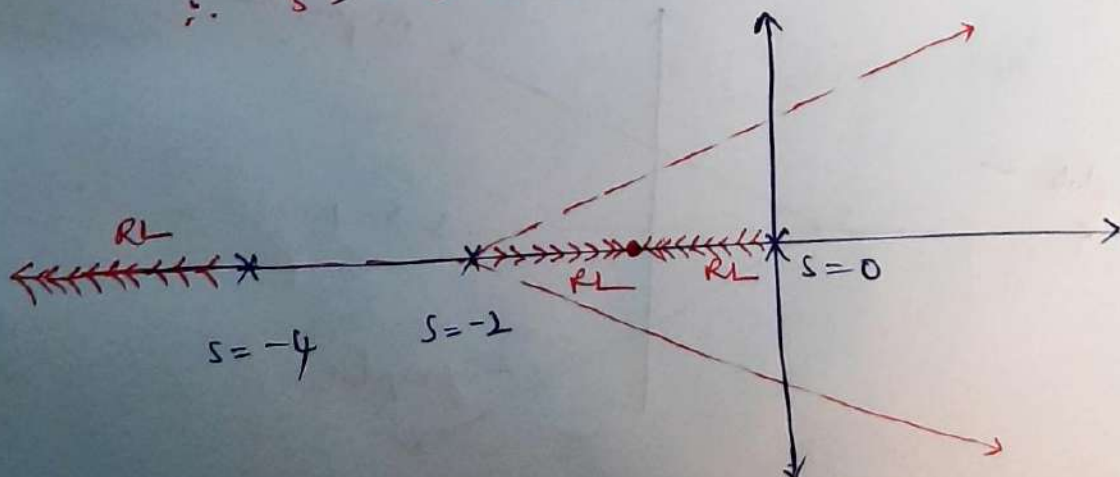
$s = -0.8, s = -3.15 \rightarrow (2)$

from (1) & (2)

for $s = -0.8 \Rightarrow K$ will be +ve. ✓

for $s = -3.15 \Rightarrow K$ will be -ve. ✗

$\therefore s = -0.8$ is a valid point.



R-H criterion

167

$$CE \Rightarrow 1 + G(s)H(s) = 0 \Rightarrow 1 + \frac{K}{s(s+2)(s+3)} = 0.$$

$$1 + \frac{K}{s^3 + 6s^2 + 8s} = 0$$

$$s^3 + 6s^2 + 8s + K = 0$$

s^3		1	8
s^2		6	K
s		$\frac{48-K}{6}$	0
s^0		K	

For $K = 48$, s^1 row will completely zero.

$$\Rightarrow AE \Rightarrow 6s^2 + K = 0$$

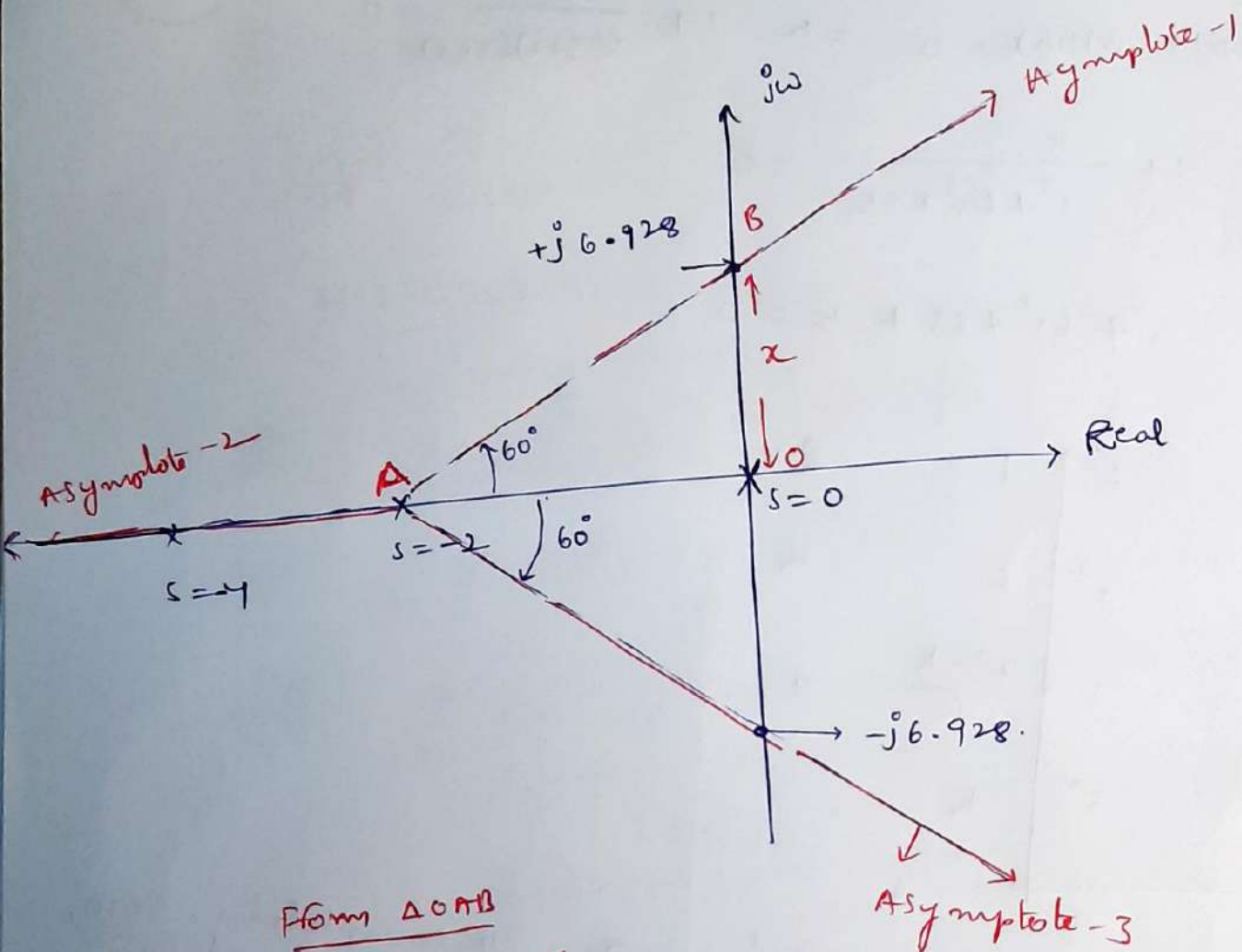
$$6s^2 + 48 = 0$$

$$s^2 + 8 = 0$$

$$s = \pm j\sqrt{8}$$

$$s = \pm j 2\sqrt{2}$$

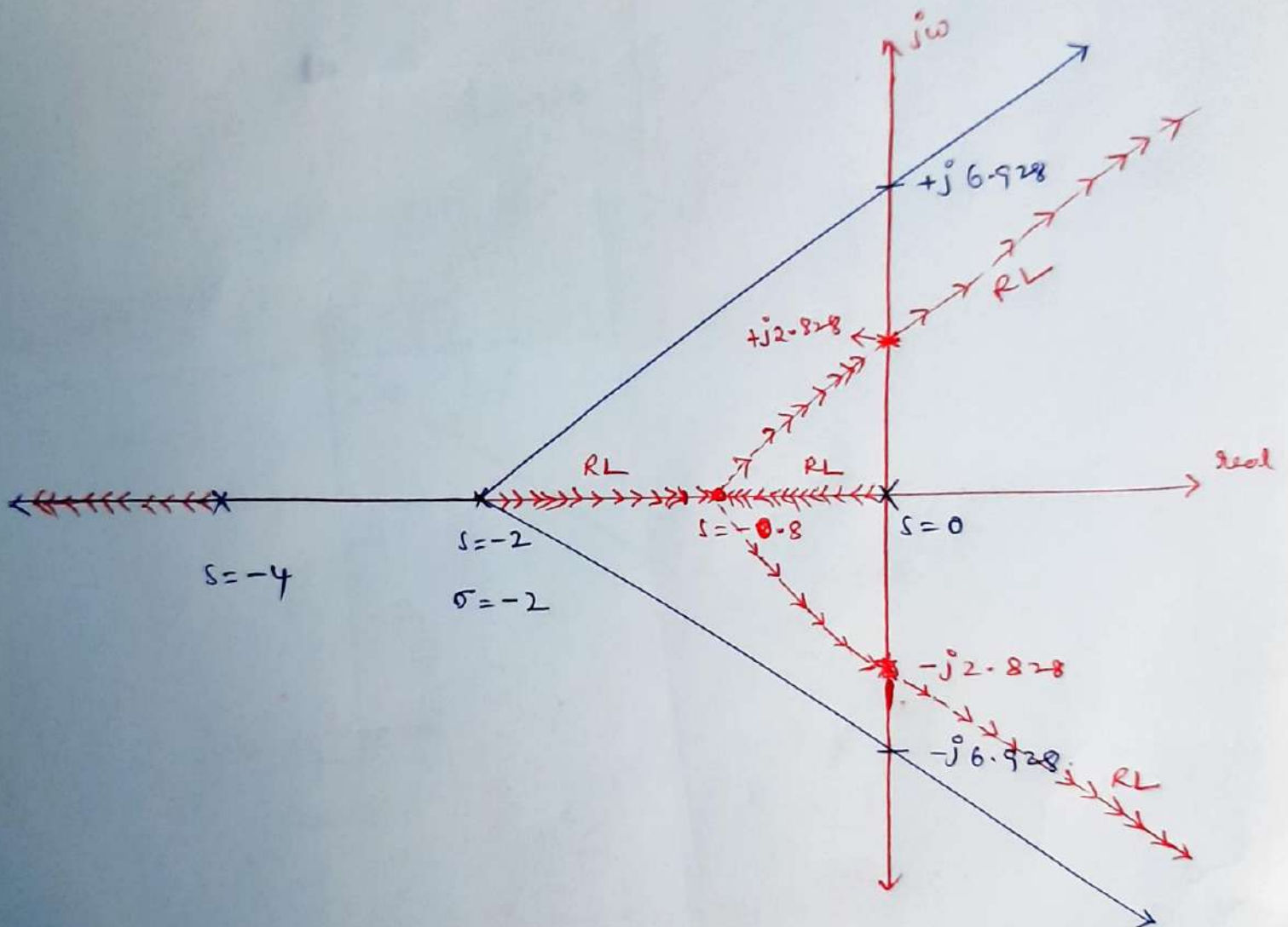
$$s = \pm j 2.828.$$



$$\Rightarrow \frac{\text{From } \triangle OAB}{\tan 60^\circ = \frac{x}{2}}$$

$$\sqrt{3} = \frac{x}{2} \Rightarrow x = 2\sqrt{3}$$

$$x = 6.928.$$



At $s = 0$

170

$$|G(s)H(s)| = 1 \leftarrow \text{Magnitude condition}$$

$$\left| \frac{K}{s(s+2)(s+4)} \right|_{s=0} = 1$$

$$\frac{K}{|0|} = 1 \Rightarrow \boxed{K = 0}$$

At $s = -2$

$$|G(s)H(s)| = 1$$

$$\left| \frac{K}{s(s+2)(s+4)} \right|_{s=-2} = 1$$

$$\left| \frac{K}{(-2)(0)(2)} \right| = 1 \Rightarrow \boxed{K = 0}$$

At $s = -4$

$$|G(s)H(s)| = 1$$

$$\left| \frac{K}{(-4)(-2)(0)} \right| = 1 \Rightarrow \boxed{K = 0}$$

Q $G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+4)}$, phase 180°

Root Locus diagram

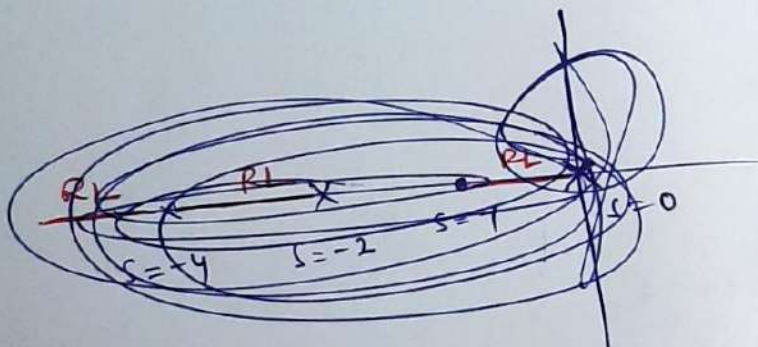
② $P=3, Z=1, P-Z=3-1=2$

$\Rightarrow$ No. of RL branches = 3

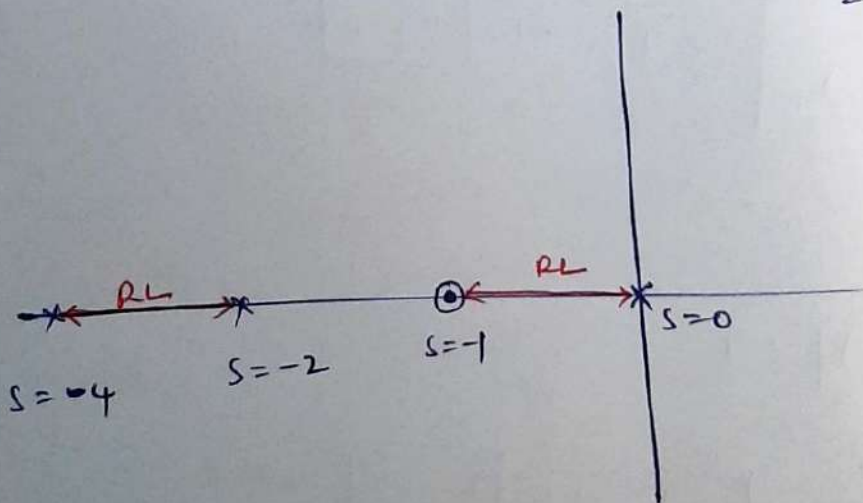
$\Rightarrow$ No. of RL branches terminating at zero = $Z=1$
 " at infinity = $P-Z=3-1=2$.

