

18/01/2022

UNIT-2 DISCRETE TIME FOURIER TRANSFORM

DTFT:-

$x(t)$ is a signal if we apply FT on $x(t)$ is $F\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt$
this is in the time domain and which is continuous.

$x[n] \rightarrow$ discrete aperiodic signal then the discrete time fourier transform for a signal $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jn\omega}$$

$X(e^{j\omega})$ is a complex quantity, $X(e^{j\omega}) = X_r(e^{j\omega}) + jX_i(e^{j\omega})$

Polar form, $X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$ (Phase)
(magnitude)

Properties that hold by DTFT :- $= M(\omega)e^{j\theta(\omega)}$

1) if $e^{j\omega}$ is periodic it holds periodicity.

$\cos\omega + jsin\omega$ is periodic with $2\pi k$.

$e^{j\omega} = e^{j(\omega \pm 2\pi k)}$ So, we can say $e^{j\omega}$ is periodic

2) Magnitude is even ($|X(e^{j\omega})| \rightarrow \text{even}$) [$X_r(e^{j\omega})$ is even]

if $x[n]$ is even function then $x[n] = x[-n]$

$$|X(e^{-j\omega})| = |X(e^{j\omega})| \quad \left[\begin{array}{l} \because |a+ib| = \sqrt{a^2+b^2} \\ |a-ib| = \sqrt{a^2+b^2} \end{array} \right]$$

3) Phase is an odd function ($\angle X(e^{j\omega}) \rightarrow \text{odd}$)

if $x[n]$ is odd function then $x[n] = -x[-n]$

$X_i(e^{j\omega})$ is also odd

4) θ is also periodic.

Rule :-

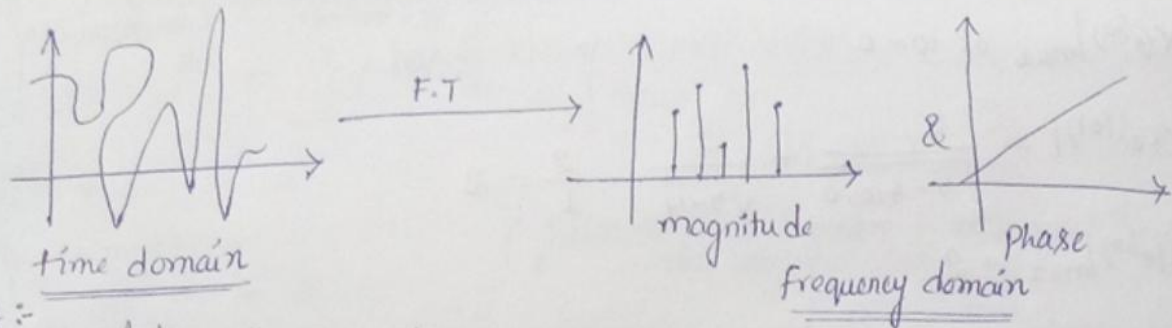
$\rightarrow$ Aperiodic in time domain $\rightarrow$ periodic in frequency domain

$\rightarrow$ Discrete in time domain $\rightarrow$ continuous in frequency domain

$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega})$ (periodic & continuous in frequency domain)
(Aperiodic & discrete in time domain)

Why frequency transform?

Every signal have different frequencies in them but in time domain we will not able to see the frequency content. that's why we are converting time domain into frequency domain.



ex:-
 $\Rightarrow$ convolution in time domain $\longleftrightarrow$ Product in frequency domain.
 it is one of the application of Fourier transform.

$y[n] = x[n] * h[n]$ it is difficult for some cases. So, we do $X(e^{j\omega}) X_H(e^{j\omega})$

$\rightarrow$ Design, analysis and synthesis are easy in frequency domain.
 $= Y(e^{j\omega}) = F^{-1}(Y(e^{j\omega})) = y[n]$

ex1:-

$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-jn\omega}$$

$$= \delta[0] e^{-j(0)\omega}$$

$$= \delta[0]$$

$$= 1$$

$$\delta[n] \xleftrightarrow{F.T.} 1$$

ex2:-

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] e^{-jn\omega}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u[n] e^{-jn\omega}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-jn\omega}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n$$

$$= 1 + \frac{1}{2} e^{-j\omega} + \left(\frac{1}{2} e^{-j\omega}\right)^2 + \dots$$

$$[\because a + ar + ar^2 + \dots = \frac{a}{1-r}]$$

$$a = 1 \quad r = \frac{1}{2} e^{-j\omega}$$

$$= \frac{a}{1-r} = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

ex3:-

Try to analyze this

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} [\cos\omega - j\sin\omega]}$$

