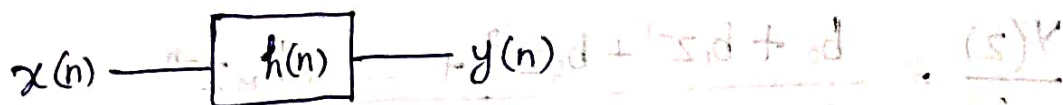


16/10

Transfer Analysis of LTI System:

- ① System function or Transfer function
- ② Impulse Response



$$y(n) = x(n) * h(n)$$

Taking Z transform on both the sides,

$$Y(z) = X(z) \cdot H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} \rightarrow \text{Transfer function / system function}$$

$Y(z)$ = Z transform of the output $y(n)$

$X(z)$ = Z transform of the input $x(n)$

$H(z)$ = Z transform of the impulse response $h(n)$

- ③ Impulse response / function

$$x(n) = 1 ; h(n) = 1$$

$$H(z) = Y(z)$$

Relationship between Transfer function and Difference equation

$$y(n) = x(n) * h(n) = \sum_{k=0}^{\infty} x(n) h(n-k)$$

$$\sum_{k=0}^N a_k y(n-k) = \sum_{k=0}^M b_k x(n-k)$$

Expanding, we get $H(z) = \frac{Y(z)}{X(z)}$

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \dots + a_N y(n-N) =$$

$$b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_M x(n-M)$$

Taking Z-transform both the sides.

$$a_0 y(z) + a_1 z^{-1} y(z) + a_2 z^{-2} y(z) + \dots + a_N z^{-N} y(z) =$$

$$b_0 x(z) + b_1 z^{-1} x(z) + b_2 z^{-2} x(z) + \dots + b_M z^{-M} x(z)$$

$$y(z) [a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}] = x(z) [b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}]$$

$$\frac{y(z)}{x(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

$$H(z) = \frac{y(z)}{x(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

Stability and Causality:

$$\text{BIBO} \therefore \sum_{n=0}^{\infty} |h(n)| < \infty$$

→ A system is said to be a causal system if the impulse response

→ If the system is causal system the ROC is lie outside the circle.

① Consider the LTI System with system function

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \cdot \frac{z}{z - \frac{1}{2}} \quad z = \frac{1}{2}$$

find the difference equation. Determine the stability, causality.

$$\underline{A_1} \quad H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = \frac{y(z)}{x(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

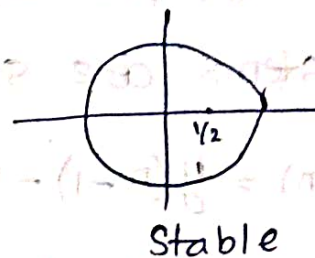
$$y(z) \left(1 - \frac{1}{2} z^{-1}\right) = x(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

Taking the inverse z -transform,

$$y(n) - \frac{1}{2} y(n-1) = x(n)$$

↓
Difference equation



② $H(z) = \frac{z+2}{2z^2-3z+4}$ find the difference equation and frequency response, causal system

A: $H(z) = \frac{z+2}{2z^2-3z+4} \Rightarrow H(z) = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

$$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2)$$

$$\Rightarrow H(z) = \frac{z+2}{2z^2-3z+4} \Rightarrow H(e^{j\omega}) = \frac{e^{j\omega}+2}{2e^{j2\omega}-3e^{j\omega}+4}$$

$$H(e^{j\omega}) = \frac{\cos \omega + j \sin \omega + 2}{2(\cos 2\omega + j \sin 2\omega) - 3(\cos \omega + j \sin \omega) + 4}$$

$$= \frac{\cos \omega + j \sin \omega + 2}{4 + (2 \cos 2\omega - 3 \cos \omega) + j(2 \sin 2\omega - 3 \sin \omega)}$$

③ Determine the system function of a discrete time system of the difference equation

$$y(n) - \frac{1}{3} y(n-1) + \frac{1}{5} y(n-2) = x(n) - 2x(n-1)$$

$$Y(z) - \frac{1}{3} z^{-1} Y(z) + \frac{1}{5} z^{-2} Y(z) = X(z) - 2z^{-1} X(z)$$

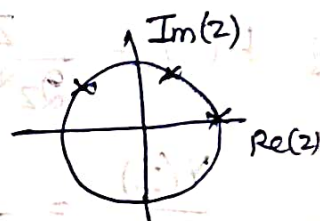
Q) Plot the pole-zero pattern and determine the following systems are stable systems.

a) $y(n] = y[n-1] - 0.8y[n-2] + x[n] + x[n-2]$

b) $y[n] = 2y[n-1] - 0.8y[n-2] + x[n] + 0.8x[n-2]$

Ans a) $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 - z^{-1} + 0.8z^{-2}} = \frac{z^2 + 1}{z^2 - z + 0.8}$

$$\frac{(z+j)(z-j)}{(z-0.5-j0.74)(z-0.5+j0.74)}$$



Stable system.