$\Rightarrow$ "

③



poles $\Rightarrow s=0, -2, -4$
 zeros $\Rightarrow s=-1$



④

$$\text{No. of Asymptotes} = p - z = 3 - 1 = 2$$

$$\sigma = 0, 1$$

$$\theta_1 = \frac{2 \times 0 + 1}{p - z} \times 180^\circ$$

$$= \frac{1}{2} \times 180^\circ = 90^\circ$$

$$\boxed{\theta_1 = 90^\circ}$$

$$\theta_2 = \frac{2 \times 1 + 1}{p - z} \times 180^\circ$$

$$= \frac{3}{2} \times 180^\circ = 270^\circ$$

$$\boxed{\theta_2 = 270^\circ}$$

⑤

$$\sigma = \frac{\sum \text{poles} - \sum \text{zeros}}{p - z}$$

$$= \frac{(0 + (-2) + (-4)) - (-1)}{2}$$

$$\sigma = -2.5$$

$$\sigma = (-2.5, 0)$$

⑥

BW / BI points

$$\textcircled{1} 1 + G(s)H(s) = 0$$

$$1 + \frac{K(s+1)}{s(s^2 + 6s + 8)} = 0$$

$$s^3 + 6s^2 + 8s + K(s+1) = 0$$

$$(2) \quad K = \frac{-(s^3 + 6s^2 + 8s)}{s+1}$$

$$(3) \quad \frac{dK}{ds} \Rightarrow \frac{d}{ds} \left[\frac{-(s^3 + 6s^2 + 8s)}{(s+1)} \right]$$

$$\Rightarrow \frac{(s+1)[-3s^2 - 12s - 8] - [-s^3 - 6s^2 - 8s](1)}{(s+1)^2}$$

$$\frac{dK}{ds} \Rightarrow \frac{-(s+1)(3s^2 + 12s + 8) + s^3 + 6s^2 + 8s}{(s+1)^2}$$

$$(4) \quad \frac{dK}{ds} = 0$$

$$\frac{-(s+1)(3s^2 + 12s + 8) + s^3 + 6s^2 + 8s}{(s+1)^2} = 0$$

$$\Rightarrow s^3 + 6s^2 + 8s - (s+1)(3s^2 + 12s + 8) = 0$$

$$\Rightarrow s^3 + 6s^2 + 8s - [3s^3 + 12s^2 + 8s + 3s^2 + 12s + 8] = 0$$

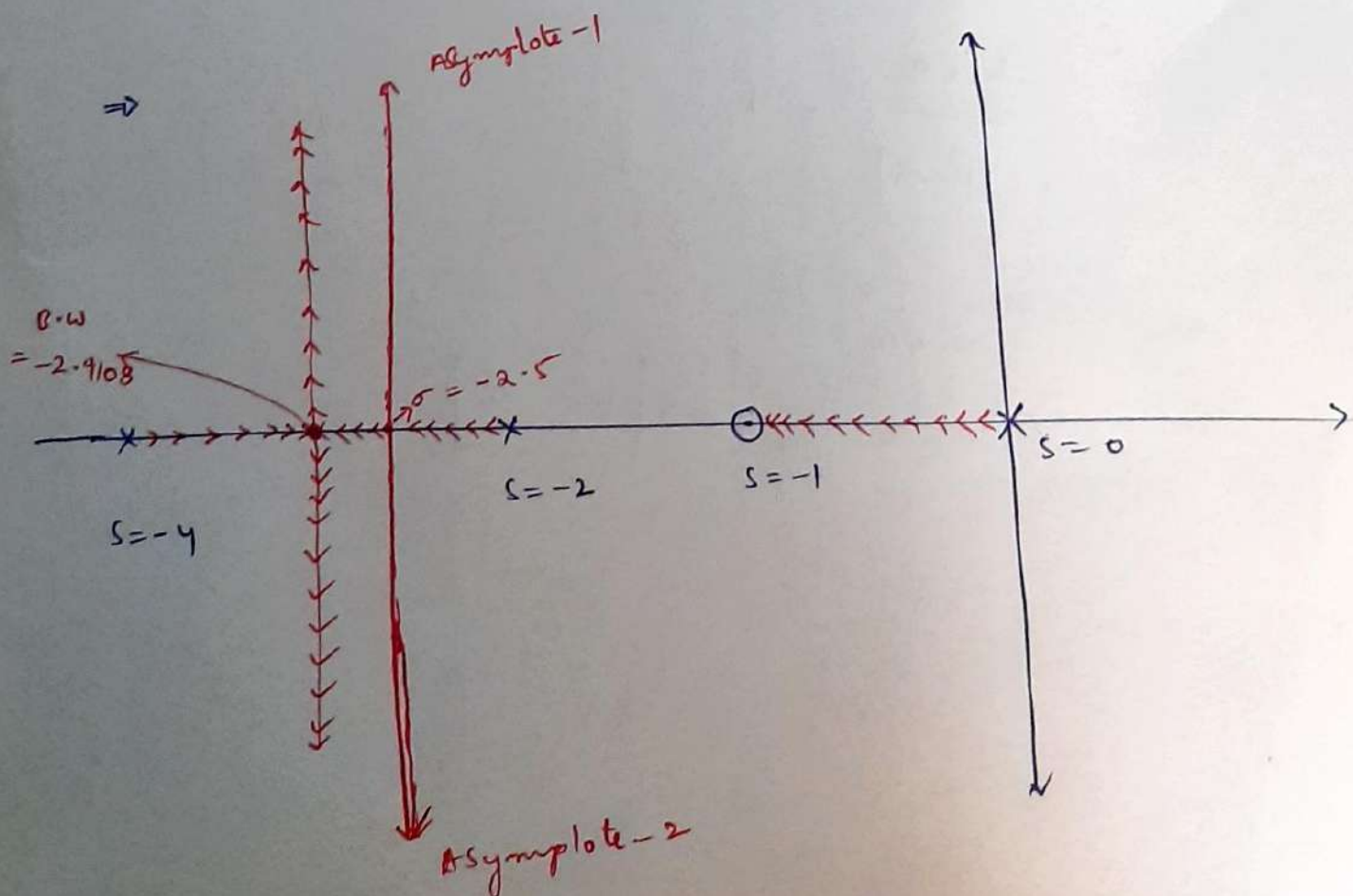
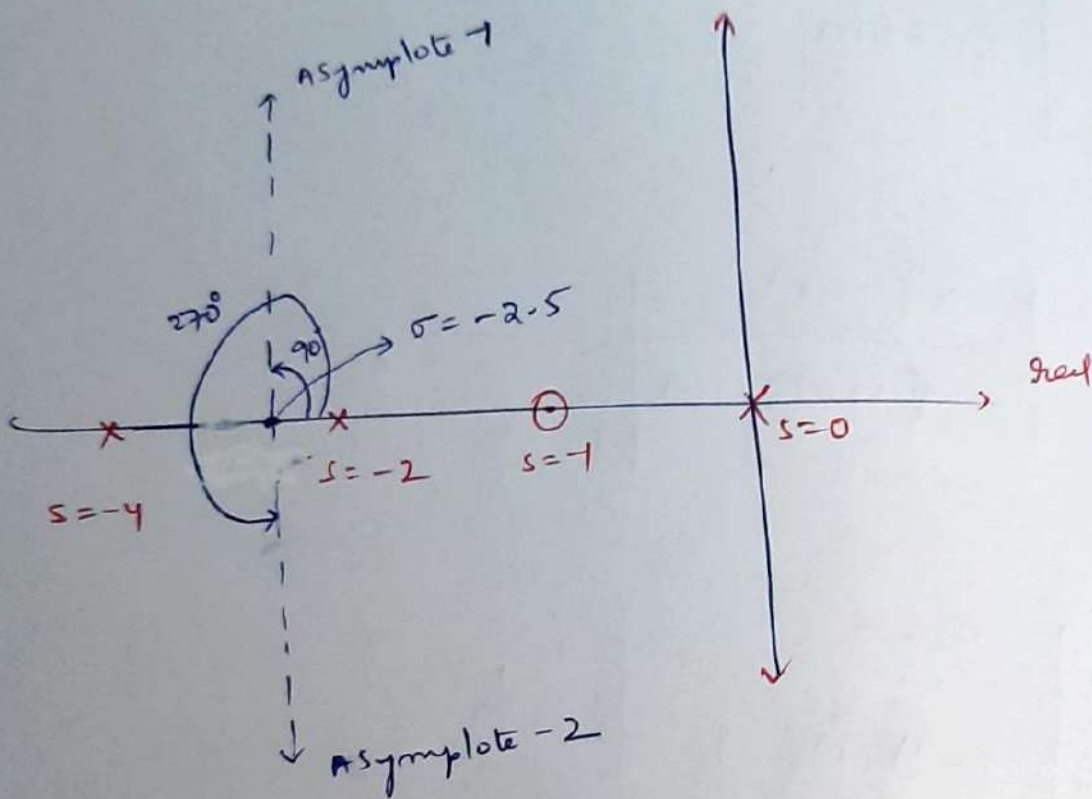
$$\Rightarrow -2s^3 - 9s^2 - 12s - 8 = 0$$

$$\Rightarrow 2s^3 + 9s^2 + 12s + 8 = 0.$$

$$s = -2.9108, \quad s = -0.7945 \pm j 0.8618$$

(174)

$\Rightarrow s = -2.9108$ is valid.



at $s = -1$

By using magnitude condition,

$$|G(s)H(s)| = 1$$

$$\left| \frac{K(s+1)}{s(s+2)(s+4)} \right|_{s=-1} = 1$$

$$\left| \frac{K \times 0}{(-1)(1)(3)} \right| = 1$$

$$\frac{K \times 0}{|-3|} = 1$$

$$K \times 0 = 3$$

$$K = 3/0 \Rightarrow \boxed{K = \infty}$$

Note (1) Let $G(s)H(s) = \frac{K(s+b)}{s(s+a)}$

Let us consider a point $s = x + jy$ is lies on the root locus.

$$\Rightarrow G(s)H(s) \Big|_{s=x+jy} = \frac{K(x+jy+b)}{(x+jy)(x+jy+a)}$$

$$\Rightarrow = \frac{K[(x+b) + jy]}{x^2 + jxy + xa + jxy + jya - y^2}$$

$$\Rightarrow = \frac{K[(x+b) + jy]}{(x^2 + xa - y^2) + j(2xy + ya)}$$

If the point lies on the root locus, it should satisfy the angle condition

$$\angle G(s)H(s) \Big|_{s=x+jy} = 180^\circ$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{(x+b)}\right) - \tan^{-1}\left(\frac{2xy + ya}{x^2 + xa - y^2}\right) = 180^\circ$$

$\downarrow$ let A $\downarrow$ let B

We know that

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B)$$

$$\tan^{-1}\left(\frac{y}{x+b}\right) - \tan^{-1}\left(\frac{2xy+ya}{x^2+ax-y^2}\right) = \tan^{-1}(180^\circ) + \tan^{-1}\left(\frac{y}{x+b}\right) + \tan^{-1}\left(\frac{2xy+ya}{x^2+ax-y^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x+b}\right) - \tan^{-1}\left(\frac{2xy+ya}{x^2+ax-y^2}\right) = 0$$

$$\Rightarrow \frac{y}{x+b} = \frac{y(2x+a)}{x^2+ax-y^2}$$

$$\Rightarrow x^2+ax-y^2 = 2x^2+2xb+ax+ab$$

$$\Rightarrow x^2+ax-y^2 - [2x^2+2xb+ax+ab] = 0$$

$$\Rightarrow -x^2 - y^2 - 2xb - ab = 0$$

$$\Rightarrow x^2 + 2xb + y^2 = -ab$$

$$\Rightarrow x^2 + 2xb + y^2 + b^2 = -ab + b^2$$

$$\Rightarrow \cancel{x+b} (x+b) + y^2 = -ab + b^2$$

$$\Rightarrow (x+b)^2 + y^2 = b(b-a) \Rightarrow \text{Circle Equation}$$

$$\Rightarrow \boxed{\text{centre} = (-b, 0)} \neq \boxed{\text{Radius} = \sqrt{b(b-a)}}$$

Note 2 BW / BE point = (Centre $\pm$ radius)

(178)

Q $G(s)H(s) = \frac{K(s+2)}{s(s+1)}$ draw the root locus diagram and comment on the stability of the system.