$$|X(e^{j\omega})| = \left| \frac{1}{(1 - \frac{1}{2}\cos\omega) + j\frac{1}{2}\sin\omega} \right|$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1 - \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} = \frac{1}{\sqrt{1 + \frac{1}{4}\cos^2\omega - \cos\omega + \frac{1}{4}\sin^2\omega}} = \frac{1}{\sqrt{1 - \cos\omega + \frac{1}{4}}}$$

$$|X(e^{j\omega})| = \frac{2}{\sqrt{5 - 4\cos\omega}}$$

$$x[n] = \alpha^n u[n] \xleftrightarrow{F.T.} \frac{1}{1 - \alpha e^{-j\omega}}$$

$$X(e^{j\omega})|_{\max} \text{ at } \omega=0$$

$$|X(e^{j(0)})| = \frac{2}{\sqrt{5-4\cos 0}} = \frac{2}{\sqrt{5-4}} = \frac{2}{1} = 2$$

$$X(e^{j\omega})|_{\max} = 2$$

$$\text{if } \omega = \pm\pi \rightarrow X(e^{j\omega}) = \frac{2}{3}$$

$$\angle \phi(e^{j\omega}) = \angle 0 - \tan^{-1} \left[\frac{\frac{1}{2} \sin \omega}{1 - \frac{1}{2} \cos \omega} \right] \quad \left[a+ib \Rightarrow \tan^{-1} \left(\frac{b}{a} \right) \right] \quad \text{Phase}$$

Inverse DTFT:-

$$x[n] \xleftrightarrow{\text{FT}} X(e^{j\omega})$$

$$X(e^{j\omega}) \xleftrightarrow{\text{IFT}} x[n]$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Consider, $x[n]$ is discrete aperiodic signal.

$$\text{DTFT of } x[n] \Rightarrow X[\omega] = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Apply integrating on both sides with limits $-\pi$ to $+\pi$ and multiplied by $e^{j\omega m}$

$$\int_{-\pi}^{\pi} X(\omega) e^{+j\omega m} d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} [x[n] e^{-j\omega n}] e^{j\omega m} d\omega$$

$x[n]$ holds the property of convergence of uniform convergence.

$$X_N(\omega) = \lim_{N \rightarrow \infty} \left[\sum_{n=-N}^{+N} x[n] e^{-j\omega n} \right] \rightarrow X(\omega)$$

convergence then we can interchange integration and summation.

R.H.S:-

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} x[n] e^{+j\omega(m-n)} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\pi}^{\pi} e^{+j\omega(m-n)} d\omega$$

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \int_{-\pi}^{\pi} e^{-j\omega(n-m)} d\omega = \int_{-\pi}^{\pi} d\omega = 2\pi$$

if $m \neq n$

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = 0 \quad \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi & m=n \\ 0 & m \neq n \end{cases}$$

RHS :- $\sum_{n=-\infty}^{+\infty} x[n] \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = 2\pi x[m]$ here, n is only exists at m
So, $x[n] = x[m]$

$$\int_{-\pi}^{\pi} x(\omega) e^{j\omega m} d\omega = 2\pi x[m]$$

$$x[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\omega) e^{j\omega m} d\omega$$

Convergence :-

Let us take $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$X_K(e^{j\omega}) = \sum_{n=-K}^{+K} x[n] e^{-j\omega n}$$

$$\lim_{K \rightarrow \infty} X_K(e^{j\omega}) \xrightarrow{\text{converges to}} X(e^{j\omega})$$

$$\lim_{K \rightarrow \infty} |X(e^{j\omega}) - X_K(e^{j\omega})| \rightarrow 0 \text{ (converge)}$$

$\downarrow$ absolute FT $\downarrow$ Kth FT

necessary condition for DTFT to exists is $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$

[absolutely summable]

Sufficient condition is $|X(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$ to exist DTFT.

With this two conditions we can say that we have a DTFT and also we can say that uniform convergence exists.

$x[n] = \frac{1}{n} u[n-1]$ whether DTFT exists or not.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{n} u[n-1] e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-j\omega n}$$

For finding existence of DTFT,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \right| < \infty \quad \lim_{n \rightarrow \infty} s_n \rightarrow \infty$$

it is divergence. So it is not ^{absolutely} summable.

$\therefore$ Not convergence and not exists.

$$* \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

* if $x[n]$ is not summable but square of $x[n]$ has fourier transform

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad (\text{Square summable})$$

Energy of a signal is finite then for that a FT exists. it is a little bit sufficient condition.

$$\lim_{K \rightarrow \infty} \underbrace{\int_{-\pi}^{\pi} \underbrace{|x_K(e^{j\omega}) - x(e^{j\omega})|^2}_{\text{error}} d\omega}_{\text{mean square error}} \xrightarrow{\text{energysignal}} 0$$

ϵ_x produce finite nature indicating presence of energy and DTFT exists.