Q) find the system function, frequency response and impulse response of the $x[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$

$$y[n] = \delta[n] - \frac{2}{3}\delta[n-1]$$

Ans System function / Transfer function,

$$H(z) = \frac{Y(z)}{X(z)}$$

Taking Z-transform on both the sides,

$$X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

$$Y(z) = 1 - \frac{2}{3}z^{-1}$$

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

$$z = e^{j\omega}$$

$$H(\omega) = \frac{z(z - \frac{2}{3})}{z^2 + \frac{1}{6}z - \frac{1}{6}}$$

$$H(\omega) = H(e^{j\omega})$$

$$H(\omega) = \frac{z(z - \frac{2}{3})}{(z - \frac{1}{3})(z + \frac{1}{2})}$$

$$H(\omega) = \frac{e^{j\omega}(e^{j\omega} - 2/3)}{(e^{j\omega} - 1/3)(e^{j\omega} + 1/2)}$$

Impulse response,

$$H(z) = \frac{z(z - 2/3)}{(z - 1/3)(z + 1/2)}$$

$$\frac{H(z)}{z} = \frac{(z - 2/3)}{(z - 1/3)(z + 1/2)} = \frac{A}{(z - 1/3)} + \frac{B}{(z + 1/2)}$$

$$\frac{H(z)}{z} = \frac{-2/5}{z - 1/3} + \frac{7/5}{z + 1/2}$$

$$\left(z - \frac{1}{3} \right) \frac{H(z)}{z} \Big|_{z=1/3} =$$

$$\frac{H(z)}{z} = -\frac{2}{5} \left[\frac{1}{z - 1/3} \right] + \frac{7}{5} \left[\frac{1}{z + 1/2} \right]$$

$$\sum_{n=0}^{\infty} a^n u(n) = \frac{z}{z-a} ; z > a$$

$$H(z) = -\frac{2}{5} \left[\frac{z}{z - 1/3} \right] + \frac{7}{5} \left[\frac{z}{z + 1/2} \right]$$

$$= -\frac{2}{5} \left(\frac{1}{3} \right)^n u(n) + \frac{7}{5} \left(-\frac{1}{2} \right)^n u(n)$$

Q) $x(n) = \left(\frac{1}{3} \right)^n u(n) - \frac{1}{5} \left(\frac{1}{3} \right)^{n-1} u(n-1)$

$$y(n) = \left(\frac{1}{2} \right)^n u(n)$$

- ① Transfer function ② Impulse response ③ frequency response
④ Causality ⑤ Stability.

Ans $X(z) = \frac{z}{z - 1/3} - \frac{1}{5} z^{-1} \left(\frac{z}{z - 1/3} \right) = \frac{z - 1/5}{z - 1/3}$

$$Y(z) = \frac{z}{z - 1/2}$$

$$\frac{H(z)}{z} = \frac{z - 1/3}{(z - 1/2)(z - 1/3)} = \frac{A}{z - 1/2} + \frac{B}{z - 1/3}$$

Relationship between S-plane and Z-plane:

Let $x(t)$ be continuous signal and $x^a(t)$ is sampled version of $x(t)$. Let the sampled period is T .

$$x^a(t) = \sum_{n=0}^{\infty} x(nT) \delta(t - nT)$$

Laplace transform

$$X^a(s) = \sum_{n=0}^{\infty} x(nT) e^{-nTs}$$

Z-transform

$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

$$z = e^{st}$$

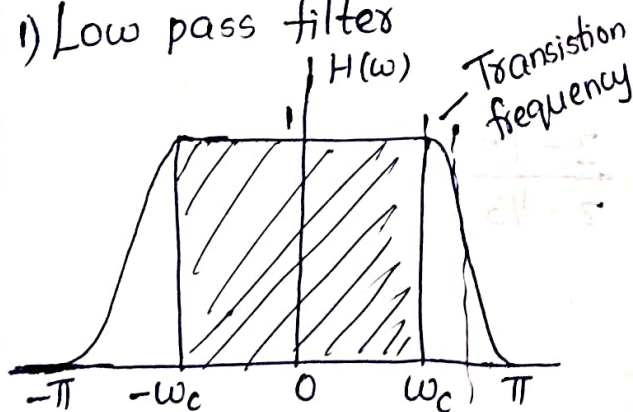
Let $z = re^{j\omega}$ and $s = \sigma + j\omega$