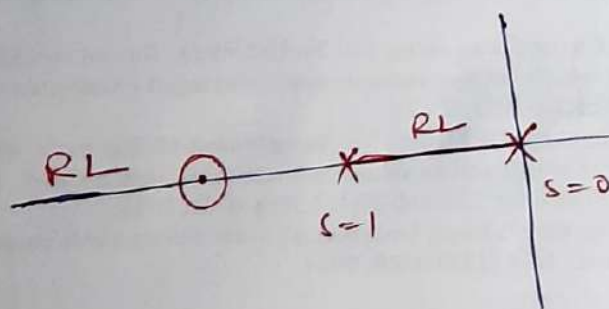
sd ② $P=2, \quad Z=1, \quad P-Z=2-1=1$

$\Rightarrow$ No. of RL branches = 2

$\Rightarrow$ No. of RL branches terminating at infinity = 1.

$\Rightarrow$ No. of asymptotes = 1.

③



poles $\Rightarrow s=0, -1$
zeros $\Rightarrow s=-2$

④

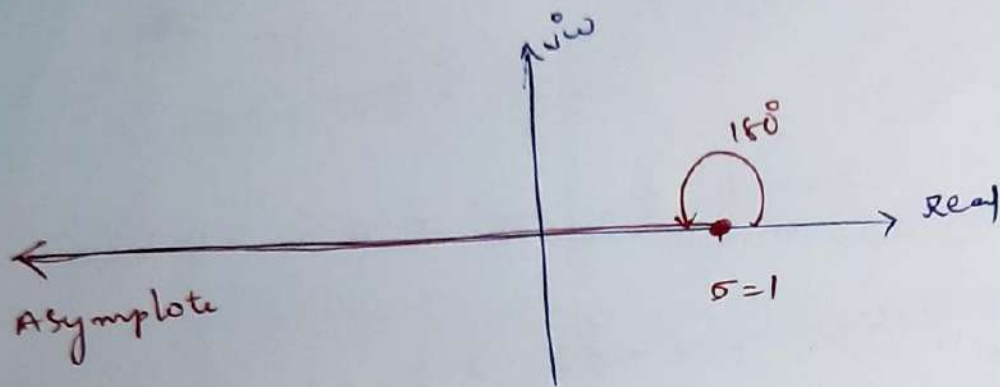
$$\theta_1 = \frac{2 \times 0 + 1}{P-Z} \times 180^\circ \Rightarrow \theta_1 = \frac{1}{1} \times 180^\circ$$

$\theta_1 = 180^\circ$

⑤

$$\sigma = \frac{\sum P - \sum Z}{P-Z} \Rightarrow \sigma = \frac{(-1) - (-2)}{1}$$

$\sigma = 1$



⑥ BW | BI points

① $1 + G(s)H(s) = 0$

$$1 + \frac{K(s+2)}{s(s+1)} = 0$$

$$s^2 + s + K(s+2) = 0$$

② $K = \frac{-(s^2 + s)}{s+2}$

③ $\frac{dK}{ds} = \frac{-[(s+2)(2s+1) - (s^2+s)(1)]}{(s+2)^2}$

$$\frac{dK}{ds} = \frac{(s^2+s) - [2s^2+s+4s+2]}{(s+2)^2}$$

$$\frac{dK}{ds} = \frac{-s^2 - 4s - 2}{(s+2)^2}$$

④ $\frac{dk}{ds} = 0$

180

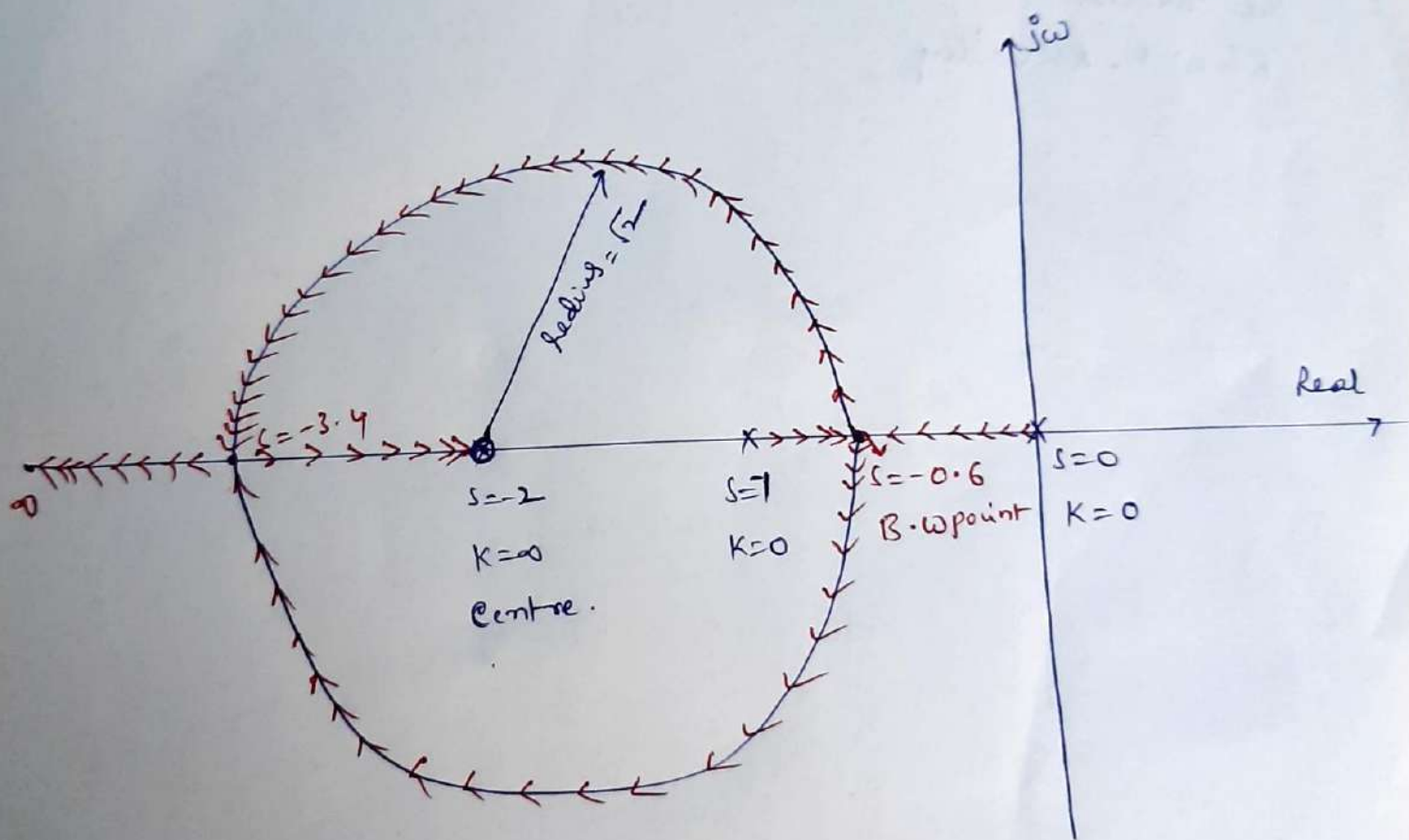
$$s^2 + 4s + 2 = 0$$

$$\Rightarrow s = -2 \pm \sqrt{2} \Rightarrow s = -0.6, -3.4$$

⑤ Both $s = -0.6$, $s = -3.4$ points are valid.

According to note ②

$$\text{BW} \mid \text{B.I point} = \underset{\substack{\downarrow \\ \text{centre}}}{-2} \pm \underset{\substack{\downarrow \\ \text{radius.}}}{\sqrt{2}}$$



Q $G(s)H(s) = \frac{K(s+2)(s+4)}{s(s+1)}$, For the given systems
Draw the Root Locus diagram and comment
on the stability.

182

Sol $G(s)H(s) = \frac{K(s+2)(s+4)}{s(s+1)}$

② $P = 2, Z = 2$

$\Rightarrow$ No. of RL branches = 2

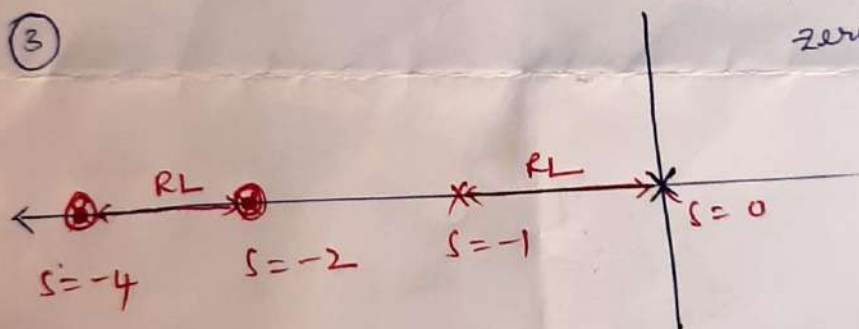
$\Rightarrow$ No. of RL branches terminating at infinity = 0

" at zeros = 2

$\Rightarrow$ "

poles $\Rightarrow s = 0, -1$
zeros $\Rightarrow s = -2, -4$

③



④ No. Asymptotes

⑤ No. centroid.

⑥ BW / BI points

① $1 + G(s)H(s) = 0$

$1 + \frac{K(s+2)(s+4)}{s(s+1)} = 0$

$$s^2 + s + K[s^2 + 6s + 8] = 0$$

$$(2) \quad K = \frac{-[s^2 + s]}{s^2 + 6s + 8}$$

$$(3) \quad \frac{dK}{ds} = - \left\{ \frac{(s^2 + 6s + 8)[2s + 1] - [s^2 + s][2s + 6]}{s^2 + 6s + 8} \right\}$$

$$(4) \quad \frac{dK}{ds} = 0$$

$$\Rightarrow (2s^3 + s^2 + 16s^2 + 6s + 16s + 8) - (2s^3 + 6s^2 + 2s^2 + 6s) = 0$$

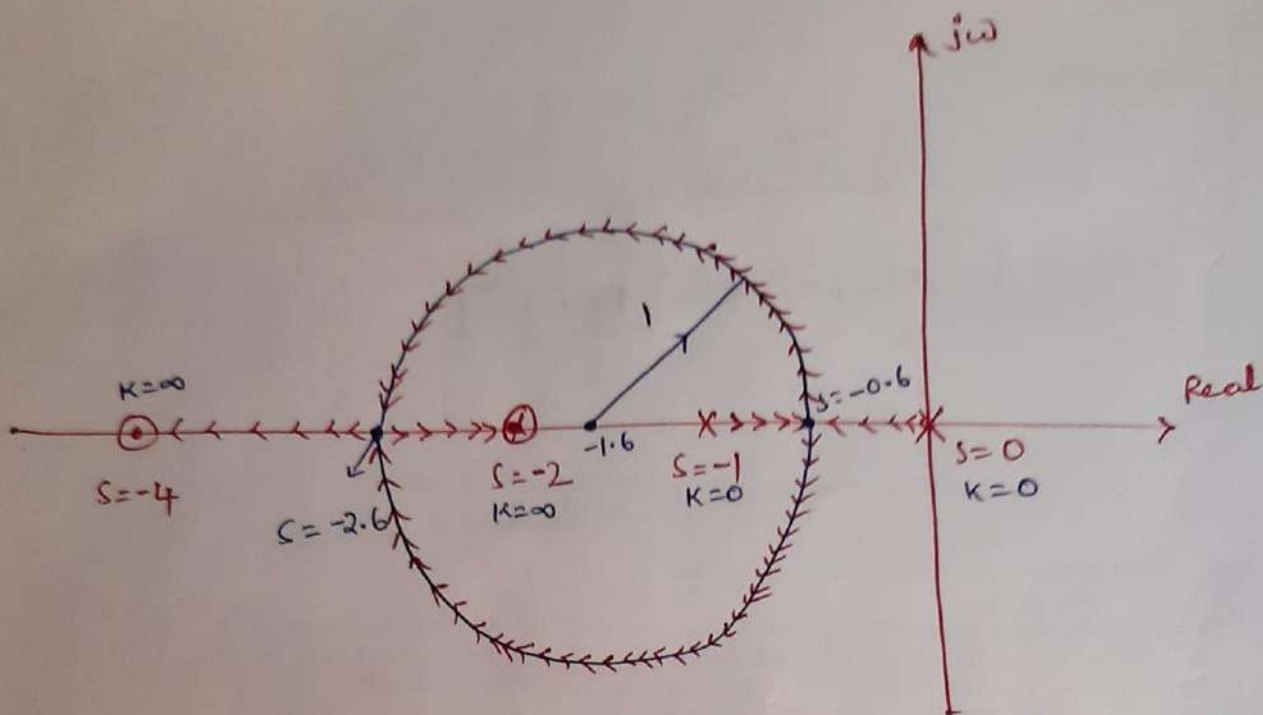
$$\Rightarrow 5s^2 + 16s + 8 = 0$$

$$\Rightarrow s = -1.6 \pm 1$$

↑
Centre
↑
radius.

$$(5) \quad s_1 = -0.6, \quad s_2 = -2.6 \text{ are valid points.}$$

↑
sw



Rule No. 7 Angle of departure and Angle of Arrival

⇒ Angle of departure is obtained when complex poles terminates at infinity. ⇒ $\theta_D = 180 + \phi$ where $\phi = \sum \phi_z - \sum \phi_p$

⇒ Angle arrival can be calculated at complex zeros. $\phi_A = 180 - \phi$

Q $G(s)H(s) = \frac{k(s+1)}{s(s+4)(s^2+2s+2)}$, For the given system draw the root locus diagram and comment on the stability.