20/01/2022

Properties of DTFT:-

1) Linearity:-

→ DTFT is a linear operation.

→ It satisfies superposition principle.

$$\text{if } F\{x_1[n]\} = X_1(e^{j\omega})$$

$$F\{x_2[n]\} = X_2(e^{j\omega})$$

then,

$$F\{ax_1[n] + bx_2[n]\} = ax_1(e^{j\omega}) + bx_2(e^{j\omega})$$

Proof :-

Consider,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (ax_1[n] + bx_2[n])e^{-j\omega n} \\ &= a \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n} \\ &= \boxed{ax_1(e^{j\omega}) + bx_2(e^{j\omega})} \end{aligned}$$

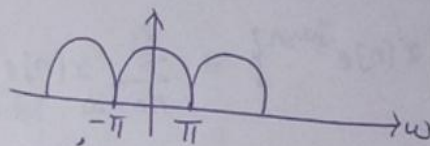
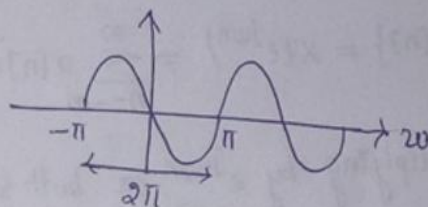
2) Periodicity:-

DTFT $\{X(e^{j\omega})\}$ is periodic in ω with period 2π .

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)})$$

where k is an integer



$$e^{j\omega} = \cos \omega + j \sin \omega$$

3) Time shifting property:-

if $F\{x[n]\} = X(e^{j\omega})$

$$F\{x[n-k]\} = e^{-j\omega k} \cdot X(e^{j\omega})$$

Shifting in time domain results in multiplying the transform by $e^{-j\omega k}$.

Proof :-

$$\begin{aligned} \text{consider, } X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{+\infty} x[n-k]e^{-j\omega n} \end{aligned}$$

$$\text{put } n-k = p$$

$$n = k+p$$

$$\begin{aligned} \text{limits :- } n = -\infty &\Rightarrow -\infty - k = p \Rightarrow p = -\infty \\ n = \infty &\Rightarrow \infty - k = p \Rightarrow p = \infty \end{aligned}$$

$$\begin{aligned}
 &= \sum_{p=-\infty}^{\infty} x[p] e^{-j\omega(p+k)} \\
 &= \sum_{p=-\infty}^{\infty} x[p] e^{-j\omega p} \cdot e^{-j\omega k} \\
 &= \boxed{x(e^{j\omega}) \cdot e^{-j\omega k}}
 \end{aligned}$$

4) Frequency shifting property :-

if $F\{x[n]\} = X(e^{j\omega})$ then

$$F\{x[n]e^{j\omega_0 n}\} = X(e^{j(\omega-\omega_0)})$$

Shifting of the signal in frequency domain results in multiplying the signal by $e^{j\omega_0 n}$.

Proof :-

$$F\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

multiplying by $e^{j\omega_0 n}$ on both sides.

$$F\{x[n]e^{j\omega_0 n}\} = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega_0 n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_1 n} \quad [\because \omega_1 = \omega - \omega_0]$$

$$= X(e^{j\omega_1}) \text{ (or) } X(e^{j(\omega-\omega_0)})$$

hence proved.

5) Time reversal :-

if $F\{x[n]\} = X(e^{j\omega})$ then

$$\boxed{F\{x[-n]\} = X(e^{-j\omega})}$$

Proof :-

$$F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

For a time reversal signal,

$$F\{x[-n]\} = \sum_{n=-\infty}^{+\infty} x[-n] e^{-j\omega[n]}$$

Put $-n = p$

$$n = -p$$

$$= \sum_{p=-\infty}^{\infty} x[p] e^{-j\omega[-p]}$$

$$= \sum_{p=-\infty}^{\infty} x[p] \cdot e^{j\omega p}$$

$$= \sum_{p=-\infty}^{\infty} x[p] e^{-j(\omega)p}$$

$$= \underline{x(e^{-j\omega})}$$

6) Differentiation in frequency:-

$$\text{if } F\{x[n]\} = x(e^{j\omega})$$

$$F\{nx[n]\} = j \frac{d}{d\omega} (x(e^{j\omega}))$$

Proof:-

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

differentiate w.r to ω on both sides.

