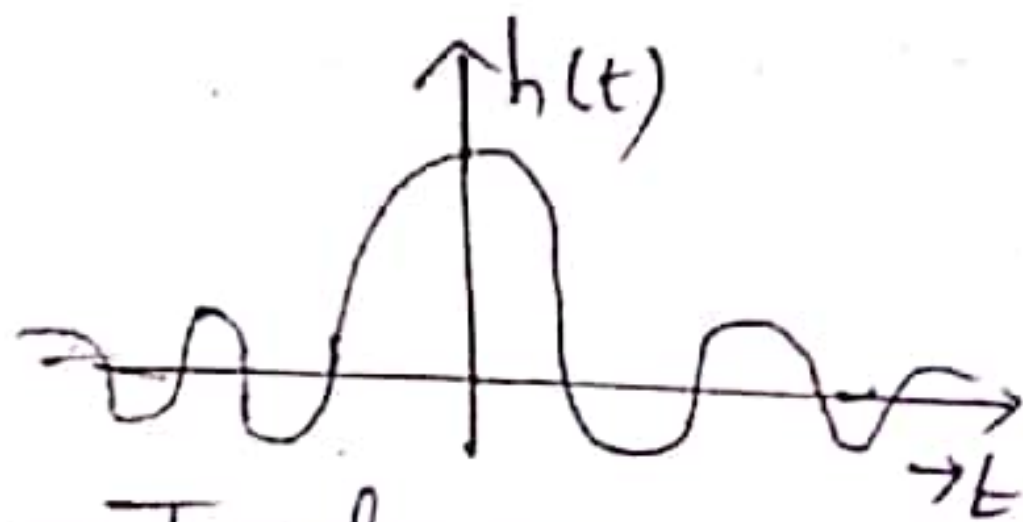


filters

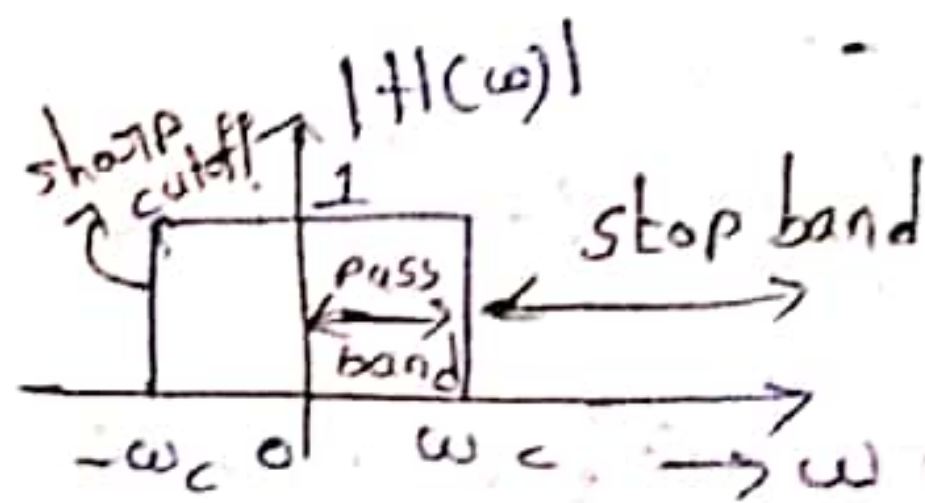
Filter: A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

Lowpass filter: (LPF) A Lowpass filter (LPF) passes all frequencies from zero to cutoff frequency & attenuate all frequencies above cutoff frequency.

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



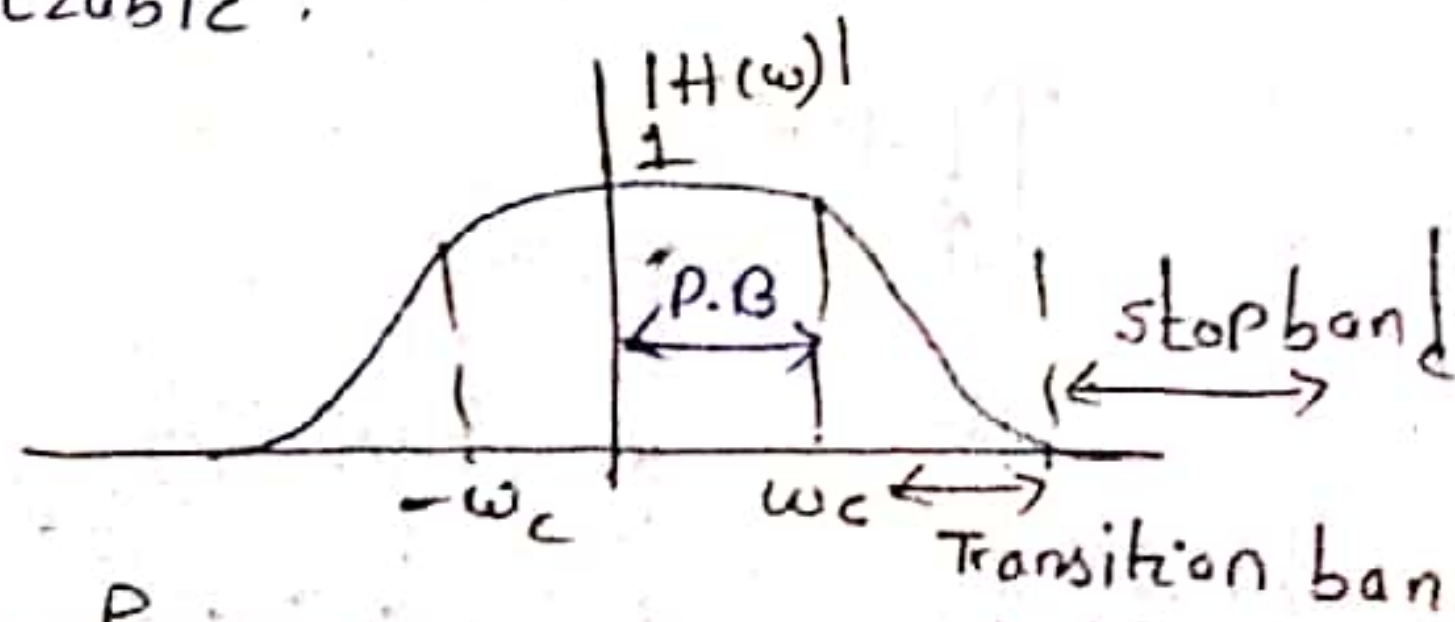
⇒ Impulse response of ideal LPF



⇒ frequency response of ideal LPF

→ Bandwidth of ideal LPF = ω_c
 ω_c → cutoff frequency (sharp)

→ The response $h(t)$ begins even before $\delta(t)$ is applied (at $t < 0$) the filter is non-causal & therefore physically unrealizable.

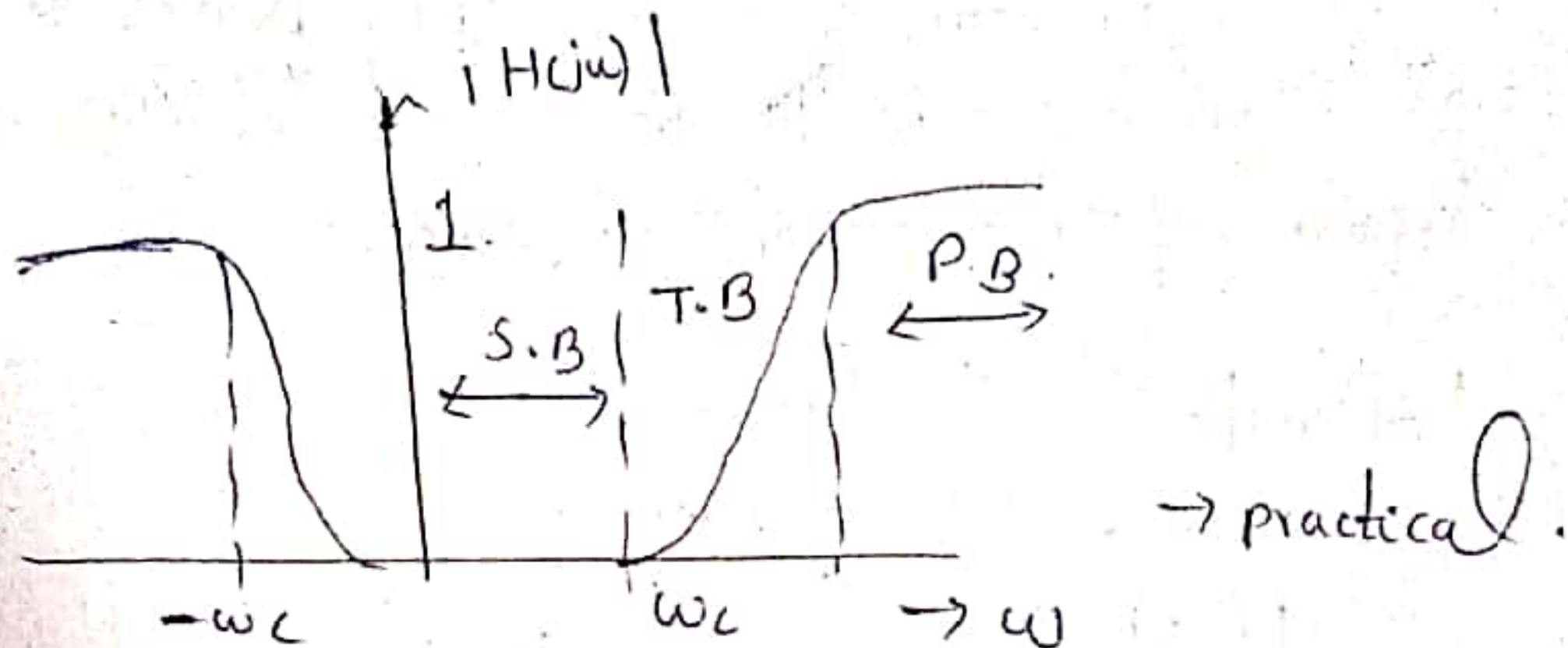
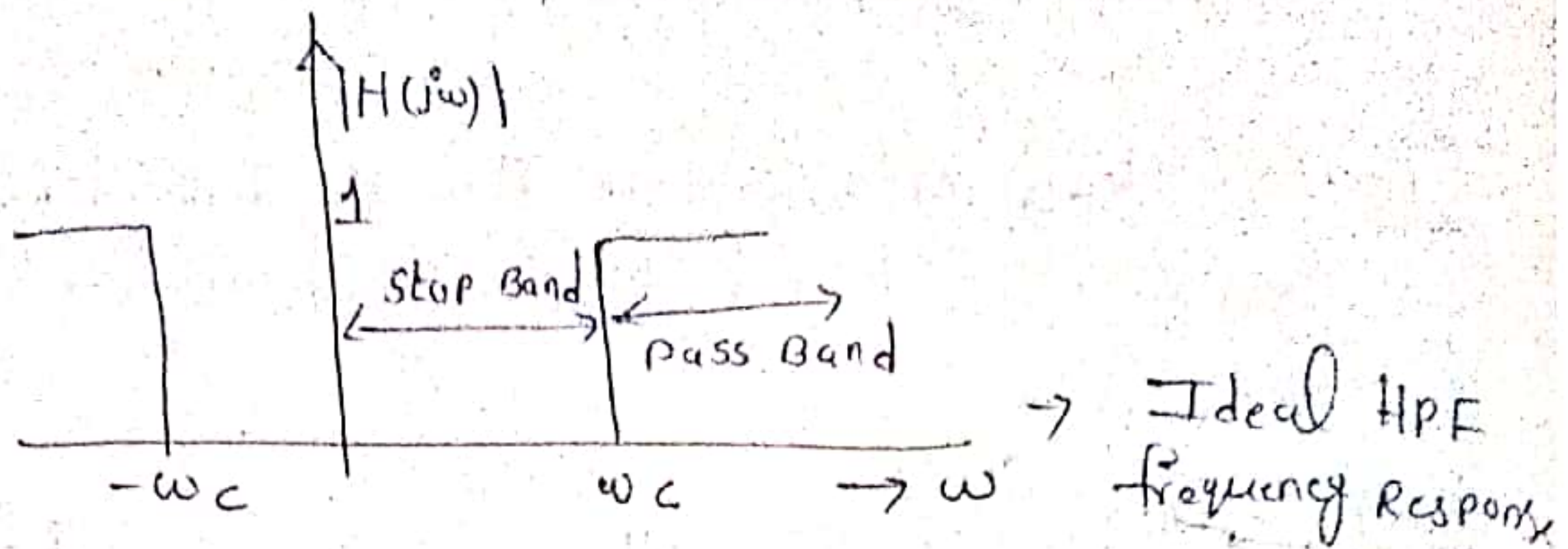


→ frequency response of practical LPF

High pass filter (HPF):

A HPF passes all frequencies above the cutoff frequency & attenuates the frequencies below cutoff frequency.

$$|H(\omega)| = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$



* B.W of ideal HPF $\approx \infty$.

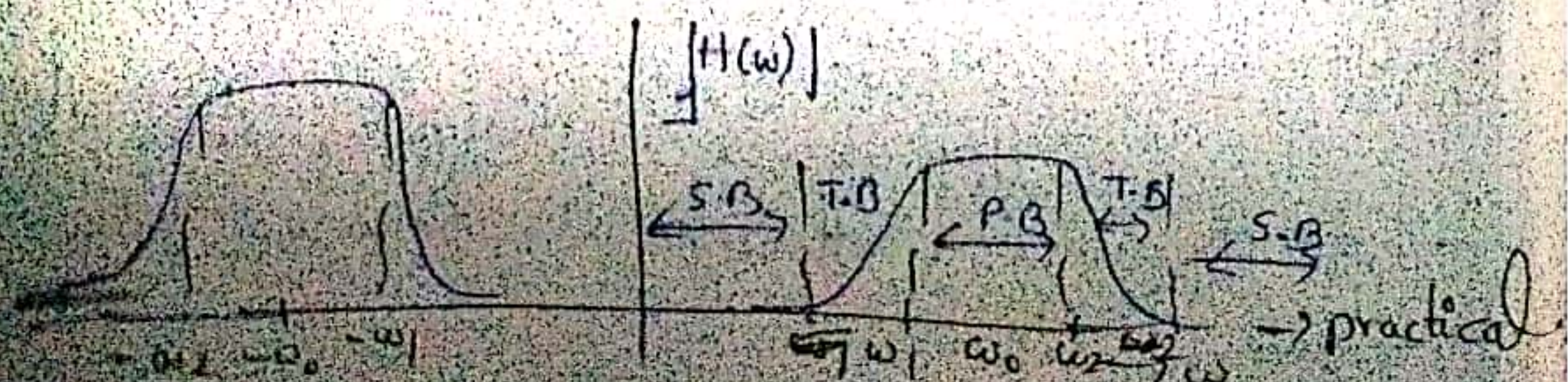
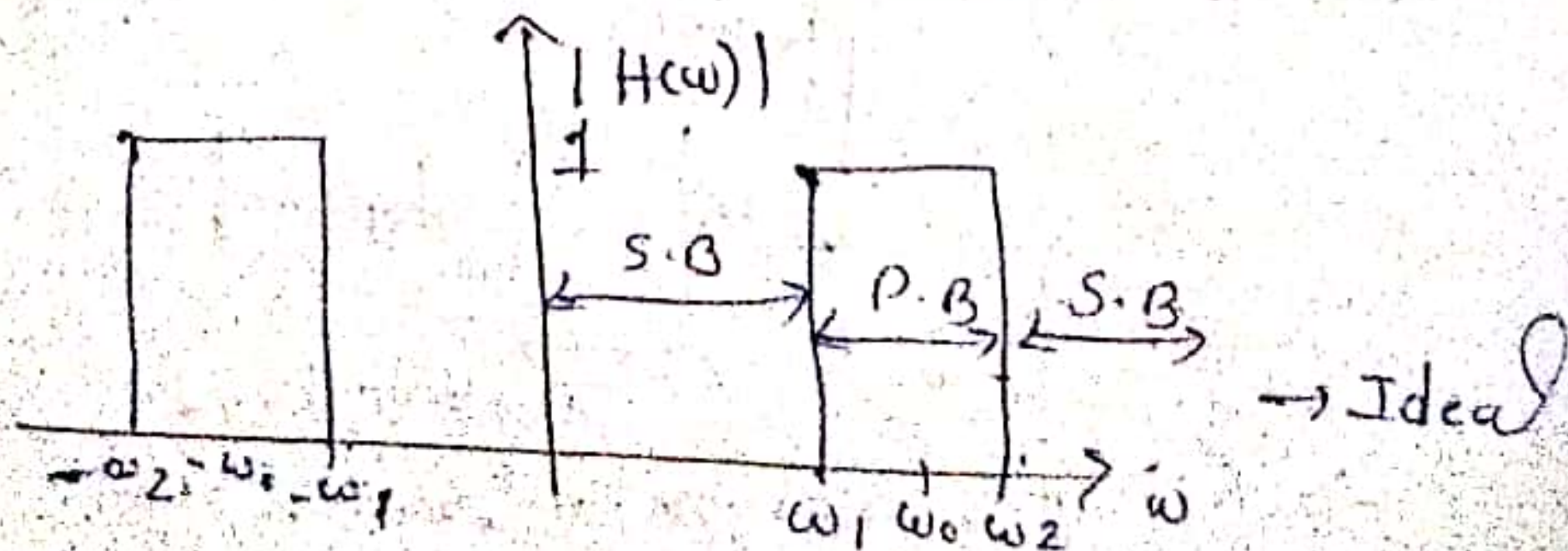
Band-pass filter (BPF):-

A BPF passes all frequencies with-in a certain frequency band & attenuates above this band.

$\omega_0 \rightarrow$ Center Frequency

$\rightarrow (\omega_2 - \omega_1)$ is B.W of BPF

$$|H(\omega)| = \begin{cases} 1 & |\omega| < \omega_1 < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

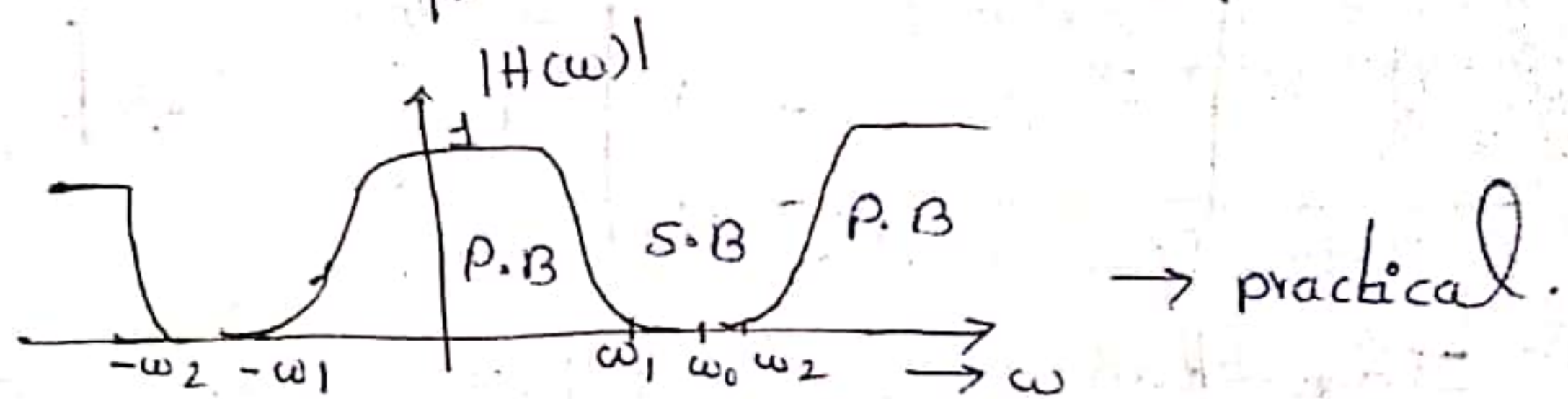
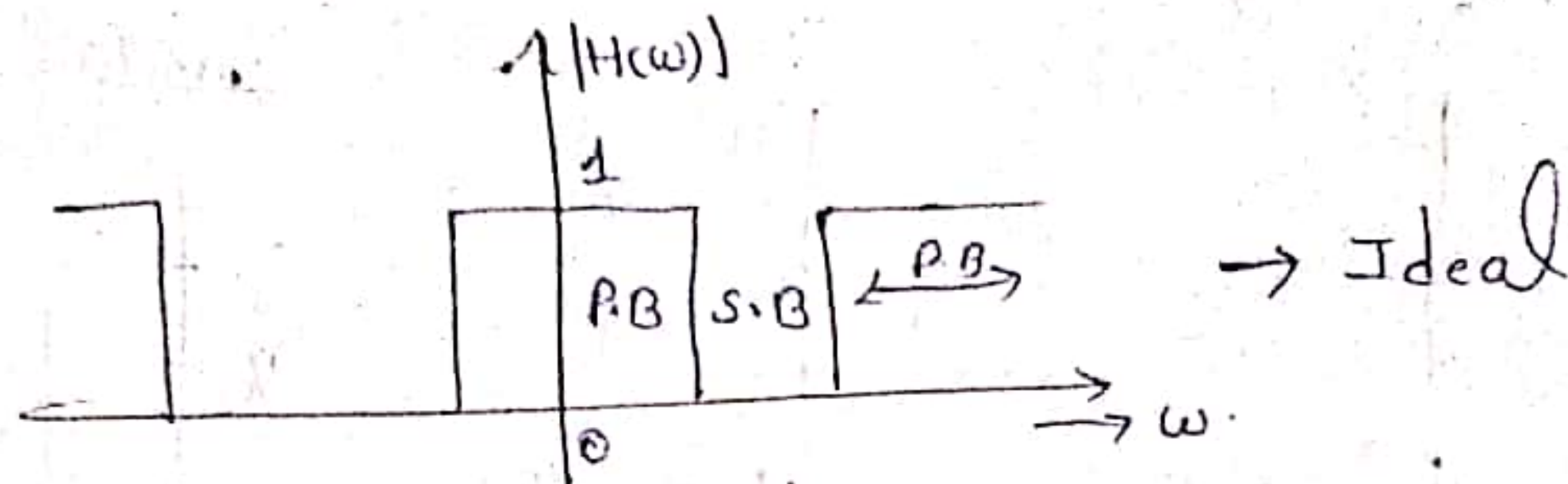


Band-stop filter (BSF) :-

A BSF Attenuates all frequencies within a band frequencies & passes all frequencies outside this band

$$B.W = (\omega_2 - \omega_1)$$

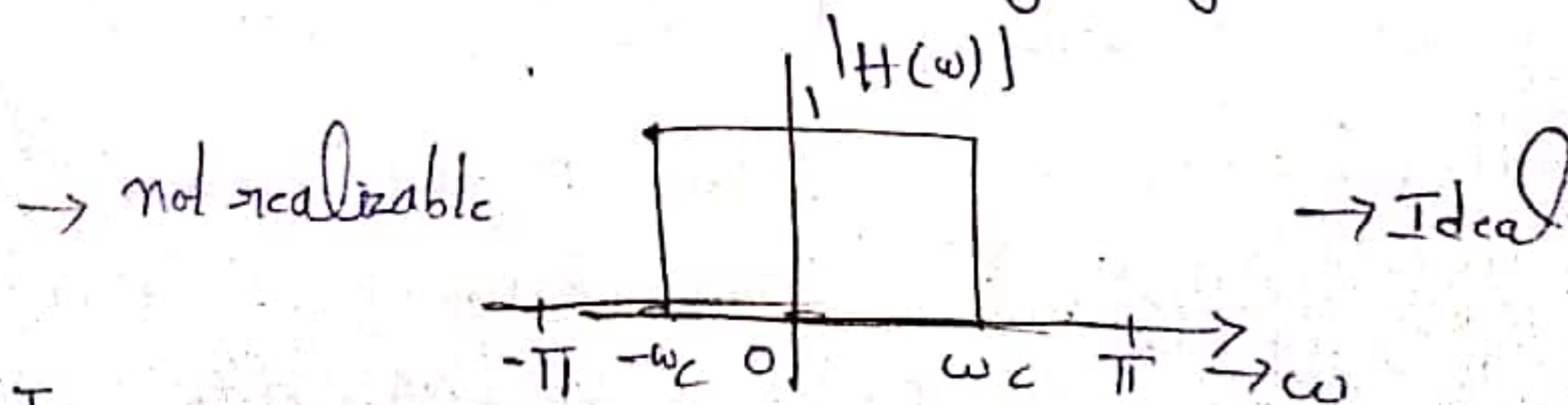
$$|H(\omega)| = \begin{cases} 0 & |\omega_1| < \omega < |\omega_2| \\ 1 & \text{otherwise} \end{cases}$$



All pass filter (APF) :-

→ A APF is a signal processing filter that passes all frequencies equally in gain, but changes the phase relationship among various frequencies

* Non-causal filters are physically unrealizable.



IDFT,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot e^{j\omega n} d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2j\pi n} [e^{j\pi n} - e^{-j\pi n}]$$

$$= \frac{1}{\pi n} \left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} \right]$$

$$\frac{\sin \omega_c n}{n\pi} \neq 0 \text{ for } n < 0$$

i.e. Non-causal system

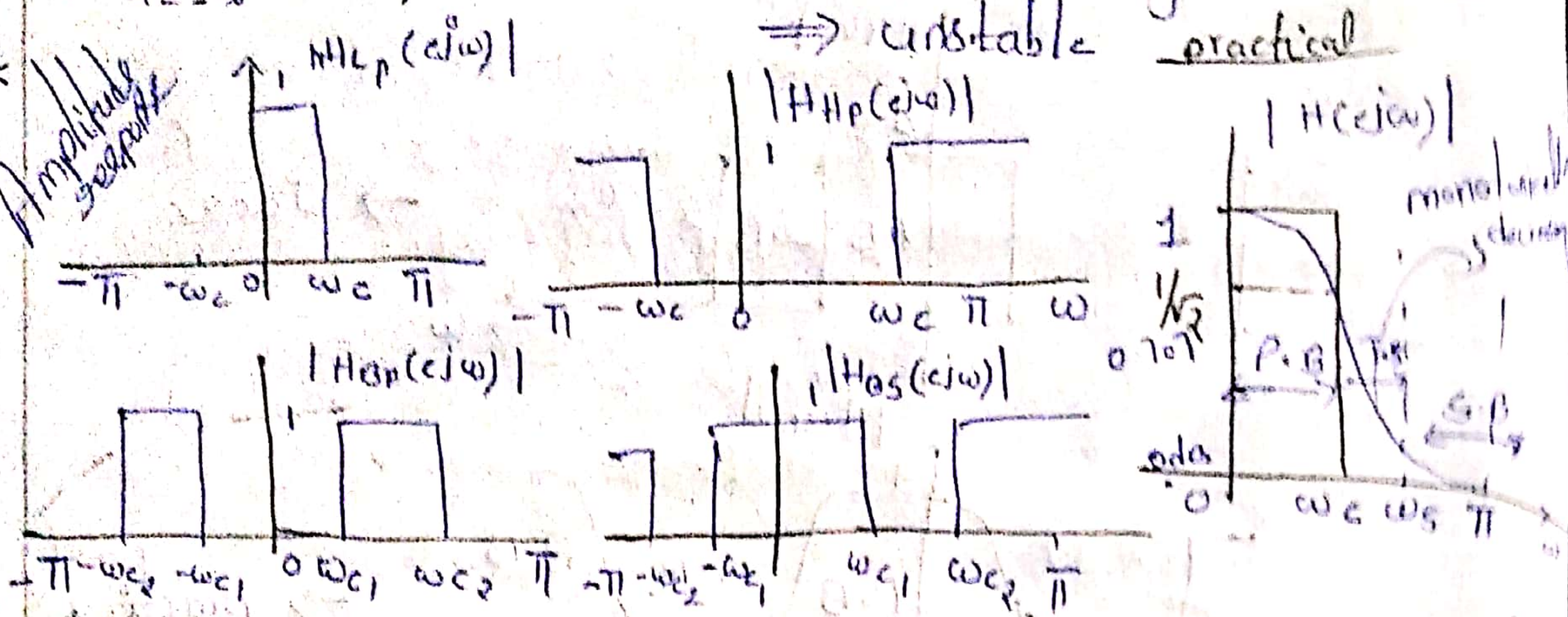
$\Rightarrow$ physically unrealizable

$$n=0 \Rightarrow h(0) = \frac{\omega_c}{\pi}$$

$$\sum_{n=-\infty}^{\infty} h(n) \neq \infty \rightarrow \text{not absolutely summable}$$

$\Rightarrow$ unstable practical

* Amplitude response



Ideal filters

• Low pass filter: passband - $0 \leq \omega \leq \omega_c$

stopband - $\omega_c < \omega \leq \pi$

• High pass filter: passband - $\omega_c \leq \omega \leq \pi$

stopband - $0 \leq \omega < \omega_c$

• Bandpass filter: passband - $\omega_{c1} \leq \omega \leq \omega_{c2}$

stopband - $0 \leq \omega < \omega_{c1}$ & $\omega_{c2} < \omega \leq \pi$

• Bandstop filter: passband - $0 \leq \omega \leq \omega_{c1}$ & $\omega_{c2} \leq \omega \leq \pi$

stopband - $\omega_{c1} < \omega < \omega_{c2}$

$\text{voltage} = A$
 $\text{power} = A^2/2 = A^2/2$
 $\text{half power} = A^2/4 = A^2/4$

$\Rightarrow$ Half power point or Half power B.W is the point at which o/p power has dropped to half of its peak value i.e. at level of approximately -3dB

$\Rightarrow$ The o/p voltage dropped to $1/\sqrt{2}$ of maximum output voltage & the power has dropped by half

$\text{in dB} = 10 \log \frac{P_1}{P_2}$
 $\text{in dB} = 20 \log \frac{V_1}{V_2}$

$x(n) \rightarrow \boxed{\text{LTT}} \rightarrow y(n) = x(n) * h(n)$
 $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$$\text{in dB} = 10 \log\left(\frac{P_2}{P_1}\right) = 10 \log\left(\frac{1}{2}\right) = -3.0103 \text{ dB}$$

$$\text{in dB} = 20 \log\left(\frac{V_2}{V_1}\right) = 20 \log\left(\frac{1}{\sqrt{2}}\right) = -3.0103 \text{ dB}$$

Note
* Narrower the transition region $\rightarrow$ the better is the filtering [go for higher order filtering] (accurate)

① Let LPF
 $x(n) \xrightarrow{\text{DTFT}} \boxed{h(n)} \xrightarrow{\text{DTFT}} y(n)$
 $H(e^{j\omega}) \rightarrow$ Frequency response

$$|H(e^{j\omega})| \approx \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Apply an i/p, $x(n) = A \cos \omega_1 n + B \cos \omega_2 n$

$\hookrightarrow$ only exist $0 < \omega_1 < \omega_c < \omega_2 < \pi$

[Eigen function of i/p form = o/p form]

$$\left[\begin{array}{l} x(n) \xrightarrow{\boxed{h(n)}} y(n) \\ e^{j\omega n} H(e^{j\omega}) = H(e^{j\omega}) e^{j\omega n} \end{array} \right]$$

change in mag & phase

$$y(n) = A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1)) + B |H(e^{j\omega_2})| \cos(\omega_2 n + \theta(\omega_2))$$

$$\text{As, } |H(e^{j\omega_1})| \approx 1, |H(e^{j\omega_2})| \approx 0$$

then o/p reduces,

$$y(n) \approx A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1))$$

$\therefore$ This system acts like a LPF

2) HPF

$$\text{The i/p } x(n) = \{ \cos(0.1n) + \cos(0.4n) \} u(n)$$

* Why we are going for filters:-

- 1) Signal separation
- 2) Signal Restoration.

DTFT

FIR filter,



$$y(n) = h(n) * x(n)$$

$$X(e^{j\omega}) \quad H(e^{j\omega}) \quad Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$H(z) \rightarrow$ transfer function

we obtain frequency response from transfer function $H(z)$

$$H(z) = H(e^{j\omega}) \big|_{z=e^{j\omega}} \quad \text{ROC: must include unit circle}$$

frequency response of the system

FIR filter (digital) $\Rightarrow h(n) = \{\alpha, \beta, \alpha\}$

$$y(n) = \sum_{k=0}^2 h(k)x(n-k)$$

response. $y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2)$

DTFT, $H(e^{j\omega}) =$



Concept of filtering

→ one application of an LTI discrete-time system is to pass certain frequency components in an i/p sequence without any distortion and to block other frequency components.

→ such systems are called digital filters.

The key to filtering process is [DTFT]

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

→ It expresses an arbitrary i/p as a linear weighted sum of an infinite no. of exponential sequences, or equivalently as a linear weighted sum of sinusoidal sequences.

→ Thus, by appropriately choosing the values of the magnitude function $|H(e^{j\omega})|$ of the LTI digital filter at frequencies corresponding to frequencies of the sinusoidal components, can be selectively heavily attenuated or filtered with respect to others.

⇒ To understand the mechanism behind the design of frequency-selective filters,

Consider a real coefficient LTI discrete-time system characterized by magnitude function

* LPF

$$|H(e^{j\omega})| \approx \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

Apply an input,

$$x[n] = A \cos \omega_1 n + B \cos \omega_2 n, \quad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system. → (exists)

→ Because of linearity, the o/p of this system is of the form

$$y[n] = A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1)) + B |H(e^{j\omega_2})| \cos(\omega_2 n + \theta(\omega_2))$$

As $|H(e^{j\omega_1})| = 1$, $|H(e^{j\omega_2})| = 0$.

The o/p reduces to

$$y[n] = A |H(e^{j\omega_1})| \cos(\omega_1 n + \theta(\omega_1))$$

→ Thus, the system act like a lowpass filter.

HPF

Let consider the design of a very simple design filter.

Ex:-

The i/p consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample & 0.4 rad/sample

$$x[n] = \{ \cos(0.1n) + \cos(0.4n) \} u[n]$$

⇒ we need to design a highpass filter that will pass the high-frequency component of the i/p but block the low-frequency component.

⇒ For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response

$$h[0] = h[2] = \alpha, \quad h[1] = \beta \quad \left\{ \alpha, \beta \in \mathbb{Z} \text{ integers} \right.$$

Transfer function, $H(z) = H(e^{j\omega}) \Big|_{z=e^{j\omega}}$ ROC must include unit circle

FIR filter (digital), $h[n] = \{\alpha, \beta, \alpha\}$

$$y[n] = \sum_{k=0}^2 h[k] x[n-k]$$

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \\ = \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

$$y[n] = \alpha/p ; x[n] = 1/p$$

Design objective:-

choose suitable values of α & β so that y/p is a sinusoidal sequence with a frequency 0.4 rad/sample.

Now, frequency response of FIR filter

$$\begin{aligned} \text{DTFT } H(e^{j\omega}) &= \sum_{n=0}^2 h(n) e^{-j\omega n} = h(0) + h(1) e^{-j\omega} + h(2) e^{-j2\omega} \\ &= \alpha + \beta e^{-j\omega} + \alpha e^{-j2\omega} \\ &= \alpha (1 + e^{-j2\omega}) + \beta e^{-j\omega} \\ &= \alpha e^{-j\omega} (e^{j\omega} + e^{-j\omega}) + \beta e^{-j\omega} \\ &= [2\alpha \cos \omega + \beta] e^{-j\omega} \end{aligned}$$

⇒ Magnitude & phase functions are

$$|H(e^{j\omega})| = 2\alpha \cos \omega + \beta ; \angle H(e^{j\omega}) = -\omega$$

⇒ In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero

$$|H(e^{j\omega_1})| = |H(e^{j0.1})| = 0$$

$$|H(e^{j\omega_2})| = |H(e^{j0.4})| = 1$$

⇒ Like wise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

Thus, the two conditions that must be satisfied are

$$|H(e^{j0.1})| = 2\alpha \cos(0.1) + \beta = 0$$

$$|H(e^{j0.4})| = 2\alpha \cos(0.4) + \beta = 1$$

$$\Rightarrow 2\alpha (0.995) + \beta = 0$$

$$\Rightarrow 2\alpha (0.9210) + \beta = 1$$

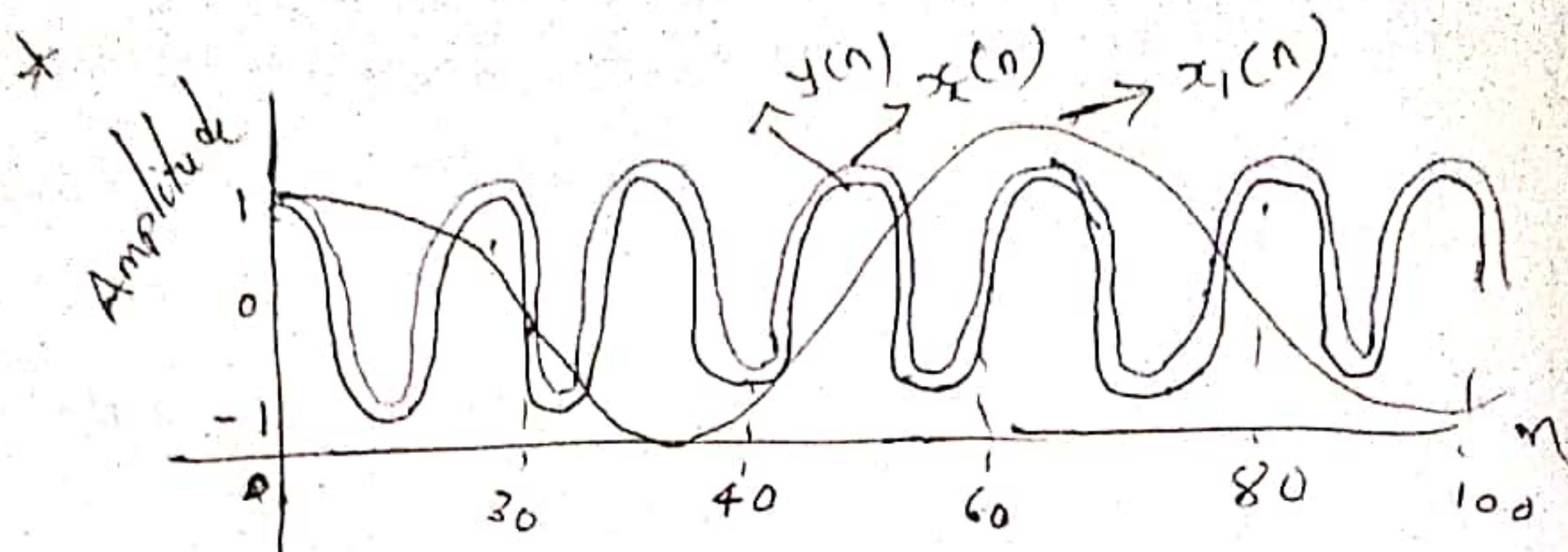
$$\left. \begin{array}{l} \text{By solving} \\ \alpha = -6.7615 \\ \beta = 13.45635 \end{array} \right\}$$

Thus, the output-input relation of the FIR filter is given by

$$Y[n] = -6.76195 [x[n] + x[n-2]] + 13.456335x[n-1]$$

where $x[n]$ is

$$x[n] = \{ \cos(0.1n) + \cos(0.4n) \} u[n]$$



⇒ The first seven samples of o/p are

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$Y[n]$
0	1.0	1.0	2.0	-13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	-0.0291995	0.8918614	0.3623572
5	0.8775825	-0.4161468	0.4614357	-0.0292002
6	0.8253356	-0.7373937	0.0879419	-0.4161467

From this table, neglecting the least significant digit

$$Y[n] = \cos(0.4(n-1)) \quad \text{for } n \geq 2$$

⇒ Computation of present value of the o/p requires the knowledge of the present & two previous i/p samples

⇒ Hence, the first two o/p samples, $Y[0]$ & $Y[1]$ are result of assumed zero i/p sample values at $n=$
 $\& n=$

So, first two o/p's samples constitute the transient part of the output.
 $\Rightarrow$ since, impulse response is of length 3, the steady state is reached at $n = N-2$.