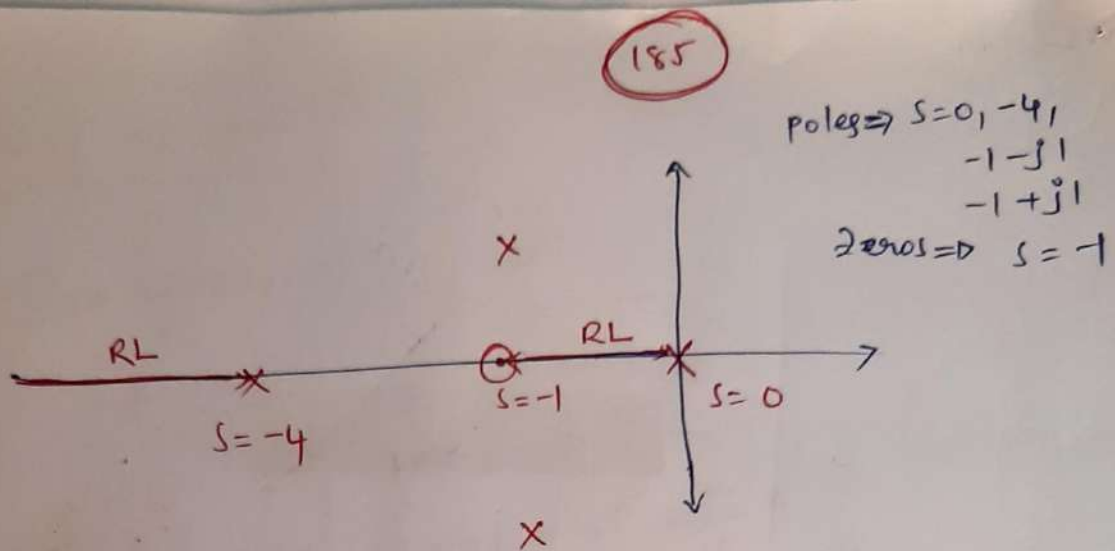
Sol ② $p = 4$, $z = 1$, $p - z = 4 - 1 = 3$.

⇒ No. of RL branches = 4

⇒ No. of RL branches terminating at zero = 1

⇒ No. of RL branches terminating at infinity = 3.

③



④ no. of asymptotes = 3

$$\theta_1 = \frac{2 \times 0 + 1}{3} \times 180^\circ = 60^\circ$$

$$\theta_2 = \frac{2 \times 1 + 1}{3} \times 180^\circ = 180^\circ$$

$$\theta_3 = \frac{2 \times 2 + 1}{3} \times 180^\circ = 300^\circ$$

⑤
$$\sigma = \frac{\sum p - \sum z}{p - z}$$

$$= \frac{[0 + (-4) + (-1-j1) + (-1+j1)] - [-1]}{3}$$

$$= -5/3 = -1.6$$

$$\sigma = (-1.6, 0)$$

⑥ Bw | BI points

① $1 + h(s)H(s) = 0$

$$1 + \frac{K(s+1)}{s(s+4)(s^2+2s+2)} = 0$$

$$(s^2+4s)(s^2+2s+2) + K(s+1) = 0$$

$$\Rightarrow s^4 + 6s^3 + 10s^2 + 8s + K(s+1) = 0$$

② $K = - \frac{s^4 + 6s^3 + 10s^2 + 8s}{s+1}$

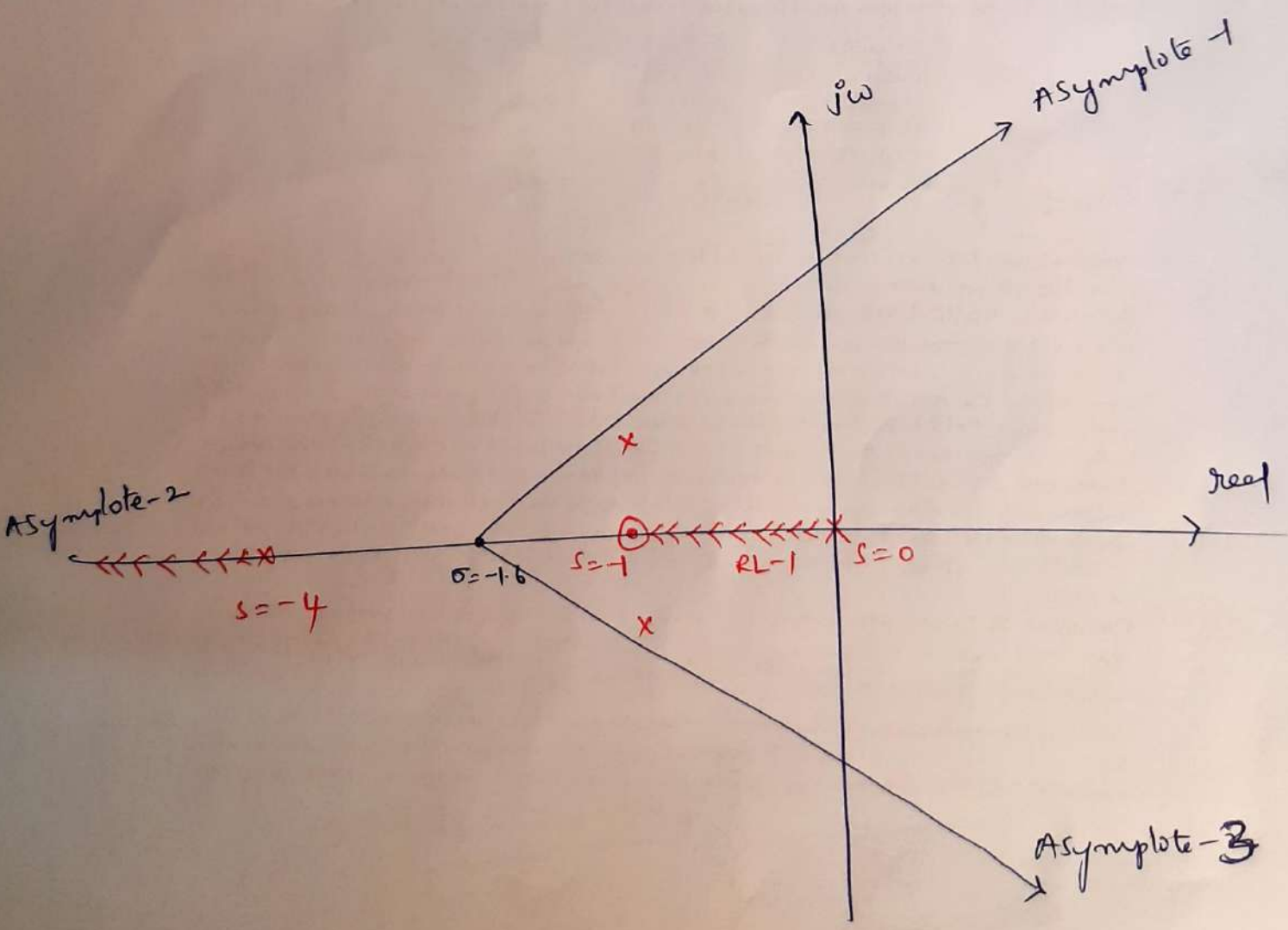
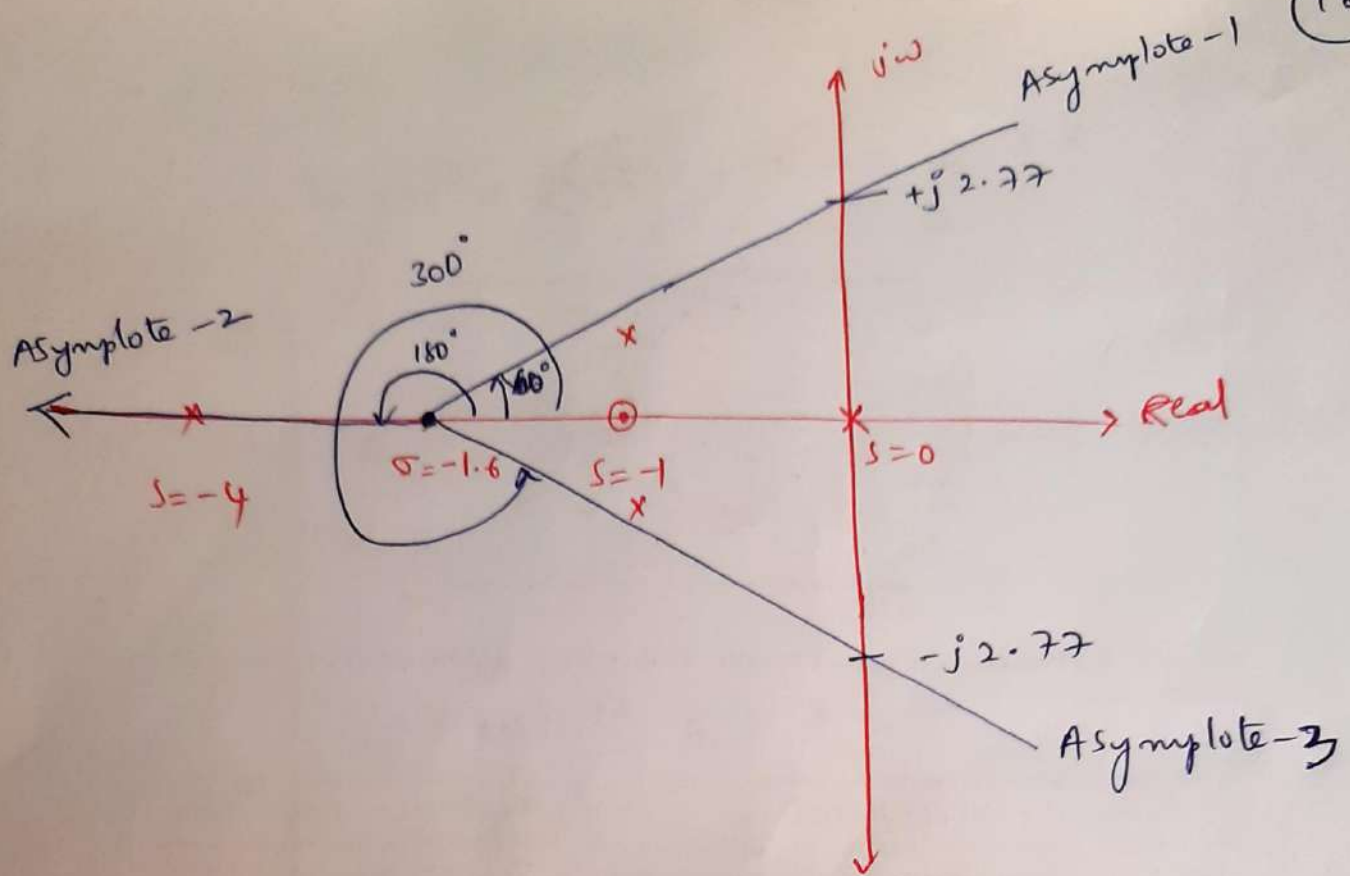
③ $\frac{dK}{ds} = - \left\{ \frac{(s+1)[4s^3 + 18s^2 + 20s + 8] - (s^4 + 6s^3 + 10s^2 + 8s)(1)}{(s+1)^2} \right\}$

④ $\frac{dK}{ds} = 0 \Rightarrow 3s^4 + 16s^3 + 28s^2 + 20s + 8 = 0$

$$s = -2, -2.4731, -0.4301 \pm j 0.5950$$

⑤ All points are not valid.

$\therefore$ There is no Bw/BI points.