$$\frac{d}{d\omega} [x(e^{j\omega})] = \frac{d}{d\omega} \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right]$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{d}{d\omega} (e^{-j\omega n})$$

$$= \sum_{n=-\infty}^{\infty} x[n] (-jn) e^{-j\omega n}$$

$$\frac{d}{d\omega} [x(e^{j\omega})] = -j \sum_{n=-\infty}^{\infty} x[n] \cdot n e^{-j\omega n}$$

$$\frac{1}{-j} \frac{d}{d\omega} (x(e^{j\omega})) = \sum_{n=-\infty}^{\infty} n \cdot x[n] e^{-j\omega n}$$

$$\frac{j}{j} x \left(-\frac{1}{j} \right) \frac{d}{d\omega} [x(e^{j\omega})] = \sum_{n=-\infty}^{\infty} nx[n] e^{-j\omega n}$$

$$\underline{j \frac{d}{d\omega} [x(e^{j\omega})] = \sum_{n=-\infty}^{\infty} nx[n] e^{-j\omega n}}$$

7) Convolution Property:- (in time domain)

$$F\{x_1[n] * x_2[n]\} \Leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

Convolution in time domain = Product of the signals in frequency domain.

$$F\{x_1[n]\} = X_1(e^{j\omega})$$

$$F\{x_2[n]\} = X_2(e^{j\omega})$$

Proof:-

$$F\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) e^{-j\omega n}$$

Convolution formula, $x_1[n] * x_2[n] = \sum_{k=-\infty}^{+\infty} x_1[k] \cdot x_2[n-k]$

$$F\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k] \cdot e^{-j\omega n}$$

put $n-k = p$

$n = p+k$

limits $p = -\infty$ to ∞ .

$$= \sum_{p=-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[p] \cdot e^{-j\omega(p+k)}$$

$$= \sum_{p=-\infty}^{\infty} x_2[p] e^{-j\omega p} \cdot \sum_{k=-\infty}^{\infty} x_1[k] e^{-j\omega k}$$

$$= \underline{X_2(e^{j\omega}) \cdot X_1(e^{j\omega})}$$

8) Convolution in frequency domain:-

$$F\{x_1[n] \cdot x_2[n]\} \Leftrightarrow X_1(e^{j\omega}) * X_2(e^{j\omega})$$

Proof:-

Consider RHS,

convolution to RHS, $\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega}) \cdot X_2(e^{j(\omega-v)}) \cdot dv \rightarrow \textcircled{C}$

$$[x_1(t) * h(t)] = \int x_1(\tau) h(t-\tau) d\tau$$

Consider LHS, $F\{x_1[n] \cdot x_2[n]\} = \sum_{n=-\infty}^{\infty} (x_1[n] \cdot x_2[n]) e^{-j\omega n} \rightarrow \textcircled{D}$

$$x_1[n] \xleftrightarrow{FT} X_1(e^{j\omega})$$

$$F^{-1}\{X_1(e^{j\omega})\} \xrightarrow{IFT} x_1[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jv}) e^{jv n} dv \rightarrow (2)$$

Substitute (2) in eq(1)

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jv}) e^{jv n} dv \cdot x_2[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jv}) e^{jv n} dv \cdot x_2[n] e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jv}) \left[\sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\omega n - v n)} \right] dv$$

$$\underbrace{\hspace{10em}}_{X_2(e^{j(\omega-v)})}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jv}) \cdot X_2(e^{j(\omega-v)}) dv \rightarrow (3)$$

Compare eq(2) and (3)

$$LHS = RHS.$$

Hence proved.

9) Modulation Theorem :-

Modulation :-

Voice signal frequency around in kHz when we try to transmit to longer distances through wireless ^{faraway} it can't transmit bcoz, frequency is very small.

So, we will take a MHz signal $\rightarrow$ carrier signal. This will carry the voice signal on its back. we will try to impose carrier signal which is in MHz's range to a voice signal and then carrier signal carry the voice signal to receiver (somewhere faraway). And this process of adding a carrier signal to voice signal is called as Modulation. Remove of carrier signal from message signal/voice signal called Demodulation.

if $F\{x[n]\} = X(e^{j\omega n})$ then

$$F\{x[n] \cdot \cos \omega_0 n\} = \frac{1}{2} [X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)})]$$

We know that, $\cos \omega = \frac{e^{j\omega} + e^{-j\omega}}{2}$

Consider LHS,

$$\begin{aligned} F\{x[n] \cos \omega_0 n\} &= \sum_{n=-\infty}^{\infty} x[n] \cos \omega_0 n e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right] e^{-j\omega n} \\ &= \frac{1}{2} \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega - \omega_0)n} + \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + \omega_0)n} \right] \\ &= \frac{1}{2} \left[X(e^{j(\omega - \omega_0)}) + X(e^{-j(\omega + \omega_0)}) \right] \\ &= \text{RHS} \end{aligned}$$

Hence proved.
21/01/2022

10) Correlation theorem :- (similarities b/w two signals)

if $F\{x_1[n]\} = X_1(e^{j\omega})$ and $F\{x_2[n]\} = X_2(e^{j\omega})$ * if we are checking similarities b/w two different signals is called cross correlation

$$F\{x_1 x_2(l)\} = X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

Correlation is similar to convolution bcoz, those two result product of signals in frequency domain.