$$z = re^{j\omega} = e^{(\sigma + j\omega)T}$$

$$z = e^{\sigma T} \cdot e^{j\omega T}$$

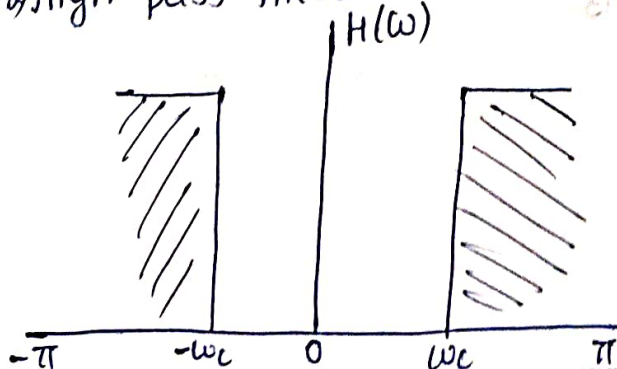
Ideal filters:-

1) Low pass filter



$$|H(\omega)| = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c \leq |\omega| \leq \pi \end{cases}$$

2) High pass filter



- 3) Band pass filter
- 4) Band stop filter
- 5) All pass filter

22/10

Relationship between transfer function and frequency response

$$H(\omega) = H(z) \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{\sum_{k=0}^N a_k e^{-j\omega k}} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \Big|_{z=e^{j\omega}}$$

$$H(\omega) = b_0 \frac{\prod_{k=1}^M (1 - z_k e^{j\omega})}{\prod_{k=1}^N (1 - p_k e^{j\omega})}$$

$$|H(\omega)| = H(\omega) H^*(\omega)$$

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^M (1 - z_k^* e^{j\omega})}{\prod_{k=1}^N (1 - p_k^* e^{j\omega})}$$

$$H^*\left(\frac{1}{z^*}\right) = b_0 \frac{\prod_{k=1}^M (1 - z_k^* z)}{\prod_{k=1}^N (1 - p_k^* z)}$$

$$H^*\left(\frac{1}{z^*}\right) = H(z^{-1})$$

$$H^*(\omega) = H(-\omega)$$

$$\begin{aligned}
 |H(\omega)|^2 &= H(\omega) H^*(\omega) \\
 &= H(\omega) H(-\omega) \\
 &= H(z) H(z^{-1})
 \end{aligned}$$

Correlation theorem :-

$$\begin{aligned}
 R_{x_1 x_2}(n) &= x_1(n) \otimes x_2(n) = X_1(z) X_1(z^{-1}) \\
 &= H(\omega) H(-\omega)
 \end{aligned}$$

Computation of the frequency response function :-

In order to evaluate the magnitude and phase response, $H(\omega)$ can be expressed in poles and zeros, such that $H(\omega)$ is in factored form

$$\begin{aligned}
 H(\omega) &= b_0 \frac{\prod_{k=1}^M (1 - z_k e^{-j\omega k})}{\prod_{k=1}^N (1 - p_k e^{-j\omega k})} = e^{-j\omega M} \\
 &= b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^M (e^{j\omega} - z_k)}{\prod_{k=1}^N (e^{j\omega} - p_k)} = e^{-j\omega N} = e^{+j\omega(N-M)}
 \end{aligned}$$

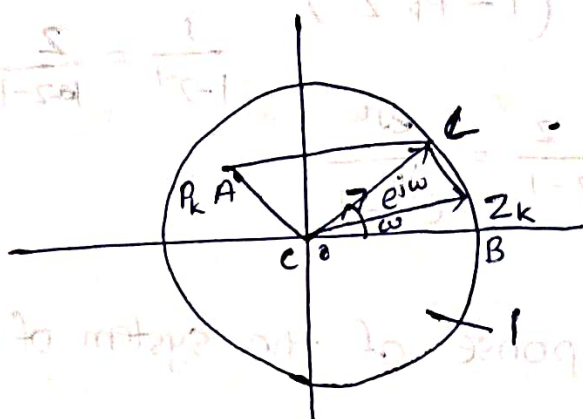
$$\left. \begin{aligned}
 e^{j\omega} - z_k &= V_k(\omega) e^{j\theta_k(\omega)} \\
 e^{j\omega} - p_k &= U_k(\omega) e^{j\theta_k(\omega)}
 \end{aligned} \right\} \begin{array}{l} \text{Polar} \\ \text{form} \end{array}$$

$$V_k(\omega) \equiv |e^{j\omega} - z_k|, \quad \theta_k(\omega) \equiv \angle(e^{j\omega} - z_k)$$