R-H criterion

$$\Rightarrow s^4 + 6s^3 + 10s^2 + (K+8)s + K = 0 \Leftarrow C.E$$

s^4	1	10	K	
s^3	6	$K+8$	0	
s^2	$\frac{52-K}{6}$	K	0	
s^1	$\frac{\left(\frac{52-K}{6}\right)(K+8) - 6K}{\left(\frac{52-K}{6}\right)}$	0	0	
s^0	K	0	0	

When $\frac{\left(\frac{52-K}{6}\right)(K+8) - 6K}{\left(\frac{52-K}{6}\right)} = 0$, s^1 row will be

completely zero.

$$(52-K)(6K+48) - 36K = 0$$

$$K^2 - 8K - 416 = 0$$

$$K = 24.78$$

✓
valid

$$K = -16.78$$

X
not valid.

$$AE \Rightarrow s^2 \left(\frac{s^2 - 12}{6} \right) + K = 0$$

$$\text{for } K = 24.78$$

$$s^2 (s^2 - 24.78) + 6(24.78) = 0$$

$$27.22s^2 + 148.68 = 0$$

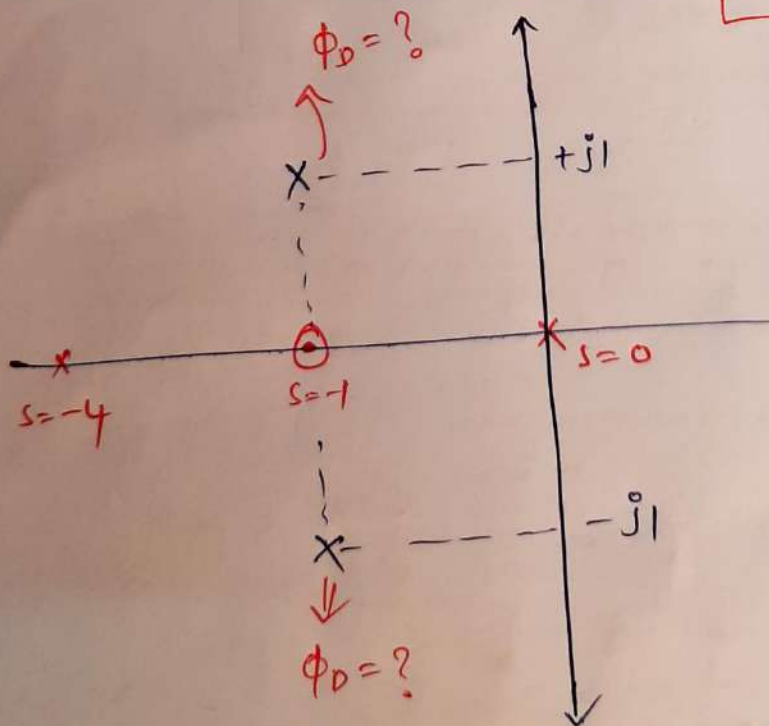
$$\Rightarrow s = \pm j 2.34$$

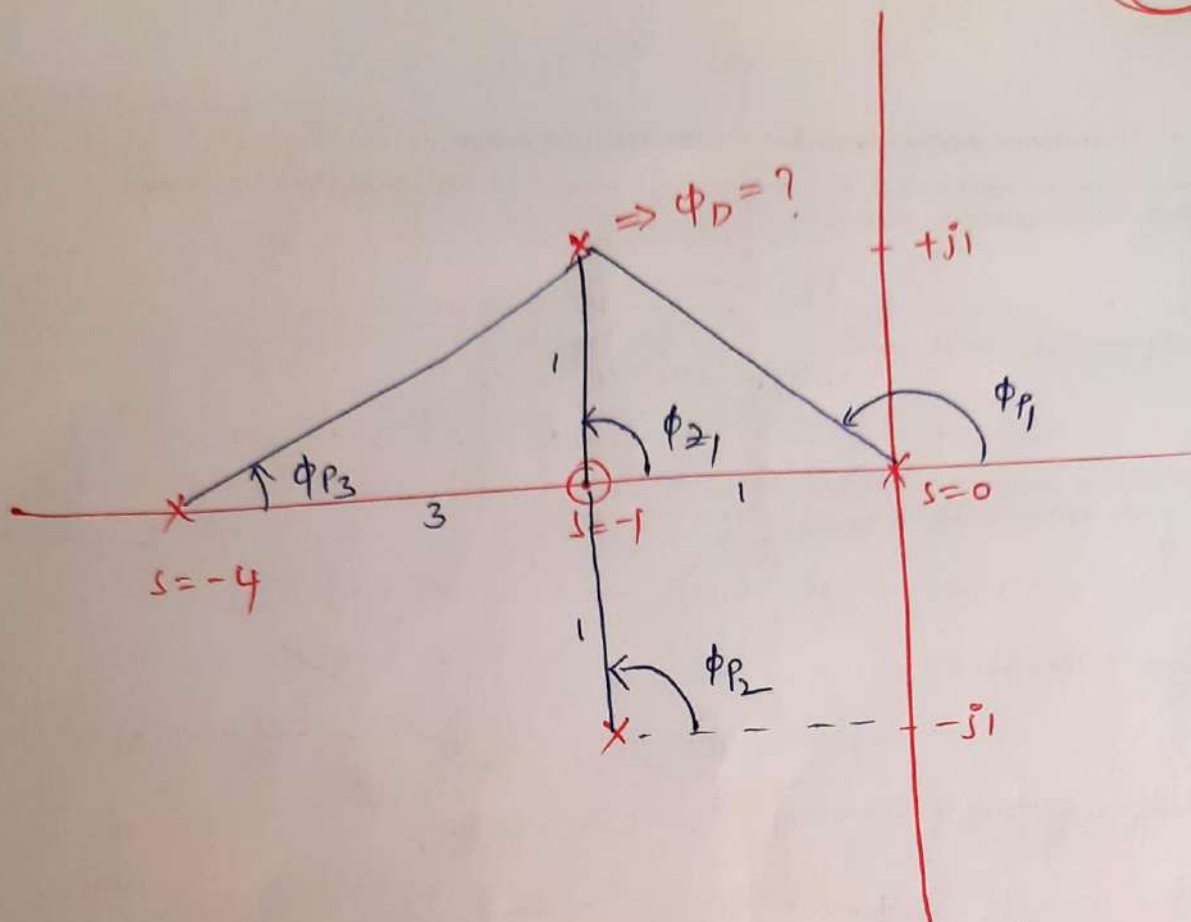
Angle of departure

$\Rightarrow$

$$\phi_D = 180 + \phi$$

$$\phi = \sum \phi_z - \sum \phi_p$$





$$\phi_{P_1} = 180 - \tan^{-1}\left(\frac{1}{1}\right) = 180 - 45^\circ = 135^\circ$$

$$\phi_{P_2} = 90^\circ$$

$$\phi_{P_3} = \tan^{-1}\left(-\frac{1}{3}\right) = 18.4^\circ$$

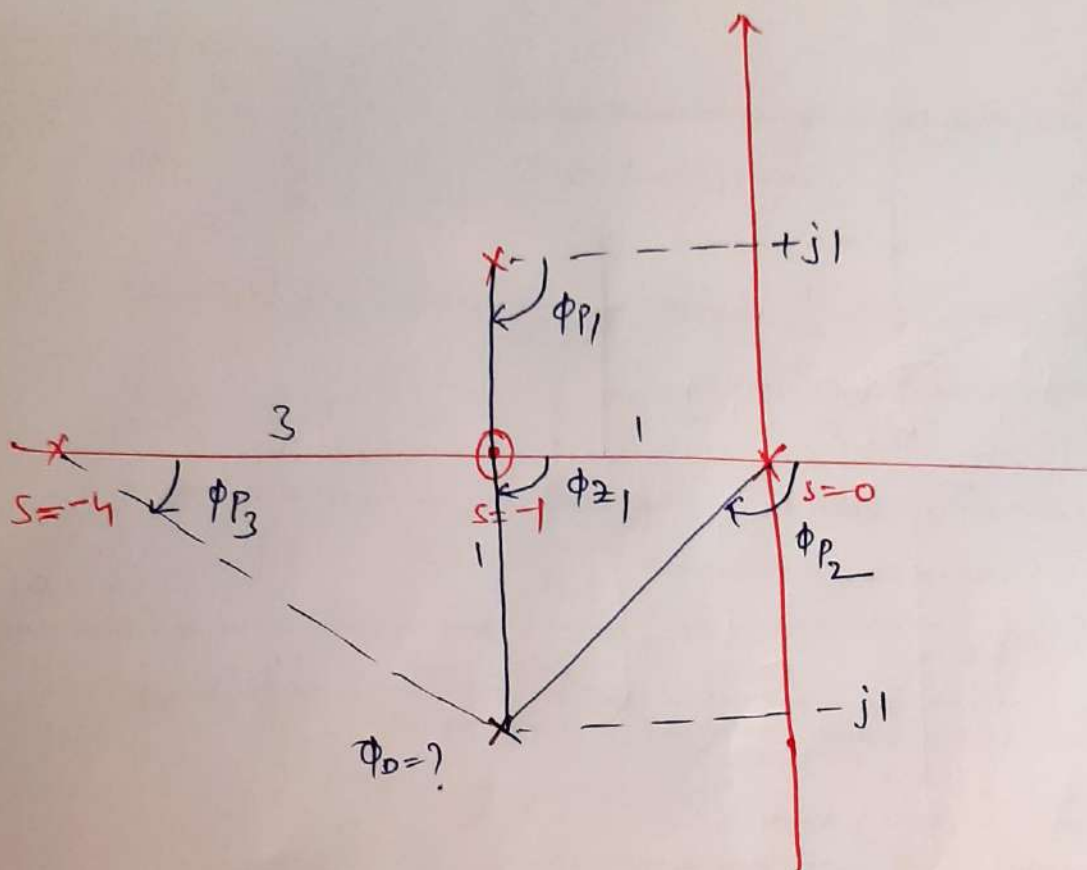
$$\phi_{Z_1} = 90^\circ$$

$$\Rightarrow \phi = \sum \phi_Z - \sum \phi_P$$

$$= 90 - [135^\circ + 90^\circ + 18.4^\circ]$$

$$\phi = -153.4^\circ$$

$$\phi_D = 180 + \phi \Rightarrow \phi_D = 180 - 153.4 \Rightarrow \boxed{\phi_D = 26.6^\circ}$$



$$\phi_{p1} = -90^\circ$$

$$\phi_{p2} = -\left[180 - \tan^{-1}\left(\frac{1}{1}\right)\right] = -135^\circ$$

$$\phi_{p3} = -\left[\tan^{-1}\left(\frac{1}{3}\right)\right] = -18.4^\circ$$

$$\phi_{z1} = -90^\circ$$

$$\phi = \sum \phi_z - \sum \phi_p$$

$$= -90 - [-90 - 135 - 18.4^\circ]$$

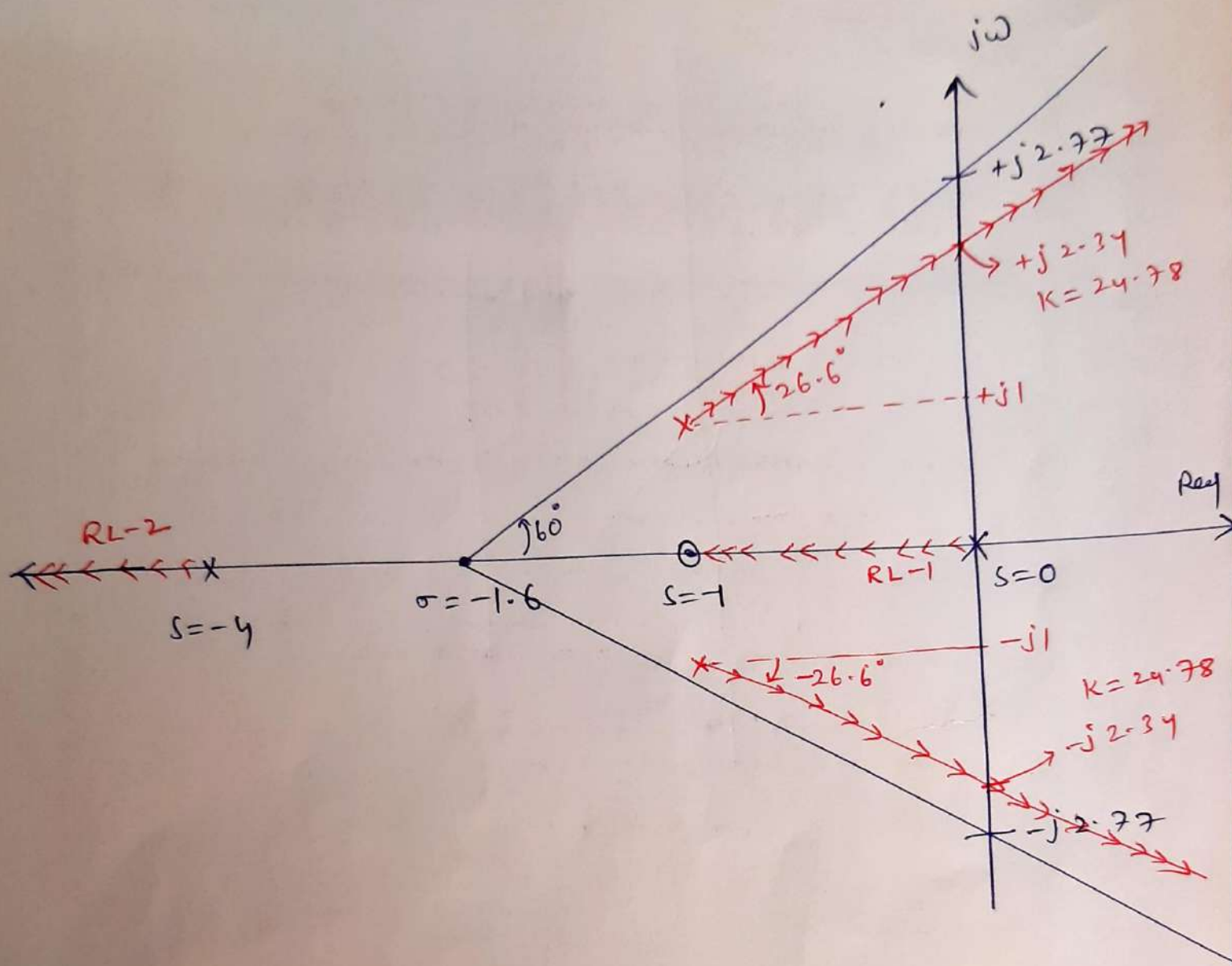
$$= +153.4^\circ$$

$$\phi_D = 180^\circ + \phi = 180^\circ + 153.4^\circ = +333.4^\circ \Rightarrow -26.6^\circ$$