* if it is b/w same sequence and its delayed version then it is called as auto correlation.

11) Symmetry properties :-

The Fourier transform $X(e^{j\omega})$ is complex function of ω .

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \rightarrow \textcircled{1}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X_R(e^{j\omega}) + jX_I(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] [\cos \omega n - j \sin \omega n]$$

$$X_R(e^{j\omega}) = x[n] \cos \omega n$$

$$X_I(e^{j\omega}) = x[n] \sin \omega n$$

Even symmetry:-

$$x_R(e^{-j\omega}) = x_R(e^{j\omega})$$

i.e., $\cos(-\omega n) = \cos \omega n$

$$\begin{aligned} x_R(e^{-j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] \cos(-\omega n) \\ &= \sum_{n=-\infty}^{\infty} x[n] \cos(\omega n) \\ &= x_R(e^{j\omega}) \end{aligned}$$

* Real part of FT is even symmetry

* Imaginary part of FT is odd symmetry

Odd symmetry:-

$$x_I(e^{-j\omega}) = -x_I(e^{j\omega})$$

i.e., $\sin(-\omega n) = -\sin \omega n$

Proof:-

$$\begin{aligned} x_I(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] \sin \omega n \\ \omega \rightarrow -\omega \quad x_I(e^{-j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] \sin(-\omega n) \\ &= - \sum_{n=-\infty}^{\infty} x[n] \sin \omega n \\ &= -x_I(e^{j\omega}) \end{aligned}$$

12) Parseval's theorem:-

It is similar to conservation of energy.

$$x[n] \xleftrightarrow{\text{DTFT pair}} X(e^{j\omega})$$

The sequence may change into different formats after Fourier transform but the energy of the sequence should be same. This is according to conservation of energy.

if $F\{x[n]\} = X(e^{j\omega})$ then

$$E = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

energy of signal before transform = energy of signal after transform

Proof:-

forward DTFT, $F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

inverse DTFT, $F^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ } DTFT pair

LHS:-

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n] x^*[n] \quad \because [(a+ib)^* = a-ib]$$

using inverse DTFT

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right]^*$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(e^{j\omega}) \cdot e^{-j\omega n} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(e^{j\omega}) e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(e^{j\omega}) \cdot \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^*(e^{j\omega}) \cdot x(e^{j\omega}) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(e^{j\omega})|^2 d\omega$$

$$= \text{RHS}$$

1. Find the DTFT of the given signal, $x[n] = \{1, -1, 2, 3\}$

Sol:- Given, $x[n] = \{1, -1, 2, 3\}$
 $\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 1 & 2 & 3 \\ x(0) & x(1) & x(2) & x(3) \end{array}$

The fourier transform of $x[n]$ is $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$X(e^{j\omega}) = x(0)(1) + x(1)e^{-j\omega} + x(2)e^{-j\omega 2} + x(3)e^{-j\omega 3}$$

$$= 1 + (-1)e^{-j\omega} + 2e^{-j\omega 2} + 3e^{-j\omega 3}$$

2. Find the DTFT of $x[n] = u[n]$. $F\{u[n]\} = ?$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u[n] e^{-j\omega n} \quad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{-j\omega 2} + e^{-j\omega 3} + e^{-j\omega 4} + \dots$$

$\left[\because \text{if } |r| < 1 \quad a + ar + ar^2 + ar^3 + \dots \text{ converges to } \frac{a}{1-r} \right]$

$$a = 1 \quad r = e^{-j\omega}$$

$$\boxed{F\{u[n]\} = \frac{1}{1 - e^{-j\omega}}}$$

3. Find the DTFT of $x[n] = u[n-k]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} u[n-k] e^{-j\omega n}$$

$$= \sum_{n=k}^{\infty} e^{-j\omega n} \quad u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$

$$= e^{-j\omega k} + e^{-j\omega(k+1)} + e^{-j\omega(k+2)} + e^{-j\omega(k+3)} + \dots$$

$$= e^{-j\omega k} [1 + e^{-j\omega} + e^{-j\omega 2} + e^{-j\omega 3} + \dots]$$

$$= e^{-j\omega k} \left[\frac{1}{1 - e^{-j\omega}} \right]$$

$$\boxed{F\{u[n-k]\} = \frac{e^{-j\omega k}}{1 - e^{-j\omega}}}$$

4. $x[n] = a^n u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= 1 + ae^{-j\omega} + (ae^{-j\omega})^2 + (ae^{-j\omega})^3 + \dots$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$\boxed{F\{a^n u[n]\} = \frac{1}{1 - ae^{-j\omega}}}$$