$\Rightarrow$ Note o/p is delayed version of high-frequency component $\cos(0.4n)$ of i/p, & delay is one sample period.

Phase delay

$\Rightarrow$ If input $x[n]$ to an LTI system $H(e^{j\omega})$ is a sinusoidal signal of frequency ω_0 .

$$x[n] = A \cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

Then, o/p $y[n]$ is also sinusoidal signal of the same frequency ω_0 but lagging in phase by $\theta(\omega_0)$ radians:

$$y[n] = A |H(e^{j\omega_0})| \cos(\omega_0 n + \theta(\omega_0) + \phi), \quad -\infty < n < \infty$$

o/p, $y[n] = A |H(e^{j\omega_0})| \cos(\omega_0 (n - \tau_p(\omega_0) + \phi))$

$$\tau_p(\omega_0) = - \frac{\theta(\omega_0)}{\omega_0} \rightarrow \text{phase delay}$$

$\Rightarrow$ The minus sign in front indicates phase lag.

$\Rightarrow y[n]$ is time-delayed version of $x[n]$.

In general, $y[n]$ will not be delayed replica of $x[n]$ unless the phase delay $\tau_p(\omega_0)$ is an int'n.

Group delay :-

$\Rightarrow$ when the i/p is composed of many sinusoidal components with different frequencies that are not harmonically related, each component will go through

different phase delays is determined using the group delay defined by

$$T_g(\omega) = -\frac{d\phi(\omega)}{d\omega}$$

M-point moving average filter [LPF]

For M-point moving-average filter,

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

Apply z-Transform on both sides,

$$Y(z) = \frac{1}{M} \sum_{n=0}^{\infty} \sum_{k=0}^{M-1} x[n-k] z^{-n}$$

$x[n] \xrightarrow{z^{-T}} X(z)$
 $x[n-k] \xrightarrow{z^{-T}} z^{-k} X(z)$

$$= \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} = \frac{1}{M} \sum_{k=0}^{M-1} (z^{-1})^k$$

$$= \frac{1}{M} \left[\frac{1 - (z^{-1})^M}{1 - z^{-1}} \right]$$

$$\Rightarrow \frac{1}{M} \left[\frac{z^M [z^M - 1]}{z^{-1} (z - 1)} \right] = \frac{1}{M} \left[\frac{z^{M-1}}{z^{M-1} (z - 1)} \right]$$

$z^M = 1 \rightarrow \text{zero}$

$z = 1, z^M - 1 = 0$

Pole $\Rightarrow z^M = e^{j2\pi k/M}$

* The pole at $z = 1$ exactly cancels zero at $z = 1$.

$\Rightarrow \text{Roc: Entire plane except at } z = 0$

$\Rightarrow$ The transfer function has M-zeros on unit circle at $z = e^{j2\pi k/M}, 0 \leq k \leq M-1$

$$H(z) = \frac{1}{M} \left[\frac{z^{M-1}}{z^{M-1} (z - 1)} \right]$$

Simple FIR Digital Filters

⇒ FIR digital filters considered here have integer valued impulse-response coefficients

Applications

→ simplicity, which makes them amenable to inexpensive hardware implementations.

Low-pass FIR Digital Filters

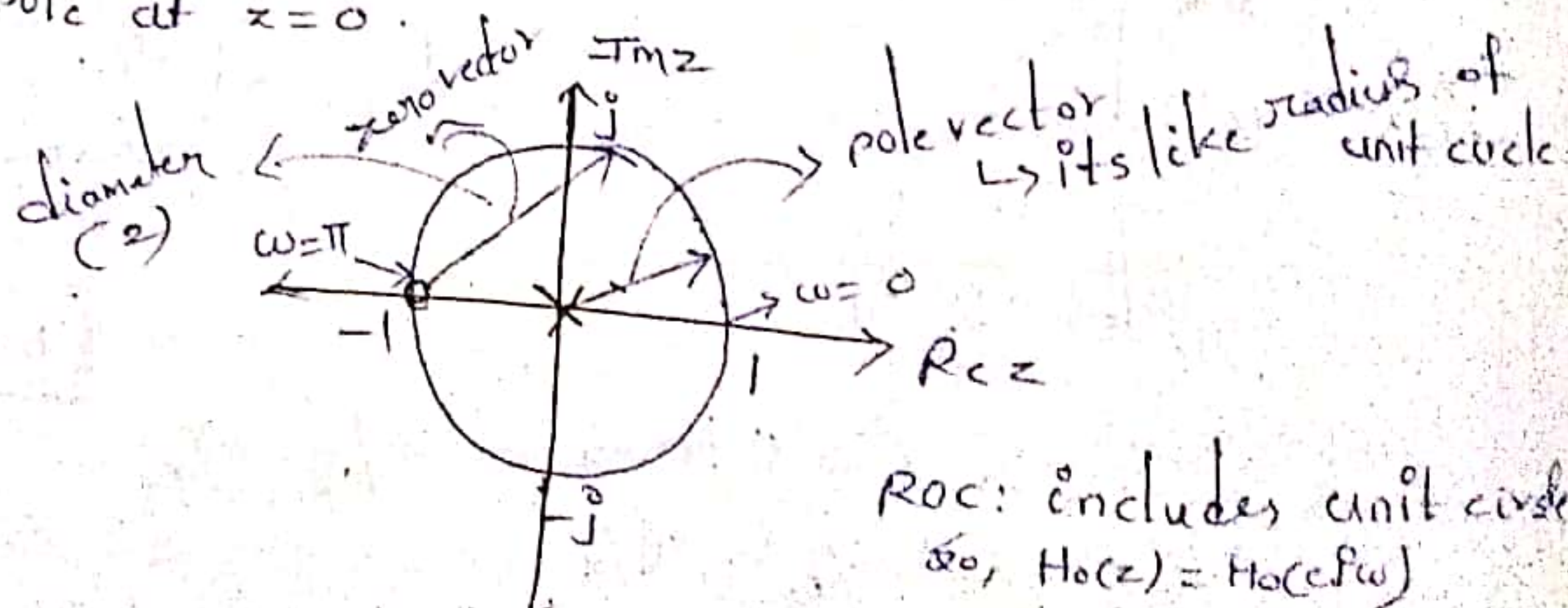
→ The simplest lowpass FIR digital filter is 2-point moving-average filter,

$$H(z) = \frac{1}{M} \left(\frac{z^M - 1}{z^{M-1}(z-1)} \right)$$

$$H(z) = \frac{1}{2} \left(\frac{z^2 - 1}{z(z-1)} \right) = \frac{1}{2} \left(\frac{(z-1)(z+1)}{(z)(z-1)} \right) = \frac{z+1}{2z}$$

$$H_0(z) = \frac{1}{2} (1 + z^{-1}) = \frac{z+1}{2z} = \frac{z^2(1+z^{-1})}{2z^2} = \frac{1}{2} (1+z)$$

→ The above transfer function has a zero at $z = -1$ & a pole at $z = 0$.



⇒ pole vector has a unity magnitude for all values of ω .

⇒ As ω increases from 0 to π , the magnitude of zero vector decreases from a value of 2, the diameter of unit circle, to 0.

⇒ so, zero vector $\downarrow \rightarrow H(e^{j\omega}) \downarrow$ as $H(e^{j\omega}) \propto$ zero vector

⇒ magnitude response $|H_0(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$.

$$H_0(z) = \frac{1}{2}(1+z^{-1})$$

$$H_0(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega})$$

→ maximum value of magnitude function is 1
at $\omega=0$

$$|H_0(e^{j0})| = \frac{1}{2}(1+e^{j0}) = \frac{2}{2} = 1 \quad (z=1 \text{ at } \omega=0)$$

→ minimum value of magnitude function is 0
at $\omega=\pi$

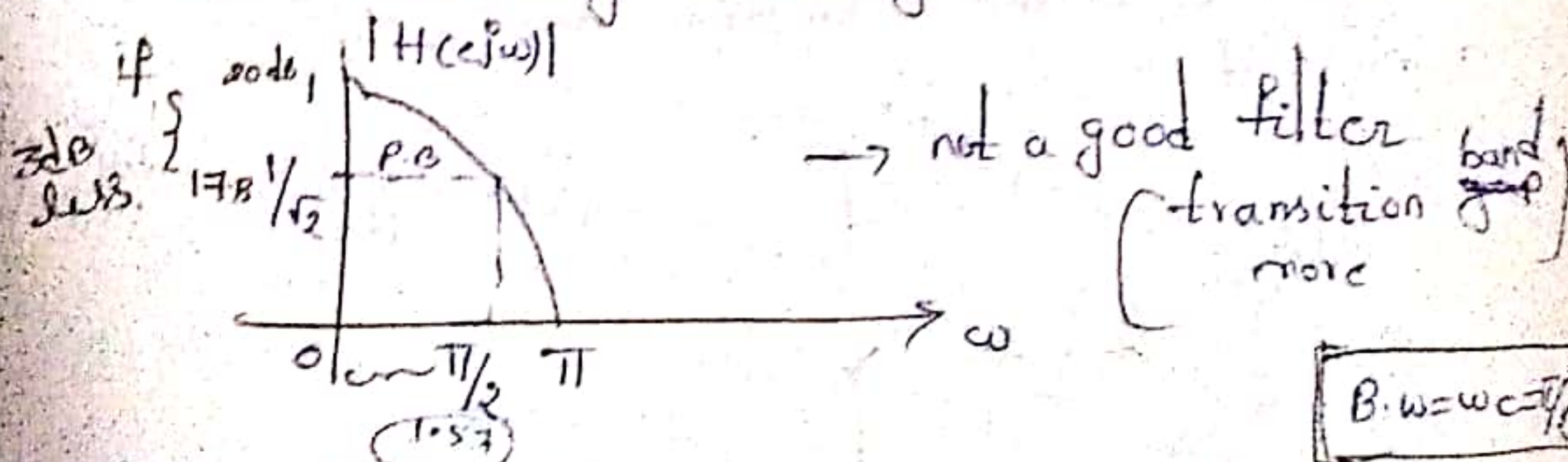
$$|H_0(e^{j\pi})| = \frac{1}{2}(1+e^{j\pi}) = \frac{1}{2}(1-1) = 0 \quad (z=-1 \text{ at } \omega=\pi)$$

→ The frequency response of above filter is given by

$$\begin{aligned} H_0(e^{j\omega}) &= \frac{1}{2}(1+e^{-j\omega}) = \frac{e^{-j\omega/2}}{2}(e^{j\omega/2}+e^{-j\omega/2}) \\ &= e^{-j\omega/2} \cos(\omega/2) \end{aligned}$$

* magnitude response $|H_0(e^{j\omega})| = |e^{-j\omega/2}| |\cos \omega/2| = \cos \omega/2$

→ monotonically decreasing



$$\cos(\omega/2) = \frac{1}{\sqrt{2}} \text{ (half power)} \Rightarrow V_m/\sqrt{2} \Rightarrow 1/\sqrt{2}$$

$$\Rightarrow \omega/2 = \cos^{-1}(1/\sqrt{2}) = \pi/4$$

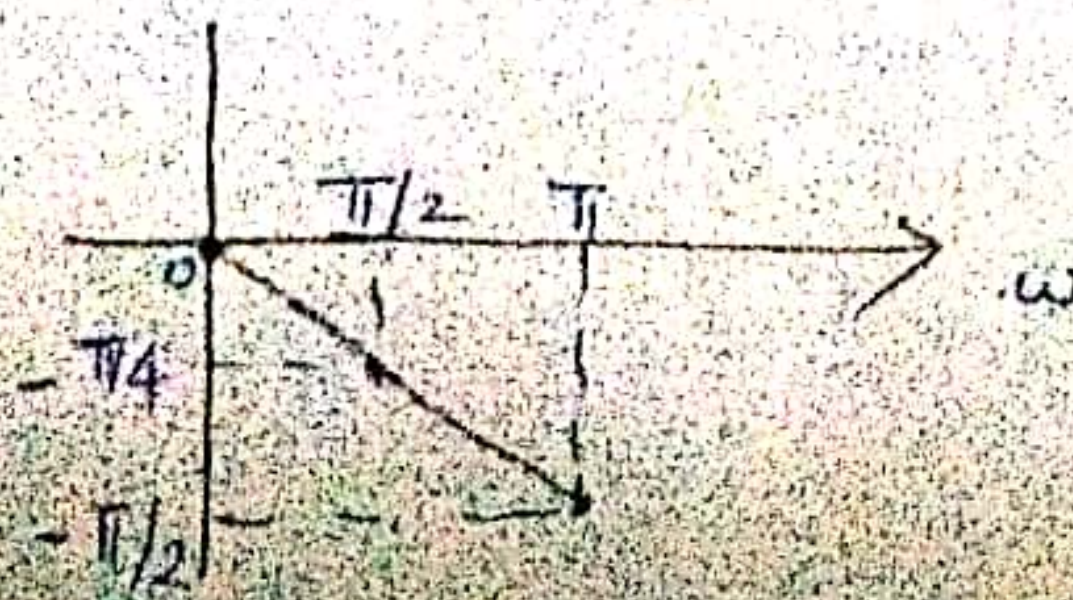
$$\Rightarrow \omega = 2\pi/4 = \pi/2$$

$$\Rightarrow \omega_c = \pi/2 \rightarrow \text{cutoff frequency}$$

→ The phase response

$$\angle H_0(e^{j\omega}) = -\omega/2 + \angle \cos \omega/2$$

0°
+ve for $\omega < \pi/2$



$$H_0(z) = \frac{1}{2}(1+z^{-1})$$

$$H_0(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega})$$

⇒ maximum value of magnitude function is 1
at $\omega=0$

$$|H_0(e^{j0})| = \frac{1}{2}(1+e^{j0}) = \frac{2}{2} = 1 \quad (z=1 \text{ at } \omega=0)$$

⇒ minimum value of magnitude function is 0
at $\omega=\pi$

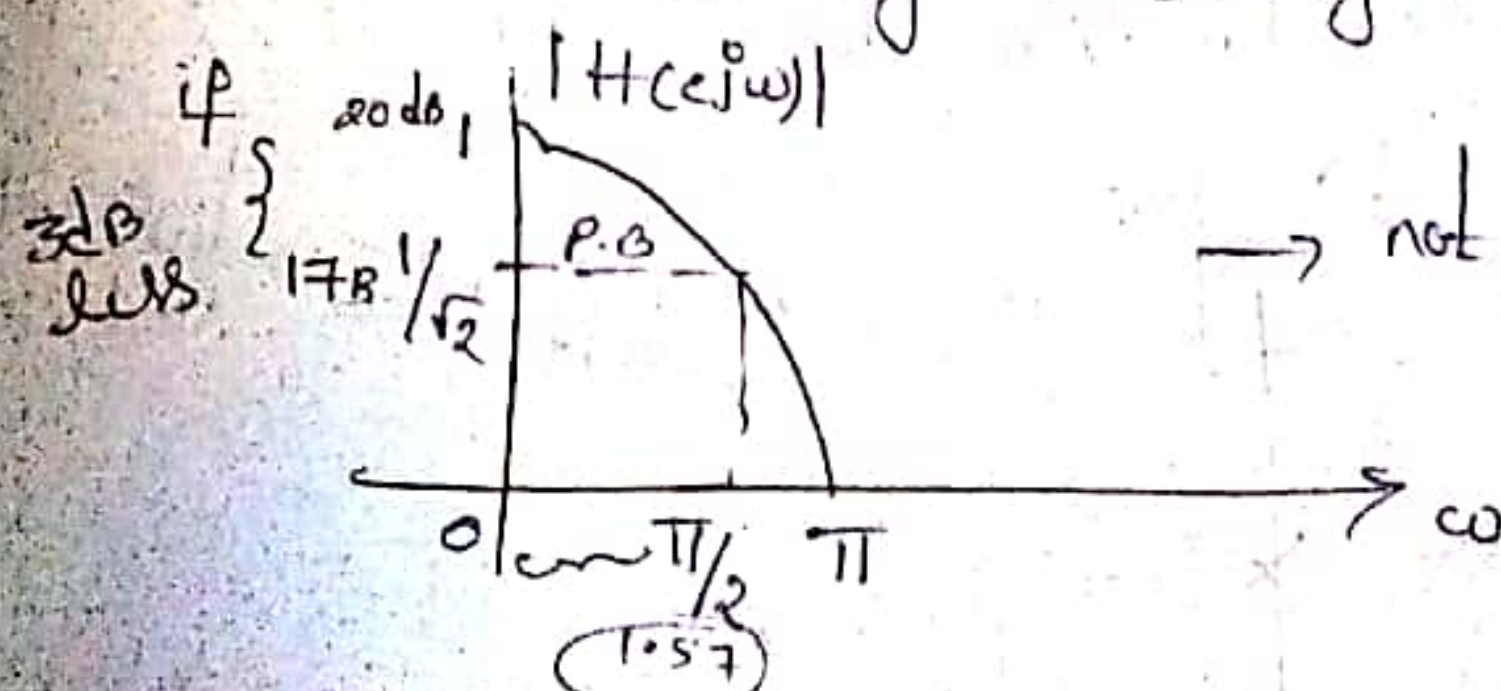
$$|H_0(e^{j\pi})| = \frac{1}{2}(1+e^{j\pi}) = \frac{1}{2}(1-1) = 0 \quad (z=-1 \text{ at } \omega=\pi)$$

⇒ The frequency response of above filter is given by

$$\begin{aligned} H_0(e^{j\omega}) &= \frac{1}{2}(1+e^{-j\omega}) = \frac{e^{-j\omega/2}}{2}(e^{j\omega/2}+e^{-j\omega/2}) \\ &= e^{-j\omega/2} \cos(\omega/2) \end{aligned}$$

* magnitude response $|H_0(e^{j\omega})| = |e^{-j\omega/2}| |\cos \omega/2| = \cos \omega/2$

↪ monotonically decreasing



→ not a good filter
(transition band more)

$$B.W = \omega_c = \pi/2$$

$$\cos(\omega/2) = \frac{1}{\sqrt{2}} \text{ (half power)} \Rightarrow v_m/\sqrt{2} \Rightarrow 1/\sqrt{2}$$

$$\Rightarrow \omega/2 = \cos^{-1}(1/\sqrt{2}) = \pi/4$$

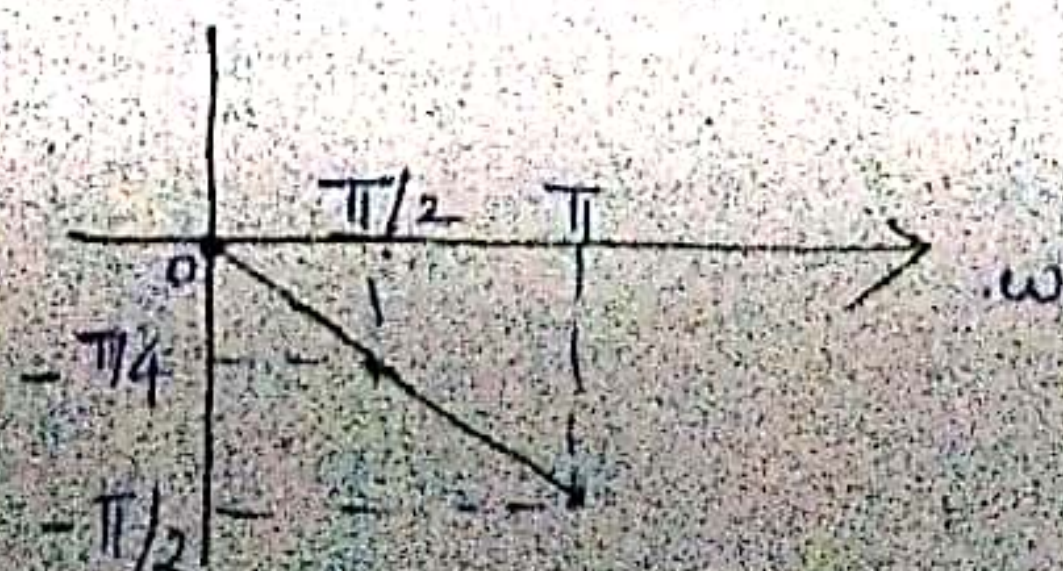
$$\Rightarrow \omega = 2\pi/4 = \pi/2$$

$$\Rightarrow \omega_c = \pi/2 \rightarrow \text{cutoff frequency}$$

⇒ The phase response

$$\angle H_0(e^{j\omega}) = -\omega/2 + \angle \cos \omega/2$$

0°
+ve for < 0°



The frequency at $\omega = \omega_c$ at which

$$|H_0(e^{j\omega_c})| = \frac{1}{\sqrt{2}} |H_0(e^{j0})| \quad \checkmark \quad V_m \Rightarrow (1)$$

Gain in dB,

$$G(\omega_c) = 20 \log_{10} |H(e^{j\omega_c})|$$

$$= 20 \log_{10} \frac{1}{\sqrt{2}} |H_0(e^{j0})|$$

$$= 20 \log_{10} |H_0(e^{j0})| - 20 \log_{10} \sqrt{2} \approx -3 \text{ dB}$$

at $\omega = 0$
dc gain

$$\checkmark G(0) = 20 \log_{10} |H(e^{j0})| = 0 //$$

$\Rightarrow$ Hence, gain $G(\omega)$ at $\omega = \omega_c$ is approx 3dB less than that gain at $\omega = 0$.

$\Rightarrow \omega_c \rightarrow$ 3-dB cutoff frequency.

To find ω_c ,

$$|H_0(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \cos \omega_c = \pi/2$$

$\Rightarrow$ The 3-dB cutoff frequency ω_c can be considered as a passband edge frequency.

$\Rightarrow$ passband width = $\pi/2$ (approx.)

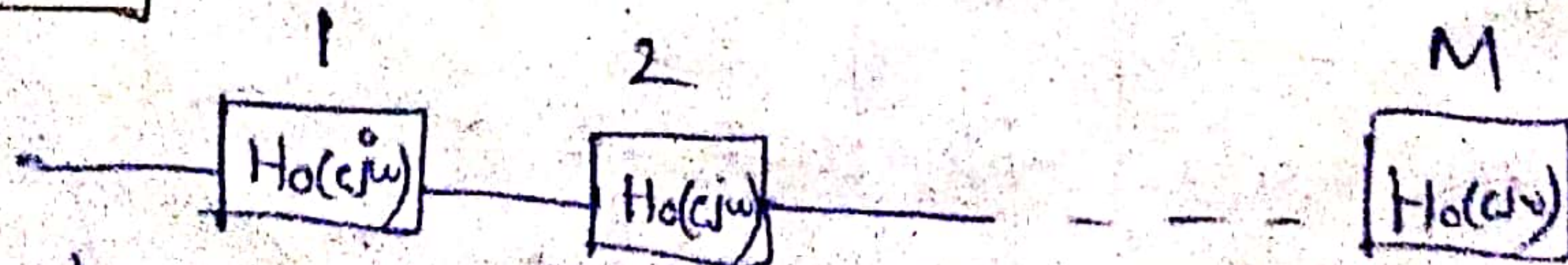
Stop band $\Rightarrow \pi/2$ to π .

Note $H_0(z)$ has a zero at $z = -1$ or $\omega = \pi$, which is in the stop band of the filter.

$\Rightarrow$ This is a bad filter.

$\Rightarrow$ For better filter $\leftarrow$ cascading of simple filters
increasing M-value.

Cascading

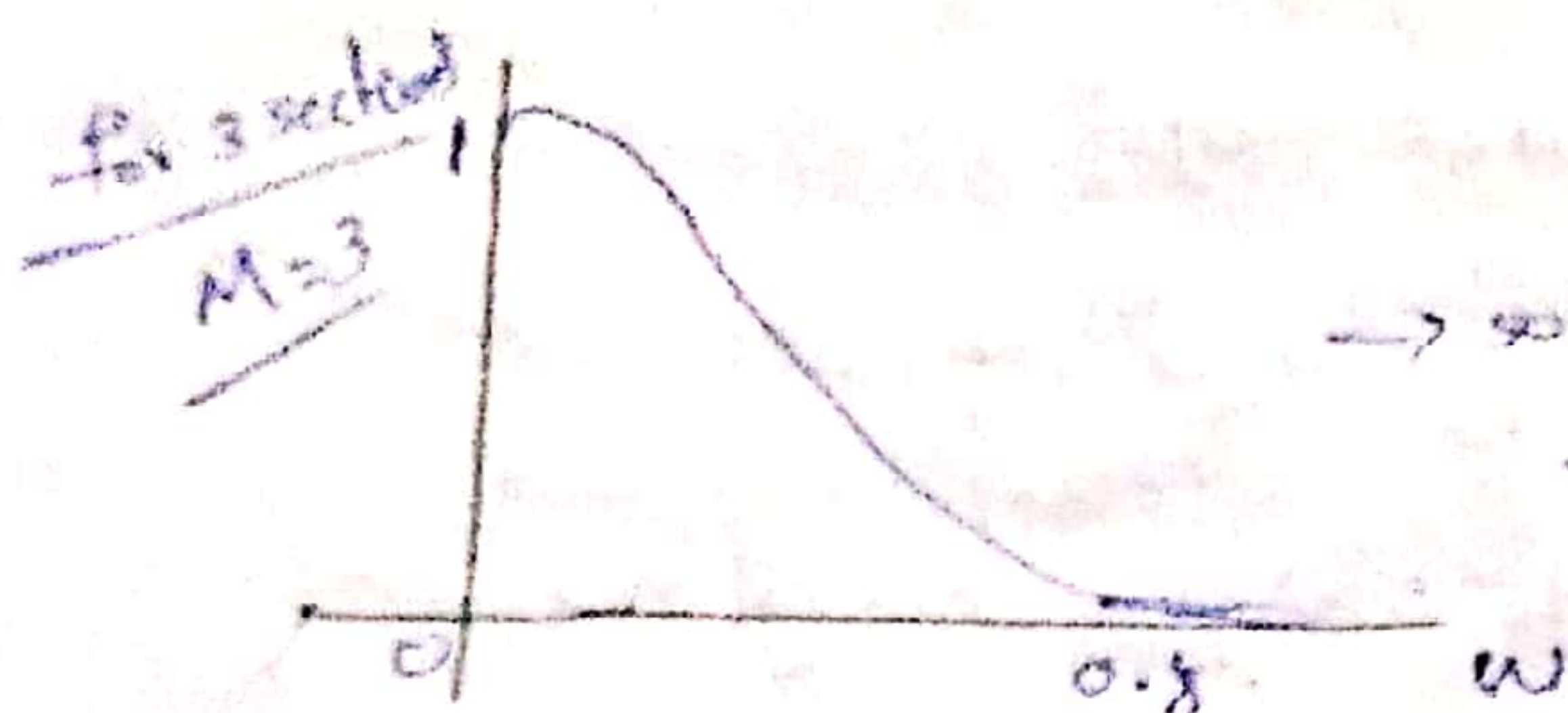


Cascading of M-filters results in multiplication of M-filters for total response.

A cascade of simple FIR filter,

$$H_0(z) = \frac{1}{2} (1+z^{-1})$$

results in an improved lowpass frequency response



→ somewhat good filter
→ improvement there

⇒ The 3-dB cutoff frequency of a cascade of M -sections,

$$|H_0(e^{j\omega})|^M$$

at $\omega = \omega_c \rightarrow |H_0(e^{j\omega})|^M = \frac{1}{\sqrt{2}} \times 1 (H(e^{j0}))^M$

$$\left(\cos\left(\frac{\omega_c}{2}\right)\right)^M = \frac{1}{\sqrt{2}}$$

$$\cos\left(\frac{\omega_c}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)^{1/M} = \left(\frac{1}{2^{1/2}}\right)^{1/M}$$

$$\cos\left(\frac{\omega_c}{2}\right) = 2^{-1/2M}$$

$$\frac{\omega_c}{2} = \cos^{-1}\left(2^{-1/2M}\right)$$

$$\boxed{\omega_c = 2 \cos^{-1}\left(2^{-1/2M}\right)}$$

* For $M=3$,

$$\omega_c = 2 \cos^{-1}\left(2^{-1/2(3)}\right) = \frac{0.9429}{\pi} \times \pi = 0.302\pi$$

⇒ Cascade of first-order sections yields a sharper magnitude response but at expense of a decrease in width of the passband.

*

⇒ A better approximation to the ideal lowpass filter is given by higher-order moving average filter.

⇒ Signals with rapid fluctuations in sample values are generally associated with higher frequency components.

⇒ These high-frequency components are essentially removed by an moving-average filter resulting in a smoother

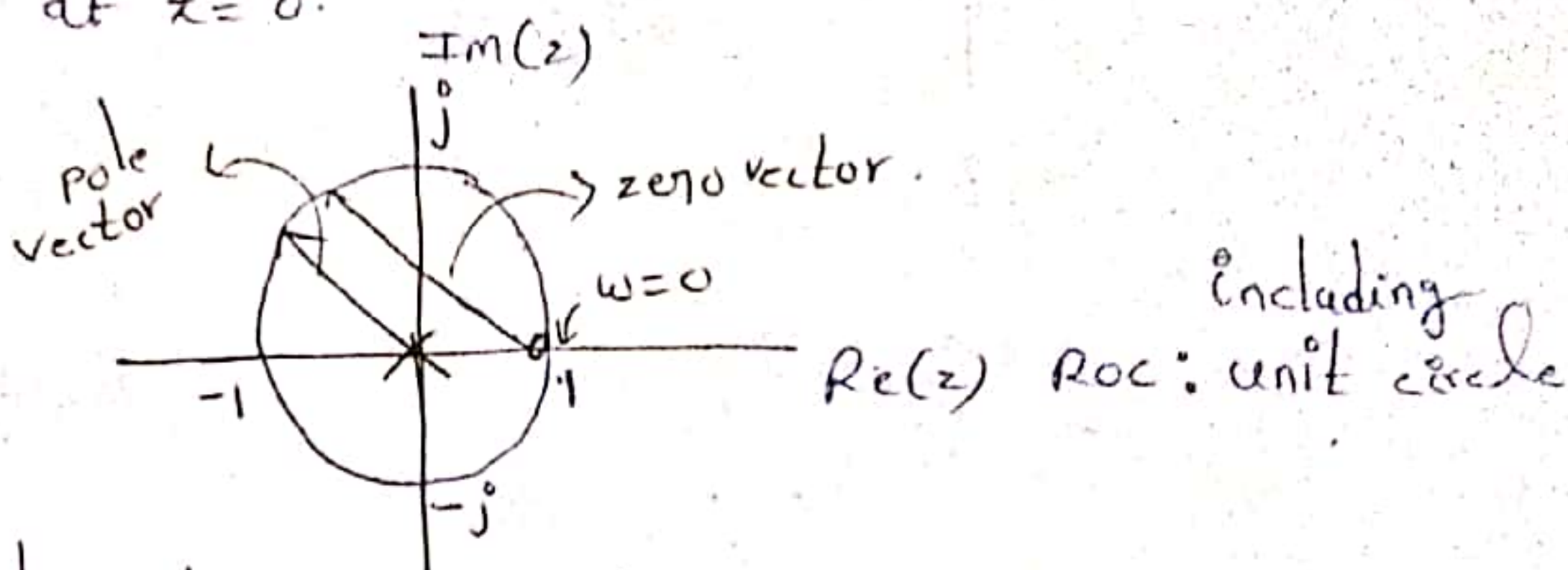
output waveform.

High-Pass FIR Digital filters

→ The simplest high-pass FIR filter is obtained from the simplest low-pass FIR filter by replacing (x) with $(-x)$

$$H_1(z) = \frac{1}{2}(1 - z^{-1}) = \frac{1 - z^{-1}}{2z} = \frac{z(1 - z^{-1})}{2z} = \frac{(1 - z^{-1})}{2}$$

→ The above transfer function has a zero at $z = +1$ & pole at $z = 0$.



→ A pole vector with unity magnitude.

→ As ω increases from 0 to π , the magnitude of zero vector decreases from a value of 2, the diameter of a unit circle, to π .

→ zero vector $\uparrow \rightarrow H_1(e^{j\omega}) \uparrow$

→ magnitude response $|H_1(e^{j\omega})|$ is a monotonically increasing function of ω from $\omega = 0$ to $\omega = \pi$

$$H_1(z) = \frac{1}{2}(1 - z^{-1})$$

$$H_1(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega})$$

→ maximum value of a magnitude function is 1 at $\omega = \pi$

$$|H_1(e^{j0})| = \frac{1}{2}(1 - e^{-j0}) = 0 \quad (z=1) \rightarrow \text{minimum value at } \omega = 0$$

$$|H_1(e^{j\pi})| = \frac{1}{2}(1 - e^{-j\pi}) = 1 \quad (z=-1)$$

→ The frequency response of above filter,

$$H_1(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = \frac{e^{-j\omega/2}}{2} (e^{j\omega/2} - e^{-j\omega/2}) \times \frac{j}{j} \\ = j e^{-j\omega/2} \sin \omega/2$$

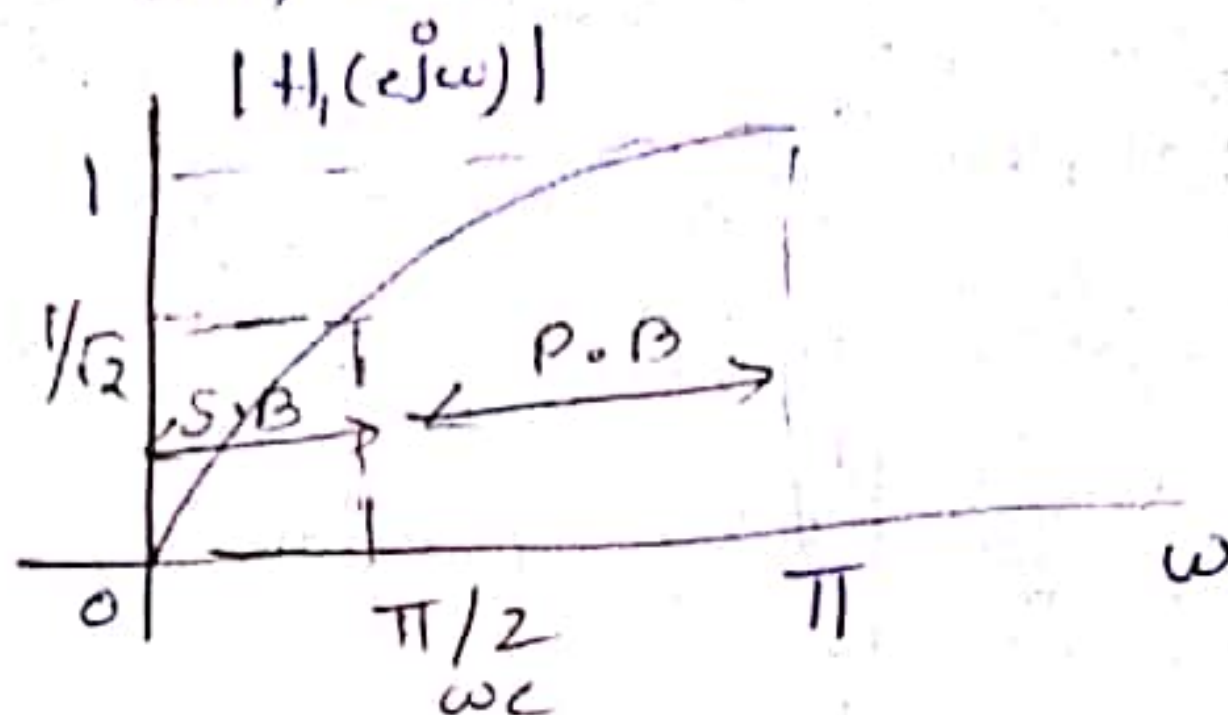
magnitude response:-

$$|H_1(e^{j\omega})| = |je^{j\omega/2} \sin \omega/2|$$

$$= \sin \omega/2$$

At $\omega=0$, $|H_1| = 0$

$\omega=\pi$, $|H_1| = 1$



$$\sin \omega_c/2 = 1/\sqrt{2} \times (1)$$

$$\Rightarrow \omega_c = \pi/2 \rightarrow \text{cutt off}$$

Bandwidth = $\pi - \omega_c$

$$= \pi - \pi/2$$

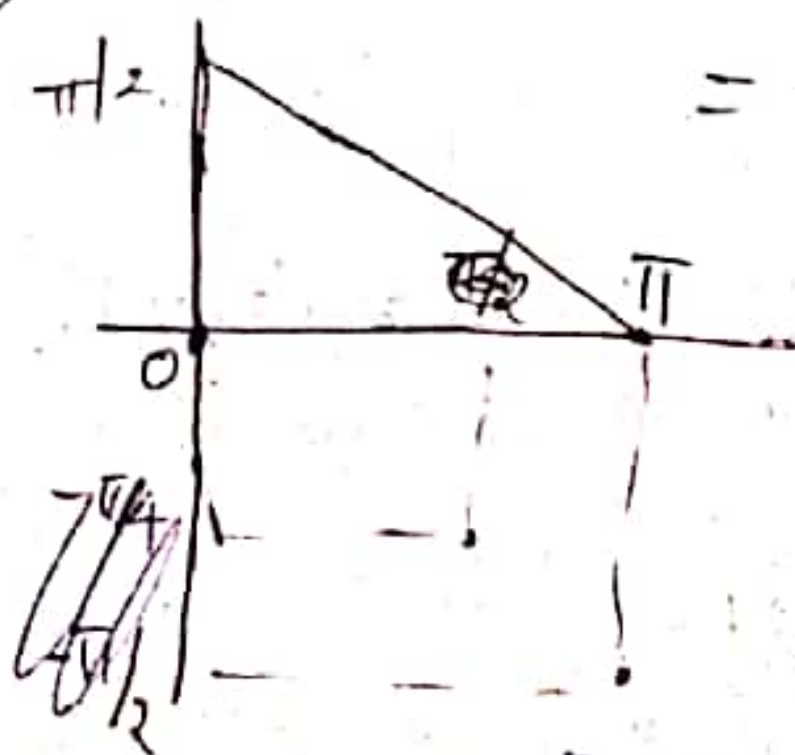
$$= \pi/2$$

Phase response:-

$$\angle H_1(e^{j\omega}) = \angle je^{j\omega/2} \sin \omega/2 = -\omega/2 + \angle \sin \omega/2$$

$$= -\omega/2 + \pi/2$$

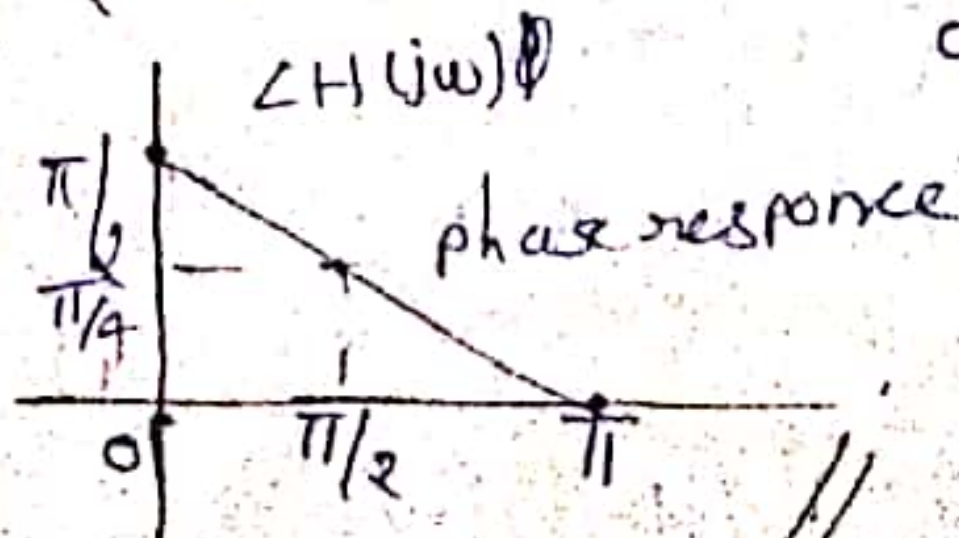
↓ +ve
0°



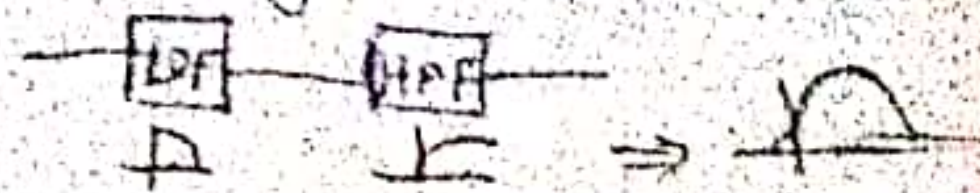
$\omega=0 \Rightarrow \angle H_1(\omega) = \pi/2$