$$\boxed{\phi_D = -26.6^\circ}$$

Anticlockwise
↗

clockwise



Q $G(s)H(s) = \frac{K(s^2 - 2s + 5)}{(s+2)(s-0.5)}$, draw the root

193

Locus and comment on the stability of the system.

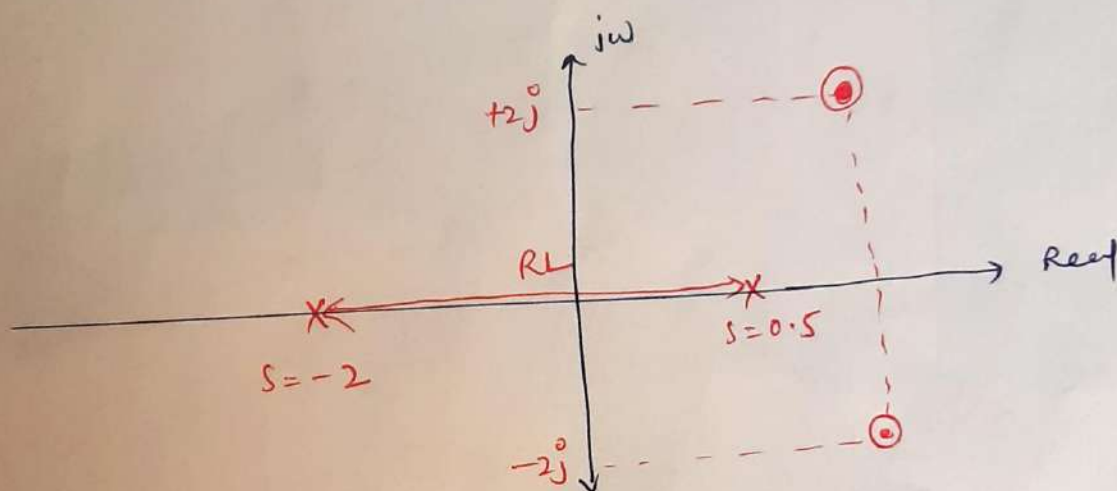
Sol (2) $P = 2, \quad Z = 2$

$\Rightarrow$ No. of RL branches = 2

$\Rightarrow$ No. of RL branches terminating at Zero = 2

$\Rightarrow$ No. of RL " " at infinity = 0.

(3)



poles $\Rightarrow s = -2, 0.5$

zeros $\Rightarrow s = 1 - 2j, 1 + 2j$

(4) No Asymptotes

(5) No Centroid.

(6) BW/BZ points

$\Rightarrow$ (1) $1 + G(s)H(s) = 0$

$s^2 + 1.5s - 1 + K(s^2 - 2s + 5) = 0$

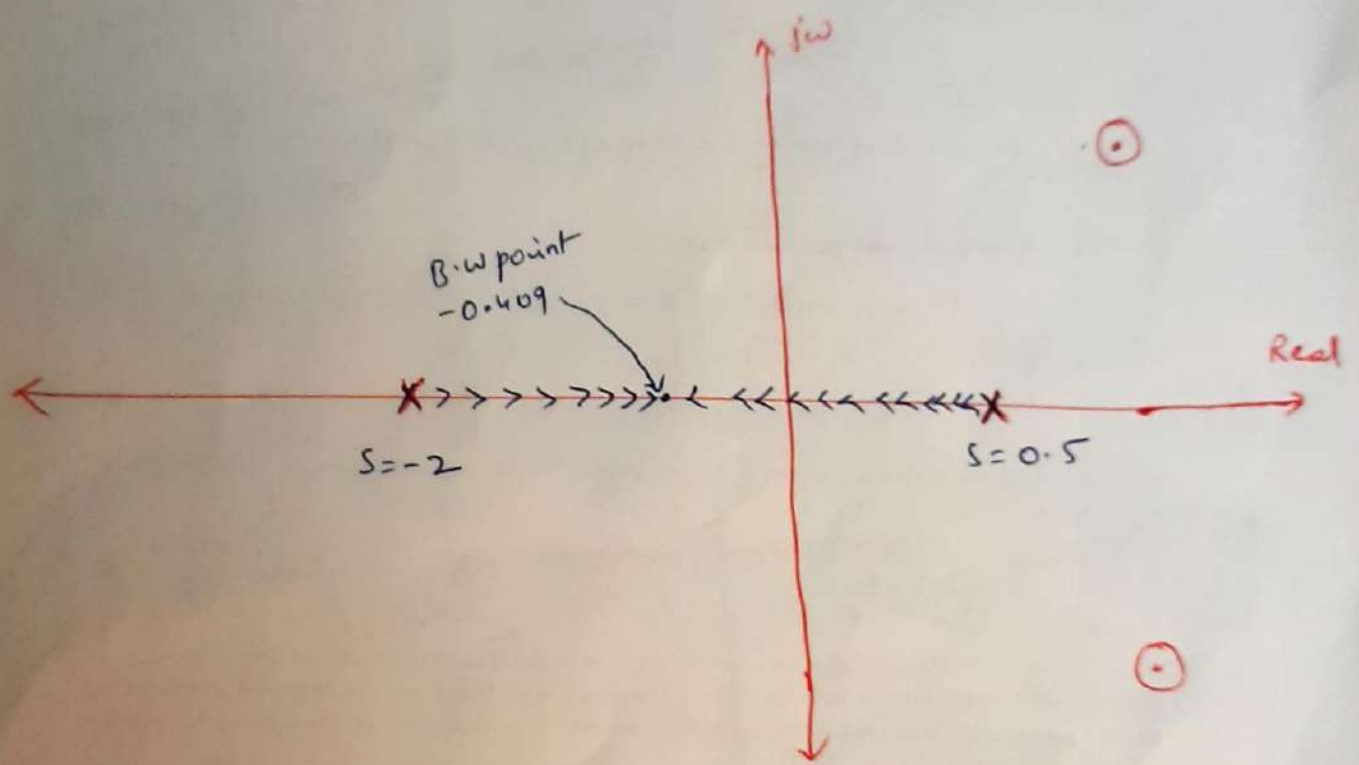
② $K = \frac{-s - 1.5s + 1}{s^2 - 2s + 5}$

③ } $\frac{dK}{ds} = 0 \Rightarrow 3.5s^2 - 12s - 5.5 = 0$
④ }

$\Rightarrow s = -0.409, s = 3.83801$

← not valid

X not valid



$\Rightarrow$ R-H criterion

$s^2(1+K) + (1.5-2K)s + (5K-1) = 0 \Leftarrow C.E$

s^2	$1+K$	$5K-1$
s	$1.5-2K$	0
s^0	$5K-1$	

for $1.5 - 2K = 0$, s now will be completely zero.

$$\Rightarrow \boxed{K = 0.75} \Rightarrow K_{\text{margin}} = 0.75$$

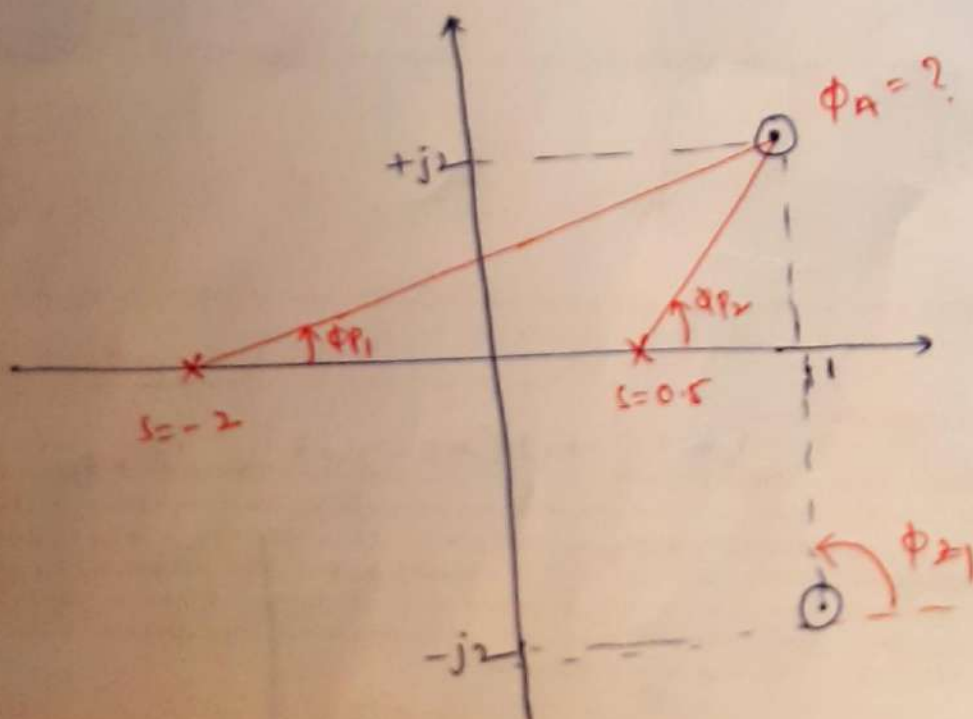
$$\Rightarrow A.E \Rightarrow (1+K)s^2 + (5K-1) = 0$$