$$U_k(\omega) \equiv |e^{j\omega} - p_k|, \quad \theta_k(\omega) \equiv \angle(e^{j\omega} - p_k)$$

$$|H(\omega)| = |b_0| \frac{V_1(\omega) + V_2(\omega) + \dots + V_M(\omega)}{U_1(\omega) + U_2(\omega) + \dots + U_N(\omega)}$$

$$e^{j\omega(N-M)} = 1$$



$$CA = P_k$$

$$CB = Z_k$$

$$CL = e^{j\omega}$$

$$CL = CA + AL$$

$$CL = CB + BL$$

$$e^{j\omega} = P_k + AL$$

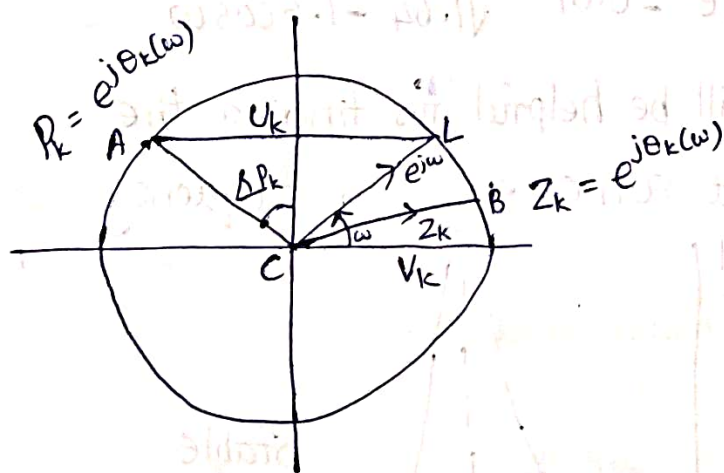
$$e^{j\omega} = Z_k + BL$$

$$AL = e^{j\omega} - P_k$$

$$BL = e^{j\omega} - Z_k$$

$$AL = e^{j\omega} - P_k = U_k(\omega) e^{j\theta_k(\omega)}$$

$$BL = e^{j\omega} - Z_k = V_k(\omega) e^{j\theta_k(\omega)}$$



Linear time invariant as frequency selective filters

① All poles should be placed inside the unit circle in order to have filter to be stable. However, zeros can be placed anywhere in the z-plane.

② If complex zeros and poles must be in the form complex conjugate pairs So that we can able to find filters

coefficients to be real.

$$H(\omega) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 \prod_{k=1}^M (1 - z_k z^{-1})}{\prod_{k=1}^N (1 - p_k z^{-1})}$$

$$\boxed{|H(\omega_0)| = 1}$$

$$\frac{z}{z-1} = \frac{e^{j\omega}}{e^{j\omega} - 1}$$

$$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Q) Evaluate the frequency response of the system of the given transfer function.

A $H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$

A: $H(\omega) = \frac{e^{j\omega}}{e^{j\omega} - 0.8}$

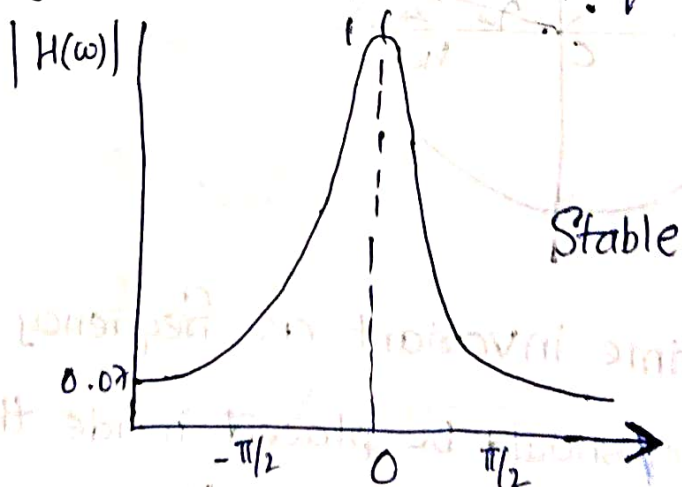
$$|H(\omega)| = \frac{|e^{j\omega}|}{|e^{j\omega} - 0.8|} = \frac{1}{|e^{j\omega} - 0.8|} = \frac{1}{\sqrt{1.64 - 1.6 \cos \omega}}$$

b_0 = gain constant which will be helpful for finding the normalized frequency at some specified frequency.