$\omega=\pi \Rightarrow \angle H_1(\omega) = 0$

$\omega=\pi/2 \Rightarrow -\pi/4 + \pi/2 = \pi/4$



* BPF Band pass filter $\Rightarrow$ cascading of LPF & HPF



$$H_2(z) = \frac{1}{2}(1 - z^{-2})$$

$$H_2(e^{j\omega}) = \frac{1}{2}(1 - e^{-j2\omega}) = \frac{e^{-j\omega}}{2}(e^{j\omega} - e^{-j\omega}) \times \frac{1}{j}$$

$$H_2(e^{j\omega}) = j e^{-j\omega} \sin \omega$$

* LPF & HPF $\Rightarrow$ BPF

$$\frac{1}{2}(1+z^{-1}) \cdot \frac{1}{2}(1-z^{-1}) \Rightarrow \frac{1}{4}(1-z^{-2}) = \frac{1}{4}(1+z^{-1})(1-z^{-1})$$

$$\Rightarrow \frac{2}{4}(1-z^{-2}) = \frac{1}{2}(1-z^{-2})$$

for getting max value = 1

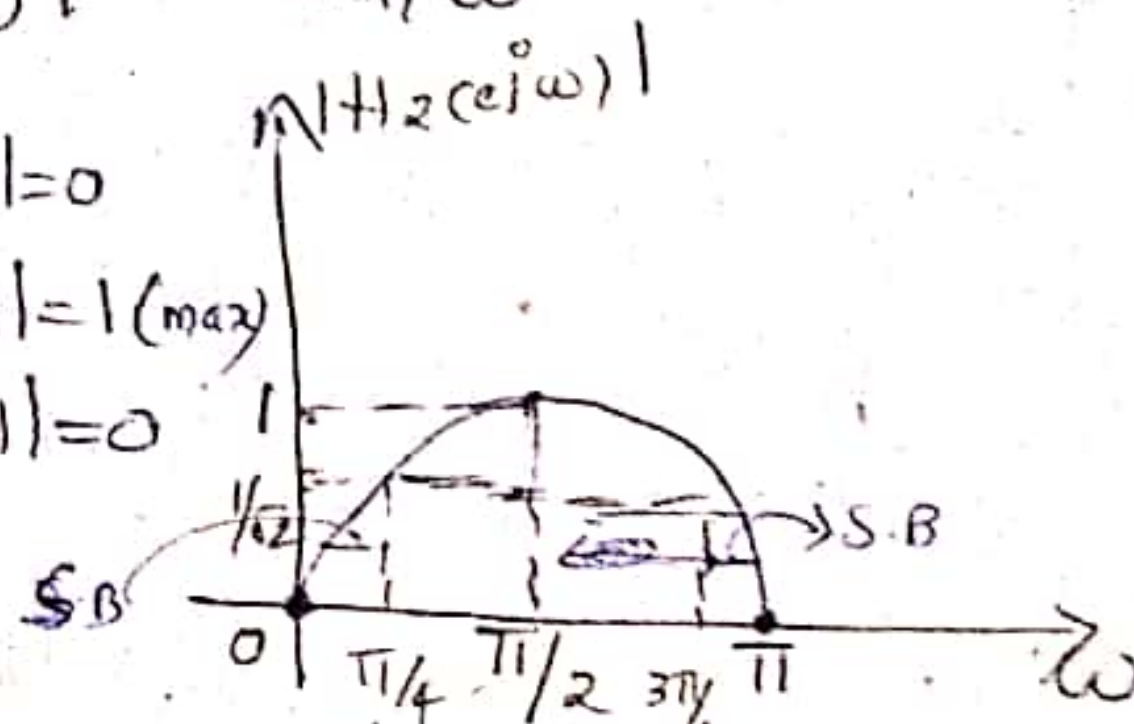
magnitude response:-

$$|H_2(e^{j\omega})| = \sin \omega$$

$$\Rightarrow \omega = 0 \Rightarrow |H| = 0$$

$$\Rightarrow \omega = \pi/2 \Rightarrow |H| = 1 (\text{max})$$

$$\Rightarrow \omega = \pi \Rightarrow |H| = 0$$



$$\sin \omega = \frac{1}{\sqrt{2}} \quad \text{P.B.}$$

$$\Rightarrow \omega_c = \sin^{-1}(\frac{1}{\sqrt{2}}) = \pi/4, 3\pi/4$$

$$\Rightarrow \omega_c = \pi/4, 3\pi/4$$

$\Rightarrow$ In this case, Center frequency = B.W //

$$\text{B.W} = \pi/2$$

Bandwidth =

higher freq - lower freq

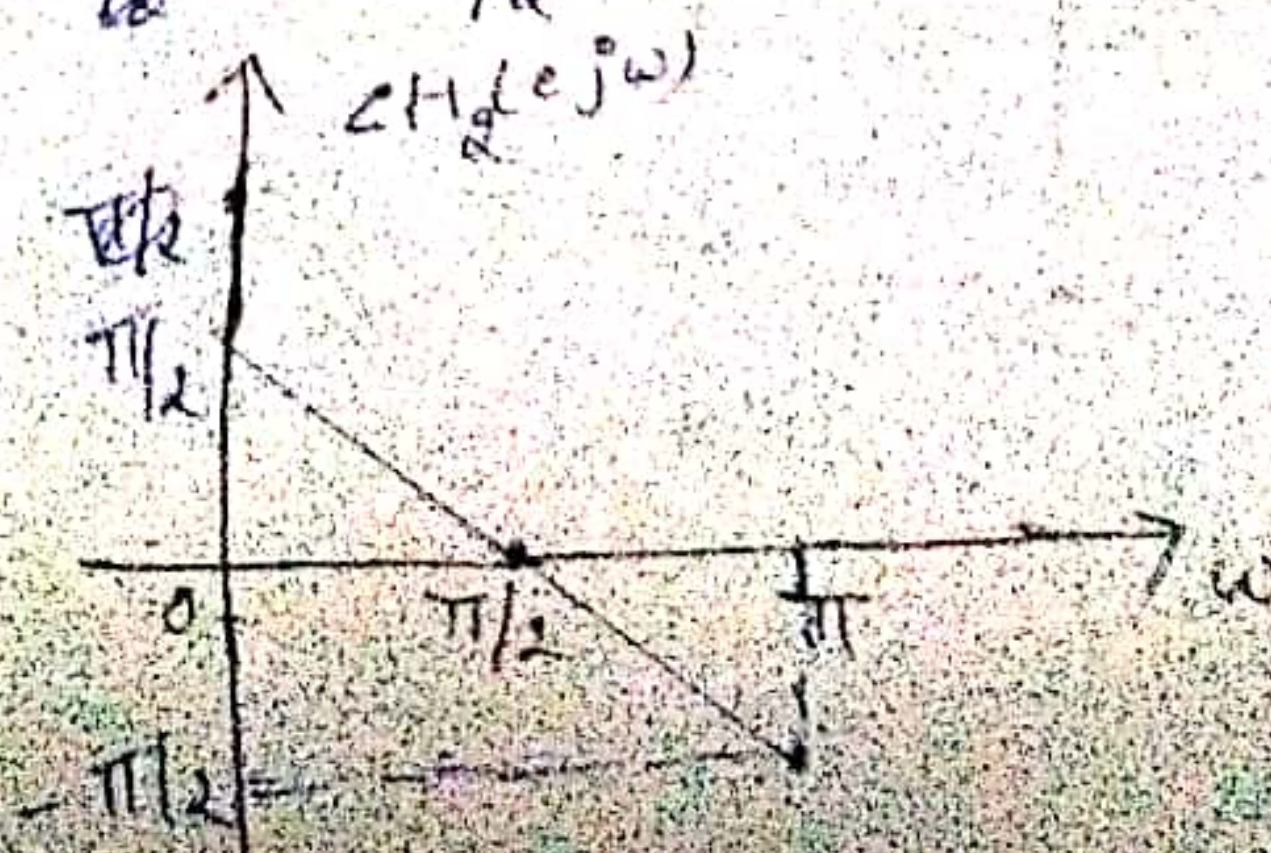
$$= \omega_{c2} - \omega_{c1}$$

$$= 3\pi/4 - \pi/4$$

$$= \pi/2 //$$

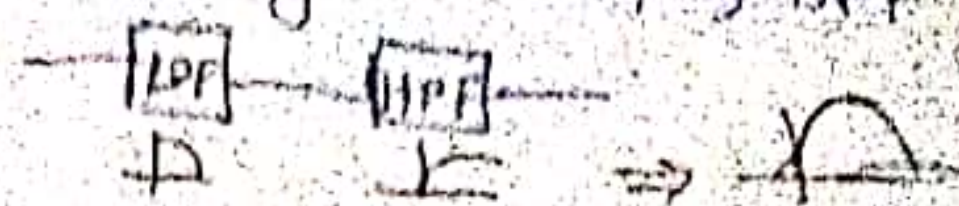
Phase characteristics:-

$$\angle H_2(e^{j\omega}) = -\omega + \pi/2$$



* BPF Band pass Filter $\Rightarrow$ cascading of LPF & HPF

$$H_2(z) = \frac{1}{2} (1 - z^{-2})$$



$$H_2(e^{j\omega}) = \frac{1}{2} (1 - e^{-j2\omega}) = \frac{e^{-j\omega}}{2} (e^{j\omega} - e^{-j\omega}) \times \frac{j}{j}$$

$$H_2(e^{j\omega}) = j e^{-j\omega} \sin \omega$$

* LPF & HPF $\Rightarrow$ BPF

$$\frac{1}{2} (1 + z^{-1}) \cdot \frac{1}{2} (1 - z^{-1}) \Rightarrow \frac{1}{4} (1 - z^{-2}) = \frac{1}{4} (1 + z^{-1})(1 - z^{-1})$$

$$\Rightarrow \frac{2}{4} (1 - z^{-2}) = \frac{1}{2} (1 - z^{-2})$$

$\hookrightarrow$ for getting max value = 1

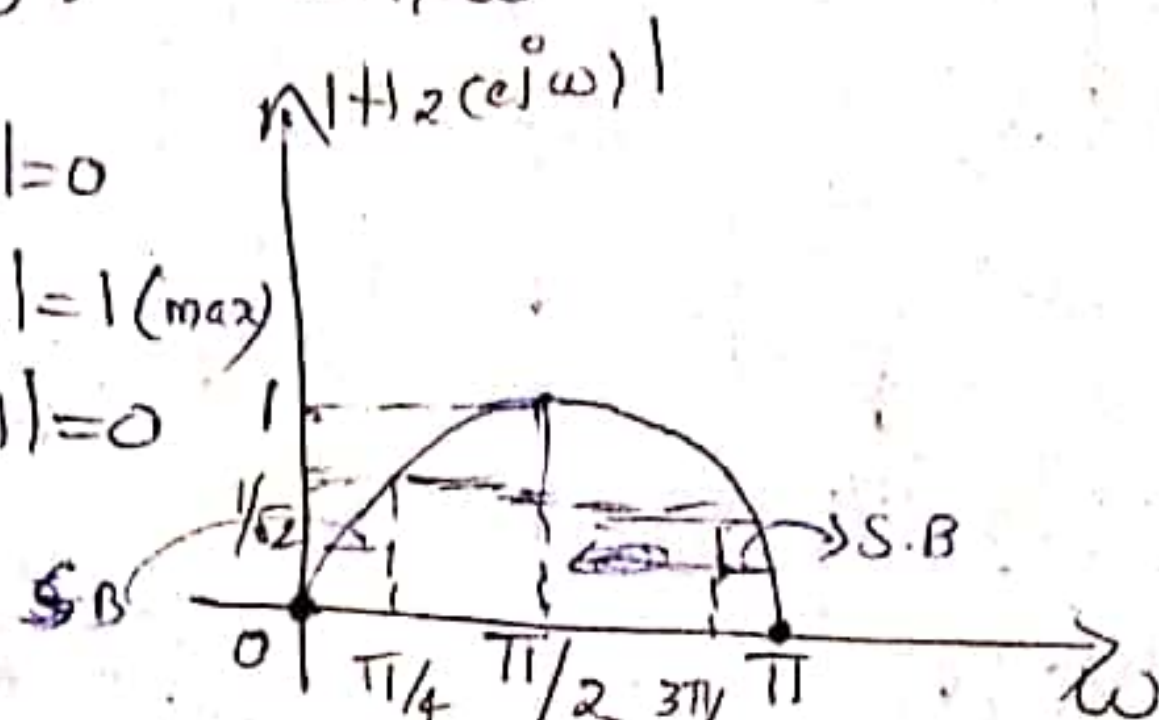
magnitude response:-

$$|H_2(e^{j\omega})| = \sin \omega$$

$$\Rightarrow \omega = 0 \Rightarrow |H| = 0$$

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$$\Rightarrow \omega = \pi \Rightarrow |H| = 0$$



$$\sin \omega = 1/\sqrt{2} \quad \text{P.B.}$$

$$\Rightarrow \omega_c = \sin^{-1}(1/\sqrt{2}) = \pi/4, 3\pi/4$$

$$\Rightarrow \omega_c = \pi/4, 3\pi/4$$

Bandwidth =

higher freq - lower freq

$$= \omega_{c2} - \omega_{c1}$$

$$= 3\pi/4 - \pi/4$$

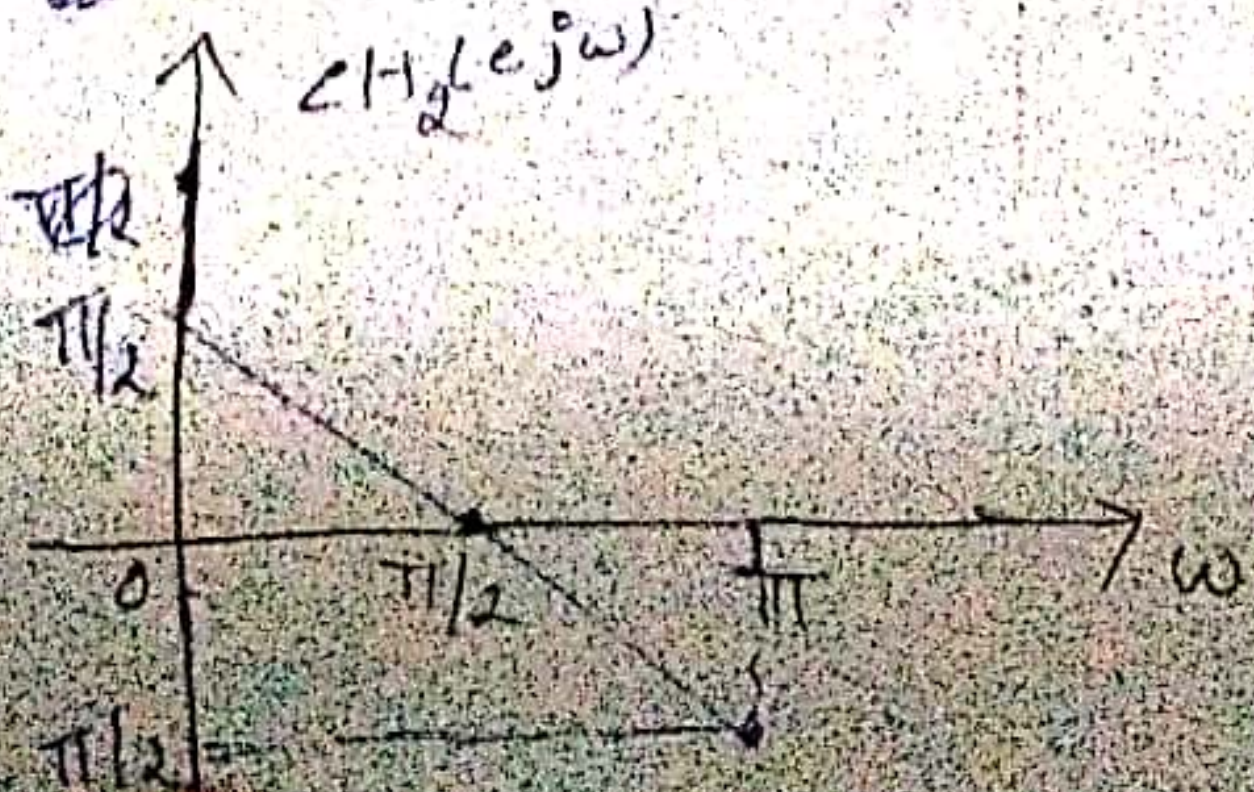
$$= \pi/2 //$$

$\Rightarrow$ In this case, center frequency = B.W //

$$\text{B.W} = \pi/2$$

Phase characteristics :-

$$\angle H_2(e^{j\omega}) = -\omega + \pi/2$$



BSF Band stop filter

$$H_3(z) = \frac{1}{2} (1 - (-z^{-2})) = \frac{1}{2} (1 + z^{-2})$$

$$H_3(e^{j\omega}) = \frac{1}{2} (1 + e^{-j2\omega}) = \frac{1}{2} e^{-j\omega} (e^{j\omega} + e^{-j\omega})$$

$$= e^{-j\omega} \cos \omega$$

$$H_3(e^{j\omega}) = e^{-j\omega} \cos \omega$$

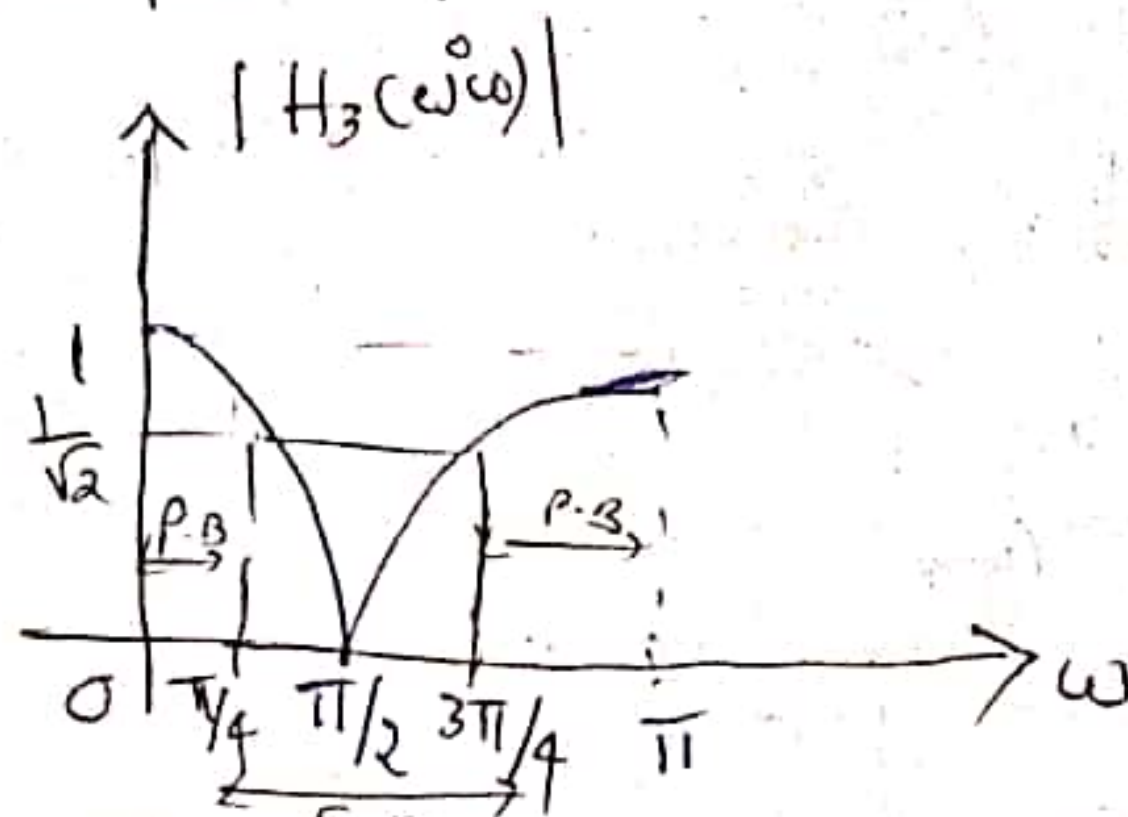
magnitude response:-

$$|H_3(e^{j\omega})| = |\cos \omega|$$

$$\omega = 0 \Rightarrow |H| = 1$$

$$\omega = \pi \Rightarrow |H| = 1$$

$$\omega = \pi/2 \Rightarrow |H| = 0$$



$$\cos \omega_c = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \omega_c = \frac{\pi}{4} \rightarrow \text{cutoff frequency}$$

$$= 3\pi/4$$

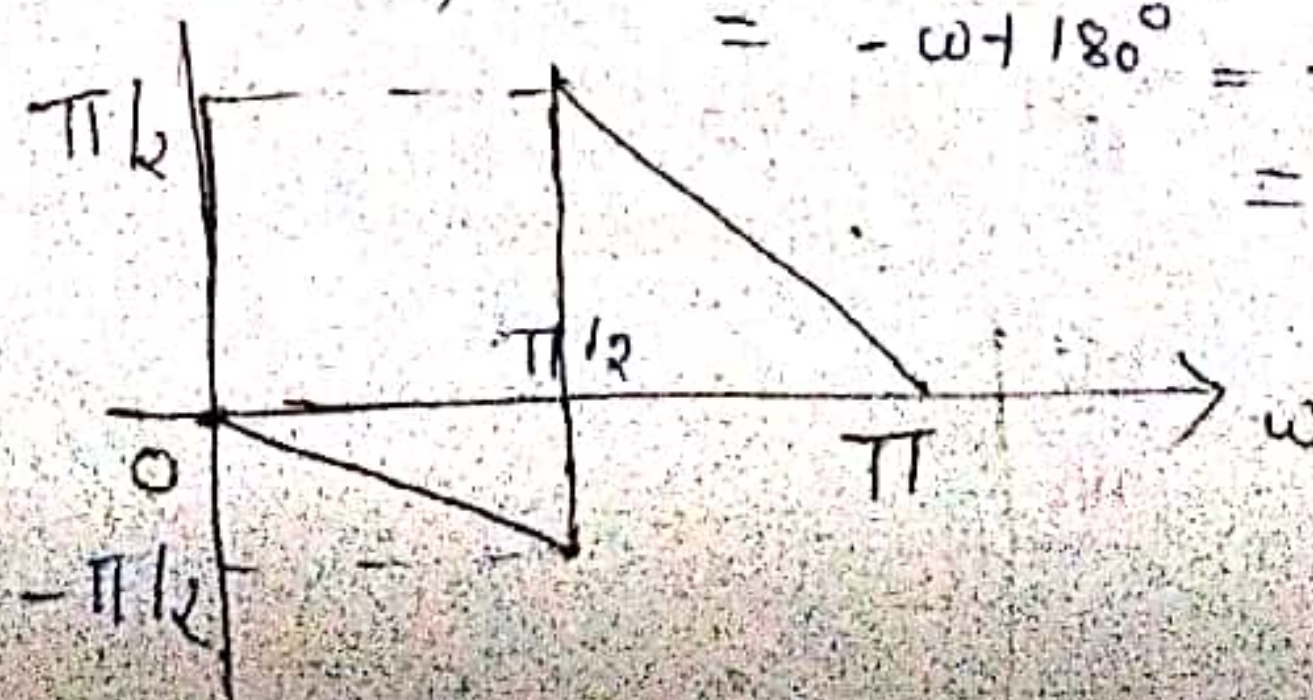
$$\omega_c = \pi/2 \rightarrow \text{Notch frequency}$$

$\Rightarrow$ So, BSF also called notch filter

$$\text{Stop Band: B.W} = 3\pi/4 - \pi/4 = \pi/2 //$$

Phase response:-

$$\angle H_3(e^{j\omega}) = \angle e^{-j\omega} \cos \omega = -\omega + \angle \cos \omega$$



$$= -\omega + 180^\circ = -\omega + \pi \quad (\pi/2 \text{ to } \pi)$$

$$= -\omega + \angle \cos \omega$$

$$\angle \cos \omega = \begin{cases} 0 & 0 \text{ to } \pi/2 \\ 180^\circ & \pi/2 \text{ to } \pi \end{cases}$$

Note: In FIR filters, especially in case of moving average filter.

- $\Rightarrow$ there are no poles, there is no pole-zero cancellation
- $\Rightarrow$ there are poles only at '0' ($p=0$)

IIR

$\Rightarrow$ there is no pole-zero cancellation.

$\rightarrow$ At least one non-zero pole of $H(z)$ is not cancelled by zero. i.e. there is denominator along with numerator.

Eg. $\rightarrow h(n) = a^n u(n) \rightarrow 0$ to $\infty \rightarrow$ infinite impulse response
 $\Rightarrow h(n) = u(n)$

Simple IIR Digital Filters

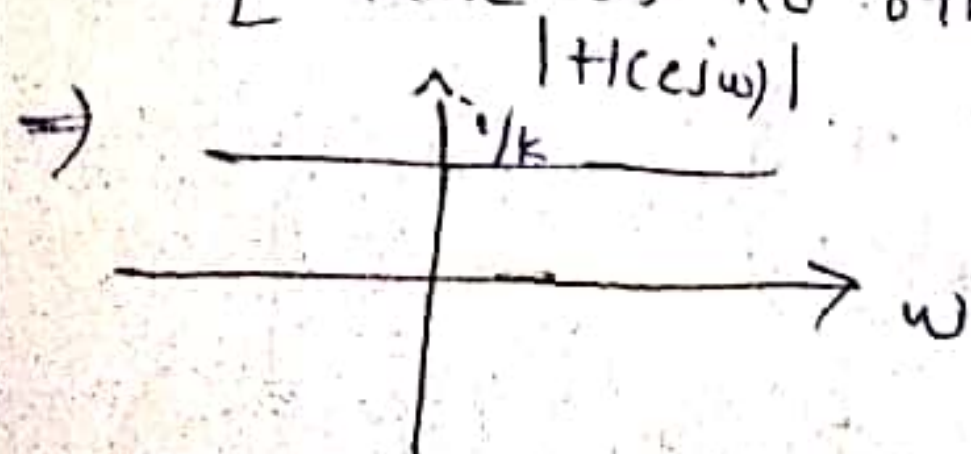
FIR All pass filter:-

$\rightarrow$ simple delay

$$H(z) = z^{-N}$$

$\Rightarrow$ magnitude is one/constant over all frequencies

$\Rightarrow$ For Amplification, just multiply by factor with magnitude
[there is no other digital amplifier]



$$|H_{AP}(z)|_{z=e^{j\omega}} = |H_{AP}(e^{j\omega})|$$

$$|(e^{j\omega})^{-N}| = |e^{-j\omega N}| = 1$$

$\rightarrow$ magnitude is not depending on ω

$\Rightarrow \phi = \angle H_{AP}(z) = \angle e^{-j\omega N} = -\omega N$

$\phi \propto -\omega \rightarrow$ linear phase.

[phase is directly proportional to frequency]

$\Rightarrow$ Angle is different at different frequencies

$\Rightarrow$ Angle is linear w.r.t to frequency

Group delay $\rightarrow \tau_g(\omega) = -\frac{d\phi}{d\omega} = N$

Note: In FIR filters, especially in case of moving average filter.

- $\Rightarrow$ there are no poles, there is no pole-zero cancellation
- $\Rightarrow$ there are poles only at '0' (P.O)

IIR

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Eg: $\Rightarrow h(n) = a^n u(n) \rightarrow 0$ to $\infty \rightarrow$ infinite impulse response
 $\Rightarrow h(n) = u(n)$

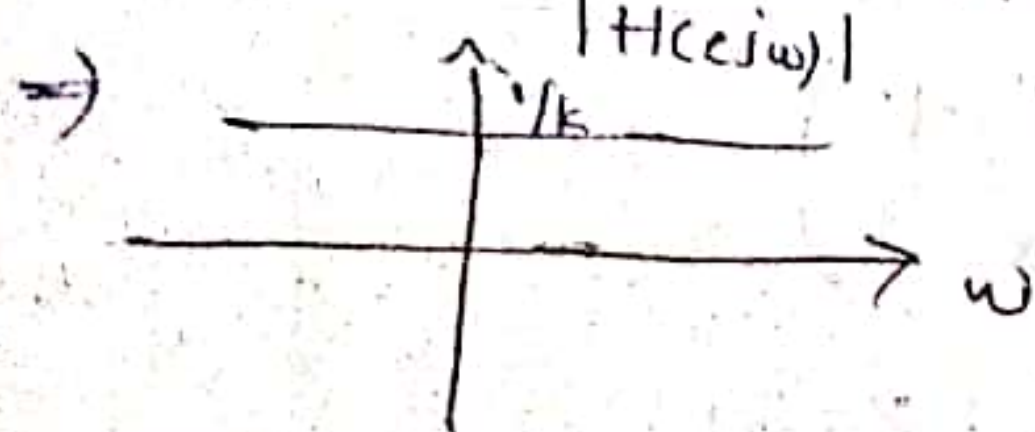
Simple IIR Digital Filters

FIR All pass filter:-

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[phase is directly proportional to frequency]

$\Rightarrow$ Angle is different at different frequencies.

$\Rightarrow$ Angle is linear w.r.t to frequency.

Group delay $\tau_g(\omega) = -\frac{d\phi}{d\omega} = N$

- Group delay is a delay suffered by group of frequencies
- if group delay is constant → Linear phase signal
- if there are group of frequencies fixed over a system channel (system) whose T.F is z^{-n} then all of these frequencies shall arrive at the opp after n samples later.
- If group delay is not constant then there is a distortion.

Simple IIR filter:-

At least one non-zero pole of $H(z)$ is not canceled by zero. Eg:- $h(n) = \alpha^n u(n)$ → infinite length sequence

Simple IIR LPF:-

$$H_L(z) = \frac{1}{2} \frac{(1+z^{-1})}{(1-\alpha z^{-1})}$$

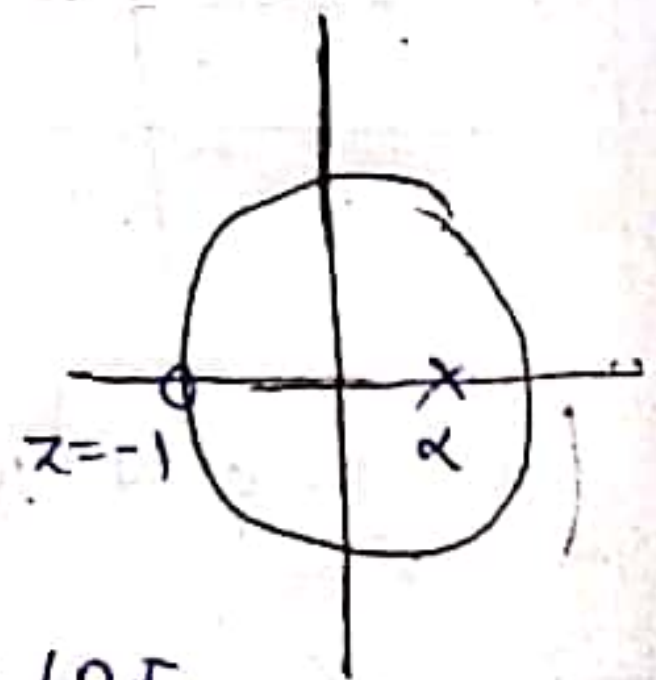
Order:-
order of z .

→ delay of z in time domain.

→ multiply by $(1-\alpha)$ to get magnitude = 1 at $\omega=0$ [Adding a pole to FIR LPF to get IIR filter characteristics]

$$= \left(\frac{1-\alpha}{2} \right) \left(\frac{1+z^{-1}}{1-\alpha z^{-1}} \right)$$

Pole: at $z = \alpha$, zero at $z = -1$
for stability $|\alpha| < 1$



Magnitude response:-

* $(1+z^{-1})$ → FIR LPF
we must add a pole to get IIR LPF
 $\left(\frac{1-\alpha}{2} \right)$ → to normalize the filter, DC response to 1 at $\omega=0$

frequency,
impulse response,

$$\frac{1+z^{-1}}{1-\alpha z^{-1}} = \frac{1+e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{1+1}{1-\alpha} = \frac{2}{1-\alpha}$$

$$H(e^{j\omega}) = \frac{1-\alpha}{2} \left(\frac{1+e^{-j\omega}}{1-\alpha e^{-j\omega}} \right)$$

$H_{LP}(z)$ has a real pole at $z = \alpha$

Magnitude response:-

$$|H_{LP}(e^{j\omega})| = \left| \frac{1-\alpha}{2} \cdot \frac{1+e^{-j\omega}}{1-\alpha e^{-j\omega}} \right|$$

$$z=1 \Rightarrow \omega=0 \Rightarrow |H_{LP}(e^{j0})| = \left(\frac{1-\alpha}{2} \right) \left(\frac{2}{1-\alpha} \right) = 1 \text{ (max)}$$

$$z=-1 \Rightarrow \omega=\pi \Rightarrow |H_{LP}(e^{j\pi})| = \left(\frac{1-\alpha}{2} \right) \left(\frac{1-1}{1-\alpha} \right) = 0 \text{ (min)}$$

$\Rightarrow |H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega=0$ to $\omega=\pi$.

$$\omega=0 \Rightarrow 1-\alpha e^{j\omega} = 1-\alpha$$

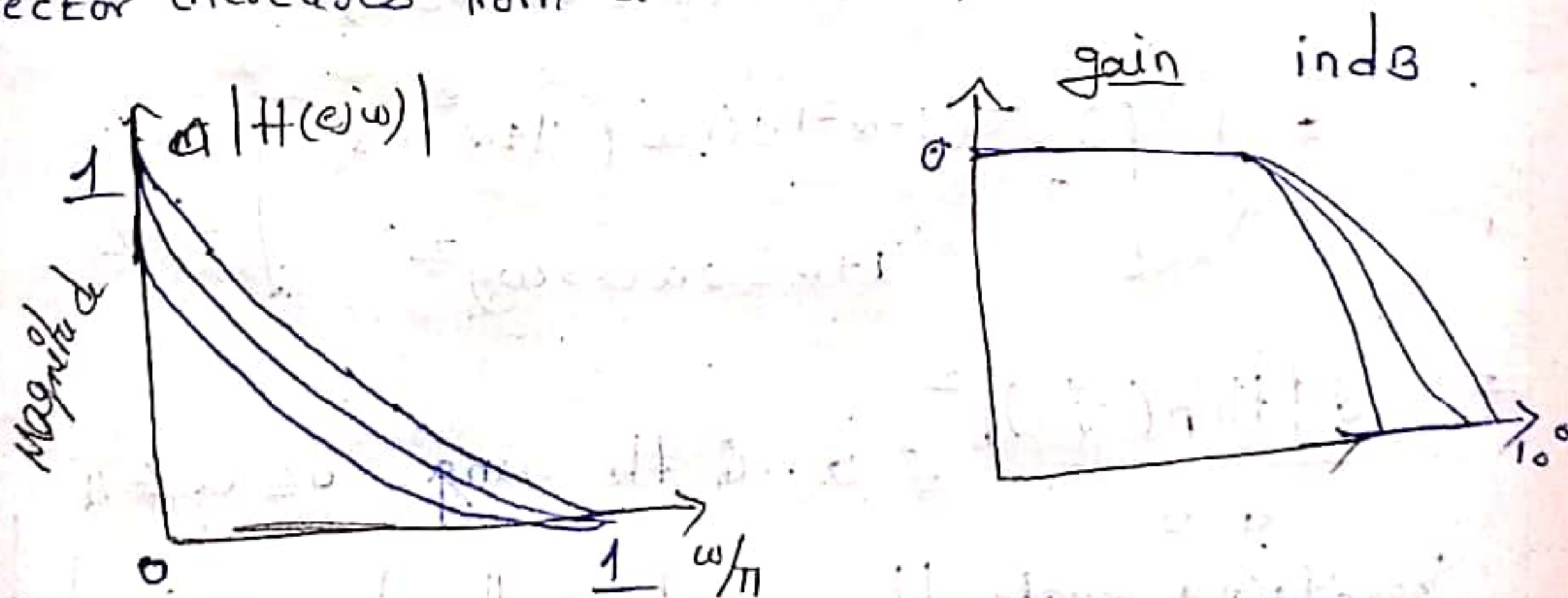
$$\omega=\pi \Rightarrow 1-\alpha e^{j\omega} = 1-(-\alpha) = 1+\alpha \quad \left. \begin{array}{l} \omega=0 \\ \omega=\pi \end{array} \right\} \text{ pole vector.}$$

~~$\Rightarrow$ pole vector~~

$\Rightarrow$ magnitude response to have a zero at $\omega=\pi$ in the stopband of the filter.