Inverse DTFT :-

5. Given $X(e^{j\omega}) = e^{j2\omega} + 3e^{j\omega} + 5 + 2e^{-j\omega} - e^{-j2\omega}$ Find $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-2}^2 x[n] e^{-j\omega n}$$

$$X(e^{j\omega}) = x[-2]e^{j2\omega} + x[-1]e^{j\omega} + x[0](1) + x[1]e^{-j\omega} + x[2]e^{-j2\omega} \rightarrow (1)$$

Compare eq(1) and given $x(e^{j\omega})$,

$$x[0] = 5$$

$$x[1] = 2$$

$$x[2] = 1$$

$$x[-1] = 3$$

$$x[-2] = 1$$

$$x[n] = \{1, 3, 5, 2, -1\}$$

$$x[n] = \{1, 3, 5, 2, -1\}$$

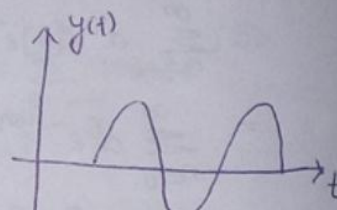
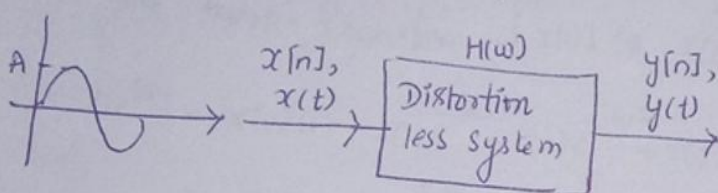
Signal transmission through LTI system :-

- Linearity and Nonlinearity
- Time invariant
- impulse response
- Causal, stability
- t, n, MIMO (multiple input multiple output)
- Distortionless transmission system

Already done in 1st unit

Distortionless system :-

A system is said to be distortionless if the output of the system is faithful reproduction of input.



→ it is delayed and change in amplitude.

- 1) The output amplitude may change.
- 2) The output is delayed w.r to input.

The o/p of distortionless system is given by

Taking DTFT on both sides,

$$Y(e^{j\omega}) = K \text{DTFT}\{x[n-n_0]\}$$

$$Y(e^{j\omega}) = K e^{-j\omega n_0} X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = K e^{-j\omega n_0}$$

$$\text{T.F of system, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = K e^{-j\omega n_0}$$

magnitude = K

$$y[n] = K x[n-n_0]$$

K = constant

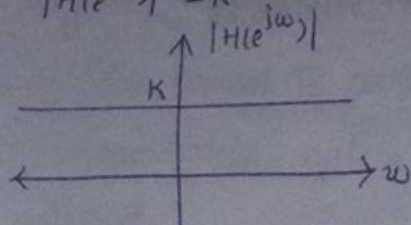
$$\cos \omega n_0 - j \sin \omega n_0$$

$$\theta = \tan^{-1} \left(\frac{-\sin \omega n_0}{\cos \omega n_0} \right)$$

Phase $\theta = \omega n_0$

Magnitude response

$$|H(e^{j\omega})| = K$$

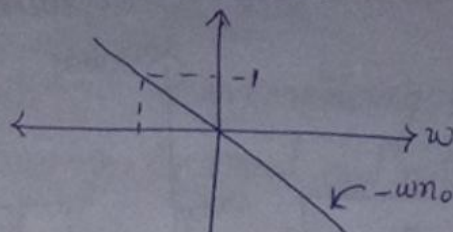


$$\rightarrow |H(e^{-j\omega})| = |H(e^{j\omega})|$$

even function

Phase response

$$\angle H(e^{j\omega}) = -\omega n_0$$



$$\rightarrow \angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$$

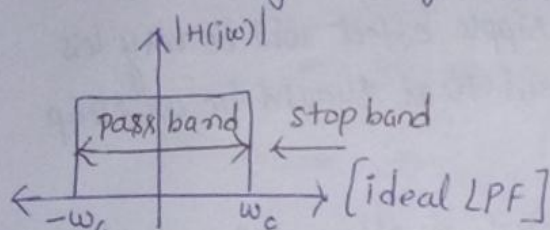
odd function

In generalised format we can say that A DTFT of a Aperiodic signal $x[n]$ its magnitude is even function and phase is odd function.