$$(1+0.75)s^2 + (5 \times 0.75 - 1) = 0$$

$$\Rightarrow s = \pm j 1.25$$

Rule No. 7

Angle of arrival



$$\phi_{P1} = \tan^{-1} \left(\frac{2}{3} \right) = 34^\circ$$

$$\phi_{P_2} = \tan^{-1} \left(\frac{2}{0.5} \right) = 76^\circ$$

$$\phi_{Z_1} = 90^\circ$$

$$\phi = \sum \phi_z - \sum \phi_p$$

$$= 90 - [34^\circ + 76^\circ]$$

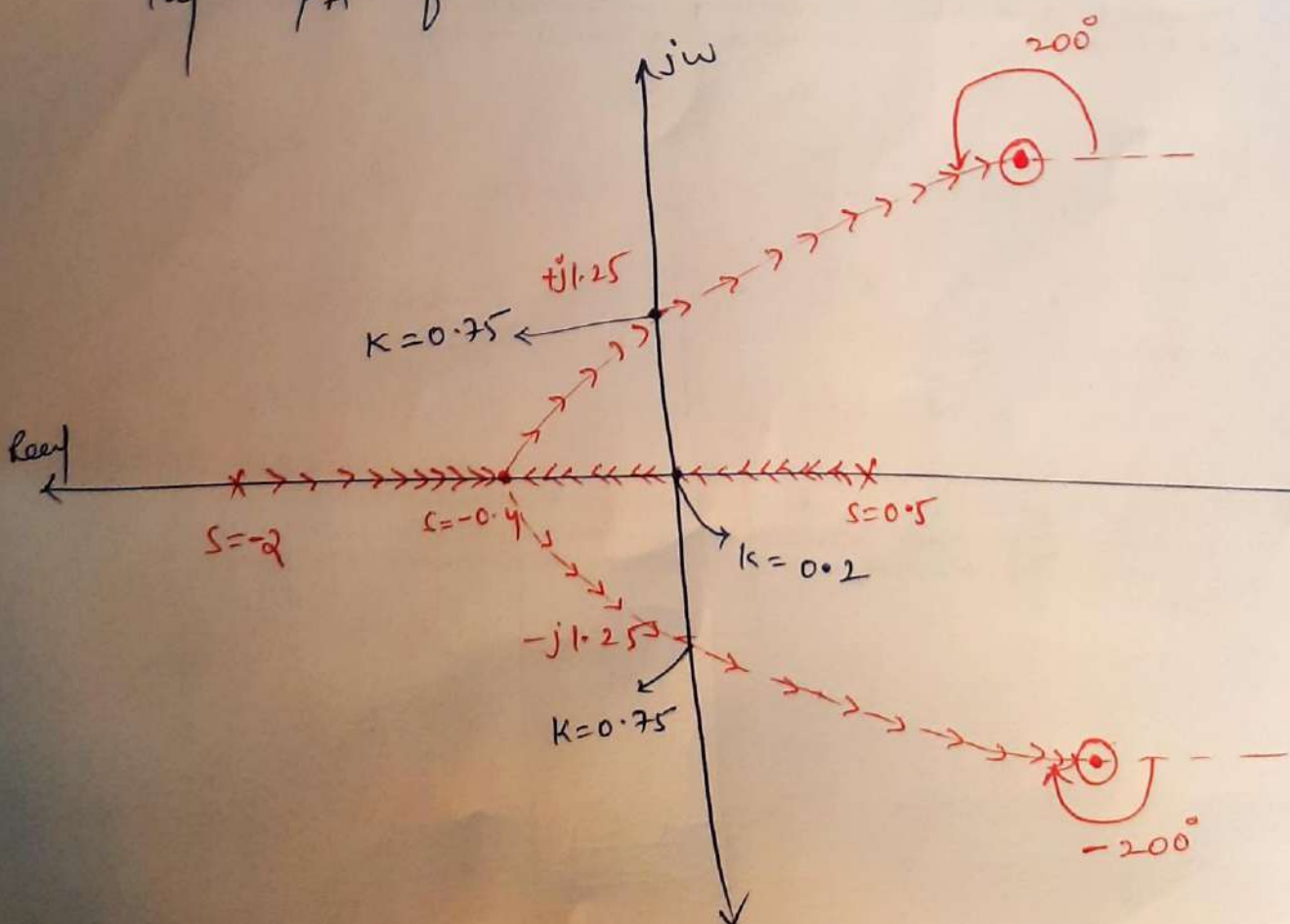
$$= -20^\circ$$

$$\phi_A = 180 - \phi$$

$$= 180 - (-20)$$

$$\phi_A = 200^\circ$$

14 ϕ_A of $-2j+1$ zero is $= -200^\circ$



$\Rightarrow$ For $0.2 < K < 0.75 \Rightarrow$ System is stable.

$\Rightarrow$ For $K = 0.2, 0.75 \Rightarrow$ System marginally stable.

$\Rightarrow$ For $K > 0.75 \Rightarrow$ System unstable.

Q $G(s)H(s) = \frac{K}{s(s+4)(s^2+s+20)}$, draw the root locus diagram and comment on the stability of the system

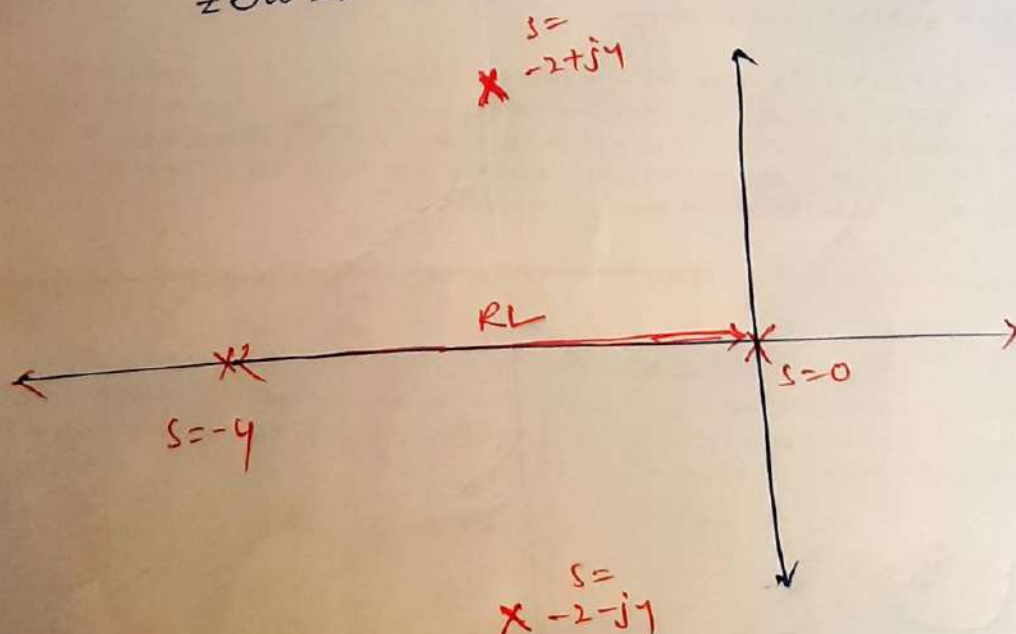
Sol ② $P = 4, \quad Z = 0, \quad P - Z = 4$

No. of RL branches = 4

No. of RL branches terminating at infinity = 4

No. of Asymptotes = 4

③ poly $\Rightarrow s = 0, -4, -2+j4, -2-j4$
zero $\Rightarrow$ none.

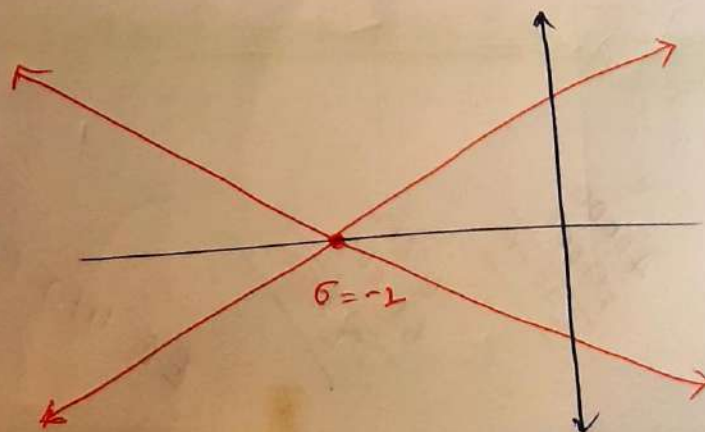
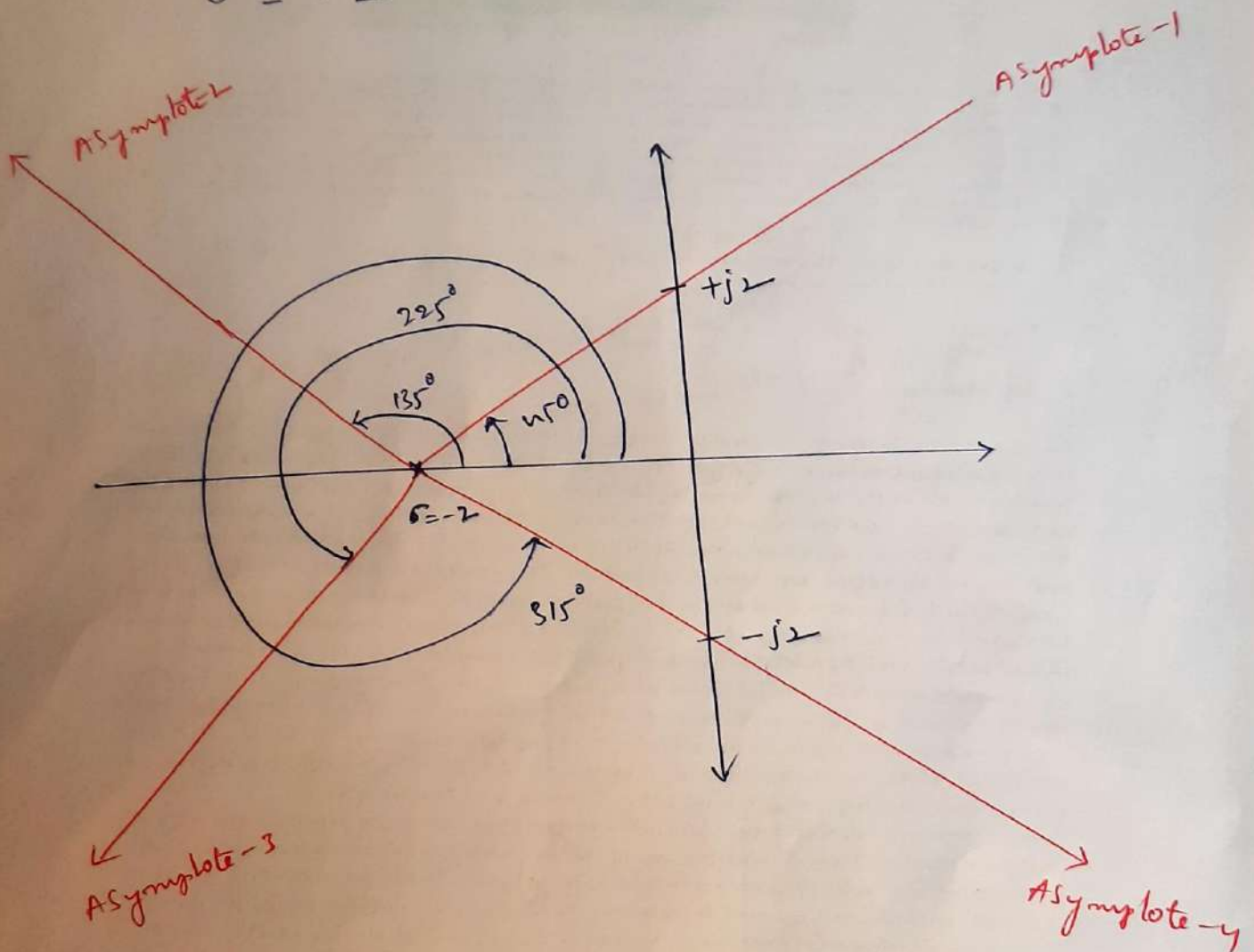


③ $\theta_1 = 45^\circ, \theta_2 = 135^\circ, \theta_3 = 225^\circ, \theta_4 = 315^\circ$

④ $\sigma = \frac{\sum p - \sum z}{p - z}$

$$\sigma = \frac{[0 + (-4) + (-2 + j4) + (-2 - j4)] - [0]}{4}$$

$$\sigma = -2$$



① $1 + n(s) H(s) = 0$

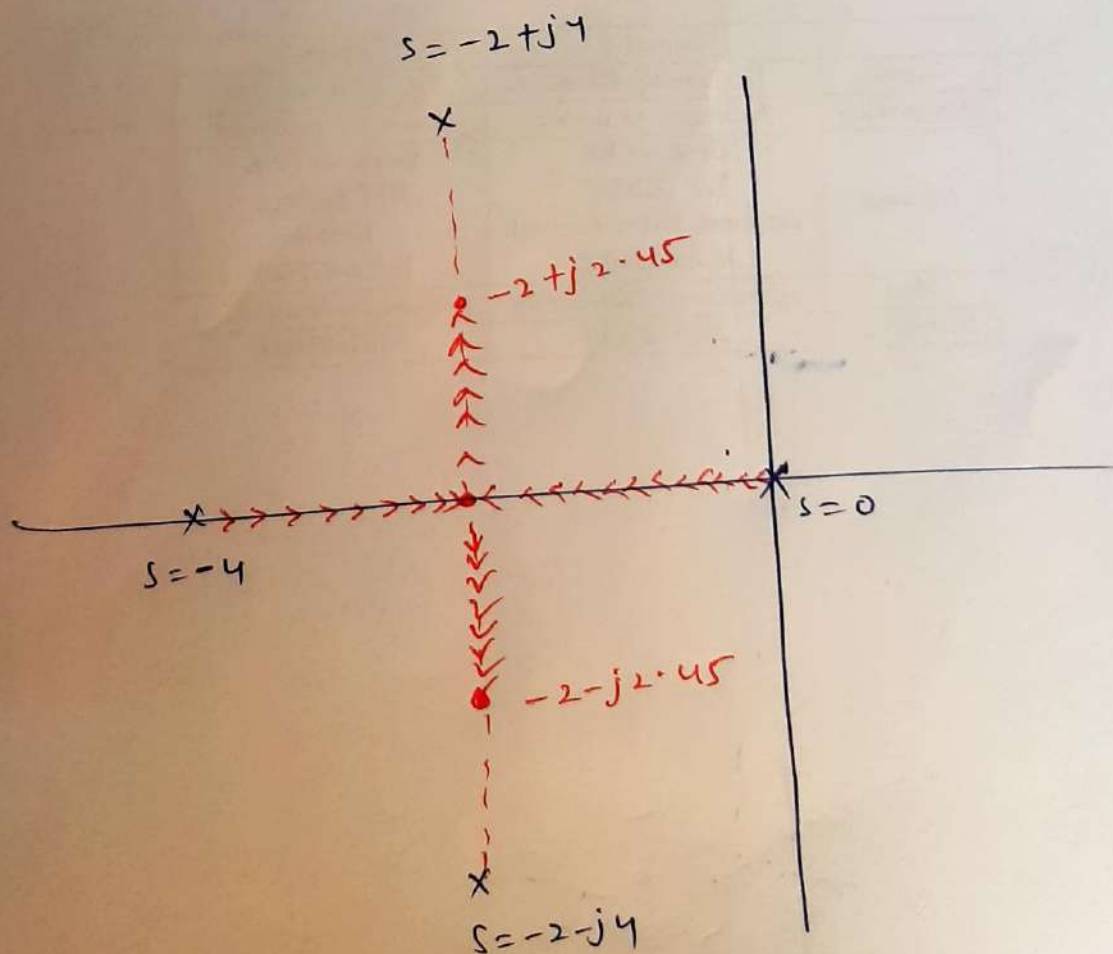
$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