$\Rightarrow$ As ω increases from 0 to π , the magnitude of the zero vector decreases from a value of 2 to 0 for

$\Rightarrow$ For a positive value of α , the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$



$\Rightarrow$ The squared magnitude function is given by

$$|H_{LP}(e^{j\omega})|^2 = \left(\frac{1-\alpha}{2} \right)^2 \frac{|1+e^{-j\omega}|^2}{|1-\alpha e^{-j\omega}|^2}$$

$$= \left(\frac{1-\alpha}{2} \right)^2 \frac{|1+\cos\omega - j\sin\omega|^2}{|1-\alpha\cos\omega + j\alpha\sin\omega|^2}$$

$$|x+iy| = \sqrt{x^2+y^2}$$

$$= \left(\frac{1-\alpha}{2} \right)^2 \frac{\left| \sqrt{(1+\cos\omega)^2 + \sin^2\omega} \right|^2}{\left| \sqrt{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega} \right|^2}$$

$$= \left(\frac{1-\alpha}{2} \right)^2 \frac{(1+\cos\omega)^2 + \sin^2\omega}{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega}$$

$$= \left(\frac{1-\alpha}{2} \right)^2 \frac{1 + \cos^2\omega + 2\cos\omega + \sin^2\omega}{1 + \alpha^2\cos^2\omega - 2\alpha\cos\omega + \alpha^2\sin^2\omega}$$

$$= \left(\frac{1-\alpha}{2} \right)^2 \frac{2 + 2\cos\omega}{1 + \alpha^2 - 2\alpha\cos\omega}$$

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2 (1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

$$\frac{d|H_{LP}(e^{j\omega})|^2}{d\omega} = \frac{1}{2} \frac{(1+\alpha^2-2\alpha\cos\omega)(1-\alpha^2)(-\sin\omega) - (1-\alpha^2)(1+\cos\omega)(-2\alpha\sin\omega)}{(1+\alpha^2-2\alpha\cos\omega)^2}$$

$$= \frac{1}{2} \left[\frac{-(1-\alpha^2)\sin\omega [1+\alpha^2-2\alpha\cos\omega + 2\alpha + 2\alpha\cos\omega]}{(1+\alpha^2-2\alpha\cos\omega)^2} \right]$$

$$= \frac{1}{2} \left[\frac{-(1-\alpha^2)\sin\omega (1+\alpha^2+2\alpha)}{(1+\alpha^2-2\alpha\cos\omega)^2} \right] \leq 0$$

$$\Rightarrow \frac{d|H_{LP}(e^{j\omega})|^2}{d\omega} \leq 0 \text{ in the range } 0 \leq \omega \leq \pi$$

Verifying again the monotonically decreasing behavior of magnitude function.

* To determine the 3-dB cutoff frequency, ω_c

$$|H_{LP}(e^{j\omega})|^2 = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

$$\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = \frac{1}{2}$$

$$\Rightarrow (1-\alpha)^2(1+\cos\omega_c) = (1+\alpha^2-2\alpha\cos\omega_c)$$

$$\Rightarrow \text{if } 1+\cos\omega_c = \alpha^2$$

$$\Rightarrow (1-\alpha)^2 + (1-\alpha)^2\cos\omega_c = 1+\alpha^2-2\alpha\cos\omega_c$$

$$\Rightarrow (1-\alpha)^2\cos\omega_c + 2\alpha\cos\omega_c = 1+\alpha^2-(1-\alpha)^2$$

$$\Rightarrow [(1-\alpha)^2 + 2\alpha]\cos\omega_c = 1+\alpha^2-(1-\alpha)^2$$

$$\Rightarrow [1+\alpha^2-2\alpha+2\alpha]\cos\omega_c = 1+\alpha^2-1-\alpha^2+2\alpha$$

$$\Rightarrow (1+\alpha^2)\cos\omega_c = 2\alpha$$

$$\cos\omega_c = \frac{2\alpha}{1+\alpha^2} \rightarrow (1)$$

$$\boxed{\cos\omega_c = \frac{2\alpha}{1+\alpha^2}}$$

$$\boxed{\omega_c = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

$\Rightarrow$ The above quadratic equation can be solved for α yielding two solutions.

$\alpha = ?$ from (1).

$$(1+\alpha^2)\cos\omega_c - 2\alpha = 0$$

$$\Rightarrow \alpha^2\cos\omega_c - 2\alpha + \cos\omega_c = 0$$

$$\Rightarrow \alpha^2 - \frac{2\alpha}{\cos\omega_c} + 1 = 0$$

$$\alpha = \frac{\frac{2}{\cos\omega_c} \pm \sqrt{\left(\frac{2}{\cos\omega_c}\right)^2 - 4}}{2}$$

$$= \frac{\frac{2}{\cos\omega_c} \pm \sqrt{\frac{4 - 4\cos^2\omega_c}{\cos^2\omega_c}}}{2}$$

$$= \frac{1}{\cos\omega_c} \pm \sqrt{\frac{1 - \cos^2\omega_c}{\cos^2\omega_c}}$$

$$= \frac{1}{\cos\omega_c} \pm \frac{\sin\omega_c}{\cos\omega_c} = \frac{1 \pm \sin\omega_c}{\cos\omega_c}$$

$$\frac{(1-\alpha)^2 (1+\cos \omega_c)}{\alpha^2 (1+\alpha^2 - 2\alpha \cos \omega_c)} = \frac{1}{2}$$

$$\Rightarrow (1-\alpha)^2 (1+\cos \omega_c) = (1+\alpha^2 - 2\alpha \cos \omega_c)$$

$$\Rightarrow 1 + \cos \omega_c = \alpha^2$$

$$\Rightarrow (1-\alpha)^2 + (1-\alpha)^2 \cos \omega_c = 1 + \alpha^2 - 2\alpha \cos \omega_c$$

$$\Rightarrow (1-\alpha)^2 \cos \omega_c + 2\alpha \cos \omega_c = 1 + \alpha^2 - (1-\alpha)^2$$

$$\Rightarrow [(1-\alpha)^2 + 2\alpha] \cos \omega_c = 1 + \alpha^2 - (1-\alpha)^2$$

$$\Rightarrow [1 + \alpha^2 - 2\alpha + 2\alpha] \cos \omega_c = 1 + \alpha^2 - 1 - \alpha^2 + 2\alpha$$

$$\Rightarrow (1 + \alpha^2) \cos \omega_c = 2\alpha$$

$$\cos \omega_c = \frac{2\alpha}{1 + \alpha^2} \quad \rightarrow (1)$$

$$\boxed{\cos \omega_c = \frac{2\alpha}{1 + \alpha^2}}$$

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$\alpha = ?$ from (1).

$$(1 + \alpha^2) \cos \omega_c - 2\alpha = 0$$

$$\Rightarrow \alpha^2 \cos \omega_c - 2\alpha + \cos \omega_c = 0$$

$$\Rightarrow \alpha^2 - \frac{2\alpha}{\cos \omega_c} + 1 = 0$$

$$\alpha = \frac{\frac{2}{\cos \omega_c} \pm \sqrt{\left(\frac{2}{\cos \omega_c}\right)^2 - 4}}{2}$$

$$= \frac{\frac{2}{\cos \omega_c} \pm \sqrt{\frac{4 - 4 \cos^2 \omega_c}{\cos^2 \omega_c}}}{2}$$

$$= \frac{1}{\cos \omega_c} \pm \sqrt{\frac{1 - \cos^2 \omega_c}{\cos^2 \omega_c}}$$

$$= \frac{1}{\cos \omega_c} \pm \frac{\sin \omega_c}{\cos \omega_c} = \frac{1 \pm \sin \omega_c}{\cos \omega_c}$$

$$\alpha = \frac{1 \pm \sin \omega_c}{\cos \omega_c}$$

Choose, $\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}$ $|\alpha| < 1$

Bounded Real (BR)

→ A causal, stable real coefficient Transfer function $H(z)$ is defined as a Bounded Real (BR) Transfer function if $|H(e^{j\omega})| \leq 1$ for all ω .

$$* |H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{\alpha(1+\alpha^2-2\alpha\cos\omega)} \quad \text{that}$$

$H_{LP}(z)$ is a BR function for $|\alpha| < 1$

Simple IIR HPF

A first order causal highpass IIR digital filter has a T.F given by.

$$H_{HP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-1}}{1+\alpha z^{-1}} \quad |\alpha| < 1 \text{ for stability}$$

$\Rightarrow z = -\alpha$ pole

[or]

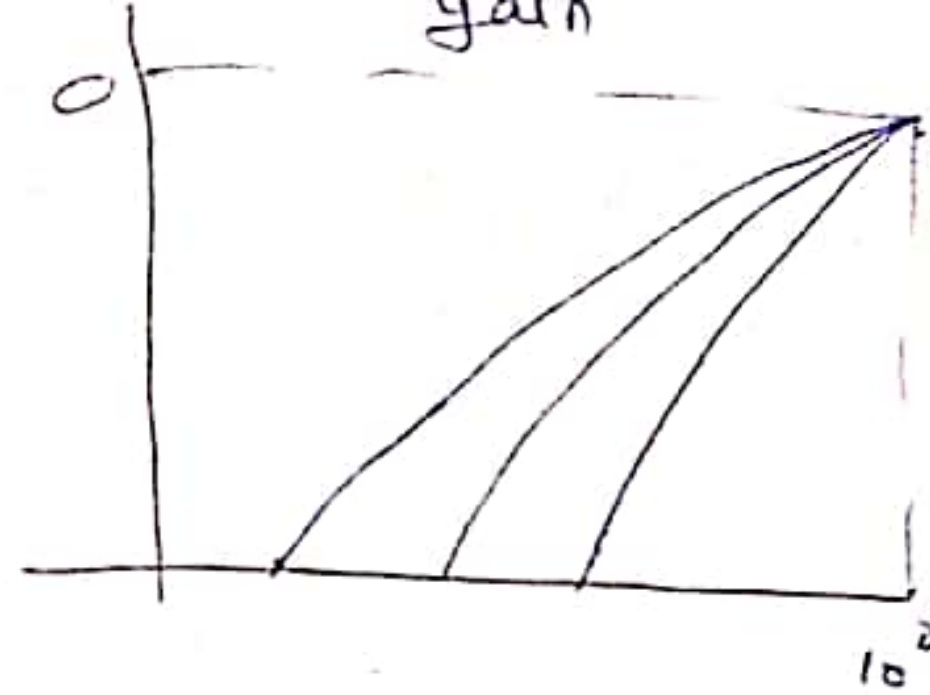
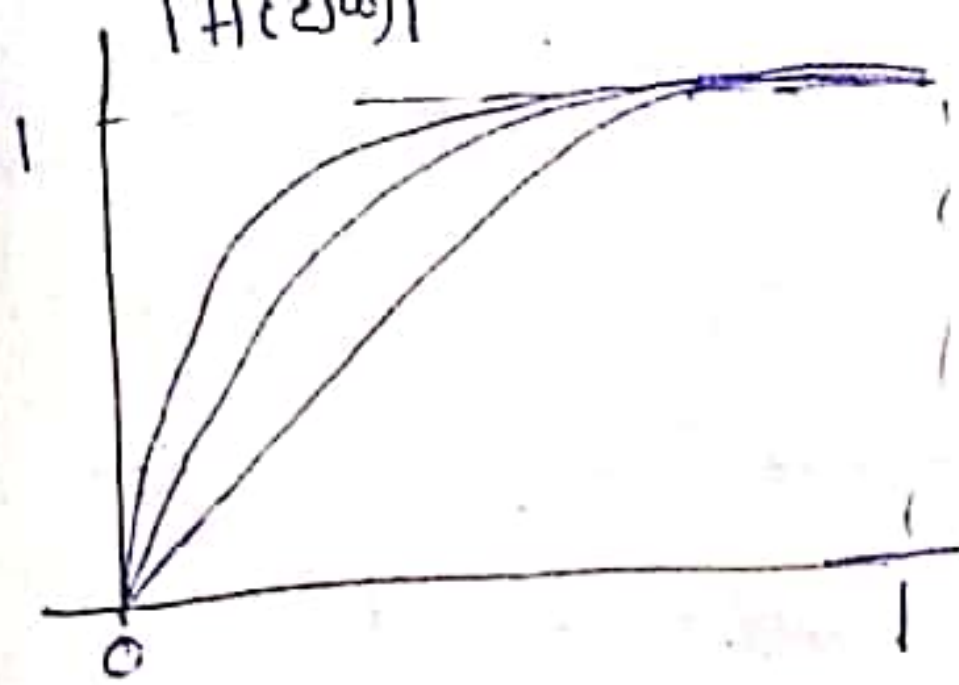
$$H_{HP}(z) = \left(\frac{1+\alpha}{2} \right) \left(\frac{1-z^{-1}}{1-\alpha z^{-1}} \right) \quad \text{pole at } z = \alpha$$

→ For above Transfer function has a zero at $z=1$ i.e. $\omega=0$ which is in stop band.

* The 3-dB cutoff frequency ω_c is same as $H_{LP}(z)$.

$$\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c}, \quad \omega_c = \cos^{-1} \left(\frac{2\alpha}{1+\alpha^2} \right)$$

* Magnitude & gain response of $H_{HP}(z)$



$\Rightarrow H_{HP}(z)$ is a BR function for $|\alpha| < 1$

Eg:- Design a first-order high pass digital filter with a 3 dB cut off frequency of 0.8π

$$\alpha = \frac{1 - \sin \omega_c}{\cos \omega_c} = \frac{1 - \sin(0.8\pi)}{\cos(0.8\pi)} = \frac{1 - 0.587785}{1 - 0.80902}$$

$$\alpha = -0.5095245$$

$$\begin{aligned} H_{HP}(z) &= \frac{1+\alpha}{2} \left(\frac{1-z^{-1}}{1-\alpha z^{-1}} \right) \\ &= \frac{1-0.5095245}{2} \left(\frac{1-z^{-1}}{1-(-0.5095245)z^{-1}} \right) \\ &= 0.245238 \left(\frac{1-z^{-1}}{1+0.5095245z^{-1}} \right) // \end{aligned}$$

Simple FIR BPF [Bandpass filter].

$\Rightarrow$ we have to use 2nd-order only becoz BPF should have zeroes at two points.

* A 2nd-order bandpass digital transfer function is given by,

$$H_{BP}(z) = \frac{1-\alpha}{2} \left(\frac{1-z^{-2}}{1-\beta(H_2)z^{-1}+\alpha z^{-2}} \right)$$

$\Rightarrow$ The above T.F has two zeroes at $z = 1$.

Its squared magnitude function is,

$$|H_{BP}(e^{j\omega})|^2$$

$$|H_{BP}(e^{j\omega})|^2 = \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \right)^2$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{1-e^{-j2\omega}}{1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-j2\omega}} \right)^2$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{1-\cos 2\omega + j\sin 2\omega}{1-\beta(1+\alpha)(\cos \omega - j\sin \omega) + \alpha(\cos 2\omega - j\sin 2\omega)} \right)^2$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \left(\frac{(1-\cos 2\omega)^2 + \sin^2 2\omega}{(1-\beta(1+\alpha)(\cos \omega - j\sin \omega) + \alpha(\cos 2\omega - j\sin 2\omega))^2} \right)$$

$$= \left(\frac{1-\alpha}{2}\right)^2 \frac{(2-2\cos 2\omega)}{[1-\beta\cos \omega + \beta\sin \omega - \beta\alpha\cos \omega + \beta\alpha j\sin \omega + \alpha\cos 2\omega - \alpha j\sin 2\omega]^2}$$

$$= \frac{\left(\frac{1-\alpha}{2}\right)^2}{4} \frac{2(1-\cos 2\omega)}{[1-\beta\cos \omega + \beta\sin \omega - \beta\alpha\cos \omega + \alpha\cos 2\omega + j(\beta\alpha\sin \omega - \alpha\sin 2\omega)]^2}$$

$$= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{[1-\beta\cos \omega + \beta\sin \omega - \beta\alpha\cos \omega + \alpha\cos 2\omega]^2 + [\beta\alpha\sin \omega - \alpha\sin 2\omega]^2}$$

$$= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{2(1-\beta(1+\alpha)(\cos \omega - j\sin \omega) + \alpha(\cos 2\omega - j\sin 2\omega))^2}$$

$$= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{2(1-\beta(1+\alpha)(\cos \omega - j\sin \omega) + \alpha(\cos 2\omega - j\sin 2\omega))^2}$$

$$= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{2(1+\beta^2)}$$

$$\begin{aligned}
&= \frac{(1-\alpha)^2}{2} \frac{1-\cos 2\omega}{[1-\beta(1+\alpha)\cos\omega + j\beta(1+\alpha)\sin\omega + \alpha\cos 2\omega - j\alpha\sin 2\omega]^2} \\
&= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{[1-\beta(1+\alpha)\cos\omega + \alpha\cos 2\omega]^2 + [\beta(1+\alpha)\sin\omega - \alpha\sin 2\omega]^2} \\
&= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{1^2 + [\beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega] + \beta^2(1+\alpha)^2\sin^2\omega - 2[\beta(1+\alpha)\cos\omega + \alpha\cos 2\omega] + \alpha^2\sin^2 2\omega - 2\beta(1+\alpha)\alpha\sin\omega\sin 2\omega} \\
&= \frac{(1-\alpha)^2}{2} \frac{(1-\cos 2\omega)}{1 + \beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega + \beta^2(1+\alpha)^2\sin^2\omega + \alpha^2\sin^2 2\omega + 2[\beta(1+\alpha)\cos\omega + \alpha\cos 2\omega] - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega - 2[\beta(1+\alpha)\cos\omega + \alpha\cos 2\omega]} \\
&= \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1 + \beta^2(1+\alpha)^2 + \alpha^2 - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega + 2[\beta(1+\alpha)\cos\omega + \alpha\cos 2\omega] - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega]} \\
&= \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1 + \beta^2(1+\alpha)^2 + \alpha^2 - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega + 2\beta(1+\alpha)\alpha\cos 2\omega\cos\omega + 2\beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega]} \\
&= \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1 + \beta^2(1+\alpha)^2 + \alpha^2 - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega + 2\beta(1+\alpha)\alpha\cos 2\omega\cos\omega + 2\beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega]} \\
&= \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1 + \beta^2(1+\alpha)^2 + \alpha^2 - 2\beta\alpha(1+\alpha)\sin\omega\sin 2\omega + 2\beta(1+\alpha)\alpha\cos 2\omega\cos\omega + 2\beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega]}
\end{aligned}$$

$$= \frac{(1-\alpha)^2 (1-\cos 2\omega)}{2 [1+\beta^2(1+\alpha)^2 + \alpha^2 - 2\alpha\beta(1+\alpha)\cos\omega +$$

$$+ 2\beta^2(1+\alpha)^2\cos^2\omega + \alpha^2\cos^2 2\omega]$$

$$= \frac{(1-\alpha)^2 (1-\cos 2\omega)}{2 [1+\beta^2(1+\alpha)^2 - 2\alpha\beta(1+\alpha)^2\cos\omega + 2\alpha\cos 2\omega]}$$

$|H_{BP}(e^{j\omega})|^2$ goes to zero at $\omega=0$ & $\omega=\pi$
 $\Rightarrow$ maximum value of 1 at $\omega=\omega_0$ called the
 Central frequency of bandpass signal filter,

$$\omega_0 = \cos^{-1}(\beta)$$

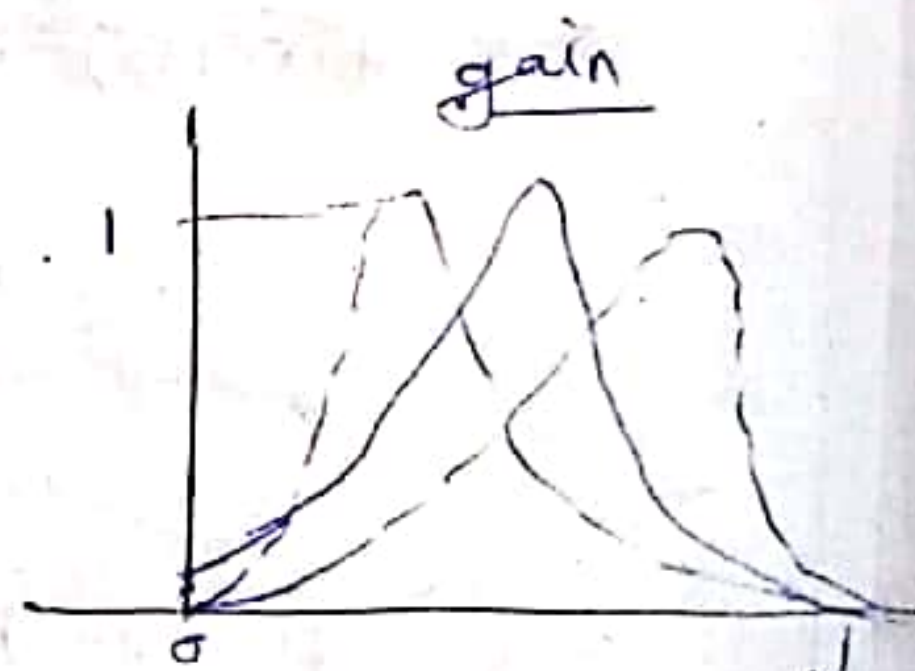
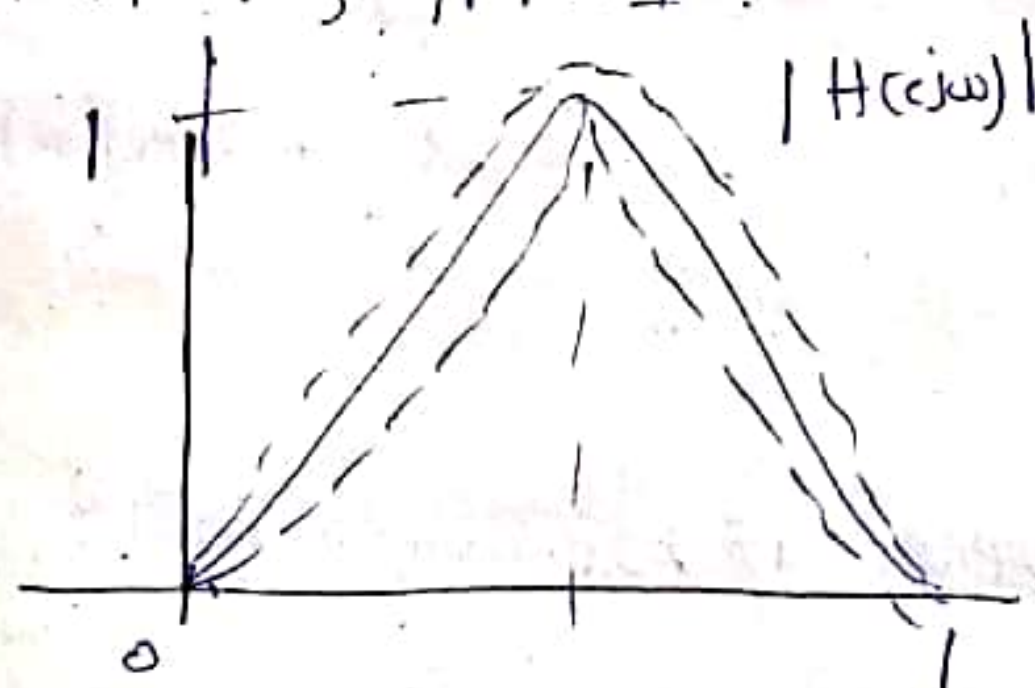
$$\beta = \cos\omega_0$$

$\Rightarrow$ The frequencies ω_{c1} & ω_{c2} where $|H_{BP}(e^{j\omega})|^2$
 becomes $1/2$ are called 3-dB cutoff frequencies.

$\Rightarrow$ The difference between the two cutoff frequencies,
 assuming $\omega_{c2} > \omega_{c1}$ is called 3-dB bandwidth

$$BW = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

$\Rightarrow$ The transfer function $H_{BP}(z)$ is a BR function if
 $|\alpha| < 1$ & $|\beta| < 1$.



Ex:- Design a 2nd order bandpass digital filter with
 center frequency at 0.4π & a 3-dB bandwidth of

$$0.1\pi, \quad \beta = \cos\omega_0 = \cos(0.4\pi) = 0.309017$$

$$\frac{2\alpha}{1+\alpha^2} = \cos(BW)$$

$$\cos(0.1\pi) = 0.9510565$$

$$\Rightarrow \alpha = \frac{1 - \sin \omega_0}{1 + \cos \omega_0} = \frac{1 - 0.30901699}{0.9510565} = 0.72654253$$

$$\alpha = \frac{1 + 0.30901699}{0.9510565} = 1.376382$$

T.F,

$$H_{BP}(z) = \frac{1-\alpha}{2} \cdot \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \left. \vphantom{\frac{1-\alpha}{2}} \right\} \alpha = 1.376382$$

$$= -0.18819 \frac{1-z^{-2}}{1-0.7343424z^{-1}+1.376382z^{-2}}$$

$$H_{BP}^{II}(z) = \frac{1-\alpha}{2} \cdot \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \left. \vphantom{\frac{1-\alpha}{2}} \right\} \alpha = 0.72654253$$

$$= 0.13673 \cdot \frac{1-z^{-2}}{1-0.533531z^{-1}+0.72654253z^{-2}}$$

$\Rightarrow$ The poles of $H_{BP}(z)$ are at $z = 0.3671712 \pm j1.1425636$

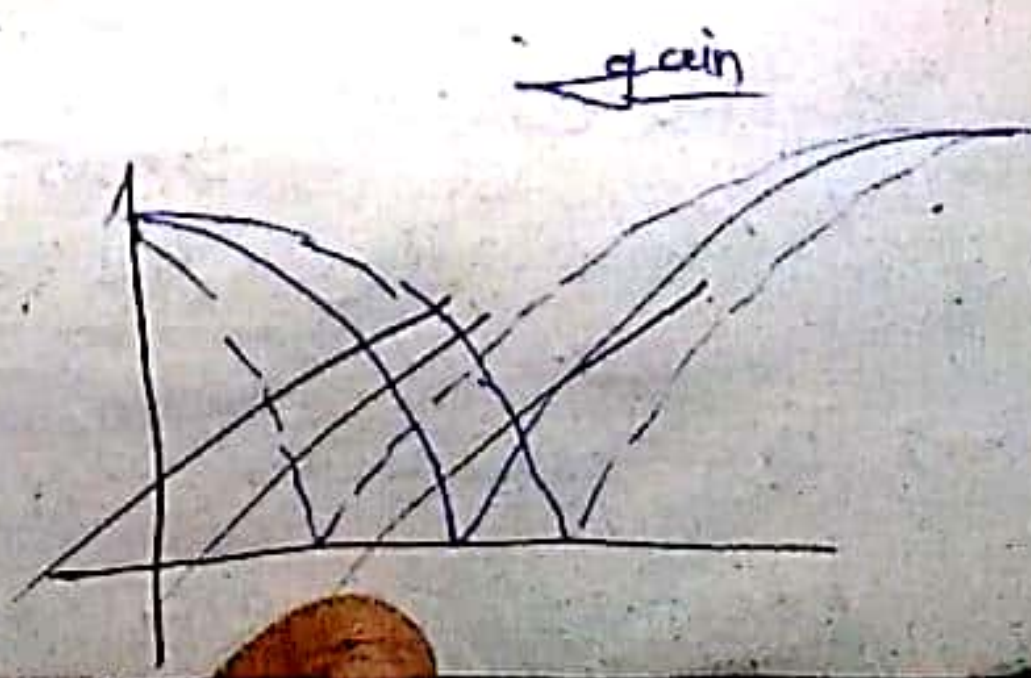
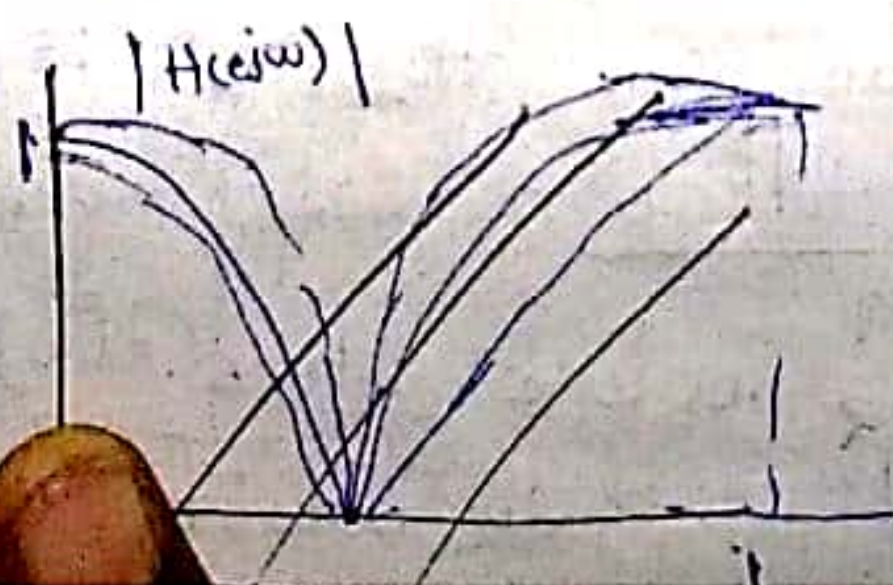
& have a magnitude > 1

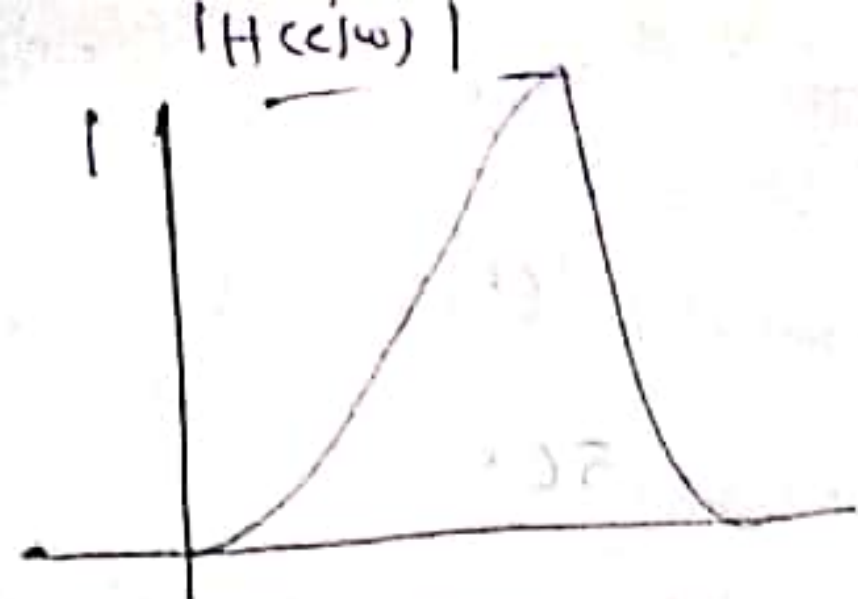
$\Rightarrow$ So, poles of $H_{BP}(z)$ are outside the unit circle making the transfer function unstable.

$\Rightarrow$ The poles of $H_{BP}^{II}(z)$ are at $z = 0.2667655 \pm j0.8095546$

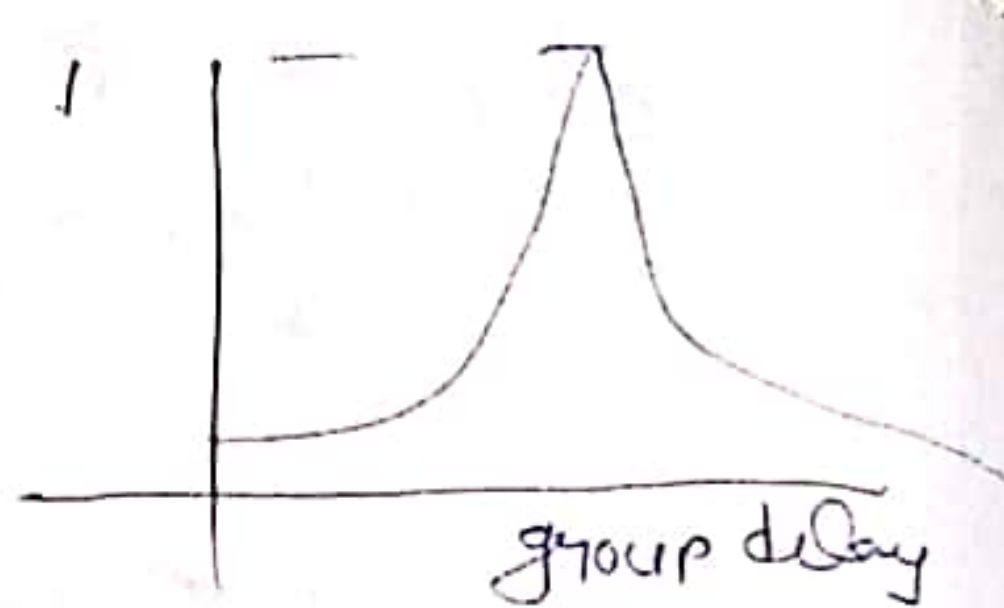
& have a magnitude of 0.8523746. (inside unit circle)

$\Rightarrow H_{BP}^{II}(z)$ is BIBO stable





→ magnitude function.



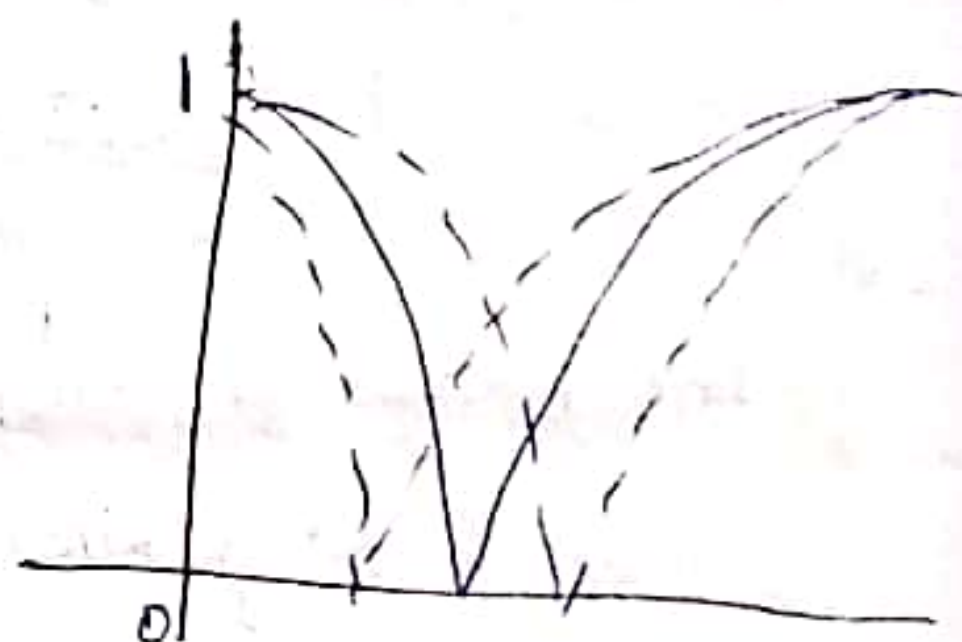
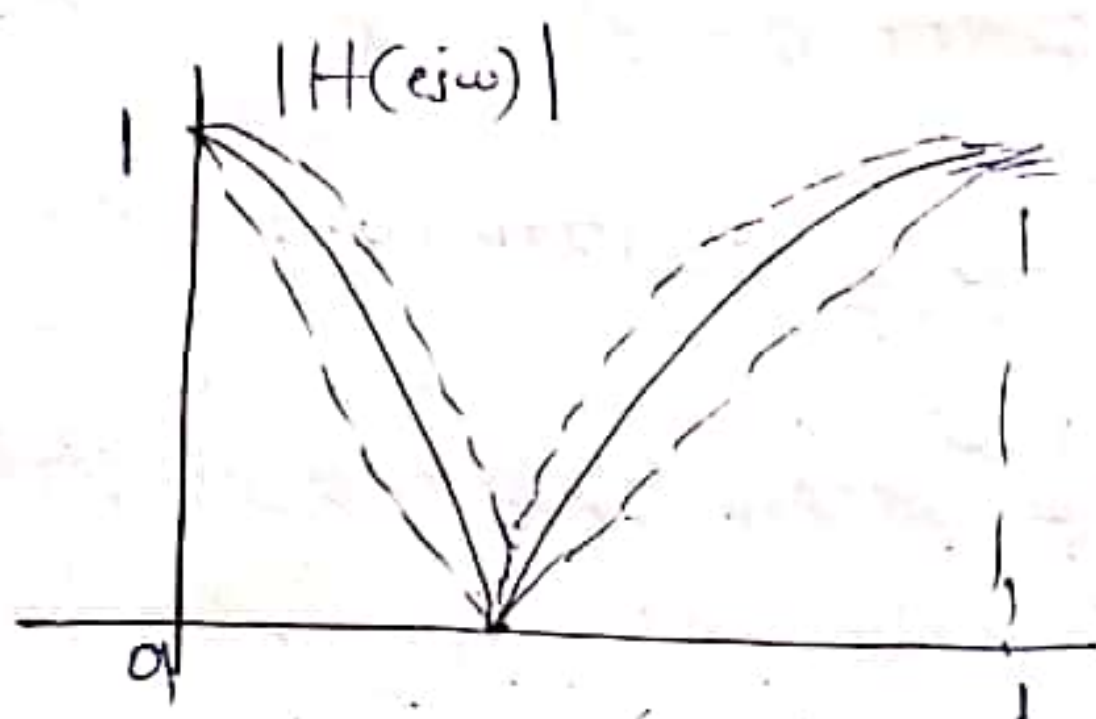
Simple IIR BSF [Bandstop filter]

⇒ A 2nd-order bandstop digital filter has a transfer function given by

$$H_{BS}(z) = \frac{1+\alpha}{2} \left(\frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \right)$$

→ The Transfer function $H_{BS}(z)$ is a BR function

$H_{BS}(z)$ is a BR function if $|\alpha| < 1$ & $|\beta| < 1$



⇒ Here, the magnitude function makes the maximum value of 1 at $w=0$ & $w=\pi$

⇒ It goes to 0 at $w=w_0$, where w_0 called notch frequency.

$$\begin{aligned} w_0 &= \cos^{-1}(\beta) \\ \beta &= \cos w_0 \end{aligned}$$

⇒ The digital transfer function $H_{BS}(z)$ is more commonly called "notch filter"

⇒ The frequencies w_1 & w_2 where $|H_{BS}(e^{jw})|^2$ becomes $1/2$ are called 3-dB cutoff frequencies

→ The difference between the two cutoff frequencies, assuming $\omega_c2 > \omega_c1$, is called 3-dB notch bandwidth & is given by

$$Bw = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

$$\cos Bw = \frac{2\alpha}{1+\alpha^2}$$

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Bounded Real T.F (BR)

(1) A causal stable real coefficient transfer function $H(z)$ is defined as a bounded real (BR) transfer function.

If $|H(e^{j\omega})| \leq 1$ for all values of ω

$$|H(e^{j\omega})|^2 \leq 1$$

$$\frac{|Y(e^{j\omega})|^2}{|X(e^{j\omega})|^2} \leq 1$$

$$\Rightarrow |Y(e^{j\omega})|^2 \leq |X(e^{j\omega})|^2$$

$$x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$y(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega})$$

$$h(n) \xleftrightarrow{\text{DTFT}} H(e^{j\omega})$$

By Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} y^2(n) \leq \sum_{n=-\infty}^{\infty} x^2(n)$$

$$\sum_{n=-\infty}^{\infty} |y(n)|^2 \leq \sum_{n=-\infty}^{\infty} |x(n)|^2$$

(2) For all finite energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as passive structure.

(3) If $|H(e^{j\omega})| = 1$, then output energy is equal to the input energy, and such a digital filter is therefore a lossless system.