28/01/2022

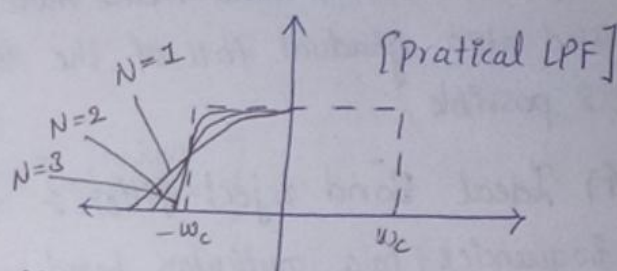
Filters :- [ideal and practical filters]

1) ideal low pass filter :- It passes all frequencies below a particular cut off frequency and rejects or attenuates other frequencies.



The frequencies within the range are passed called Pass band.

band = Group of frequencies

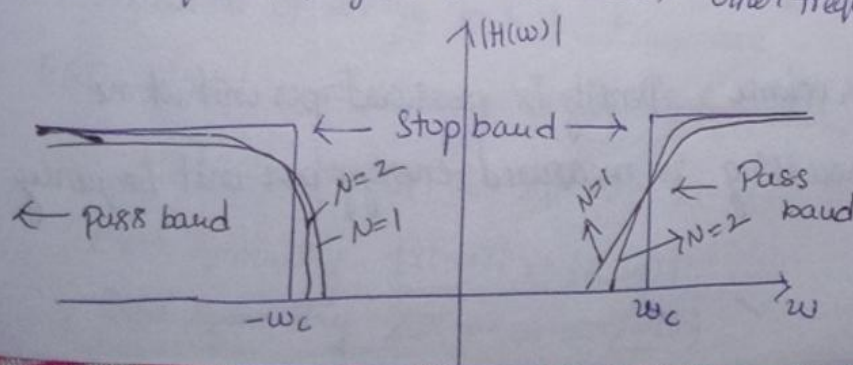


if order of filter is increases it becomes near to ideal case.

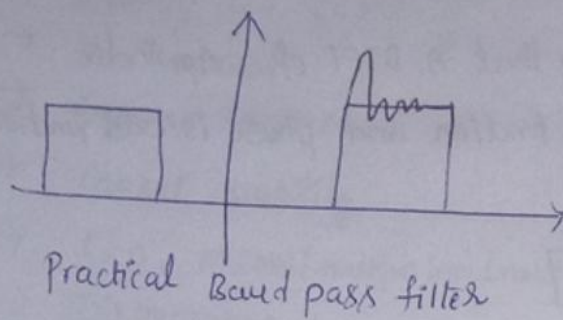
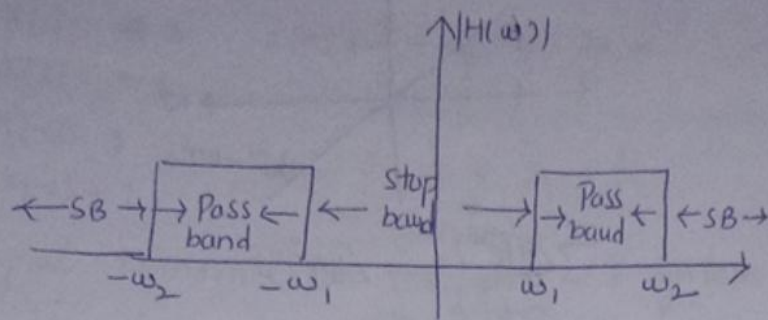
higher order - higher cut off frequency
- better filter

if order of filter increases and tends to ideal characteristics in other hand the number of components required for Particular filter drastically increases

2) ideal high pass filter :- It passes all frequencies above particular cut-off frequencies, rejects or attenuates other frequencies.



3) Ideal Band Pass filter :- It passes all the frequencies in a particular band and rejects or attenuates other frequencies.

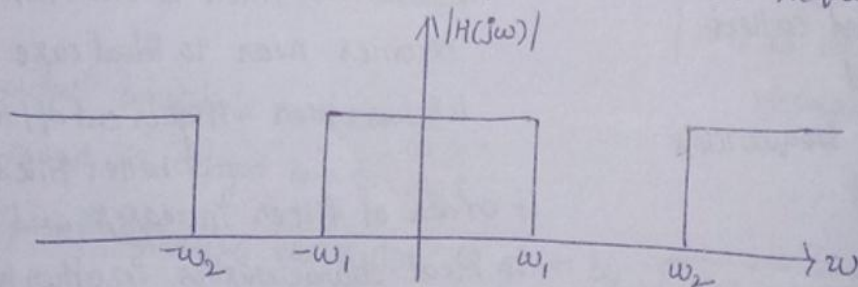


Two main differences arise for Practical and ideal cases.