② $K = -s^4 - 8s^3 - 36s^2 - 80s.$

③ } $\frac{dK}{ds} = 0 \Rightarrow -4s^3 - 24s^2 - 72s - 80 = 0$
 ④ }

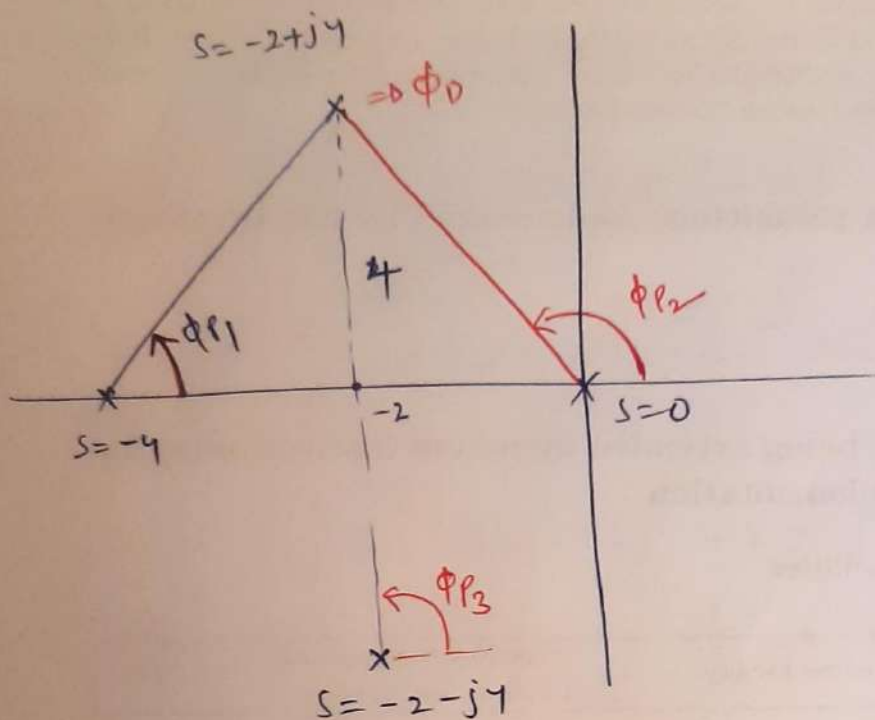
$$s = -2, -2 \pm j 2 \cdot 45$$

⑤ $s = -2, -2 \pm j 2 \cdot 45 \Rightarrow$ Both are valid.



Angle of departure

200



$$\phi_{p1} = \tan^{-1}\left(\frac{4}{2}\right) = 63.4^\circ$$

$$\phi_{p2} = 180 - \tan^{-1}\left(\frac{4}{2}\right) = 116.6^\circ$$

$$\phi_{p3} = 90^\circ$$

$$\phi = \sum \phi_z - \sum \phi_p$$

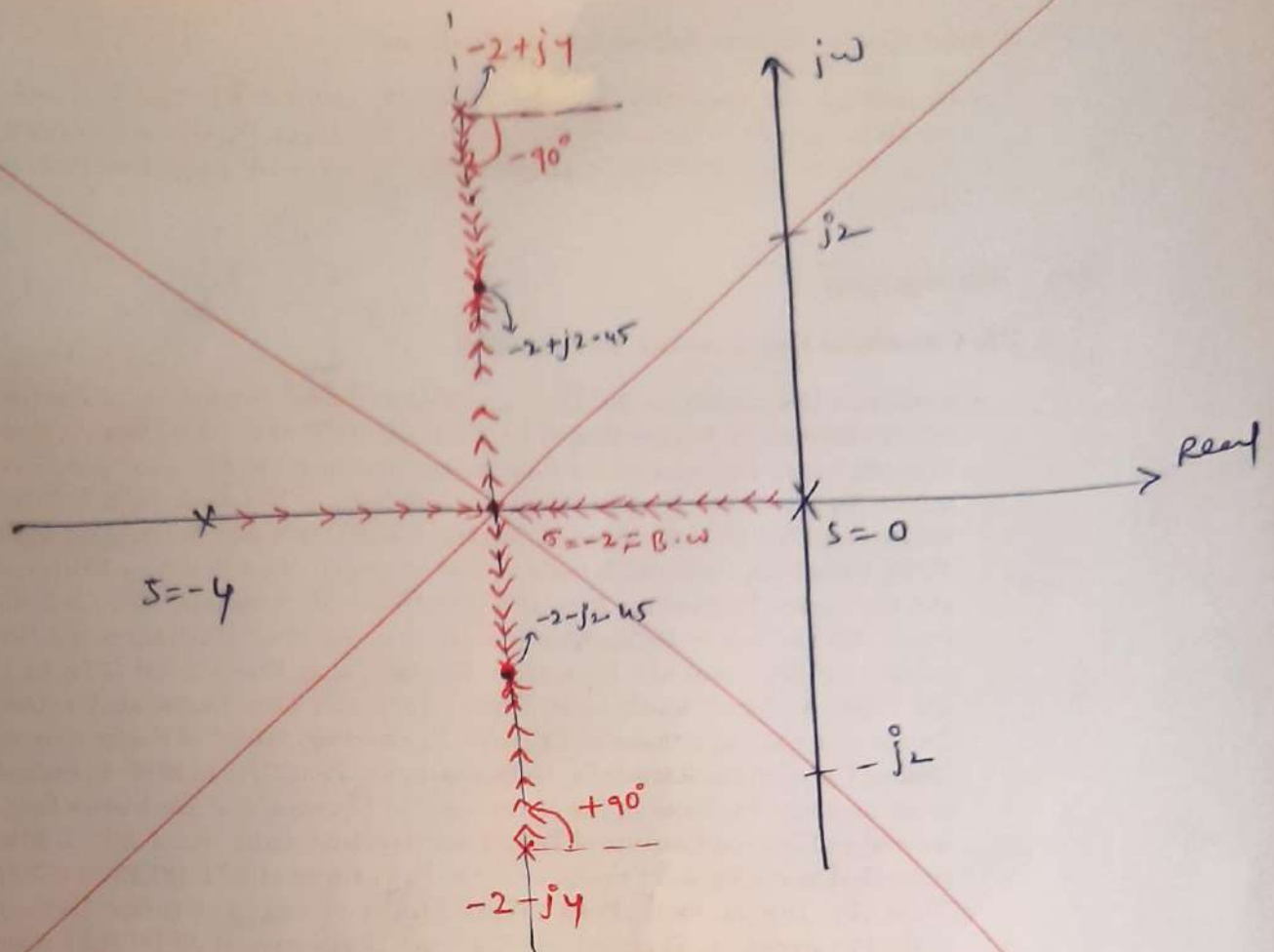
$$= 0 - [90 + 116.6^\circ + 63.4^\circ]$$

$$\phi = -270^\circ$$

$$\phi_D = 180 + \phi \Rightarrow \phi_D = 180 - 270^\circ$$

$$\boxed{\phi_D = -90^\circ}$$

My ϕ_D at $-2-j4$ will be $\boxed{\phi_D = +90^\circ}$



R-H criterion

$$CE = s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

s^4	1	36	K
s^3	8	80	0
s^2	26	K	0
s^1	$\frac{2080 - 8K}{26}$	0	
s^0	K		

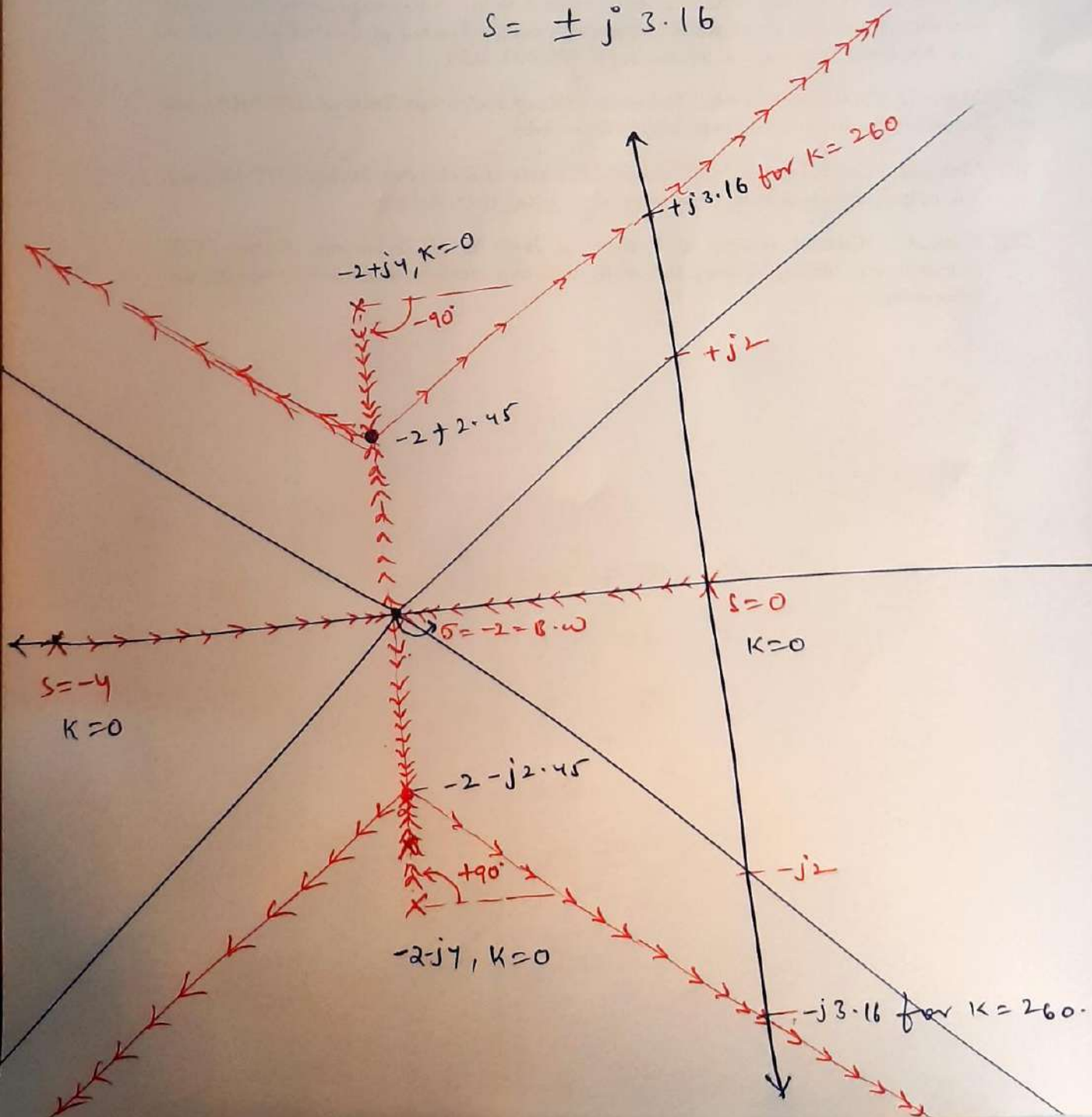
$\Rightarrow$ For $\frac{2080 - 8K}{26} = 0$, s' how will be zero.

$\Rightarrow K = 260 = K_{margin}.$

$\Rightarrow AE \Rightarrow 26s^2 + 260 = 0$

$$s = \pm j\sqrt{10}$$

$$s = \pm j^{\circ} 3.16$$



For $K=0, 260 \Rightarrow$ System marginally stable.

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For $0 < K < 260 \Rightarrow$ System stable.

For $K > 260 \Rightarrow$ System unstable.

H.W Questions

①. $G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+5)}$, draw the RL and comment on the stability

②. $G(s)H(s) = \frac{K}{s(s+3)(s^2+3s+4.5)}$, draw the RL and comment on the stability.