(4) A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a lossless bounded real (LBR) transfer function.

(5) The BR and LBR transfer functions are the keys to the realization of digital filters with low-coefficient sensitivity.

Ex) consider the causal stable IIR transfer function,
 $H(z) = \frac{k}{1 - \alpha z^{-1}}$, $0 < |\alpha| < 1$, where k & α are real constants.

Ans

$$P(z) = \frac{k}{1 - \alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

square-magnitude function is,

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1}) \Big|_{z=e^{j\omega}} = \frac{k^2}{(1 + \alpha^2) - 2\alpha \cos \omega}$$

(1) maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha \cos \omega$ in the denominator is maximum and the minimum value is obtained when $2\alpha \cos \omega$ is a minimum.

(2) For $\alpha > 0$, maximum value of $2\alpha \cos \omega$ is equal to 2α at $\omega = 0$, and minimum value is -2α at $\omega = \pi$.

(3) $\alpha > 0$, $|H(e^{j\omega})|^2 = \frac{k^2}{(1 - \alpha)^2}$ at $\omega = 0 \Rightarrow$ maximum value

$|H(e^{j\omega})|^2 = \frac{k^2}{(1 + \alpha)^2}$ at $\omega = \pi \Rightarrow$ minimum value.

$$\alpha < 0, \quad 2\alpha \cos \omega = +2\alpha \text{ at } \omega = \pi$$

$$2\alpha \cos \omega = 2\alpha \text{ at } \omega = 0$$

(4) $|H(e^{j\omega})|^2 = \frac{k^2}{(1 - \alpha)^2}$ at $\omega = \pi$ maximum value

$= \frac{k^2}{(1 + \alpha)^2}$ at $\omega = 0$ minimum value.

(5) maximum value can be made equal to 1, by choosing $k = \pm(1 - \alpha)$, in which case the

(4) A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a lossless, bounded real (LBR) transfer function.

(5) The BR and LBR transfer functions are the keys to the realization of digital filters with low-coefficient sensitivity.

Ex) consider the causal stable IIR transfer function $H(z) = \frac{k}{1-\alpha z^{-1}}$, $0 < |\alpha| < 1$, where k & α are real constants.

Ans

$$P(z) = \frac{k}{1-\alpha z^{-1}}, \quad 0 < |\alpha| < 1$$

Square-magnitude function is,

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1}) \Big|_{z=e^{j\omega}} = \frac{k^2}{(1+\alpha^2)-2\alpha\cos\omega}$$

(1) maximum value of $|H(e^{j\omega})|^2$ is obtained when max in the denominator is maximum and the minimum value is obtained when max is a minimum.

(2) For $\alpha > 0$, maximum value of $2\alpha\cos\omega$ is equal to 2α at $\omega=0$, and minimum value is -2α at $\omega=\pi$.

(3) $\alpha > 0$, $|H(e^{j\omega})|^2 = \frac{k^2}{(1-\alpha)^2}$ at $\omega=0 \Rightarrow$ maximum value

$$|H(e^{j\omega})|^2 = \frac{k^2}{(1+\alpha)^2} \text{ at } \omega=\pi \Rightarrow \text{minimum value}$$

$$\alpha < 0, \quad 2\alpha\cos\omega = +2\alpha \text{ at } \omega=\pi$$

$$2\alpha\cos\omega = 2\alpha \text{ at } \omega=0$$

$$(4) |H(e^{j\omega})|^2 = \frac{k^2}{(1-\alpha)^2} \text{ at } \omega=\pi \text{ maximum value}$$

$$= \frac{k^2}{(1+\alpha)^2} \text{ at } \omega=0 \text{ minimum value}$$

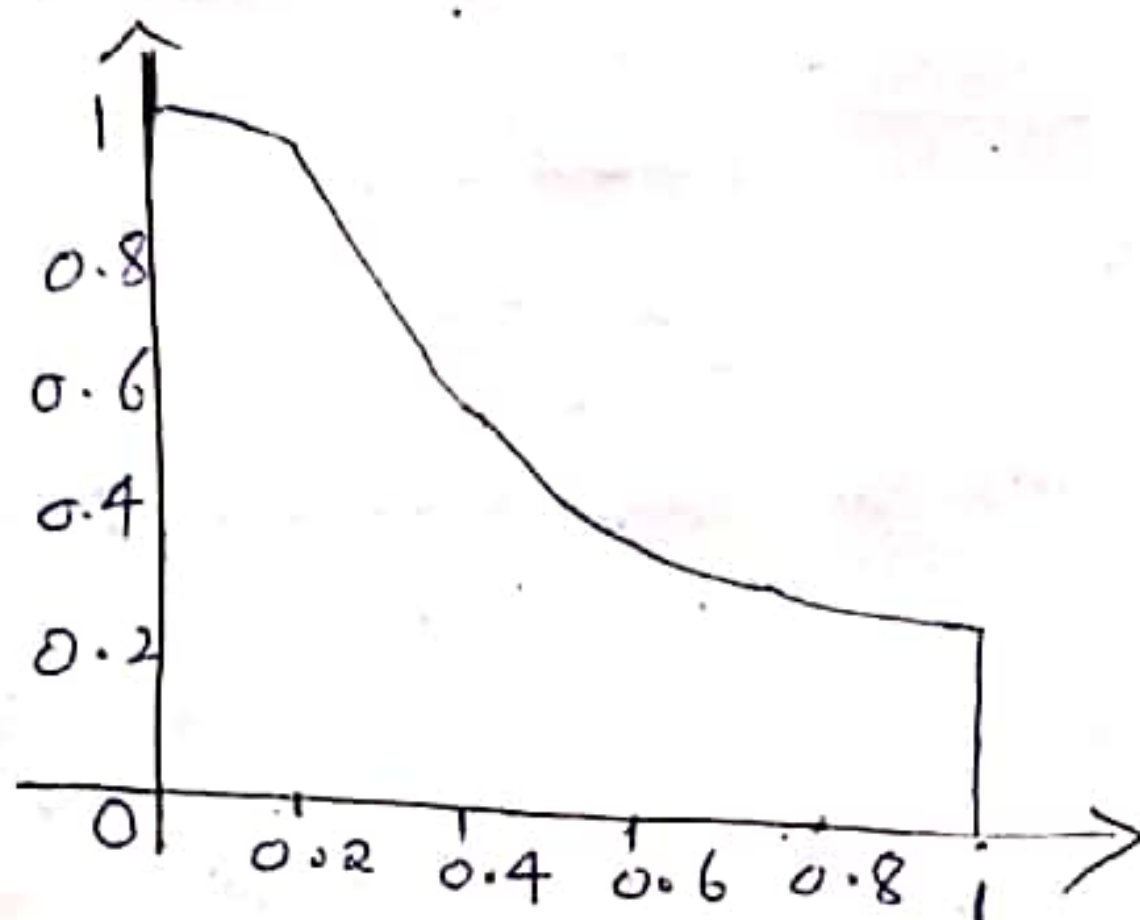
(5) maximum value can be made equal to 1, by choosing $k = \pm(1-\alpha)$, in which case the

minimum value becomes $\frac{(1-\alpha)^2}{(1+\alpha)^2}$

$$(6) H(z) = \frac{k}{1 - \alpha z^{-1}}, 0 < |\alpha| < 1$$

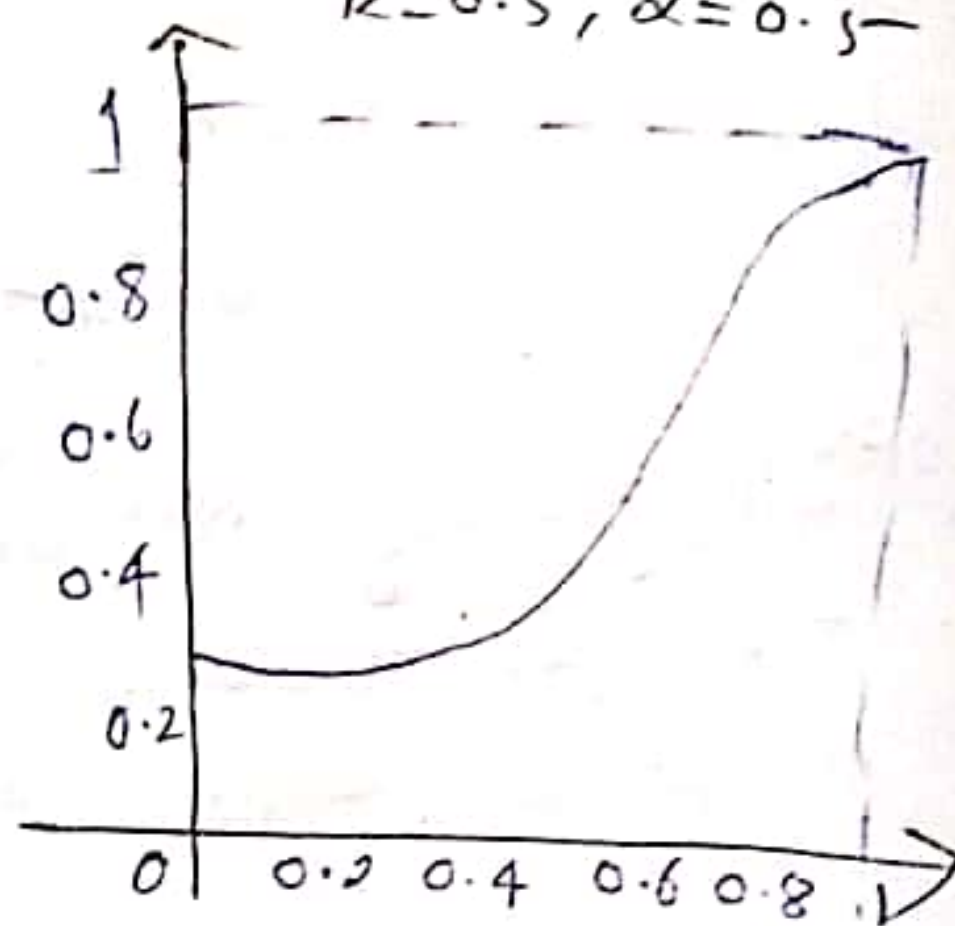
is a BR function for $k = \pm(1-\alpha)$

$$k = 0.5, \alpha = 0.5$$



Low-pass filter

$$k = 0.5, \alpha = 0.5$$



High-pass filter

All pass Transfer function

Definition:- An IIR Transfer function $A(z)$ with unity magnitude response for all frequencies; i.e.,

$$|A(e^{j\omega})|^2 = 1, \text{ for all } \omega$$

is called an "All pass-transfer function"

All pass Transfer function = $A(z)$

$$\left. \begin{aligned} |A(e^{j\omega})| &= 1 \\ |A(e^{j\omega})|^2 &= 1 \end{aligned} \right\} \text{ for all } \omega.$$

$|A(e^{j\omega})|$ = Anything (which is desired)

First order:-

one pole & one zero

$$A(z) = k \cdot \frac{1 + az^{-1}}{1 + bz^{-1}} \quad |b| < 1$$

$$\text{Let } a = 1/b, k = b$$

$$A(z) = b \cdot \left(\frac{1 + 1/b z^{-1}}{1 + b z^{-1}} \right) = \frac{b + z^{-1}}{1 + b z^{-1}} = \frac{P_1(z)}{D_1(z)}$$

$$A(z) = \frac{z^{-1} D_1(z^{-1})}{D_1(z)}$$

$$D_1(z) = 1 + bz^{-1}$$

$$D_1(z^{-1}) = 1 + bz$$

$$z^{-1}(D_1(z^{-1})) = z^{-1} + b = b + z^{-1}$$

Zeros:- $b + z^{-1} = 0$
 $\Rightarrow z = -1/b$

poles:- $1 + bz = 0$
 $z = -b$

* In general M th order causal real-coefficient all pass Transfer function

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \dots + d_1 z^{-(M-1)} + z^{-M}}{1 + d_1 z^{-1} + d_2 z^{-2} + \dots + d_{M-1} z^{-(M-1)} + d_M z^{-M}}$$

$$A_M(z) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

from numerator,

$$z^{-M} (d_1 + d_1 z + d_2 z^2 + \dots + d_{M-1} z^{M-1} + d_M z^M)$$

$$\Rightarrow z^{-M} (D_M(z^{-1}))$$

$\therefore$ poles and zeros of all pass filters occurs in reciprocal pairs. $z = a \rightarrow \text{pole}$, $z = 1/a \rightarrow \text{zero}$.

(2) The numerator of an all pass T.F is said to be mirror image polynomial of the denominator & vice versa.

First order all pass T.F $A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$

2nd order all pass T.F $A_2(z) = \frac{d_2 + d_1 z^{-1} + d_0 z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$

To show that magnitude of $A_M(e^{j\omega})$ is equal to one for all ω ,

$$A_M(z^{-1}) = \frac{z^M D_M(z)}{D_M(z^{-1})}$$

$$A_m(z) A_m(z^{-1}) = \frac{z^{-M} D_m(z^{-1})}{D_m(z)} \times \frac{z^M D_m(z)}{z^{-M} D_m(z^{-1})} = 1$$

$$|A_m(e^{j\omega})|^2 = A_m(z) A_m(z^{-1}) \Big|_{z=e^{j\omega}}$$

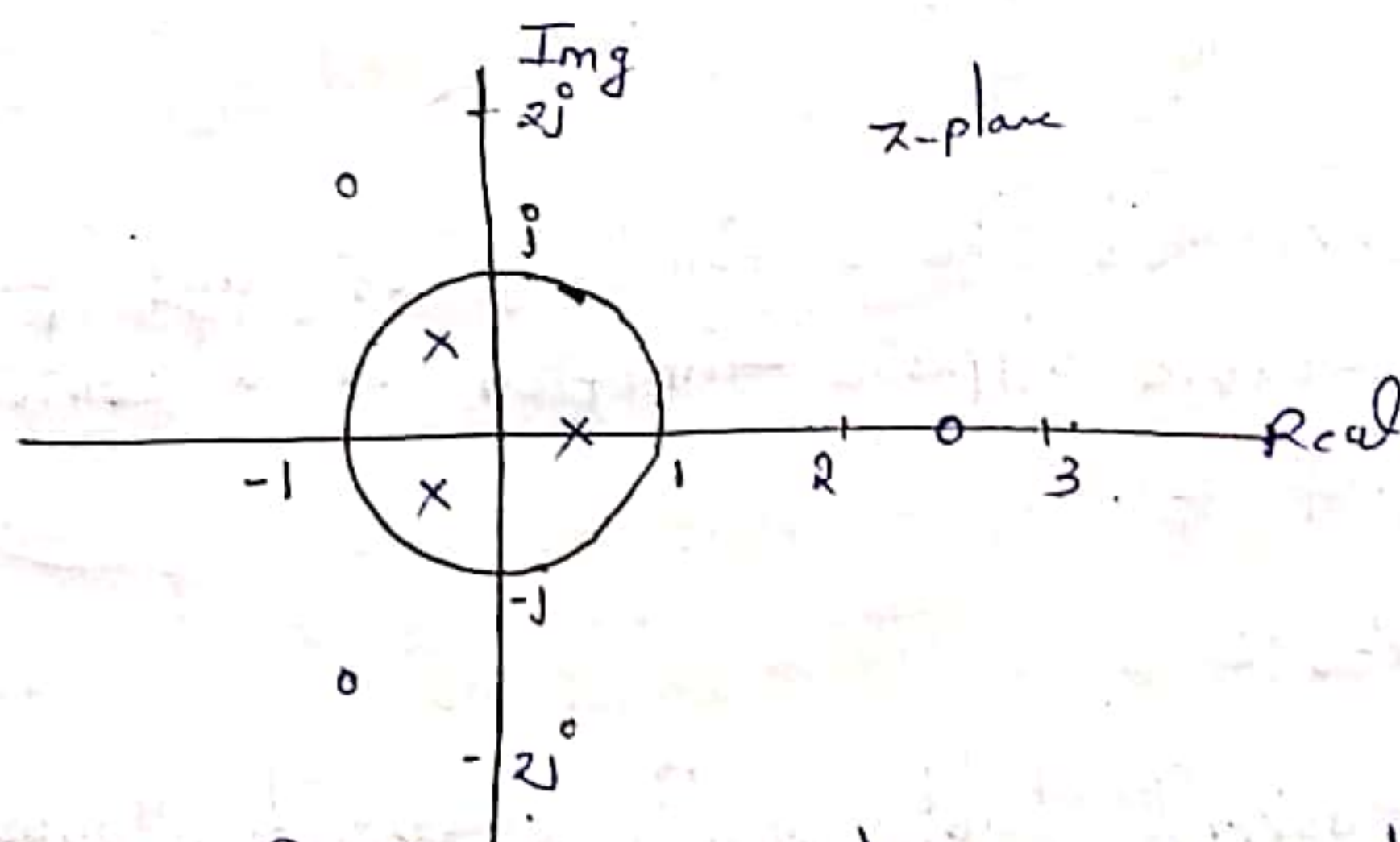
⇒ poles of a causal stable T.F must lie inside the unit circle. As a result, all zeroes of a causal stable all pass T.F lie outside the unit circle. [mirr-image with its pole situated inside the unit circle. symmetry]

Eg:- $A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$

$$A_3(z) = \frac{z^{-3}(-0.2z^3 + 0.18z^2 + 0.4z + 1)}{z^3(z^3 + 0.4z^2 + 0.18z - 0.2)}$$

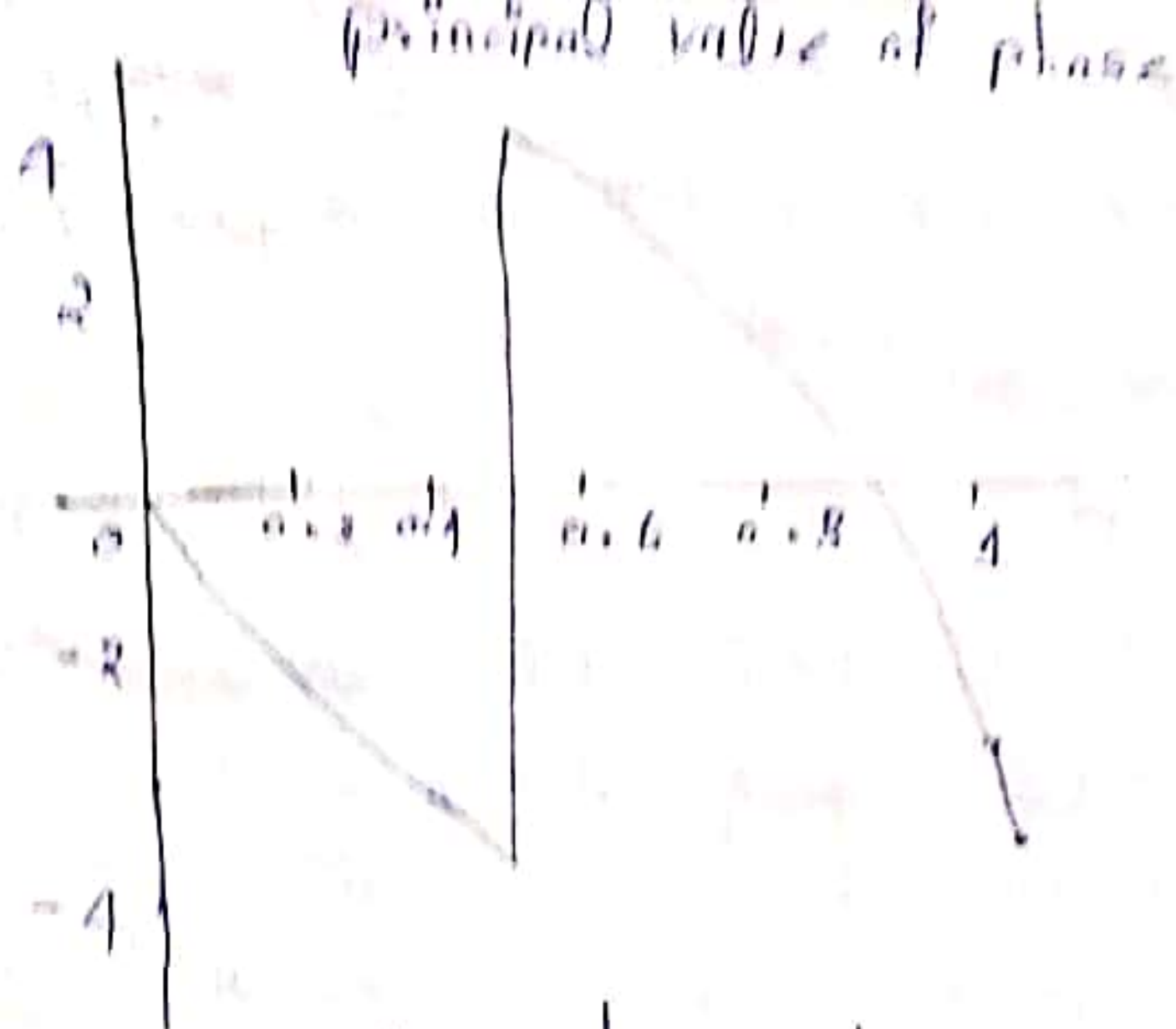
$$A_3(z) = \frac{-0.2z^3 + 0.18z^2 + 0.4z + 1}{z^3 + 0.4z^2 + 0.18z - 0.2}$$

$$A_3(z) = \frac{(z-2.5)(z+0.8-1.1661903j)(z+0.8+1.1661903j)}{(z-0.4)(z+0.4-0.583095j)(z+0.4+0.583095j)}$$



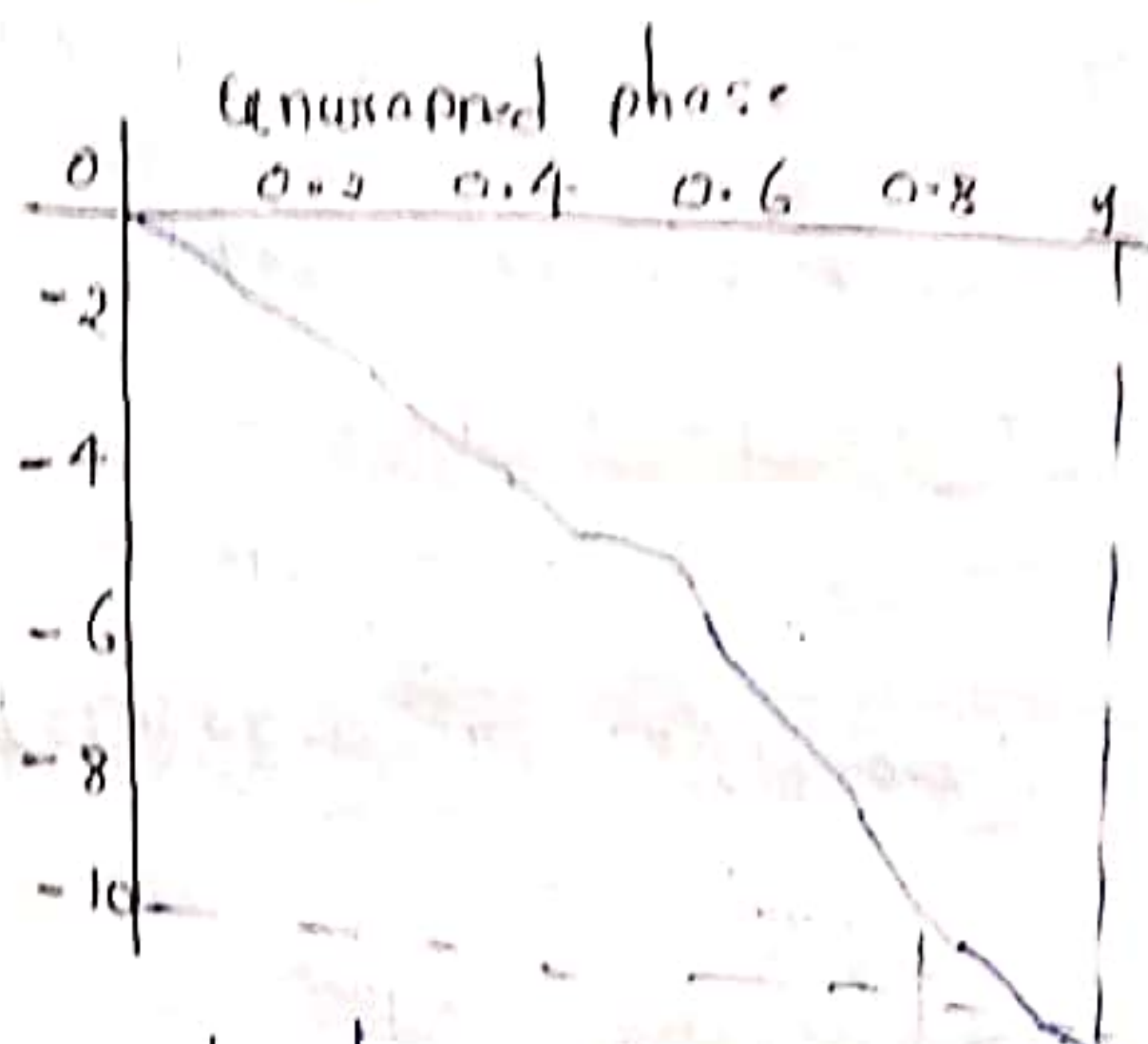
⇒ The principal value of the phase of the 3rd order all pass Transfer Function.

⇒ The discontinuity by the amount of 2π in the phase $\theta(\omega)$.



* If we unwrap the phase by removing discontinuity, we arrive at the unwrapped phase function $\theta(\omega)$ indicated below.

→ The unwrapped phase function is a continuous function of ω .



* The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω .

Properties:-

- (i) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently a causal stable allpass filter is a lossless structure.
- (2) The magnitude function of a stable all pass function $A(z)$ satisfies:

$$|A(z)| = \begin{cases} < 1, & \text{for } |z| > 1 \\ = 1, & \text{for } |z| = 1 \\ > 1, & \text{for } |z| < 1 \end{cases}$$

(3) Let $T_g(\omega)$ denote group delay function of allpass filter $A(z)$ i.e.,

$$T_g(\omega) = -\frac{d}{d\omega} [\theta_c(\omega)]$$

* The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $T_g(\omega)$ is everywhere positive in the range $0 < \omega < \pi$

* The group delay of an M -th order stable real-coefficient allpass transfer function satisfies:

$$\int_0^\pi T_g(\omega) d\omega = M\pi$$

7/4/21

Linear phase filter :- T.F used as filters also
Linear phase $\rightarrow$ phase & frequency response

$\Rightarrow$ This system has constant magnitude response, linear phase, & constant group delay.

In general, (mostly used one) whose frequency response is given by

$$H(e^{j\omega}) = e^{-j\omega D}$$

eg $\rightarrow$ which has linear phase response from $\omega=0$ to ω_c

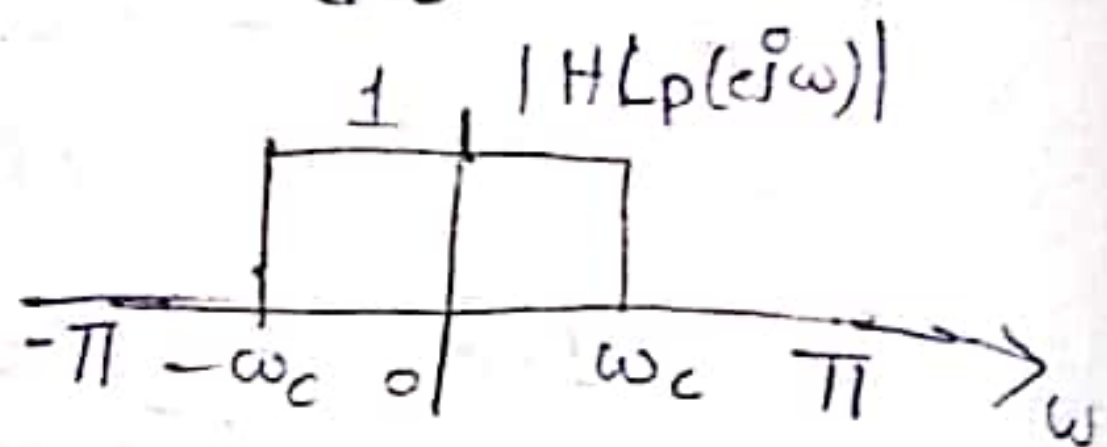
$$|H(e^{j\omega})| = 1, \angle H(e^{j\omega}) = -\omega D$$

$$\text{group delay, } T_g(\omega) = \frac{-d\angle H(e^{j\omega})}{d\omega} = D$$

Cases:-

i) For Low pass filter

$\Rightarrow$ magnitude response

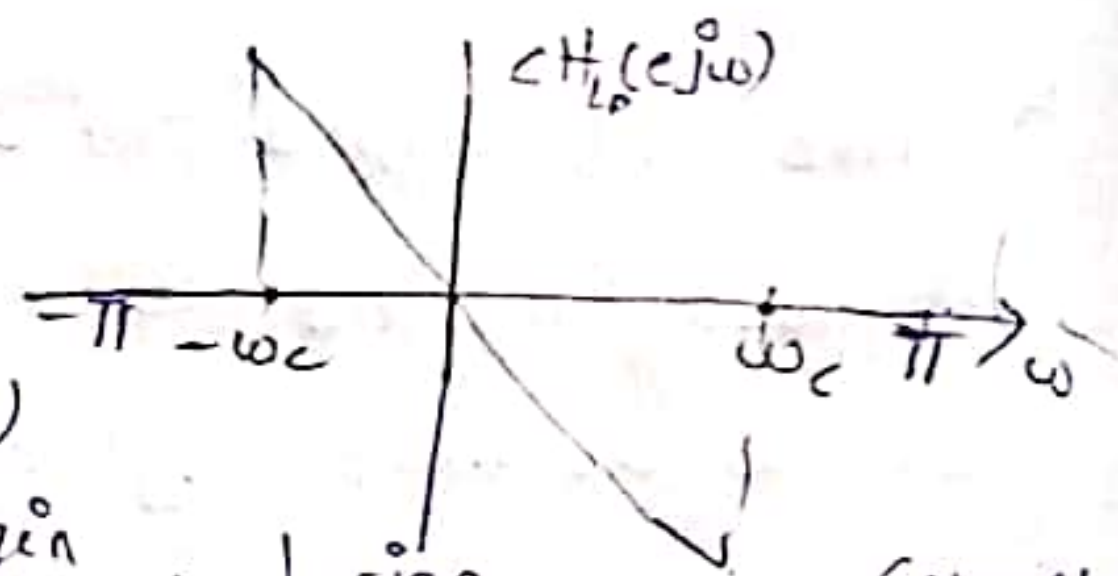


$$\Rightarrow \angle H(e^{j\omega}) = -\omega D$$

$$y = mx$$

$$m = -D \text{ (slope -ve)}$$

$\rightarrow$ passing through origin



Note:-

$\Rightarrow$ Linear phase is ^{always} possible only when FIR T.F. with an exact linear phase response, while it is nearly impossible to design a linear phase response IIR T.F. (and also not possible with analog filters)

ii) let us consider A causal FIR T.F. of length $N+1$

$$H(z) = \sum_{n=0}^N h(n)z^{-n}$$

$\Rightarrow$ if this T.F. has a linear phase if its impulse response $h(n)$ is either symmetric or is Anti-symmetric

*Symmetric:-

$$h(n) = h(N-n), 0 \leq n \leq N$$

*Anti-symmetric:-

$$h(n) = -h(N-n), 0 \leq n \leq N$$

Based on this, there are four types of linear phase filters:-

$\Rightarrow$ since length of the impulse response can be either even or odd, we can define 4 types

Type 1 Linear phase filter:-

→ with symmetric impulse response & with odd length

→ N is even

→ length $(N+1)$ is odd.

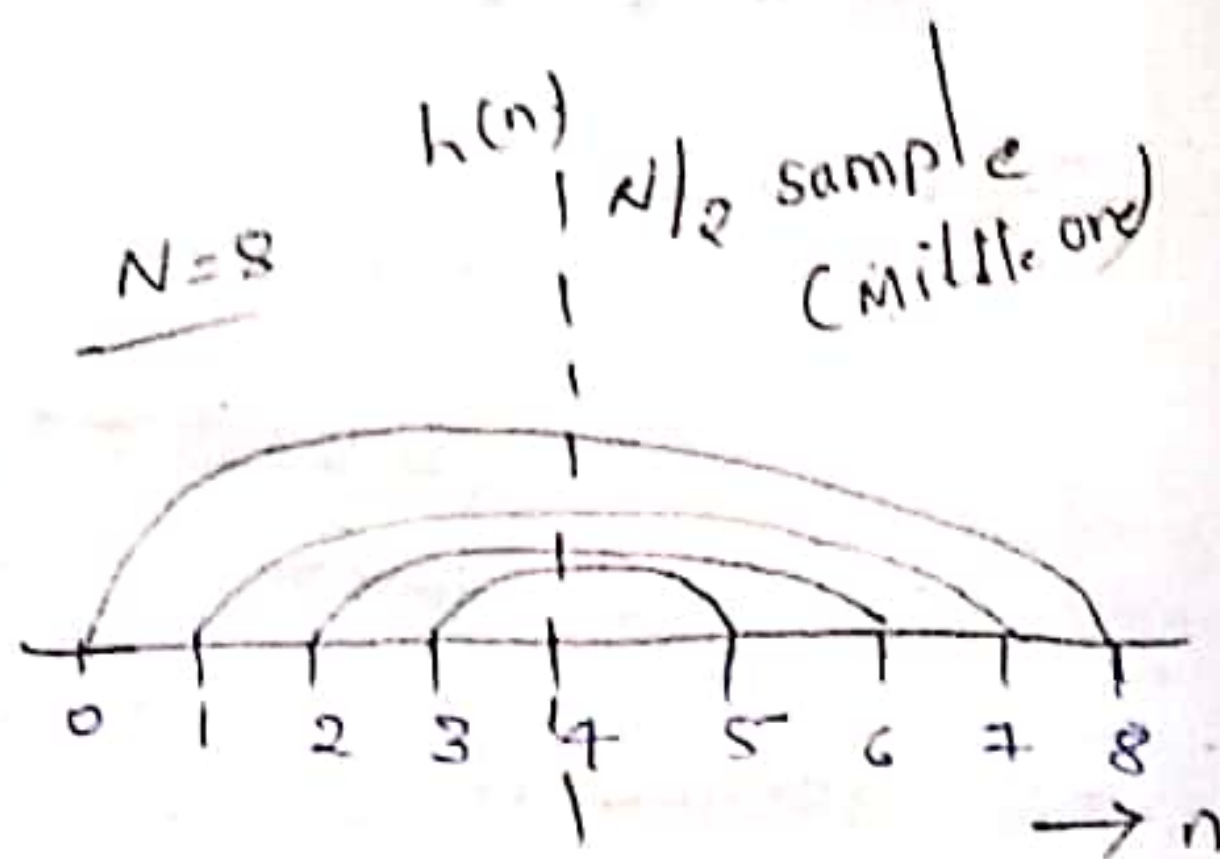
Let $N=8$,

Length $\Rightarrow N+1=9$ (odd)

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

Multipliers
→ difficult to design.



For $N=8$, $h(0)=h(8)$, $h(1)=h(7)$, $h(2)=h(6)$, $h(3)=h(5)$

$$H(z) = h(0)[1 + z^{-8}] + h(1)[z^{-1} + z^{-7}] + h(2)[z^{-2} + z^{-6}] + h(3)[z^{-3} + z^{-5}] + h(4)z^{-4}$$

$\Rightarrow$ For this case, we need 8 multipliers.

* Because of symmetry no. of multipliers are going to required is reduced [here reduced to 5 multipliers]

$$H(z) = z^{-4} [h(0)[z^4 + z^{-4}] + h(1)[z^3 + z^{-3}] + h(2)[z^2 + z^{-2}] + h(3)[z + z^{-1}] + h(4)z^0]$$

* Frequency response:- $z = e^{j\omega}$

$$H(e^{j\omega}) = e^{-j\omega 4} [2h(0)\cos(4\omega) + 2h(1)\cos(3\omega) + 2h(2)\cos(2\omega) + 2h(3)\cos(\omega) + h(4)] \rightarrow (1)$$

phase
- $\omega 4$

real term

$\rightarrow$ cosine term may be +ve or -ve

$\hookrightarrow$ positive $\rightarrow 0^\circ$ phase

$\hookrightarrow$ negative $\rightarrow 180^\circ$ phase

Type 1: Linear Phase Filter:-

→ with symmetric impulse response & with odd length

→ N is even

→ length $(N+1)$ is odd.

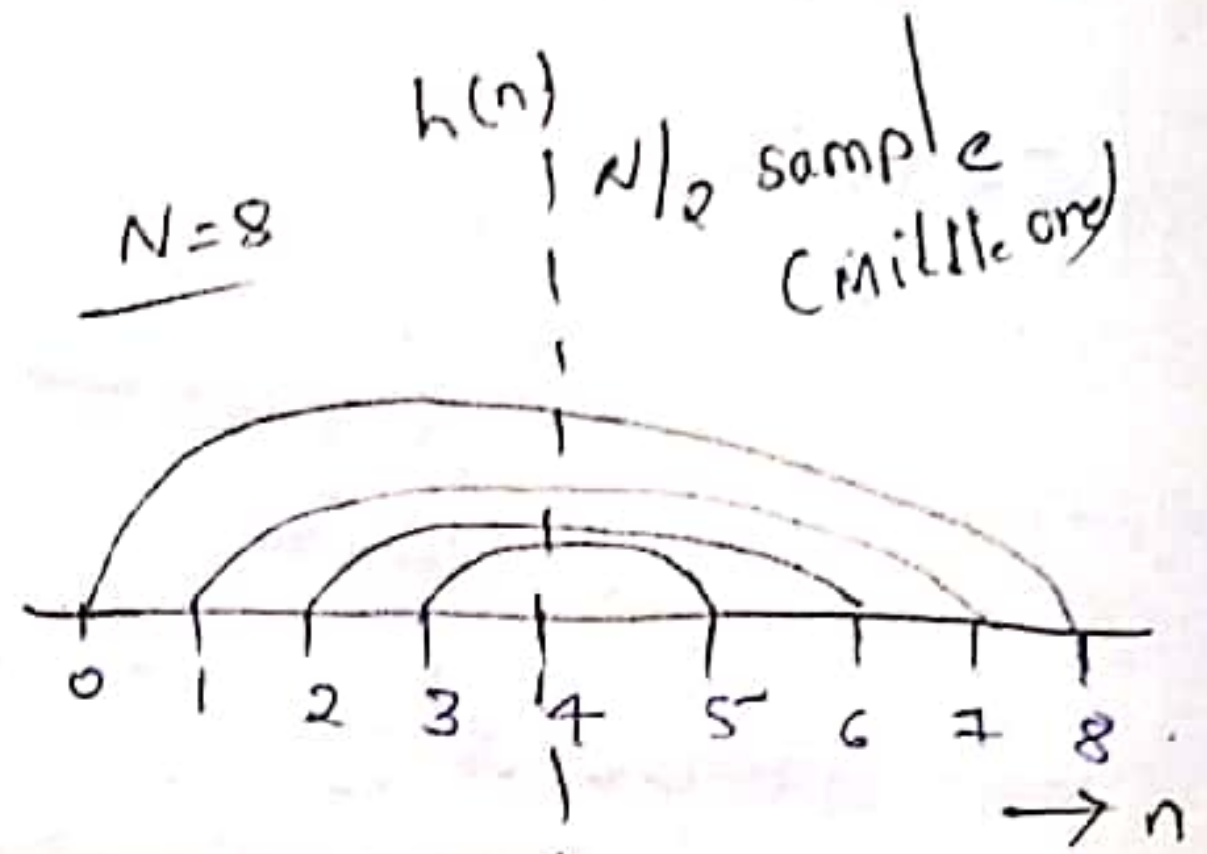
Let $N=8$,

length $\Rightarrow N+1=9$ (odd)

$$H(z) = \sum_{n=0}^8 h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

Multipliers
 $\hookrightarrow$ difficult to design.