$\omega = 0 = 5 f$

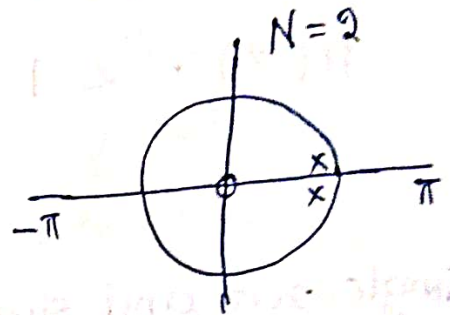
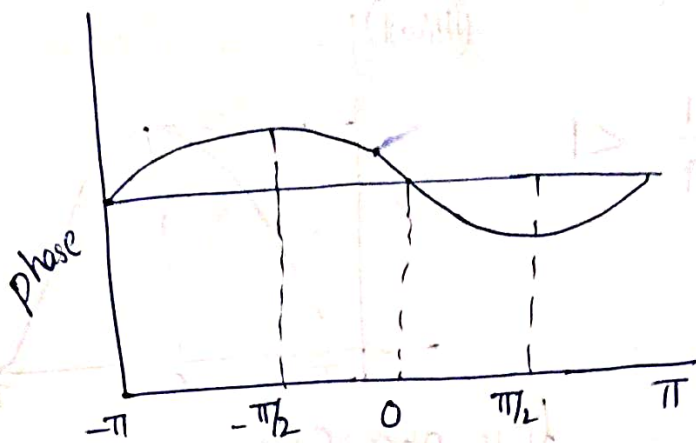
$\omega = \pi/2 = 0.7$

$\omega = \pi/2 = 0.81$



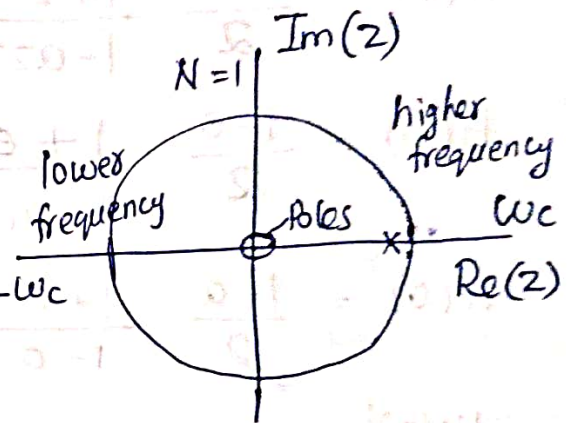
phase $\rightarrow$

$$\theta(\omega) = \omega - \tan^{-1} \frac{\sin \omega}{\cos \omega - 0.8}$$



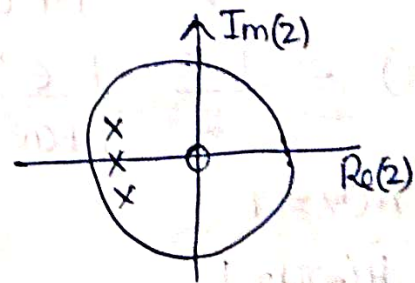
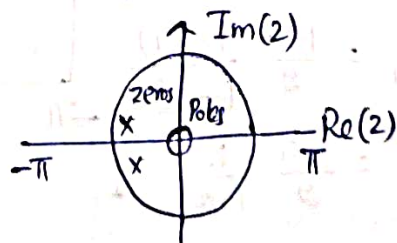
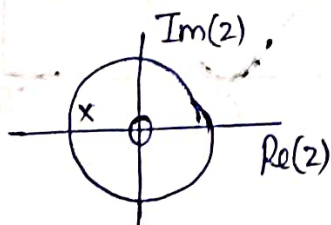
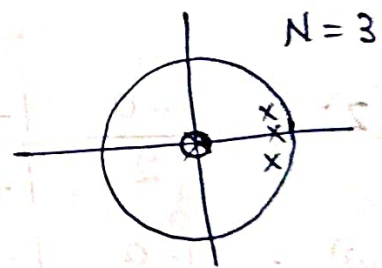
Simple Low pass filter

① In the design of the low pass filter, poles should be placed near to the unit circle corresponding to $\omega = 0$ (low frequency) and zeros should be placed near to the unit circle corresponding to $\omega = \pi$ (high frequency).



Simple high pass filter

Poles must be placed near to the unit circle $\omega = 0$ and zero should be placed near to the unit circle corresponding to the $\omega = -\pi$ (low ~~high~~ frequency).