1. cutoff frequency will be distributed in practical case and also the gain which needs to be constant will not be constant and it will be ripple.

When we design a FIR filter is two conditions should be as optimum as possible. we should make sure the ripple effect will be very less and also gradual fall of the tails and head should be as steep as possible.

4) Ideal Band reject filter :- It rejects or attenuates all frequencies in a particular band, passes all other frequencies.



* Ideal filter has sharp cut-off frequency and gain of a Ideal filter will be flat. practical filter has ripple. ripple means it will not constant.

Spectral Density :-

→ Density is mass per volume. Density is measured per unit volume.

→ Density in Signal processing is measured energy per unit frequency [J/Hz].

→ Spectral density of a signal characterizes the distribution of signal's energy or power in the frequency domain.

→ The concept is particularly important when considering filtering in communication system while evaluating the signal and noise at filter output.

→ The energy spectral density [ESD] or The power spectral density [PSD] is used in evaluation.

$$\text{Spectral density of signal is } \frac{|m(t)|^2}{f}$$

Energy spectral density [ESD] of a Aperiodic Signal :-

We know that energy of a discrete time signal $x[n]$ is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \cdot e^{-j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \cdot X(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

(Energy Spectral Density) $|X(\omega)|^2 = S_{xx}(\omega)$

Where $S_{xx}(\omega)$ is called energy density spectrum. It tells you the distribution of energy w.r. to frequency. $|X(\omega)|^2 \rightarrow \text{J/rad}$

→ ESD donot contain any phase information. $|x(f)|^2 \rightarrow \text{J/Hz}$

$$|X^*(\omega)| = |X(\omega)| \quad \& \quad X^*(\omega) = X(-\omega)$$

$$\text{Even symmetry } |X(-\omega)| = |X(\omega)|$$

$$\text{Odd symmetry } \angle \theta(-\omega) = -\angle \theta(\omega)$$

Considering these two

$$S_{xx}(-\omega) = S_{xx}(\omega) \quad \therefore S_{xx}(\omega) = |X(\omega)|^2$$

(even symmetry)

Spectral Density is even function. So, find $S_{xx}(\omega)$ from $0 \rightarrow \pi$ then

$$S_{xx}(\omega) \Big|_{-\pi \rightarrow 0} = S_{xx}(\omega) \Big|_{0 \rightarrow \pi}$$

example :-

For one sided from $0 \rightarrow \pi$ SD is $2J/\text{Hz}$ if they ask what is SD from $-\pi$ to 0 it will be also $2J/\text{Hz}$.

Conclusion :- The period is $-\pi \leq \omega \leq \pi$.

→ from the above properties, we need to find out $X(\omega)$ in $0 \leq \omega \leq \pi$.

and the other is can be observed by using these symmetry property.

→ Energy Spectral Density (J/Hz) denoted by S_{xx} it is a even function.
 $S_{xx}(-\omega) = S_{xx}(\omega)$.

Power density spectrum :-

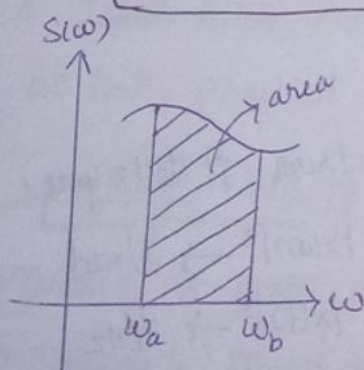
It characterizes a stationary random process in the frequency domain.

$$\begin{array}{ccc} F\{r[n]\} & \xleftrightarrow{\text{DTFT}} & S(\omega) \\ \downarrow & & \downarrow \\ \text{auto correlation function} & & \text{power spectral density} \end{array}$$

It's a easiest way of power spectral density is consider a auto correlation function perform a fourier transform on that we will get a power spectral density. Inverse also occur.

$$S(\omega) = \sum_{n=-\infty}^{\infty} r[n] e^{-j\omega n}$$

$$r[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j\omega n} d\omega$$



Power of a spectral density

$$P_{ab} = \frac{1}{2\pi} \int_{\omega_a}^{\omega_b} S(\omega) d\omega$$

avg power contributed b/w ω_a and ω_b .

Properties :-

1. Units :- Power/rad , Power/Hz

$$2. \gamma[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) d\omega = F\{x^2[n]\} \quad (\text{total power})$$

if $n=0$ in a auto correlation function then we will get power of a sequence.

$$\gamma[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{-j\omega n} d\omega$$

$$\gamma[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) d\omega \quad [\because e^{-j\omega(0)} = 1]$$

(overall power of a signal)

$$3. S(\omega) \geq 0$$

$$4. S(\omega) \text{ is real.}$$