For $N=8$, $h(0)=h(8)$, $h(1)=h(7)$, $h(2)=h(6)$, $h(3)=h(5)$

$$H(z) = h(0)[1 + z^{-8}] + h(1)[z^{-1} + z^{-7}] + h(2)[z^{-2} + z^{-6}] + h(3)[z^{-3} + z^{-5}] + h(4)z^{-4}$$

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$$H(z) = z^{-4} [h(0)[z^4 + z^{-4}] + h(1)[z^3 + z^{-3}] + h(2)[z^2 + z^{-2}] + h(3)[z + z^{-1}] + h(4)z^0]$$

* Frequency response:- $xe^{j\omega}$

$$H(e^{j\omega}) = e^{-j\omega 4} [2h(0)\cos(4\omega) + 2h(1)\cos(3\omega) + 2h(2)\cos(2\omega) + 2h(3)\cos(\omega) + h(4)] \rightarrow (1)$$

phase
 $-\omega 4$

real term

$\rightarrow$ cosine term may be +ve or -ve

$\hookrightarrow$ positive $\rightarrow 0^\circ$ phase

$\hookrightarrow$ negative $\rightarrow 180^\circ$ phase

Note The quantity inside the braces in the above expression is a real function of ω & can assume positive or negative values in the range $0 \leq |\omega| \leq \pi$.

The phase is given by,

$$\angle H(e^{j\omega}) = \theta(\omega) = -4\omega + \beta \quad \beta \text{ is either } 0 \text{ or } \pi$$

$y = mx + c$

$\Rightarrow$ Hence it is a linear function of ω

Group delay, $\tau_g(\omega) = \frac{-d\theta(\omega)}{d\omega} = 4$ (which is constant)

$\Rightarrow$ A constant group-delay of 4-samples.

In general type-1 FIR filters, $\rightarrow$ Tilt

$$H(e^{j\omega}) = e^{-jN\omega/2} \tilde{H}(\omega)$$

$\tilde{H}(\omega) \rightarrow$ Amplitude response / zero-phase response
[± 1]

Phase $\rightarrow$ multiples of π
 0° or 180°

$$\tilde{H}(\omega) = h(N/2) + 2 \sum_{n=1}^{N/2} h[\frac{N}{2} - n] \cos(\omega n)$$

$\downarrow$ This is general form for Eq (1)

Type-2 Linear Phase FIR filter

$\rightarrow$ with symmetric impulse response with even length

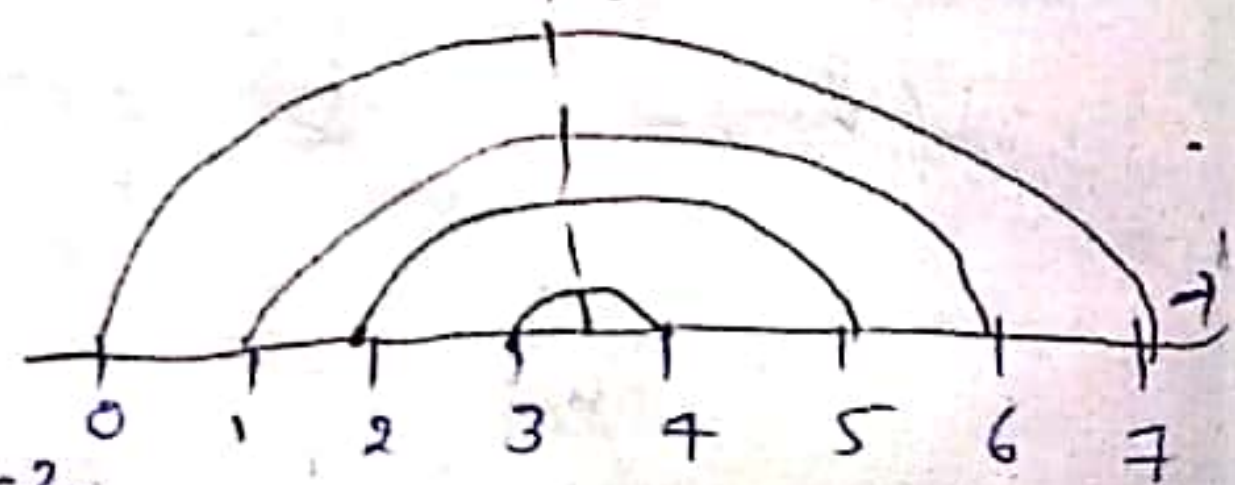
N is odd $\rightarrow$ Let $N=7$

$N+1 \Rightarrow$ Even

length $\Rightarrow N+1=8$

$N/2 \Rightarrow 3.5$ sample
does not exist.
0 sample

$$H(z) = \sum_{n=0}^7 h(n) z^{-n}$$



$$= h(0) + h(1)z^{-1} + h(2)z^{-2}$$

$$+ h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$

For $N=7$,

$$h(0)=h(7), h(1)=h(6), h(2)=h(5), h(3)=h(4)$$

$$H(z) = h(0)[1+z^{-7}] + h(1)[z^{-1}+z^{-6}] + h(2)[z^{-2}+z^{-5}] + h(3)[z^{-3}+z^{-4}]$$

$$H(z) = z^{-7/2} \left[h(0)[z^{7/2}+z^{-7/2}] + h(1)[z^{5/2}+z^{-5/2}] + h(2)[z^{3/2}+z^{-3/2}] + h(3)[z^{1/2}+z^{-1/2}] \right]$$

Frequency response: $z = e^{j\omega}$

$$H(e^{j\omega}) = e^{-j7\omega/2} \left[2h(0)\cos\left(\frac{7\omega}{2}\right) + 2h(1)\cos\left(\frac{5\omega}{2}\right) + 2h(2)\cos\left(\frac{3\omega}{2}\right) + 2h(3)\cos\left(\frac{\omega}{2}\right) \right] \quad \text{--- (2)}$$

Real term
→ phase 0° or π

$$\theta(\omega) = -7/2 \omega + \beta, \quad \beta = 0 \text{ or } \pi$$

$$y = mx + c$$

$\tau_g(\omega) \rightarrow$ Hence it is also a linear function of ω

Group delay, $\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = +7/2$

⇒ A constant Group delay of $7/2$ samples.

*Note In general, type-2 FIR filters, FR

$$H(e^{j\omega}) = e^{-jN\omega/2} \tilde{H}(\omega)$$

The Amplitude response,

$$\tilde{H}(\omega) = 2 \sum_{n=0}^{(N+1)/2} h\left(\frac{N+1}{2} - n\right) \cos[\omega(n-1/2)]$$

↓ This is general form of Eq (2)

Type:3 Linear phase FIR filter

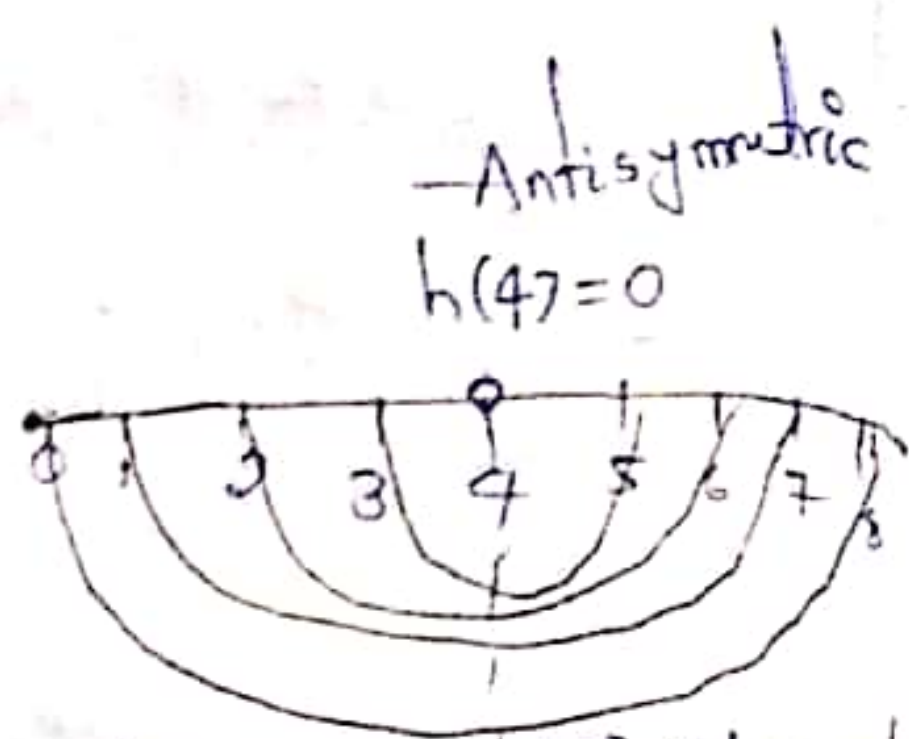
→ with anti-symmetric impulse response, with odd length

$N \rightarrow \text{Even}$
 $\text{length} \Rightarrow N+1 \rightarrow \text{odd}$

$N=8$
 $\text{length} \Rightarrow N+1=9$

$$H(z) = \sum_{n=0}^8 h(n) z^{-n}$$

$$H(z) = h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$



Antisymmetric
 $h(4)=0$

$h(0) = h(-8)$
 $h(1) = h(-7)$
 $h(2) = h(-6)$
 $h(3) = h(-5)$

$$H(z) = h(0)[1 - z^{-8}] + h(1)[z^{-1} - z^{-7}] + h(2)[z^{-2} - z^{-6}] + h(3)[z^{-3} - z^{-5}] + \underbrace{h(4)z^{-4}}_0$$

$$= z^{-4} [h(0)[z^4 - z^{-4}] + h(1)[z^3 - z^{-3}] + h(2)[z^2 - z^{-2}] + h(3)[z - z^{-1}]]$$

Frequency response:

$$H(e^{j\omega}) = e^{-j4\omega} [2j h(0) \sin(4\omega) + 2j h(1) \sin(3\omega) + 2j h(2) \sin(2\omega) + 2j h(3) \sin(\omega)]$$

$$= e^{-j4\omega} j [2h(0) \sin(4\omega) + 2h(1) \sin(3\omega) + 2h(2) \sin(2\omega) + 2h(3) \sin(\omega)]$$

$$= e^{-j4\omega} e^{j\pi/2} [2h(0) \sin(4\omega) + 2h(1) \sin(3\omega) + 2h(2) \sin(2\omega) + 2h(3) \sin(\omega)]$$

Phase $\theta(\omega) = -4\omega + \pi/2 + \beta$ real term $\rightarrow \beta$

$\beta = 0 \text{ or } \pi$

$\therefore$ Hence it is linear function of ω .

Group delay: $T_g(\omega) = 4$ // $\rightarrow$ constant group delay of 4 samples

In general

Amplitude response

$$H(e^{j\omega}) = j e^{-jN\omega/2} \tilde{H}(\omega)$$

$$\tilde{H}(\omega) = 2 \sum_{n=1}^{N/2} h[\frac{N}{2} - n] \sin(\omega n)$$

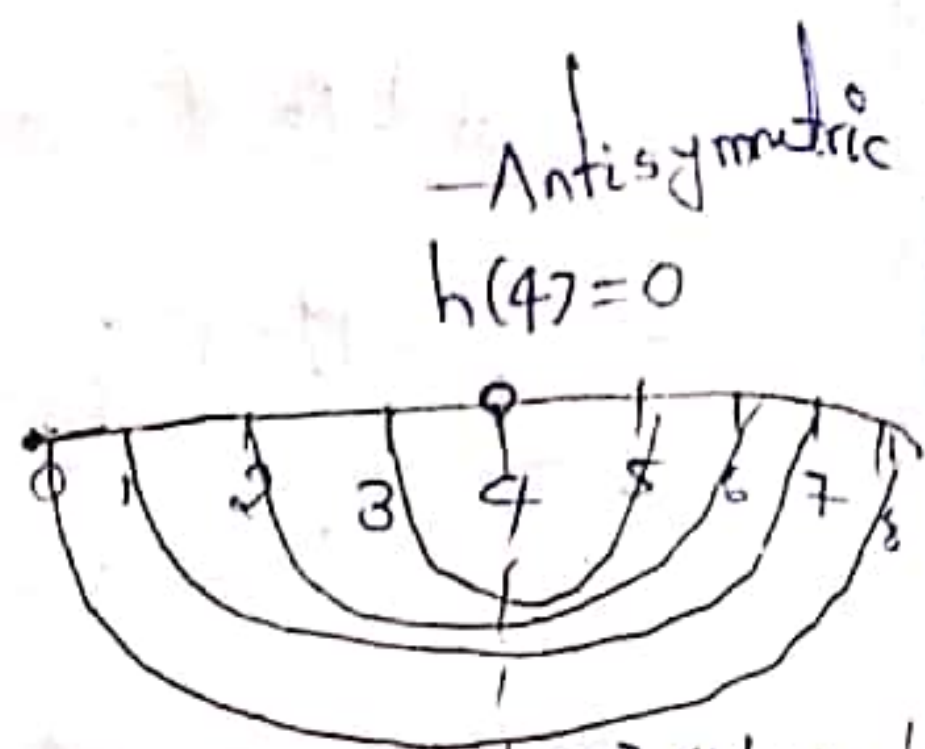
$\rightarrow$ general form of $\tilde{H}(\omega)$

$N \rightarrow \text{Even}$
 $\text{length} \Rightarrow N+1 \rightarrow \text{odd}$

Let $N=8$
 $\text{length} \Rightarrow N+1=9$

$$H(z) = \sum_{n=0}^8 h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$



Antisymmetric
 $h(4)=0$

$$\begin{aligned} h(0) &= h(-8) \\ h(1) &= h(-7) \\ h(2) &= h(-6) \\ h(3) &= h(-5) \end{aligned}$$

$$\begin{aligned} H(z) &= h(0)[1 - z^{-8}] + h(1)[z^{-1} - z^{-7}] + h(2)[z^{-2} - z^{-6}] + h(3)[z^{-3} - z^{-5}] + \underbrace{h(4)z^{-4}}_0 \\ &= z^{-4} [h(0)[z^4 - z^{-4}] + h(1)[z^3 - z^{-3}] + h(2)[z^2 - z^{-2}] + h(3)[z - z^{-1}]] \end{aligned}$$

Frequency response:

$$\begin{aligned} H(e^{j\omega}) &= e^{-j4\omega} [2j^0 h(0) \sin(4\omega) + 2j^1 h(1) \sin(3\omega) + 2j^2 h(2) \sin(2\omega) + 2j^3 h(3) \sin(\omega)] \\ &= e^{-j4\omega} j^0 [2h(0) \sin(4\omega) + 2j h(1) \sin(3\omega) - 2h(2) \sin(2\omega) - 2j h(3) \sin(\omega)] \\ &= e^{-j4\omega} e^{j\pi/2} [2h(0) \sin(4\omega) + 2h(1) \sin(3\omega) - 2h(2) \sin(2\omega) - 2h(3) \sin(\omega)] \end{aligned}$$

Phase

$$\theta(\omega) = -4\omega + \pi/2 + \beta$$

real term $\rightarrow \beta$

$$\beta = 0 \text{ or } \pi$$

$\therefore$ Hence it is linear function of ω .

* Group delay: $\tau_g(\omega) = 4$ // $\rightarrow$ constant group delay of 4 samples

In general

$$H(e^{j\omega}) = j^N e^{-jN\omega/2} \tilde{H}(\omega)$$

Amplitude response

$$\tilde{H}(\omega) = 2 \sum_{n=1}^{N/2} h[\frac{N}{2} - n] \sin(n\omega)$$

$\rightarrow$ general form of $\tilde{H}(\omega)$

Type-4 Linear phase FIR filter:-

→ with Anti-symmetric impulse response, with Even length

$$N \Rightarrow \text{odd}$$

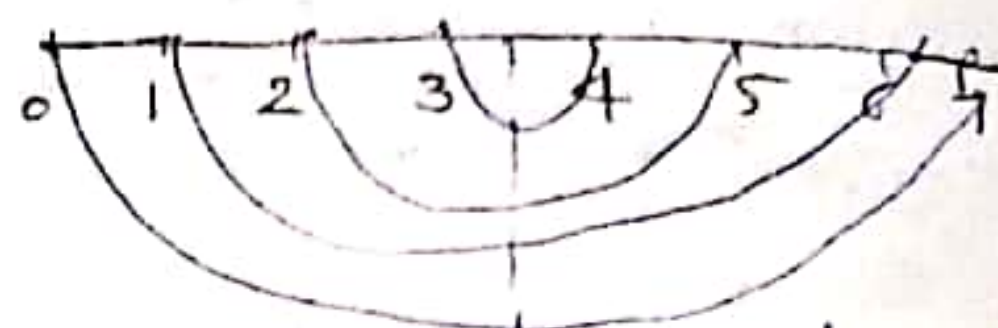
$$\text{Let } N = 7$$

$$N+1 \rightarrow \text{even}$$

$$\text{length } N+1 = 8$$

$$H(z) = \sum_{n=0}^7 h(n) z^{-n}$$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$



$$h(0) = -h(7)$$

$N/2$ sample

3.5 not exist

$$H(z) = h(0)[1 - z^{-7}] + h(1)[z^{-1} - z^{-6}] + h(2)[z^{-2} - z^{-5}] + h(3)[z^{-3} - z^{-4}]$$

$$h(1) = -h(6)$$

$$h(2) = -h(5)$$

$$h(3) = -h(4)$$

$$H(z) = z^{-7/2} [h(0)(z^{7/2} - z^{-7/2}) + h(1)(z^{5/2} - z^{-5/2}) + h(2)(z^{3/2} - z^{-3/2}) + h(3)(z^{1/2} - z^{-1/2})]$$

Frequency response:-

$$H(e^{j\omega}) = e^{-j\frac{7\omega}{2}} \underbrace{e^{j\frac{\pi}{2}} [2h(0)\sin(\frac{7\omega}{2}) + 2h(1)\sin(\frac{5\omega}{2}) + 2h(2)\sin(\frac{3\omega}{2}) + 2h(3)\sin(\frac{\omega}{2})]}_{\text{real term}} \rightarrow (4)$$

phase

$$\theta(\omega) = -\frac{7}{2}\omega + \frac{\pi}{2} + \beta \quad \beta = 0 \text{ or } \pi$$

∴ Hence it is linear phase function of ω

$$\text{Group delay} \rightarrow T_g(\omega) = 7/2 \rightarrow \text{constant}$$

* In general,

$$H(e^{j\omega}) = j^N e^{-jN\omega/2} \tilde{H}(\omega)$$

Amplitude response; $\left(\frac{N+1}{2}\right)$

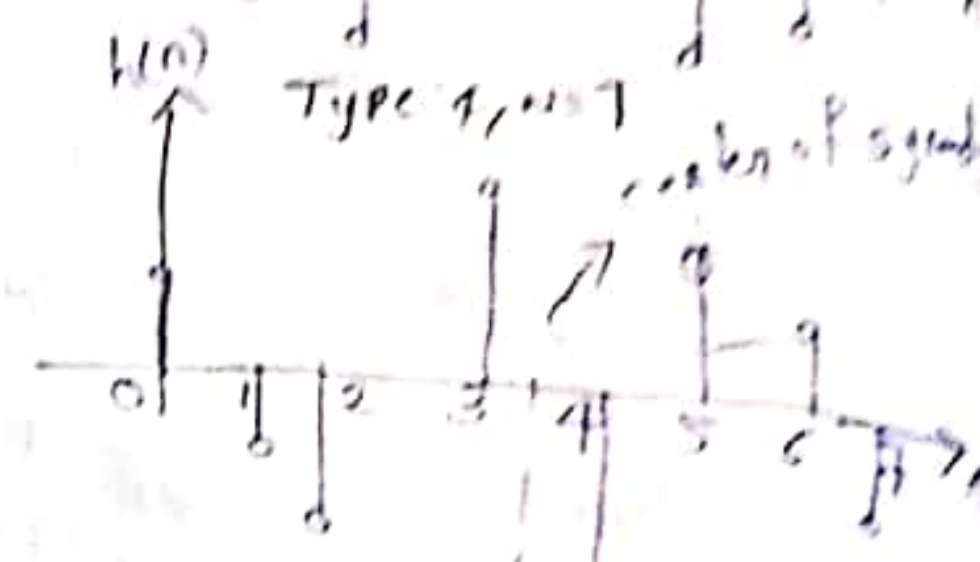
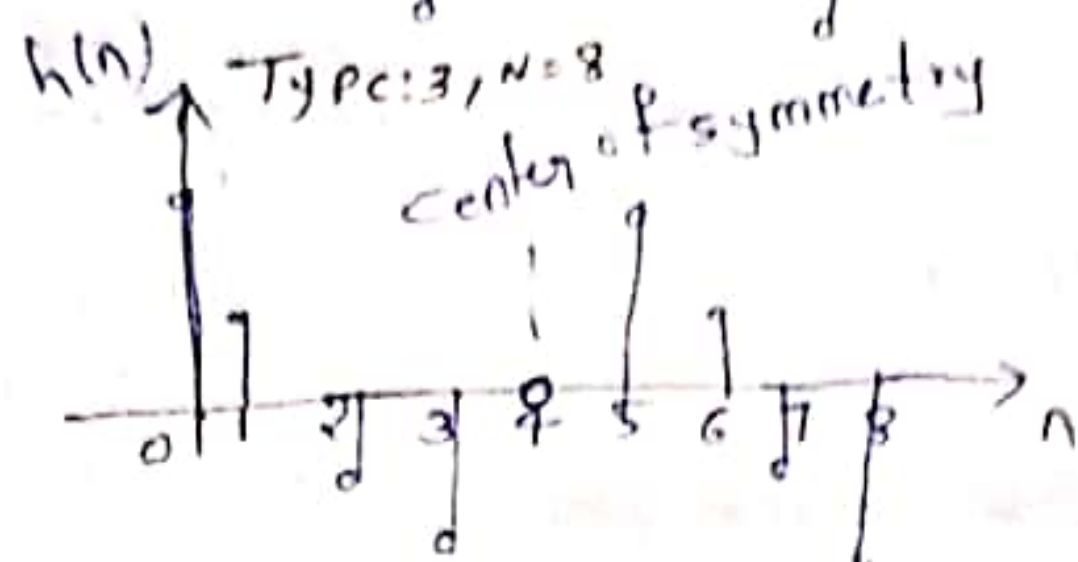
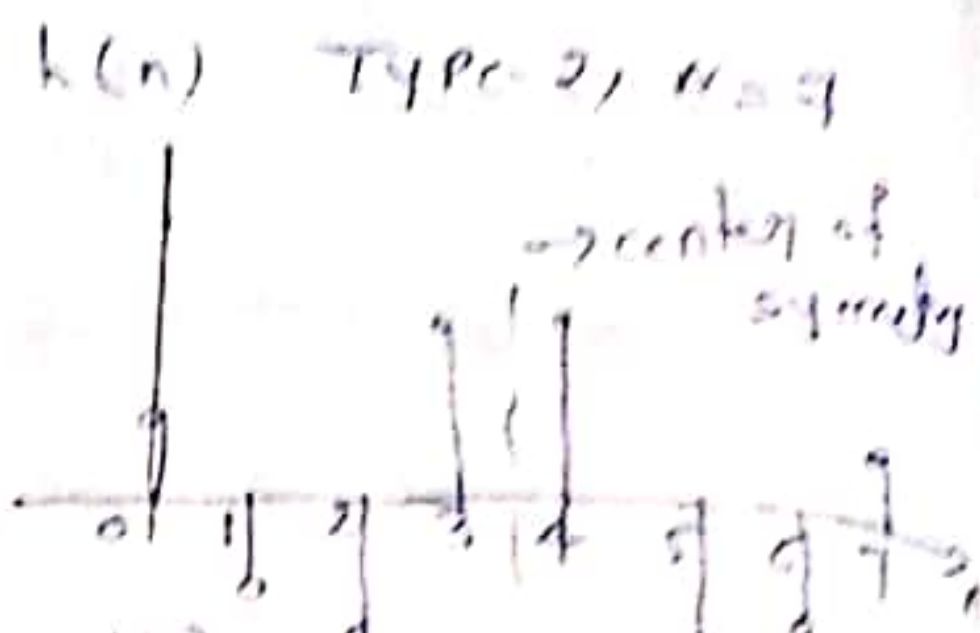
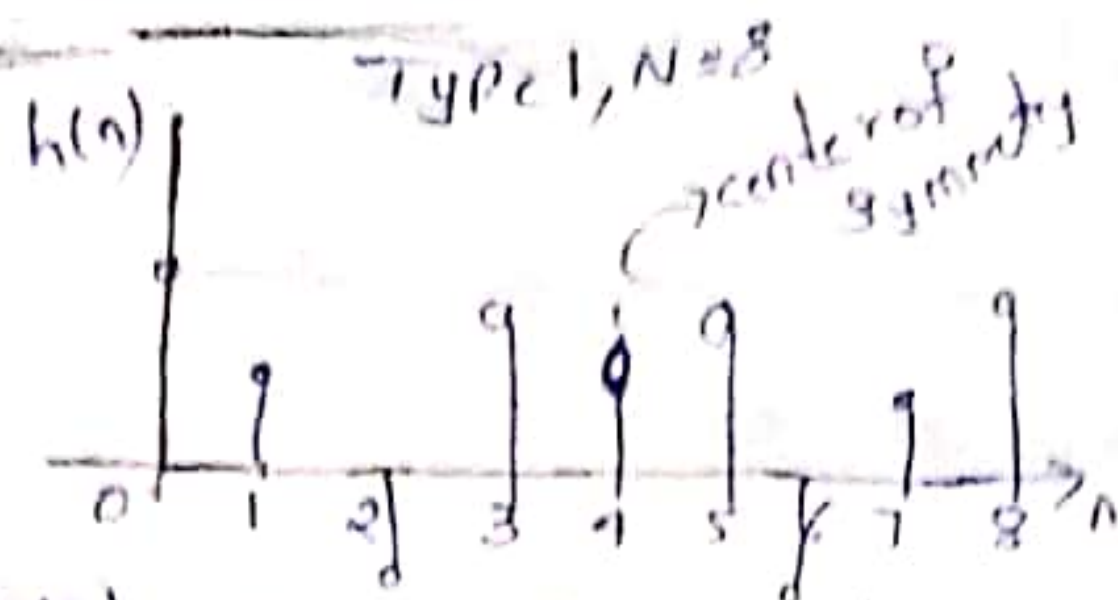
$$|\tilde{H}(\omega)| = 2 \sum_{n=1}^{\frac{N+1}{2}} h\left[\frac{N+1}{2} - n\right] \sin(\omega(n-1/2))$$

→ It is general form of Eq (4)

Conclusion

- (3) $T_g(\omega) = N/2$ [group delay] in every case
 → Even $T_g(\omega) \rightarrow \text{constant}$ since in general $|H(e^{j\omega})|$ is not constant
 the y_p waveform is not a replica of x_p waveform.

Conclusions:-



Note General Form of Frequency Response.

→ Magnitude is same for all 4-types.

① → $H(e^{j\omega}) = e^{-j\frac{N\omega}{2}} e^{j\beta} \tilde{H}(\omega)$ where $\beta = 0$ for $\frac{\pi}{2}$ and $\beta = \pi/2$ for $\frac{\pi}{2}$

→ The amplitude response $\tilde{H}(\omega)$ for each of 4-types of linear-phase FIR filters can become negative over certain frequency ranges, typically in stopband.

* The magnitude & phase response of linear-phase FIR,

$|H(e^{j\omega})| = |\tilde{H}(\omega)|$ $\theta(\omega) = \begin{cases} -\frac{N\omega}{2} + \beta + 0 & \text{for } \tilde{H}(\omega) \geq 0 \\ -\frac{N\omega}{2} + \beta - \pi & \text{for } \tilde{H}(\omega) < 0 \end{cases}$

* Zero locations of Linear-phase FIR T.F:-
[no denominator to get poles]

Consider FIR filter with a symmetric impulse response

$H(z) = \sum_{n=0}^N h(n)z^{-n} = \sum_{n=0}^N h(N-n)z^{-n}$

Let $m = N-n$ $n=0 \Rightarrow m=N$
 $n=N \Rightarrow m=0$

$H(z) = \sum_{m=0}^N h(m)z^{-(N+m)} z^{-(N-m)}$
 $= z^{-N} \sum_{m=0}^N h(m)z^m \downarrow z^{-m} \quad m=-m$

$H(z) = z^{-N} H(z^{-1})$

⇒ if $H(z)$ satisfies this $\Rightarrow$ it is called as minor image polynomial

ii) Consider FIR filter with an Anti-symmetric impulse response.

$$H(z) = \sum_{n=0}^N h(n)z^{-n} = \sum_{n=0}^N -h(N-n)z^{-n}$$

$N-n=m$

$$= -\sum_{m=0}^N h(m)z^{-(N-m)}$$

$$= -z^{-N} \sum_{m=0}^N h(m)z^m$$

$$\boxed{H(z) = -z^{-N}H(z^{-1})} \rightarrow (2)$$

→ If $H(z)$ satisfies this eq, it is called as Anti-mirror Image polynomial.

Note Conclusion from (1) & (2) Zeros

* $z = \xi_0$ (zero) $H(\xi_0) = 0$

from eq (1), $H(\xi_0) = (\xi_0)^{-N} H(\xi_0^{-1}) = 0$

$\downarrow \xi_0^{-1}$ is also a zero

⇒ If $\boxed{z = \xi_0}$ is a zero of $H(z)$ so is $\boxed{z = 1/\xi_0}$ → also zero.

⇒ Reciprocal of zero is also a zero of $H(z)$.

* polar form. $\xi_0 = \eta e^{j\theta}$ is a zero of $H(z)$
 $\xi_0^{-1} = \eta^{-1} e^{-j\theta}$ is also a zero of $H(z)$

* Complex form

⇒ when $h(n)$ is real & $\xi_0 = \eta e^{j\theta}$ is a zero of $H(z)$
 $\xi_0^* = \eta e^{-j\theta}$ will be zero of $H(z)$
 $(\xi_0^*)^{-1} = \eta^{-1} e^{j\theta}$ will also be zero of $H(z)$

⇒ A complex zero is not on the unit circle ($\eta \neq 1$)
 is associated with a set of 4-zeros → Quad

is $\underline{z = \eta e^{j\theta}}$, $\underline{z = 1/\eta e^{\pm j\theta}}$

⇒ For a zero on the unit circle, its reciprocal is also its complex conjugate.

$\underline{z = e^{\pm j\theta}}$ ($\eta = 1$) → pair

→ A real zero is paired with its reciprocal ^{also} zero

$$z = \rho, \quad z = 1/\rho$$

⇒ if, $z = \pm 1$ is its own reciprocal ⇒ it can be appear as single
 ↳ reciprocal also equal to $z = \pm 1$

Symmetric case:-

$$H(z) = z^{-N} H(z^{-1}) \rightarrow \text{For Type-I \& II}$$

$$\underline{z = -1}$$

$$H(-1) = (-1)^{-N} H((-1)^{-1}) = (-1)^{-N} H(-1)$$

i) If 'N' is even, $H(-1) = \underbrace{(-1)^{-N}}_{\pm 1} H(-1) = H(-1) \rightarrow \text{Type-I}$

ii) If 'N' is odd, $H(-1) = -H(-1) \Rightarrow \text{Type-II}$
 valid only $\rightarrow H(-1) = 0$.

→ symmetric impulse response with 'N' odd the the system must have a zero at $\boxed{z = -1}$

So, By using Type-II, we cant design HPF ^{or notch} _{type-II}

Anti-symmetric:- Type-III & Type-IV

$$H(z) = -z^{-N} H(z^{-1})$$

i) If $z = 1$, $H(1) = -(1)^{-N} H(1) = -H(1)$
 $H(1) = -H(1)$

$H(z)$ must have zero at $z = 1$ for both N is even
 & N is odd
 i.e. for both Type-III & Type-IV

ii) If $z = -1$, $H(-1) = -(-1)^{-N} H(-1)$
 $H(-1) = (-1)^{-N+1} H(-1)$

iii) N is even, $(-N+1) \Rightarrow \text{odd} \Rightarrow H(-1) = -H(-1)$ } Type-III

So, $z = -1$ must be a zero of $H(z)$ if N is

$$H(z) = (1+z^{-1})(1-z^{-1}) H_1(z) \quad \begin{matrix} \text{even} \\ \rightarrow z=1 \\ z=-1 \end{matrix} \text{ zeros}$$

ii) N is odd $(-N+1) \Rightarrow$ Even $H(-1) = H(-1)$ } Type - IV
 $z=1 \rightarrow z=0$

Example problems

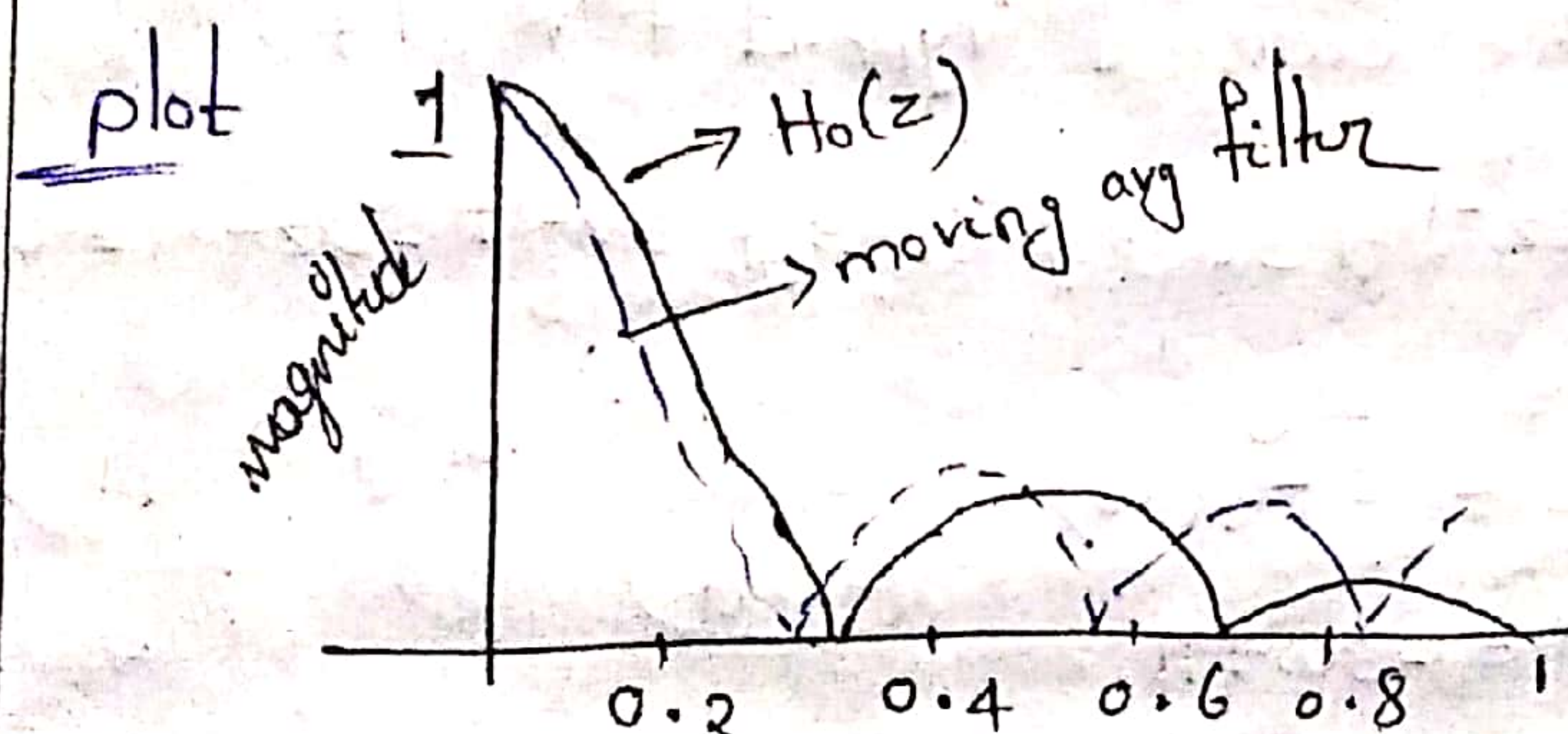
$$H(z) = (1 - z^{-1})H_1(z)$$

Type I

eg 1 Consider $H_0(z) = \frac{1}{6} \left[\frac{1}{2} + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \frac{1}{2}z^{-6} \right]$

$N=6$ which is seen to be a slightly modified version of a length-7 moving-average FIR filter.

$\Rightarrow$ The above transfer function has a symmetric impulse response and therefore a linear phase response.



$\rightarrow$ Note, the improved magnitude response ($H_0(z)$) obtained by simply changing the $\frac{1}{2}$ first & the last impulse response coefficients of a moving-average (MA) filter.

$$H_0(z) = \frac{1}{2} (1 + z^{-1}) \frac{1}{6} (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

which seen to be cascade of a 2-point MA filter with a 6-point MA filter.