Single pole example:-

$$H_1(z) = \frac{1-a}{1-az^{-1}}$$

$$H(\omega) = \frac{1-a}{1-ae^{-j\omega}}$$

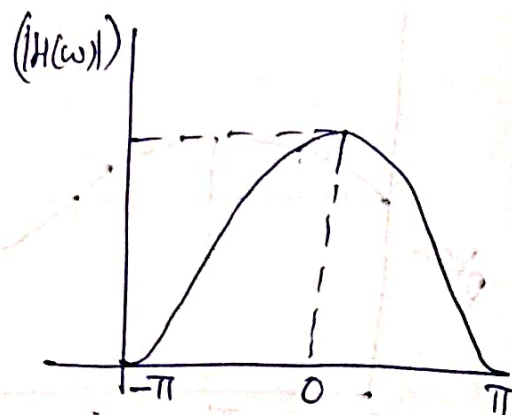
Let $a = 0.9$; $\omega = 0$

$$H(\omega) = \frac{1-0.9}{1-0.9e^0} = \frac{0.1}{0.1} = 1$$

$$H(0) = 1$$

$$H(\pi) = \frac{1-0.9}{1-0.9e^{j\pi}} = \frac{0.1}{0.9} < 1$$

$$H(-\pi) = < 1$$



low pass filter.

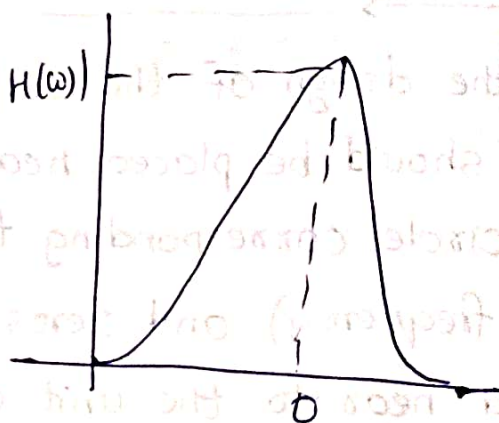
② Single zero and single pole

$$H(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}}$$

$$H(\omega) = \frac{1-a}{2} \cdot \frac{1+e^{-j\omega}}{1-ae^{-j\omega}}$$

$$H(0) = \frac{1-a}{2} \cdot \frac{1+1}{1-a} = 1$$

$$|H(\pi)| =$$



low pass filter

③

$$H(z) = \frac{1-a}{2} \cdot \frac{1-z^{-1}}{1+az^{-1}}$$

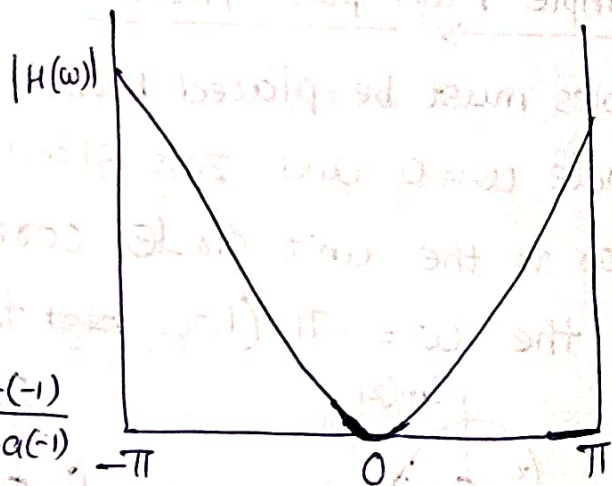
$$H(\omega) = \frac{1-a}{2} \cdot \frac{1-e^{j\omega}}{1+ae^{j\omega}}$$

$$H(0) = \frac{1-a}{2} \cdot \frac{1-1}{1+a} = 0$$

$$H(\pi) = \frac{1-a}{2} \cdot \frac{1-e^{j\pi}}{1+ae^{j\pi}} = \frac{1-a}{2} \cdot \frac{1-(-1)}{1+a(-1)} = \frac{1-a}{2} \cdot \frac{2}{1-a} = 1$$

$$H(\pi) = 1$$

$$H(-\pi) = 1$$



Q) A two-pole low pass filter has the system function

$$H(z) = \frac{b_0}{(1-pz^{-1})^2}$$

Determine the value of b_0 and frequency response $H(\omega)$

Satisfy the condition $H(0) = 1$ and $|H(\pi/4)|^2 = \frac{1}{2}$

$$\underline{A:} \quad H(z) = \frac{b_0}{(1-pz^{-1})^2}$$

$$H(\omega) = \frac{b_0}{(1-pe^{-j\omega})^2}$$

$$H(0) = \frac{b_0}{(1-p)^2} = 1$$

$$b_0 = (1-p)^2$$

$$H\left(\frac{\pi}{4}\right) = \frac{b_0}{(1-pe^{-j\pi/4})^2}$$

$$H\left(\frac{\pi}{4}\right) = \frac{(1-p)^2}{(1-pe^{-j\pi/4})^2}$$

$$H\left(\frac{\pi}{4}\right) = \frac{(1-p)^2}{(1-p\cos\frac{\pi}{4} + jp\sin\frac{\pi}{4})^2}$$

$$\Rightarrow \frac{(1-p)^4}{\left[\left(1-\frac{p}{\sqrt{2}}\right)^2 + \frac{p^2}{2}\right]^2} = \frac{1}{2}$$

$$b_0 = (1-p)^2$$

$$\Rightarrow \sqrt{2}(1-p^2) = 1 + p^2 - \sqrt{2}p$$

$$p = 0.32$$

$$= 0.46$$

$$H(z) = \frac{b_0}{(1-pz^{-1})^2} = \frac{0.46}{(1-0.32z^{-1})^2}$$

Q) Design a two-pole band pass filter that has the center of its pass band at $\omega = \pi/2$ zero in its frequency response characteristics at $\omega = 0$ and $\omega = \pi$ and its magnitude response is $\frac{1}{\sqrt{2}}$ at $\omega = \frac{4\pi}{4}$.

$$H(z) = G \frac{(z-1)(z+1)}{(z-j\delta)(z+j\delta)}$$

$$= G \frac{e^{j\pi} - 1}{e^{j\pi} + \delta^2}$$

$$H(z) = G \frac{z^2 - 1}{z^2 + \delta^2}$$

$$H\left(\frac{\pi}{2}\right) = G \frac{-2}{\delta^2 - 1} = 0$$

$$H(\omega) = G \frac{e^{2j\omega} - 1}{e^{2j\omega} + \delta^2}$$

$$G = \frac{\delta^2 - 1}{-2} \quad \left| H\left(\frac{4\pi}{4}\right) \right| = \frac{1}{\sqrt{2}}$$

$$H\left(\frac{\pi}{2}\right) = G \frac{e^{2j\pi/2} - 1}{e^{2j\pi/2} + \delta^2}$$

$$G = \frac{1 - \delta^2}{2}$$

$$\left| H\left(\frac{4\pi}{9}\right) \right|^2 = \frac{1}{\sqrt{2}}$$

$$G \cdot \frac{z^2 - 1}{z^2 + \gamma^2}$$

$$= \frac{(1 - \gamma^2)^2}{(2)^2} \cdot \frac{z^2 - 1}{z^2 + \gamma^2}$$

$$= \frac{(1 - \gamma^2)^2}{2^2} \cdot \frac{e^{2j\frac{4\pi}{9}} - 1}{e^{2j\frac{4\pi}{9}} + \gamma^2}$$

$$H\left(\frac{4\pi}{9}\right) = \frac{(1 - \gamma^2)^2}{4} \cdot \frac{2 - 2\cos\left(\frac{8\pi}{9}\right)}{1 + \gamma^4 + 2\gamma^2\cos\left(\frac{8\pi}{9}\right)} = \frac{1}{2}$$

$$1.94(1 - \gamma^2)^2 = 1 - 1.88\gamma^2 + \gamma^4$$

$$\gamma^2 = 0.7$$

$$H(z) = G \cdot \frac{z^2 - 1}{z^2 + \gamma^2}$$

$$= 0.15 \frac{z^2 - 1}{z^2 + 0.7}$$

$$\frac{1 - \gamma^2}{2} = G$$

$$H(z) = 0.15 \frac{1 - z^{-2}}{1 + 0.7z^{-2}}$$

$$\frac{1 - 0.7}{2} = G$$

$$H(\omega) = 0.15 \frac{1 - e^{-j2\omega}}{1 + 0.7e^{-j2\omega}}$$

$$= 0.15 \frac{1 - \cos 2j\omega + j \sin 2j\omega}{1 + 0.7(\cos 2j\omega - j \sin 2j\omega)}$$