$\therefore H_0(z)$ has a double zero at $z = -1$ i.e. ($\omega = \pi$)

Type II

eg Consider ^{causal} type-I FIR transfer function

$$H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6}$$

Length = 7, $N=6$

amplitude

$$\begin{aligned} H_1(\omega) &= h(N/2) + 2 \sum_{n=1}^{N/2} h(N/2 - n) \cos(\omega n) \\ &= h(3) + 2 \sum_{n=1}^3 h[3-n] \cos(\omega n) \end{aligned}$$

$$= 6 + 2 \left[h(2) \cos(\omega) + h(1) \cos(2\omega) + h(0) \cos(3\omega) \right]$$

$$= 6 + 2(-3) \cos \omega + 2(0) \cos 2\omega + 2(-1) \cos(3\omega)$$

$$\tilde{H}_1(\omega) = 6 - 6 \cos \omega + 4 \cos(2\omega) - 2 \cos(3\omega)$$

phase response,

$$\theta_1(\omega) = \frac{-\omega N}{2} + \frac{N}{2} \omega + \beta \Rightarrow$$

$$\theta_1(\omega) = -3\omega \quad (\beta = 0)$$

Next, consider the causal Type-1 FIR T.F,

$$H_2(z) = 1 - 2z^{-1} + 3z^{-2} - 6z^{-3} + 3z^{-4} - 2z^{-5} + z^{-6}$$

$$H_1(z^{-1})$$

$$\tilde{H}_2(\omega) = -\tilde{H}_1(\omega)$$

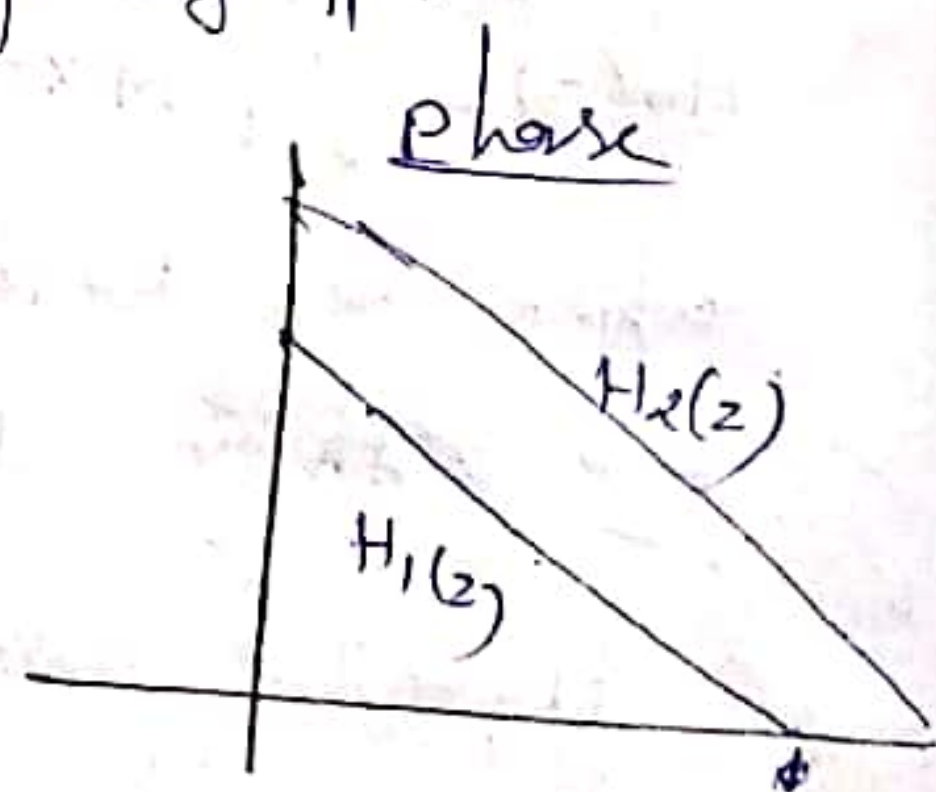
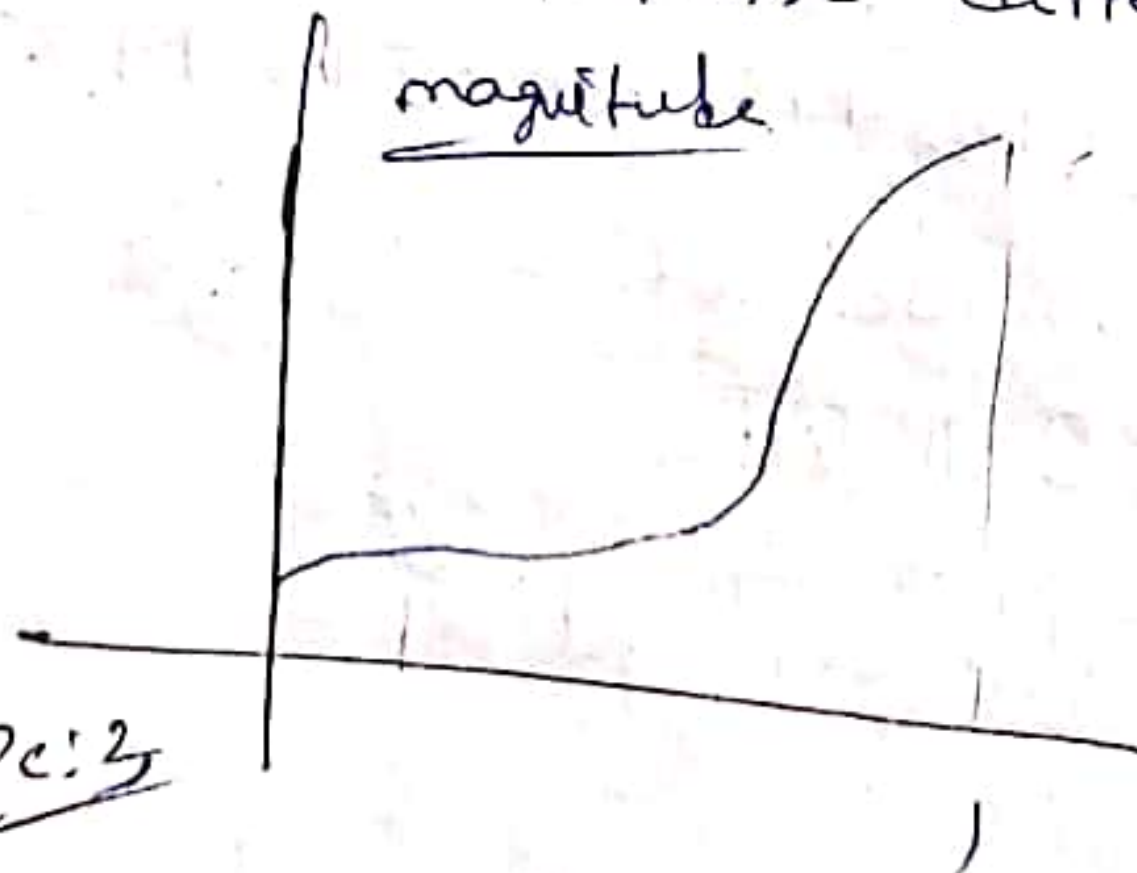
$$\theta_2(\omega) = -3\omega + \pi$$

$$(\beta = \pi)$$

Note

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$$

$\therefore$ Hence, $H_1(z)$ & $H_2(z)$ have identical magnitude responses but phase response differing by π .



Type-3

Eg - Consider causal Type-3 FIR transfer function

$$H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6}$$

$N+1=7$
 $N=6$

~~Amplitude~~ $\tilde{H}_3(z) = \left(\frac{N+1}{2} \right) \sum_{n=0}^{N/2} h\left(\frac{N+1}{2} - n\right) \cos\left(\frac{(N+1)\omega}{2}\right)$

$$\tilde{H}_3(z) = \frac{1}{2} \sum_{n=-1}^4$$

$$\tilde{H}(\omega) = \frac{1}{2} \sum_{n=-1}^{N/2} h\left[\frac{N}{2} - n\right] \sin(\omega n)$$

$$= \frac{1}{2} \sum_{n=1}^3 h[3-n] \sin(\omega n)$$

$$= \frac{1}{2} h(2) \sin(\omega) + \frac{1}{2} h(1) \sin(2\omega) + \frac{1}{2} h(0) \sin(3\omega)$$

$$= \frac{1}{2} (3) \sin(\omega) + \frac{1}{2} (-2) \sin(2\omega) + \frac{1}{2} (1) \sin(3\omega)$$

$$= 6 \sin \omega - 4 \sin(2\omega) + 2 \sin(3\omega)$$

$$\theta_3(\omega) = -\omega - 3\omega + \pi/2$$

Next, consider the causal Type-1 FIR f.f.

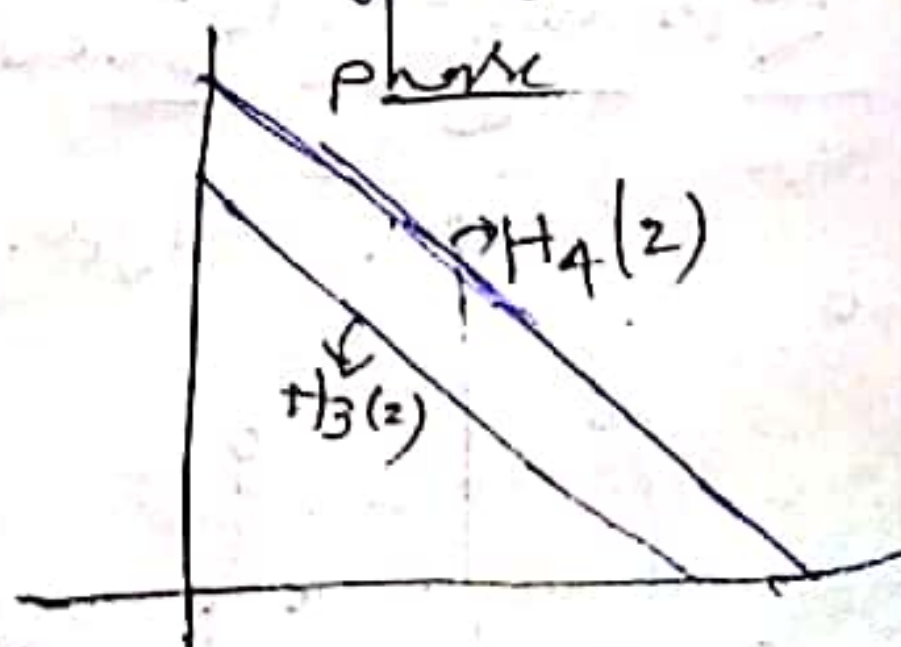
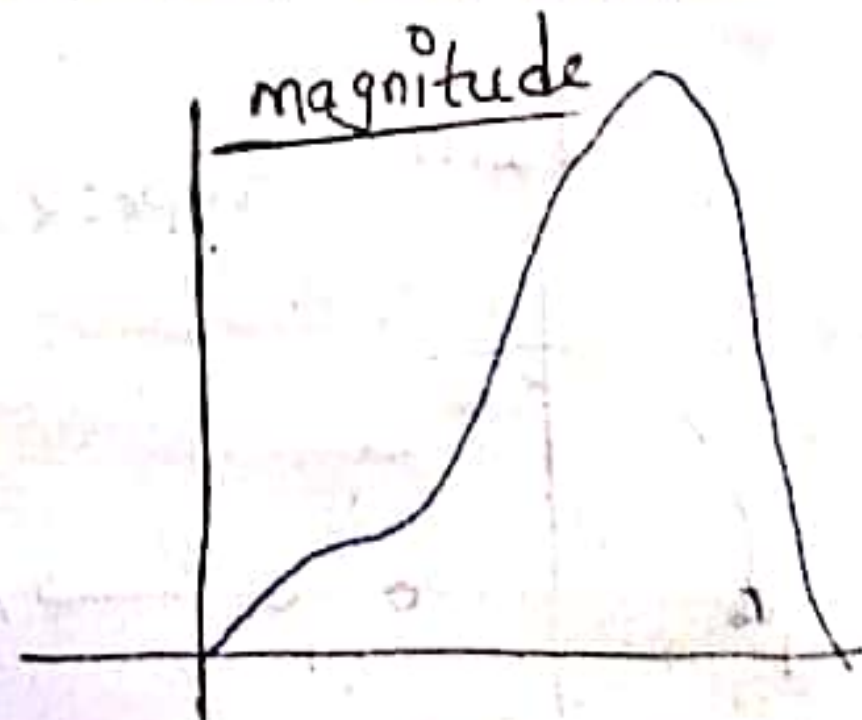
$$H_4(z) = -1 + 2z^{-1} - 3z^{-2} + 3z^{-4} - 2z^{-5} + z^{-6}$$

$$\tilde{H}_4(\omega) = -\tilde{H}_3(\omega)$$

$$\theta_4(\omega) = -3\omega - \pi/2$$

Note: $|H_3(e^{j\omega})| = |H_4(e^{j\omega})|$

* Hence $H_3(z)$ & $H_4(z)$ have identical magnitude responses but phase response differing by π .



Note Linear phase filter (4-types)

For all 4-types filters, $H(e^{j\omega}) = e^{-jN\omega/2} \cdot e^{j\tau} \tilde{H}(\omega)$

$$|H(e^{j\omega})| = |\tilde{H}(\omega)|$$

$$\tilde{H}_3(z) = \frac{1}{2} \sum_{n=-1}^1 z^n$$

$$\tilde{H}(\omega) = \sum_{n=-1}^1 h\left[\frac{N}{2} - n\right] \sin(\omega n)$$

$$= \sum_{n=1}^3 h[3-n] \sin(\omega n)$$

$$= 2h(2)\sin(\omega) + 2h(1)\sin(2\omega) + 2h(0)\sin(3\omega)$$

$$= 2(3)\sin(\omega) + 2(-2)\sin(2\omega) + 2(1)\sin(3\omega)$$

$$= 6\sin\omega - 4\sin(2\omega) + 2\sin(3\omega)$$

$$\theta_3(\omega) = -\omega - 3\omega + \pi/2$$

Next, consider the causal Type-1 FIR f.f.

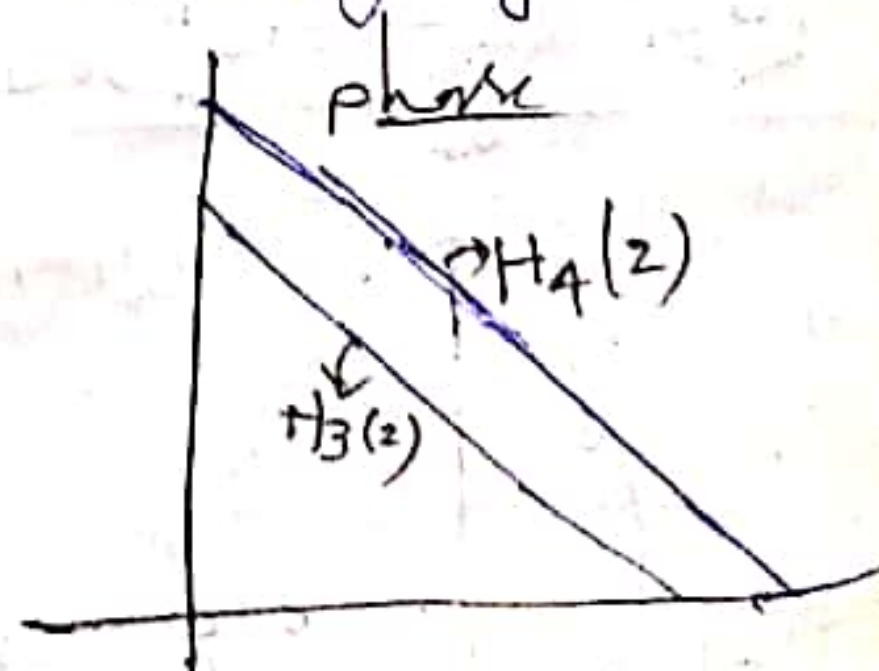
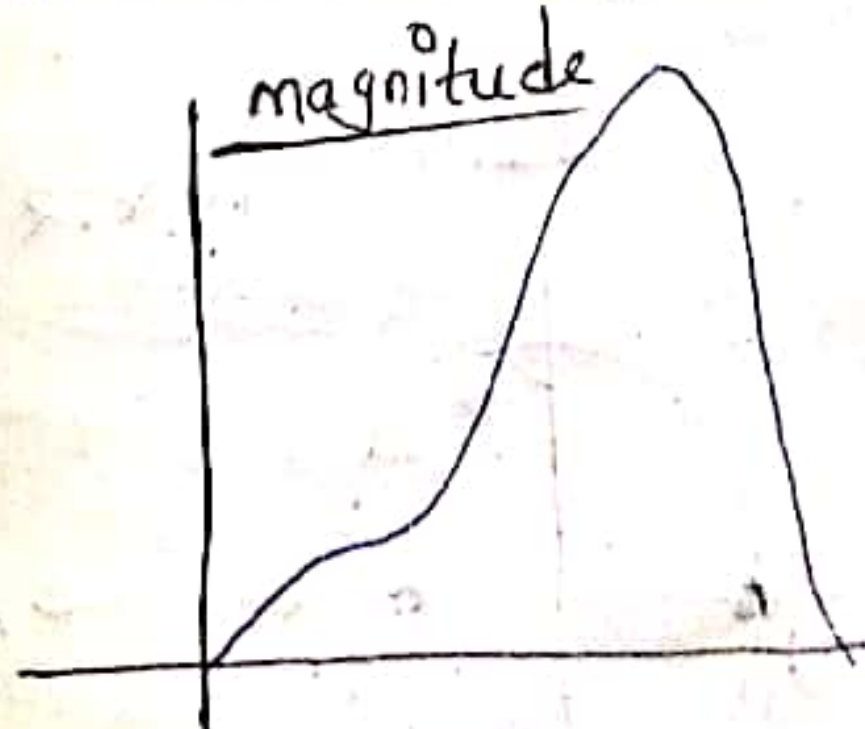
$$H_4(z) = -1 + 2z^{-1} - 3z^{-2} + 3z^{-4} - 2z^{-5} + z^{-6}$$

$$\tilde{H}_4(\omega) = -\tilde{H}_3(\omega)$$

$$\theta_4(\omega) = -3\omega - \pi/2$$

Note: $|H_3(e^{j\omega})| = |H_4(e^{j\omega})|$

* Hence $H_3(z)$ & $H_4(z)$ have identical magnitude responses but phase response differing by π .



Note Linear phase filter (4-types)

For all 4-types filters, $H(e^{j\omega}) = e^{-jN\omega/2} \cdot e^{j\tau} \tilde{H}(\omega)$

$$|H(e^{j\omega})| = |\tilde{H}(\omega)|$$

$$\angle H(e^{j\omega}) = -\frac{N\omega}{2} + \gamma + \beta$$

$\gamma = 0 \rightarrow$ for symmetric

$\gamma = \pi/2 \rightarrow$ for Antisymmetric

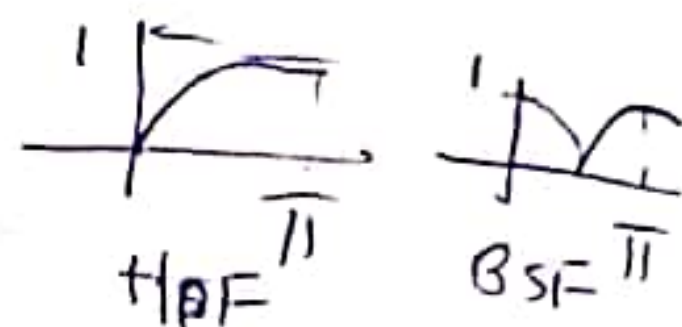
$$\left. \begin{array}{l} \beta = 0 \quad \tilde{H}(\omega) \geq 0 \\ \beta = \pi \quad \tilde{H}(\omega) < 0 \end{array} \right\} \text{for all.}$$

Note

Type: 1 No restriction, More versatile $\Rightarrow$ All filters can be design.

Type: 2 $z = -1$ must be zero. $z = e^{j\omega}$, $\omega = \pi \Rightarrow z = -1$

We cannot design HPF & BSF

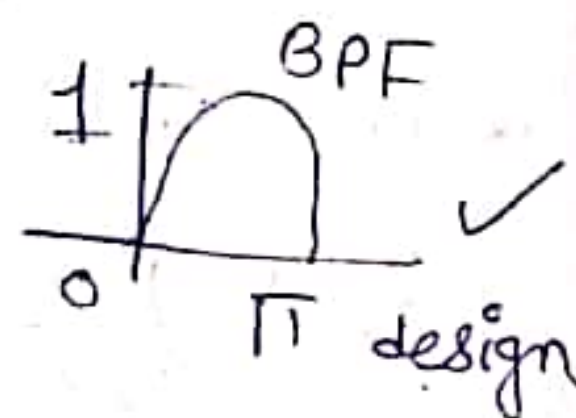


Type: 3

$z = -1$ & $z = +1$ must be zeros

$\omega = \pi$, $\omega = 0$

We cannot design LPF, HPF, BSF



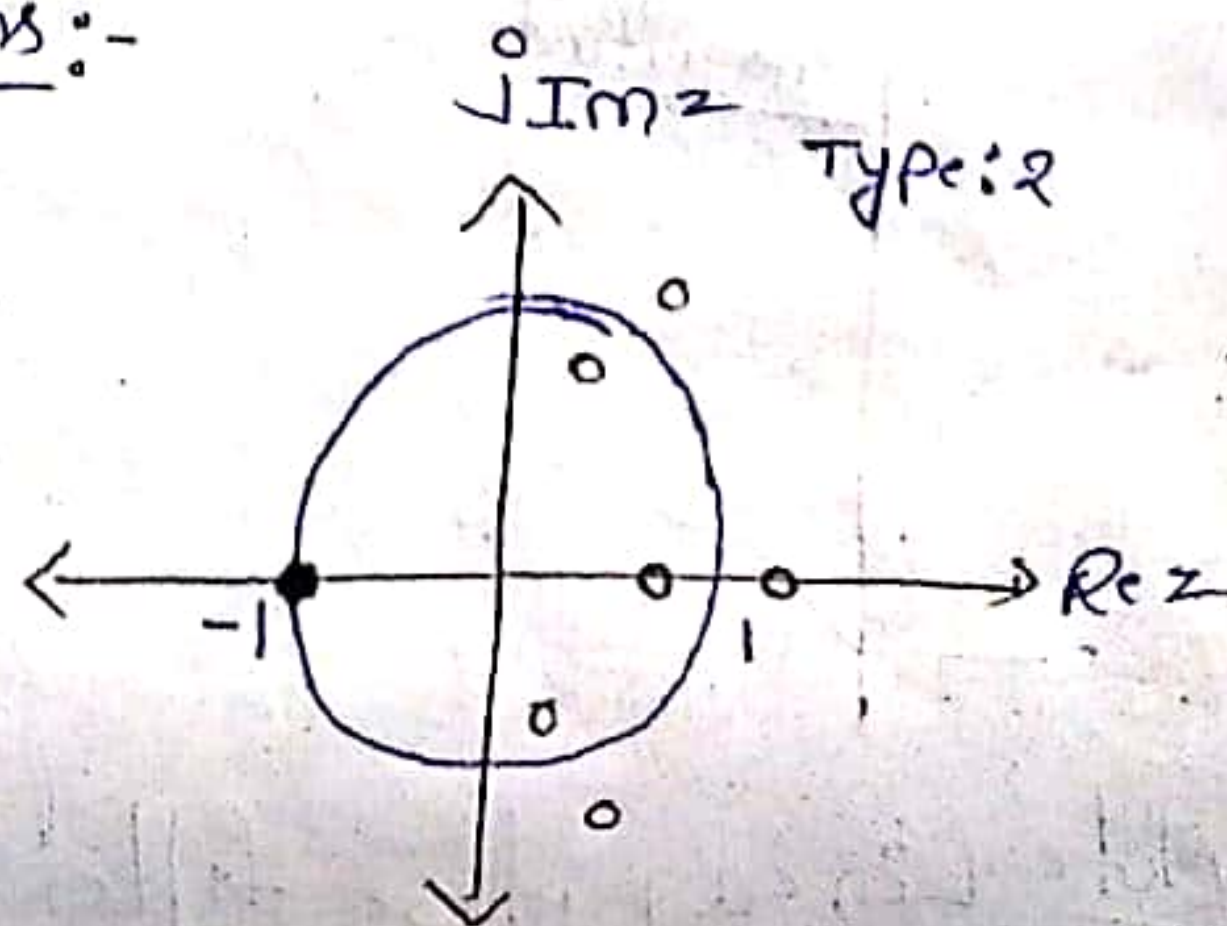
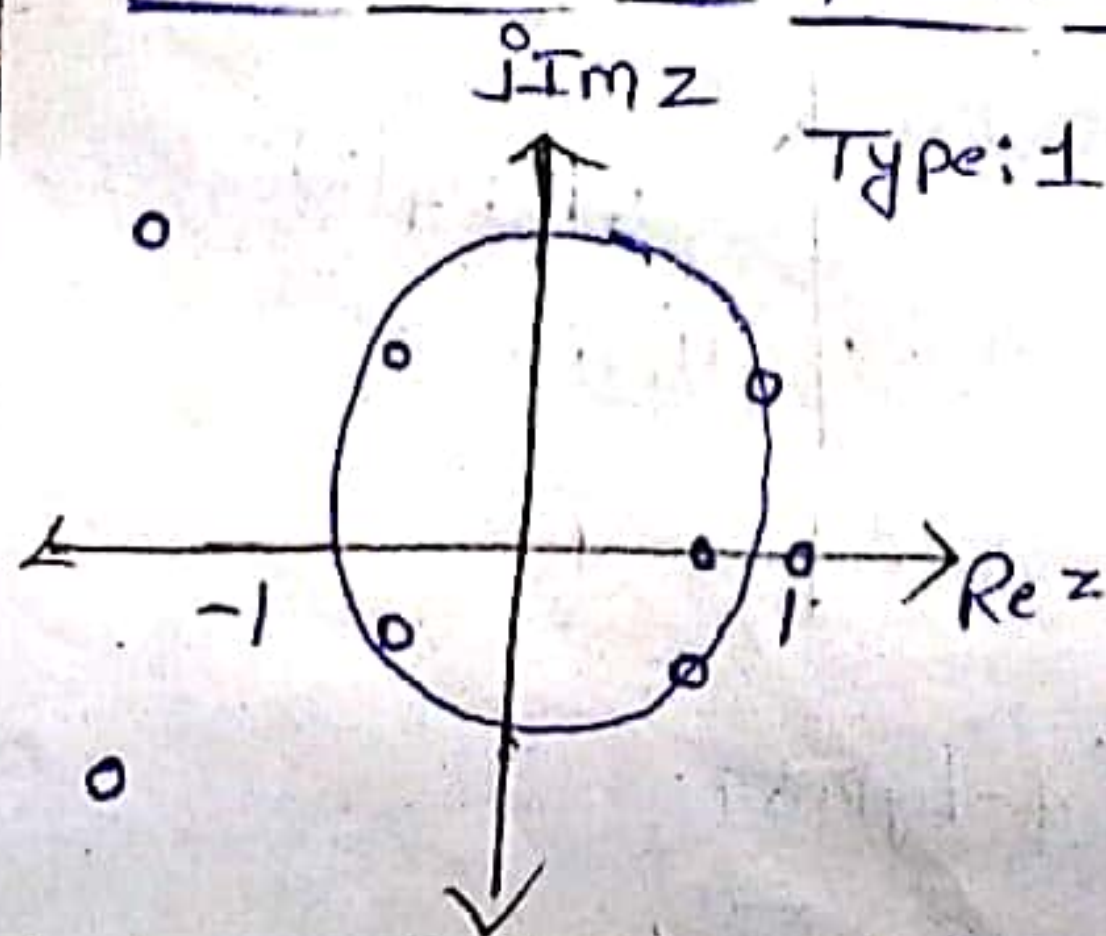
Type: 4

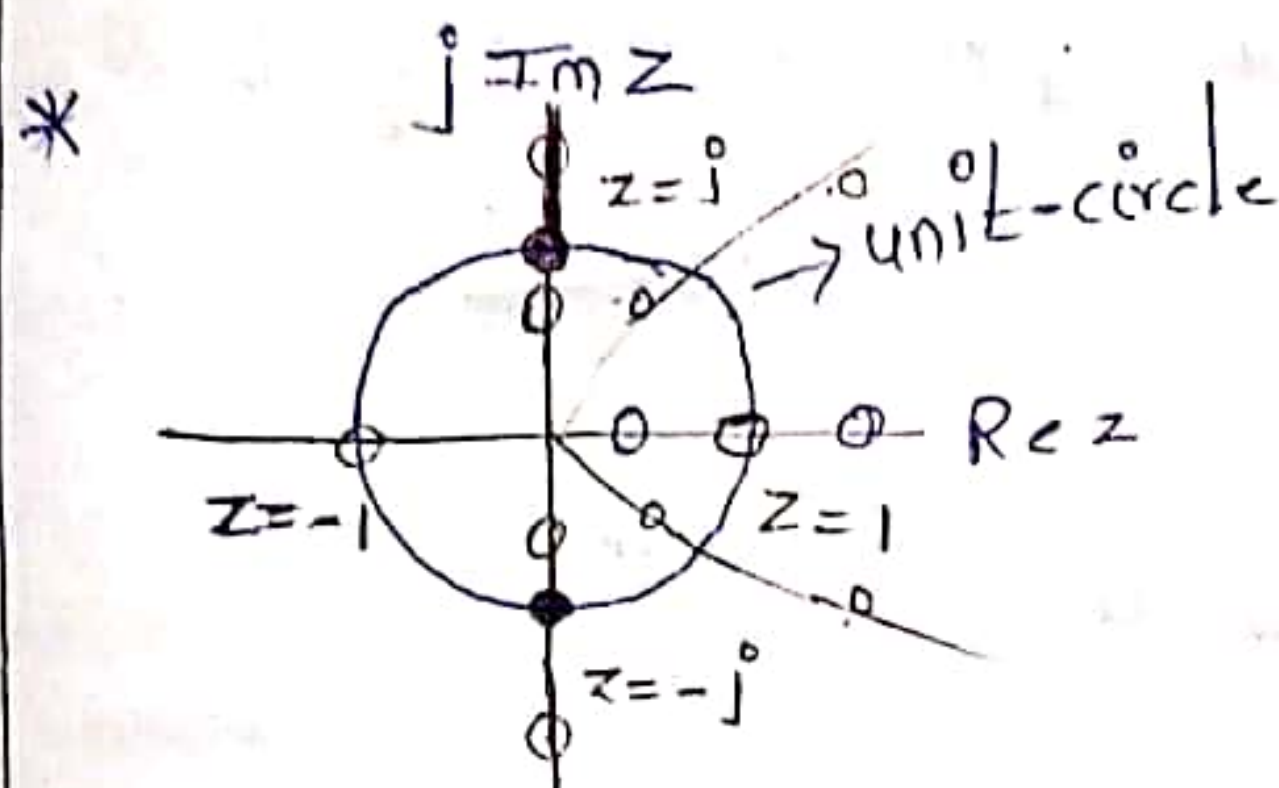
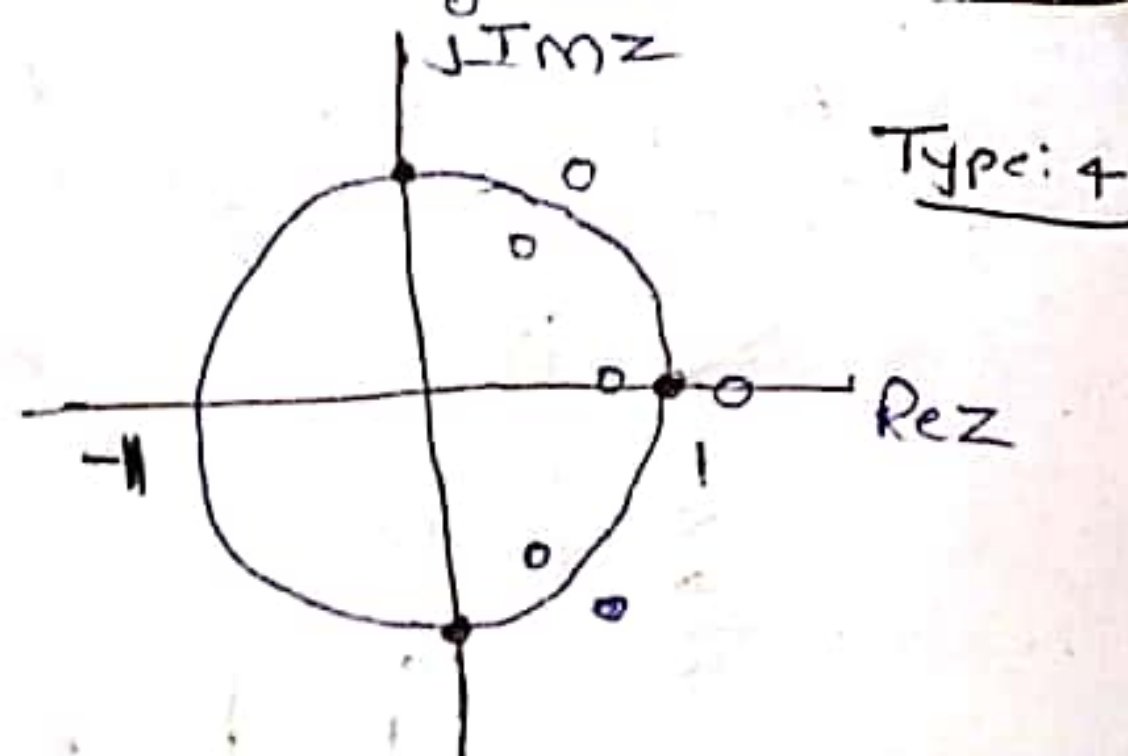
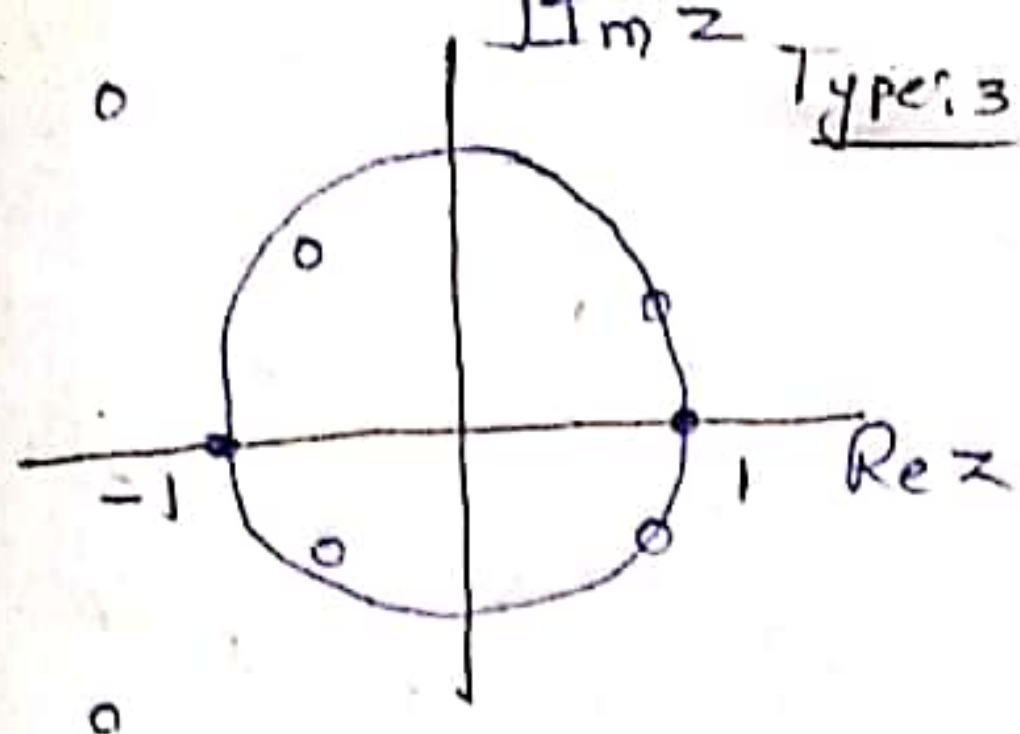
$z = 1$, $\omega = 0$, $z = e^{j\omega} = 1$
 $\hookrightarrow$ zero must be zero

$\rightarrow$ we cannot design LPF, BSF

zero-Locations of linear-phase FIR
Transfer Functions:-

Typical-zero locations:-





→ concave mirror

5th-order Linear FIR filter:-

Given, one zero $\frac{1}{2} e^{j\pi/4}$

$$(1 - \frac{1}{2} e^{j\pi/4} z^{-1})$$

$$H_5(z) = (1 + z^{-1})(1 - \frac{1}{2} e^{j\pi/4} z^{-1})(1 - \frac{1}{2} e^{-j\pi/4} z^{-1})(1 - 2 e^{j\pi/4} z^{-1})(1 - 2 e^{-j\pi/4} z^{-1})$$

$$\begin{aligned} 1 + z^{-1} &\rightarrow \text{zero at } -1 \Rightarrow \text{Type: 2} \\ 1 - z^{-1} &\rightarrow \text{zero at } 1 \Rightarrow \text{Type: 3} \end{aligned}$$

Examples

Type 1

If impulse response is $h[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$

$$H(e^{j\omega}) = \sum_{n=0}^4 e^{-j\omega n}$$

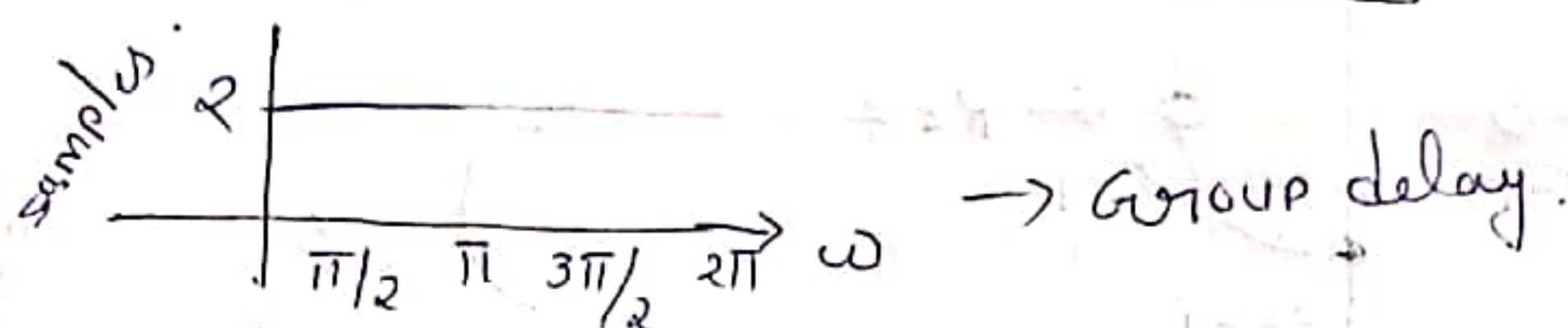
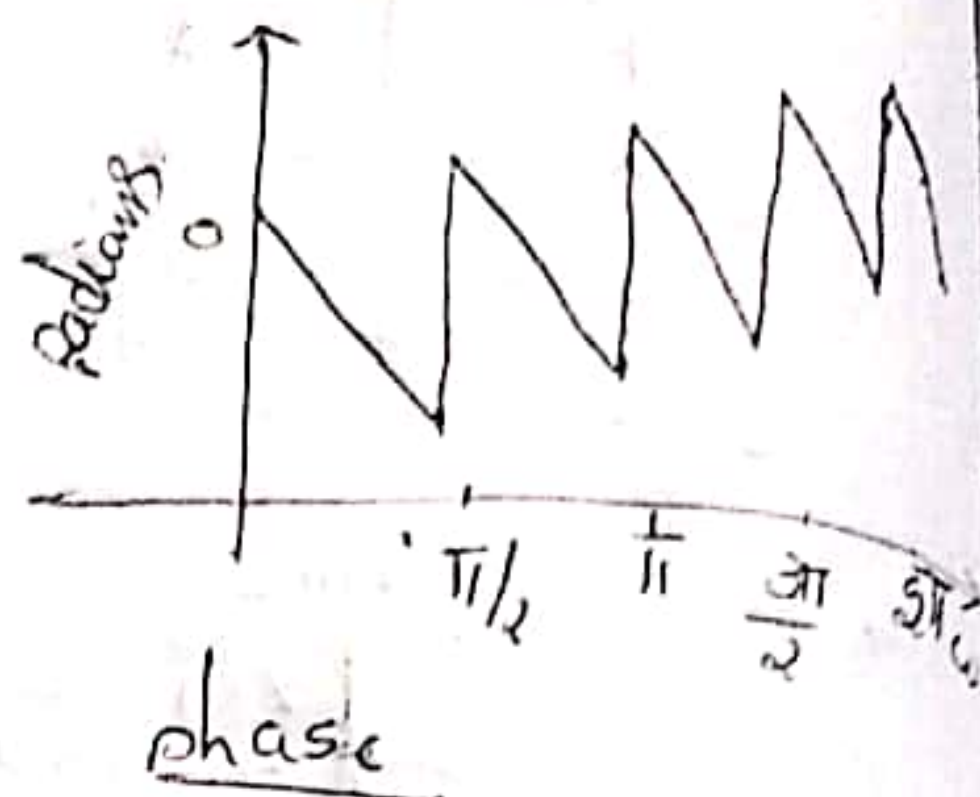
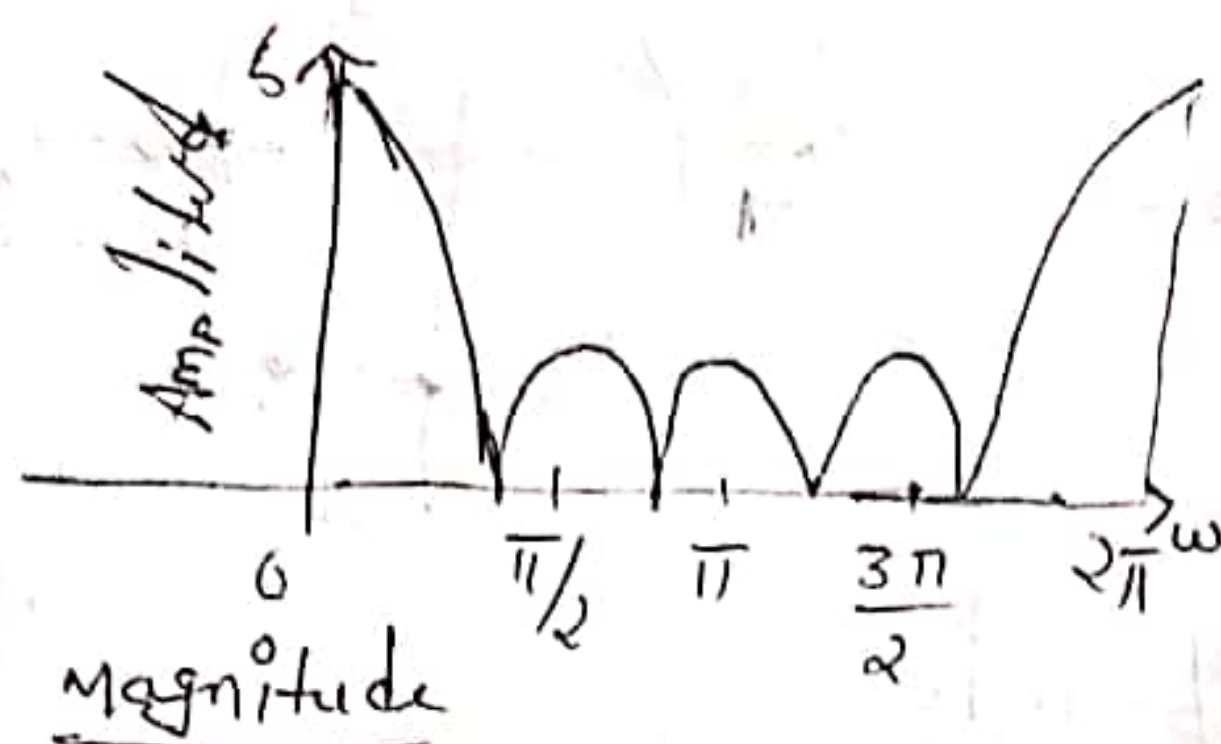
↓ $N \Rightarrow \text{even}, N+1 \rightarrow \text{odd}$
 → symmetric } Type: 1

$$= \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega 5/2} (e^{j\omega 5/2} - e^{-j\omega 5/2})}{e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})}$$

$$= e^{-j\omega(5/2 - 1/2)} \frac{\sin(5\omega/2)}{\sin(\omega/2)} = e^{-j\omega 2} \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$N=4 \rightarrow \text{Even}$, $T_g(\omega) = 2$
 Frequency response:-



Eg Type-II

If length of ^{impulse response} previous example is extended by one sample.

$$h(n) = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$N \Rightarrow \text{odd}$

$N+1 \Rightarrow \text{length (even)}$
 $\rightarrow \text{symmetric.}$

$$H(e^{j\omega}) = \sum_{n=0}^5 e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega 6}}{1 - e^{-j\omega}}$$

$$= e^{-j\omega 3} (e^{j\omega 3} - e^{-j\omega 3})$$

$$= e^{-j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})$$

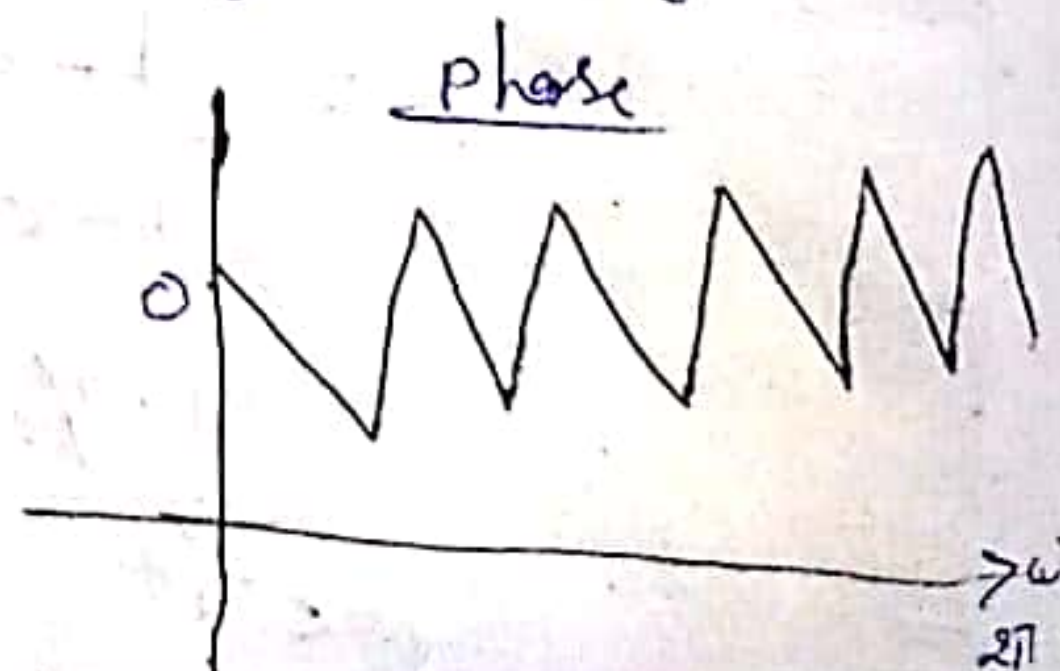
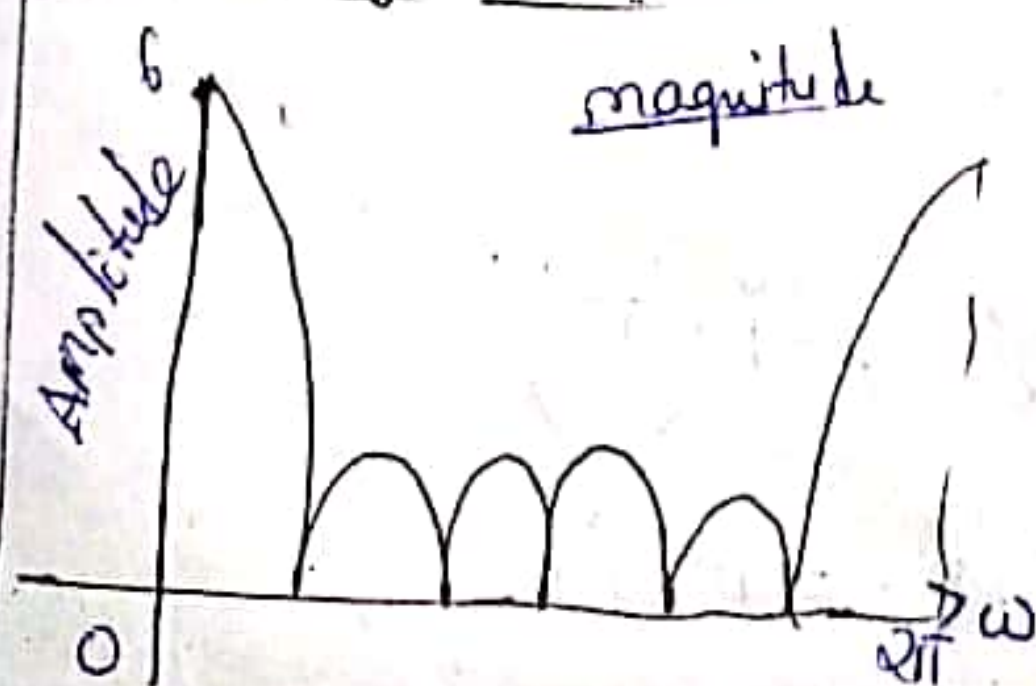
$$= e^{-j\omega 5/2} \frac{\sin(3\omega)}{\sin(\omega/2)}$$

$$= e^{-j\omega(3-1/2)} \frac{\sin(3\omega)}{\sin(\omega/2)}$$

$N=5 \rightarrow \text{odd}$,

Frequency response:-

$T_g(\omega) = 5/2 \rightarrow \text{group delay.}$



group delay $\leftarrow$



Type: III

IF impulse response is

$$h[n] = \delta[n] - \delta[n-2]$$

$$N+1=3$$

$$N=2$$

→ $N=0 \rightarrow \text{EVEN}$
 $N+1 \rightarrow \text{odd length}$
Asymmetric

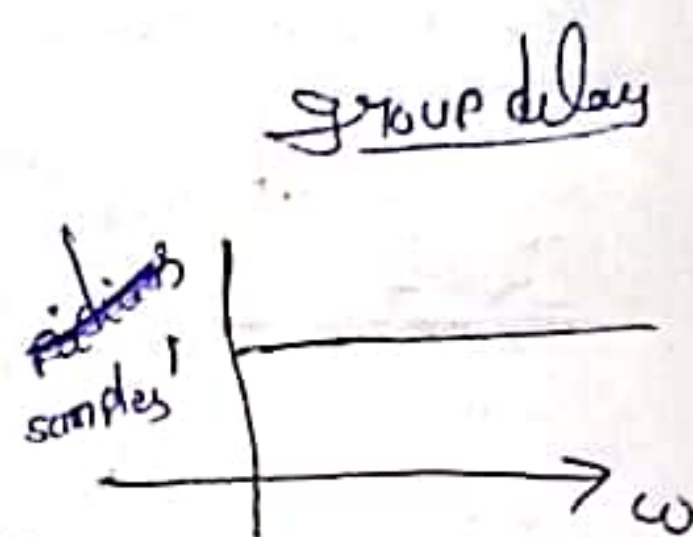
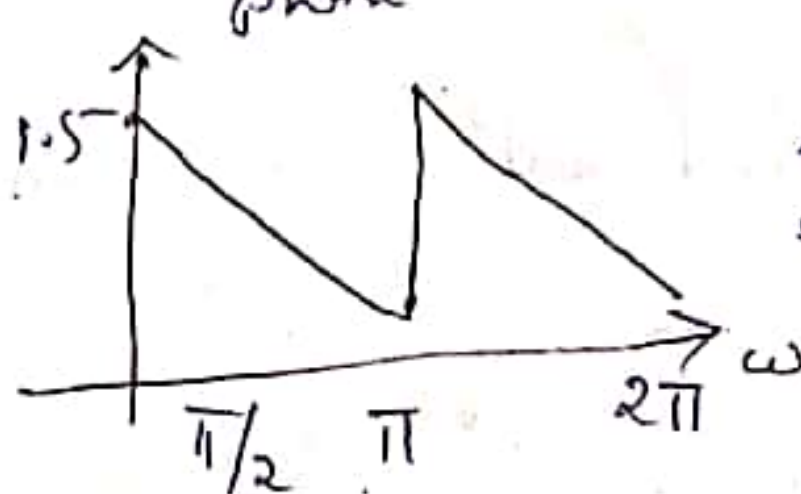
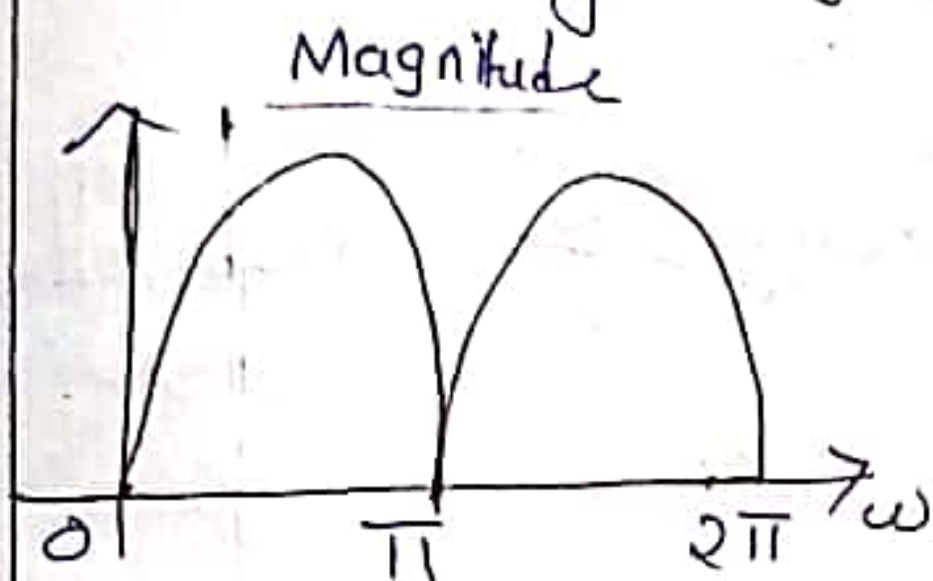
Type: 3.

$$H(e^{j\omega}) = 1 - e^{-j2\omega}$$

$$= e^{-j\omega} (e^{j\omega} - e^{-j\omega})$$

$$= j[2 \sin(\omega)] e^{-j\omega}$$

Group delay: $T_g(\omega) = 1$ // phase



Type: IV

IF impulse response is

$$h[n] = \delta[n] - \delta[n-1]$$

$$N+1=2 \text{ Even}$$

$$N=1 \rightarrow \text{odd}$$

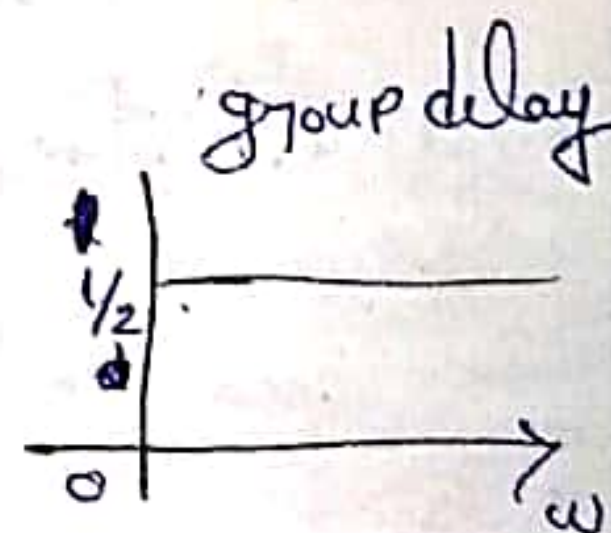
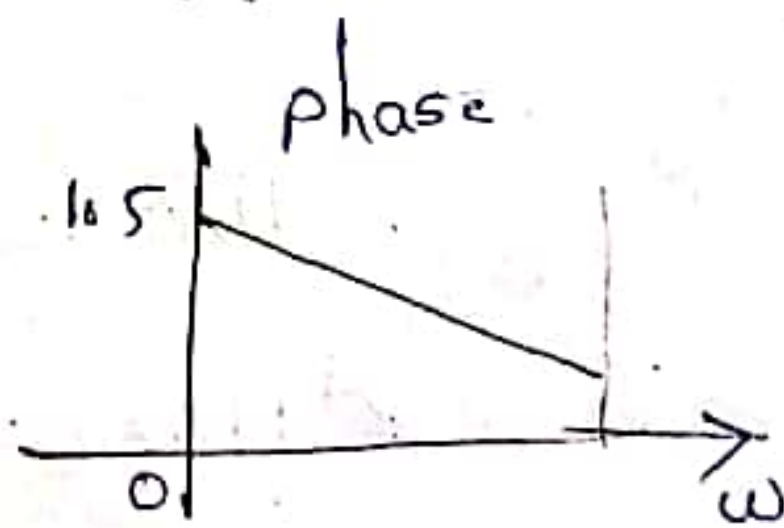
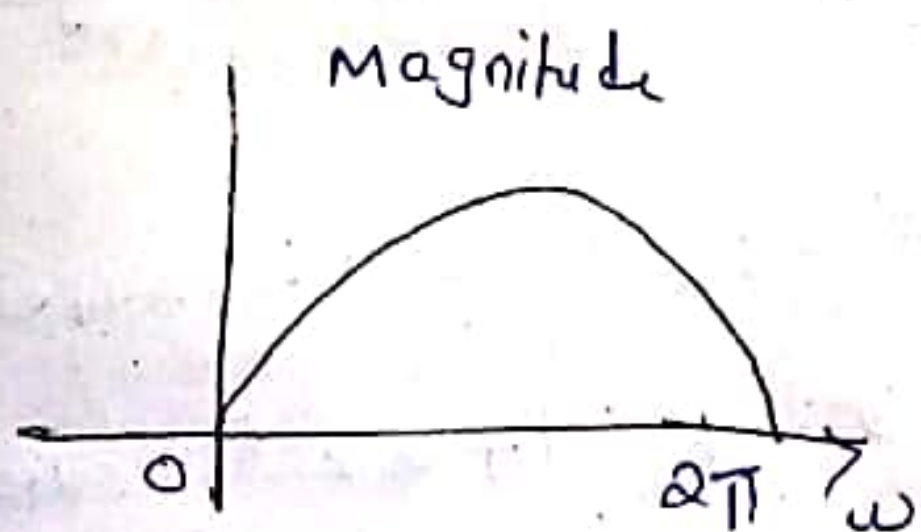
for which frequency response is,

$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

$$= e^{-j\omega/2} [e^{j\omega/2} - e^{-j\omega/2}]$$

$$= j[2 \sin(\omega/2)] e^{-j\omega/2}$$

Group delay: $T_g(\omega) = 1/2$



Zero-phase Transfer-function:-

⇒ An FIR filter with a frequency response is a real-function of ω is called as "zero-phase filter".

$$\phi = 0, \pi, 2\pi \dots$$

(multiples of π)

⇒ Such a filter must have a non-causal impulse response

⇒ In practice, we cannot design this zero-phase filter

Applications

⇒ For recorded data, we can design zero-phase filter

Eg:- $h(n) = \{\alpha, \beta, \alpha\}$

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-1}^1 h(n) e^{-j\omega n} \\ &= \alpha e^{j\omega} + \beta + \alpha e^{-j\omega} \\ &= \beta + 2\alpha \cos \omega \end{aligned}$$

$$\angle H(e^{j\omega}) = 0 + \angle \cos \omega \text{ } \{ \text{phase } 0 \text{ or } \pi \}$$

So, $\downarrow$ it is real function

∴ Hence it is a zero-phase T.F or filter

* zero phase filters can be made causal by multiplying by z^{-n}

$$\begin{aligned} H(z) &= \alpha z + \beta + \alpha z^{-1} \rightarrow \text{non causal} \\ z^1 H(z) &= \alpha + \beta z^{-1} + \alpha z^{-2} \rightarrow \text{causal} \end{aligned}$$

Eg:- $h[n] = \{1, 0, 1\}$

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-1}^1 h(n) e^{-j\omega n} \\ &= e^{j\omega} + \beta + e^{-j\omega} \\ &= 2 \cos \omega \rightarrow \text{zero phase filter.} \end{aligned}$$

$$\angle H(e^{j\omega}) = \angle \cos \omega \rightarrow 0 \text{ or } \pi$$

$$i) h[n] = \{1, 0, -1\}$$

$$H(e^{j\omega}) = e^{j\omega} - e^{-j\omega} = 2j \sin \omega$$

$$\angle H(e^{j\omega}) = \pi/2 + \angle \sin \omega \neq 0 \text{ or } \pi$$

→ Not a zero phase filter.

$$iii) h[n] = \{1, 0, -1\}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} h(n) e^{-j\omega n}$$

$$= 1 + (-1)e^{-j2\omega}$$

$$= 1 - e^{-j2\omega} \rightarrow \text{Not a zero phase filter}$$

$$\angle H(e^{j\omega}) = -2\omega \neq 0 \text{ or } \pi$$

⇒ Not in terms of cosine or sine.

Minimum phase, Maximum-phase & Mixed phase Transfer

Functions :-

including poles
↓
IIR
FIR ↑

Minimum phase T.F :-

→ A causal, stable transfer function with all zeroes inside a unit circle is called "Minimum-phase T.F"

Maximum phase T.F :-

→ A causal, stable transfer function with all zeroes outside the unit circle is called "Maximum phase T.F"

Mixed phase T.F :-

→ A system^{T.F} with some zeros inside the unit circle & the remaining zeroes outside the unit circle is called "Mixed-phase T.F"

Eg:- Let us consider 2- FIR systems.

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2})$$

zero at $z = -\frac{1}{2}$

Impulse response:-

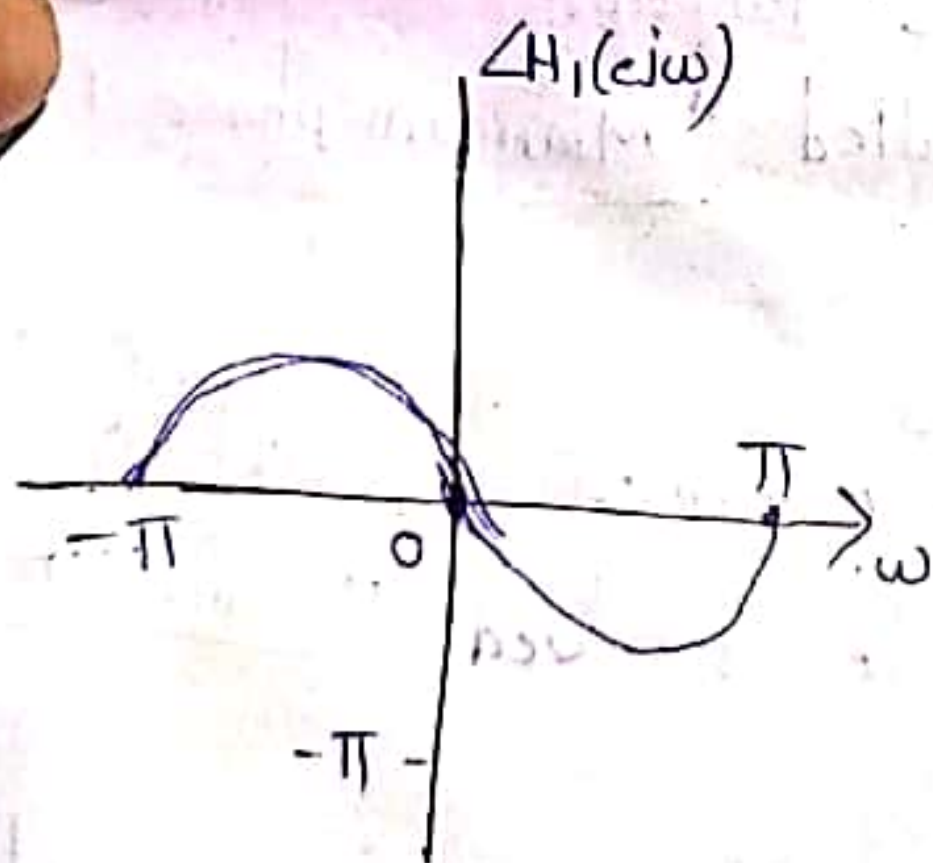
$$h(0) = 1, h(1) = \frac{1}{2}$$

$$H_1(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$$

$$\begin{aligned} H_1(e^{j\omega}) &= e^{-j\omega/2}(e^{j\omega/2} + \frac{1}{2}e^{-j\omega/2}) \\ &= e^{-j\omega/2}[\cos(\omega/2) + j\sin(\omega/2)] \\ &= e^{-j\omega/2}[(\frac{1}{2} + \cos(\omega/2)) + j\sin(\omega/2)] \end{aligned}$$

$$\begin{aligned} |H_1(e^{j\omega})| &= |e^{-j\omega/2}| \sqrt{(\frac{1}{2} + \cos(\omega/2))^2 + \sin^2(\omega/2)} \\ &= \sqrt{(\frac{1}{2} + \cos(\omega/2))^2 + \sin^2(\omega/2)} \\ &= \sqrt{\frac{1}{4} + \cos^2(\omega/2) + \sin^2(\omega/2) + \cos(\omega)} \\ &= \sqrt{\frac{5}{4} + \cos(\omega)} \end{aligned}$$

$$\angle H_1(e^{j\omega}) = -\omega/2 + \tan^{-1}\left(\frac{\sin(\omega/2)}{\frac{1}{2} + \cos(\omega/2)}\right)$$



$$\text{at } \omega = 0 \Rightarrow -0 + \tan^{-1}(0) = 0$$

$$\begin{aligned} \omega = \pi &\Rightarrow -\pi/2 + \tan^{-1}\left(\frac{\sin(\pi/2)}{\frac{1}{2} + \cos(\pi/2)}\right) \\ &= -\pi/2 + \tan^{-1}(0) \\ &= -\pi/2 + \pi - 0 \\ &= \pi/2 \end{aligned}$$

$$\angle H_1(\pi) - \angle H_1(0) = \pi/2$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1)$$

zero at $z = -2$

$$\text{I.R.}, h(0) = \frac{1}{2}, h(1) = 1$$

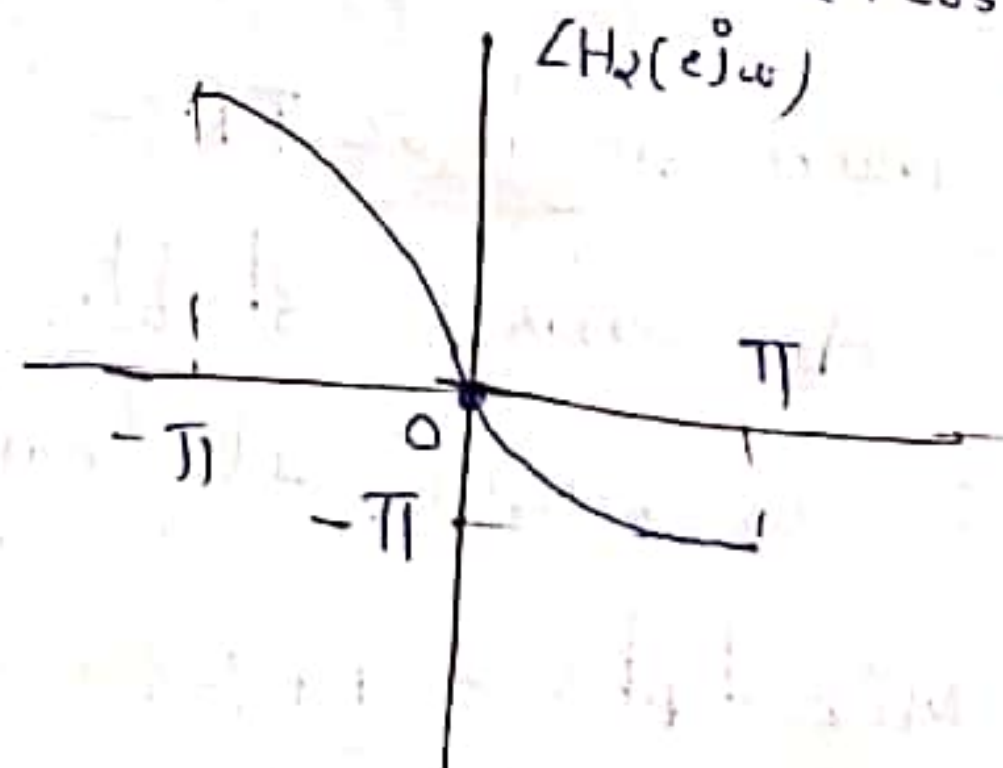
$$\begin{aligned} H_2(e^{j\omega}) &= e^{-j\omega}(\frac{1}{2}e^{j\omega} + 1) \\ &= e^{-j\omega}(\frac{1}{2}(\cos(\omega) + j\sin(\omega)) + 1) \\ &= e^{-j\omega}((\frac{1}{2} + \cos(\omega)) + j\frac{1}{2}\sin(\omega)) \end{aligned}$$

$$|H_2(e^{j\omega})| = \sqrt{(\frac{1}{2} + \cos(\omega))^2 + (\frac{1}{2}\sin(\omega))^2}$$

$$= \sqrt{\frac{1}{4} + \cos^2(\omega) + \cos(\omega) + \frac{1}{4}\sin^2(\omega)}$$

$$= \sqrt{\frac{5}{4} + \cos(\omega)}$$

$$\begin{aligned} \angle H_2(e^{j\omega}) &= -\omega + \tan^{-1}\left(\frac{\frac{1}{2}\sin(\omega)}{\frac{1}{2} + \cos(\omega)}\right) \\ &= -\omega + \tan^{-1}\left(\frac{\sin(\omega)}{2 + \cos(\omega)}\right) \end{aligned}$$



$$\text{at } \omega = 0 \Rightarrow -0 + \tan^{-1}(0) = 0$$

$$\begin{aligned} \text{at } \omega = \pi &\Rightarrow -\pi + \tan^{-1}\left(\frac{\sin(\pi)}{2 + \cos(\pi)}\right) \\ &\Rightarrow -\pi + 0 \\ &\Rightarrow -\pi \end{aligned}$$

$$\angle H_1(e^{j\pi}) - \angle H_1(e^{j0}) =$$

$$= 0 - 0 = 0$$

⇒ Minimum-phase system

→ minimum delay
[time delay]

$$\angle H_2(e^{j\pi}) - \angle H_2(e^{j0}) =$$

$$= -\pi - 0$$

$$= -\pi$$

or

other side, π

⇒ Maximum-phase system

→ maximum delay

Note

$$\angle H(e^{j\pi}) - \angle H(e^{j0}) = M\pi \rightarrow \text{maximum-phase system}$$

→ which is the largest possible phase change for FIR system with 'M' zeroes.

⇒ Minimum phase system corresponds to minimum delay (in t.D)

⇒ Maximum phase system corresponds to maximum delay (in t.D)

Examples

i) $H_1(z) = 6 + z^{-1} - z^{-2}$

$$H_1(z) = z^{-2}(6z^2 + z + 1) \rightarrow \text{zeros } z = -1/2, 1/3 \rightarrow \text{minimum phase system}$$

ii) $H_2(z) = 1 - z^{-1} - 6z^{-2}$

$$H_2(z) = z^{-2}(z^2 - z - 6) \rightarrow \text{zeros } z = -2, 3 \rightarrow \text{Maximum phase}$$

iii) $H_3(z) = 1 - 5/2 z^{-1} - 3/2 z^{-2}$

$$H_3(z) = z^{-2}(z^2 - 5/2 z - 3/2) \rightarrow \text{zeros } z = -1/2, 3 \rightarrow \text{Mixed-phase}$$

iv) $H_4(z) = 1 + 5/3 z^{-1} - 2/3 z^{-2}$

$$H_4(z) = z^{-2}(z^2 + 5/3 z - 2/3) \rightarrow \text{zeros } z = -2, 1/3 \rightarrow \text{Mixed phase}$$

$$H_4(z) = z^{-2}(z^2 + 5/3 z - 2/3)$$

Comb filter:- → there will be Band stop & Band pass frequencies

A comb filter can be viewed as a notch filter in which the nulls occur periodically across the frequency band/hence

Applications:-

→ in rejection of power-line harmonics

→ in separation of solar & lunar components from

ionospheric measurements of electron conⁿ (MTI) radars

→ suppression of clutter from fixed objects in moving-target indicators

Designing:-

Let an FIR filter,

$$H(z) = \sum_{k=0}^M h(k) z^{-k}$$

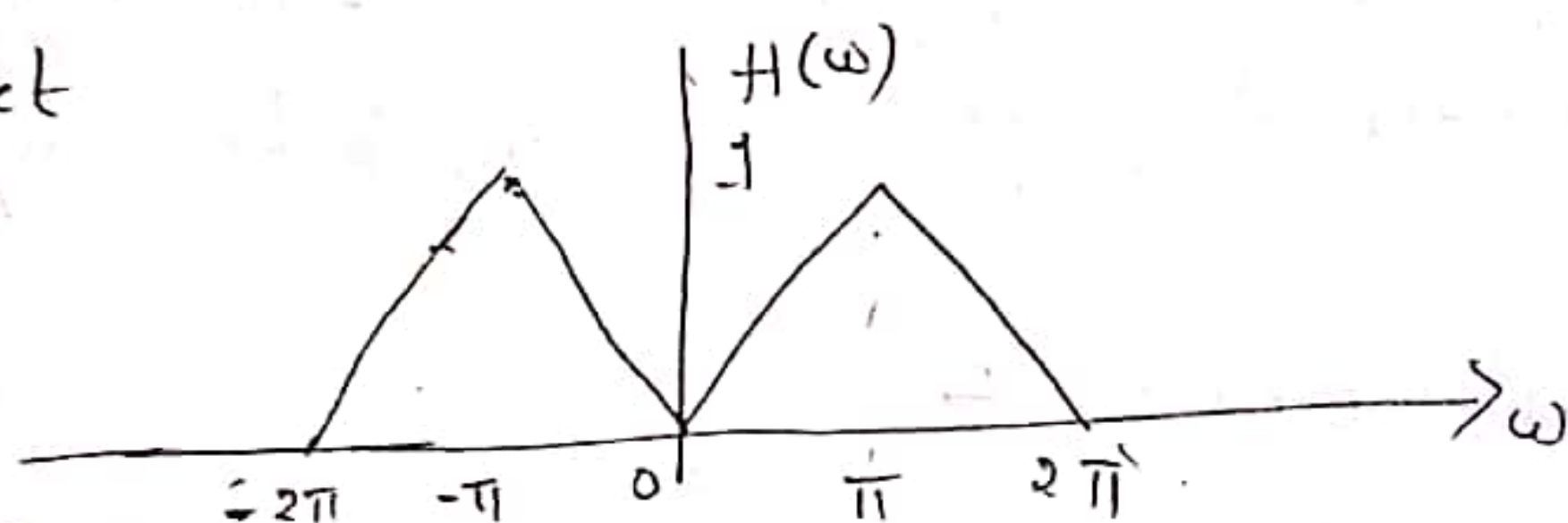
Replacing z by z^L , where L is a positive inty

$$H_L(z) = \sum_{k=0}^M h(k) z^{-kL}$$

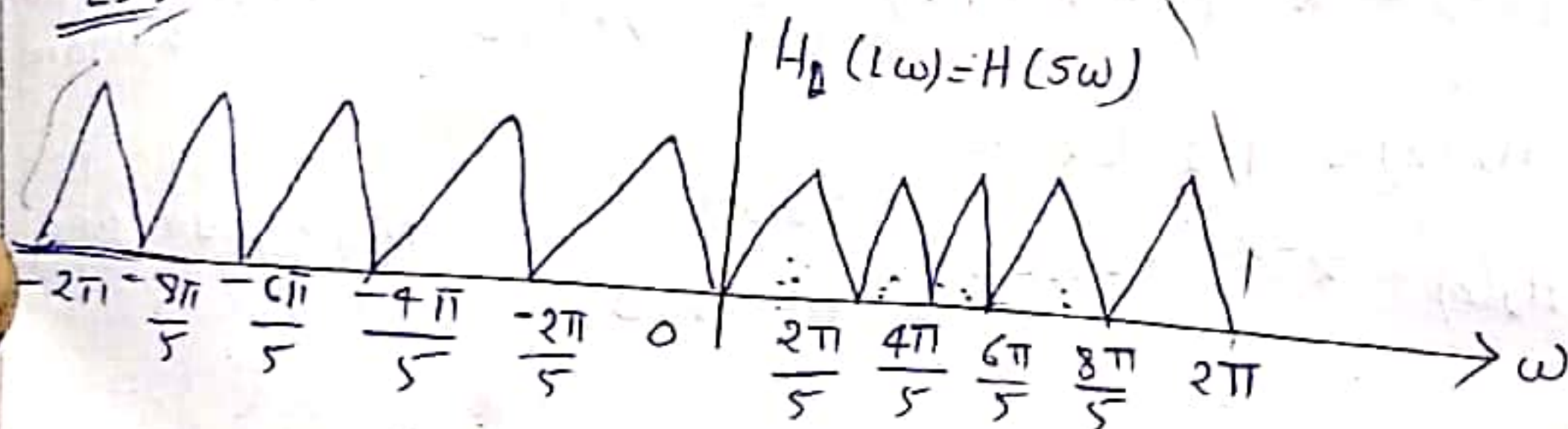
Frequency response,

$$H_L(\omega) = \sum_{k=0}^M h(k) e^{-j\omega kL} = H(L\omega)$$

Let



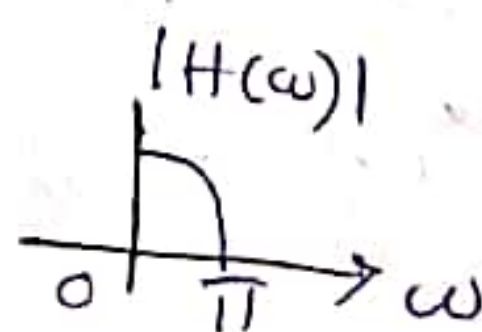
L=5



$\Rightarrow$ The frequency response characteristic $H_L(\omega)$ is simply an L -order repetition of $H(\omega)$ in the range 0 to 2π . $\rightarrow 0 \leq \omega \leq 2\pi$

Eg:- first-order LRF

$$H(z) = \frac{1}{2}(1+z^{-1})$$



$$H_L(z) = \frac{1}{2}(1+z^{-L}) \quad [z \rightarrow z^L]$$

$$H_L(e^{j\omega}) = \frac{1}{2}(1+e^{-j\omega L})$$

$$= \frac{e^{-j\omega L/2}}{2} (e^{j\omega L/2} + e^{-j\omega L/2})$$

$$= e^{-j\omega L/2} \cos \omega L/2$$

$$|H_L(e^{j\omega})| = |\cos \omega L/2|$$

highest mag = peaks
lowest mag = notches

$$|H_L(e^{j\omega})| = 1 \text{ when } |\cos(\omega L/2)| = 1$$

$$\frac{\omega L}{2} = n\pi \quad n \in \mathbb{Z}$$

$$n = 0, 1, \dots, L-1$$

→ peaks at $\omega = \frac{2\pi r}{L}$

$$L = 5$$

$$\frac{\omega L}{2} = (n+1)\frac{\pi}{2}$$

$$\omega = \frac{(2n+1)\pi}{L}$$