$$\omega = 0$$

$$H(0) = 0.15$$

$$H(0) = 0$$

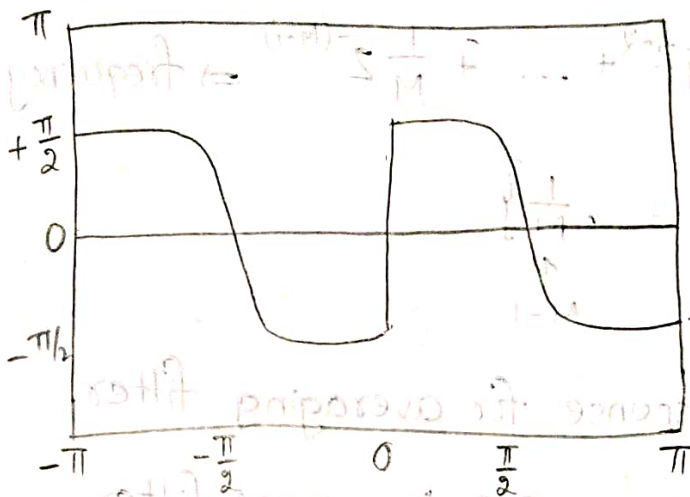
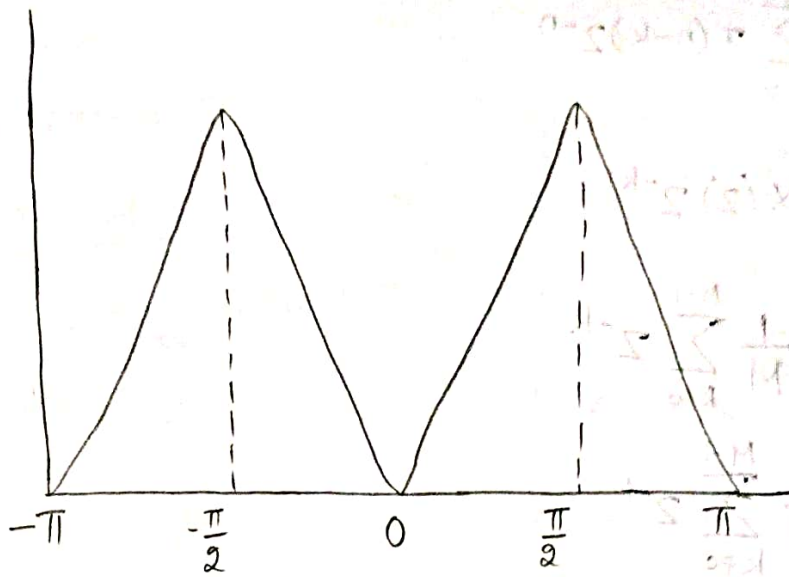
$$\omega = \frac{\pi}{2}$$

$$H\left(\frac{\pi}{2}\right) = 0.15 \frac{1 - (-1) + 0}{1 + (0.7) + 0}$$

$$H\left(\frac{\pi}{2}\right) = 0.15 \cdot \frac{2}{0.3}$$

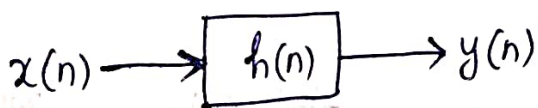
$$H\left(\frac{\pi}{2}\right) = 1$$

$$H\left(-\frac{\pi}{2}\right) = 1$$



Linear phase filter :-

Moving average filter (LPF) FIR filter :-



$$y(n) = x(n) \otimes h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$(or) = \sum_{k=-\infty}^{\infty} x(n-k) h(k)$$

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

$$Y(z) = \frac{1}{M} \sum_{n=0}^{\infty} \sum_{k=0}^{M-1} x(n-k) z^{-n}$$

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z) z^{-k}$$

$$Y(z) = X(z) \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}$$

$$H(z) = \frac{1}{M} z^0 + \frac{1}{M} z^{-1} + \frac{1}{M} z^{-2} + \dots + \frac{1}{M} z^{-(M-1)} \Rightarrow \text{frequency response.}$$

$$h(n) = \left\{ \underset{\substack{\uparrow \\ 0}}{\frac{1}{M}}, \underset{\substack{\uparrow \\ 1}}{\frac{1}{M}}, \underset{\substack{\uparrow \\ 2}}{\frac{1}{M}}, \dots, \underset{\substack{\uparrow \\ M-1}}{\frac{1}{M}} \right\}$$

$h(n) \rightarrow$ Impulse response for averaging filter.

① Moving average system is FIR low pass filter

② for FIR filter, the length of $h(n)$ is finite

③ FIR filters are always stable.

① Simple digital FIR filters

(i) LPF FIR filter

$$M=2 \quad H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^1 z^{-k}$$

$$= \frac{1}{2} [1 + z^{-1}]$$

$$H(z) = \frac{1+z^{-1}}{2}$$

$$H(\omega) = \frac{1}{2} (1 + e^{-j\omega})$$

$$H(\omega) = e^{-j\omega/2} \frac{(e^{j\omega/2} + e^{-j\omega/2})}{2}$$

$$H(\omega) = e^{-j\omega/2} \cos \frac{\omega}{2}$$

Magnitude response,

$$|H(\omega)| = |e^{-j\omega/2}| |\cos \frac{\omega}{2}|$$

$$|H(\omega)| = 1 \cdot \cos \frac{\omega}{2}$$

$$\omega = 0, |H(0)| = 1$$

$$\omega = \pi, |H(\pi)| = 0$$

cut off frequency

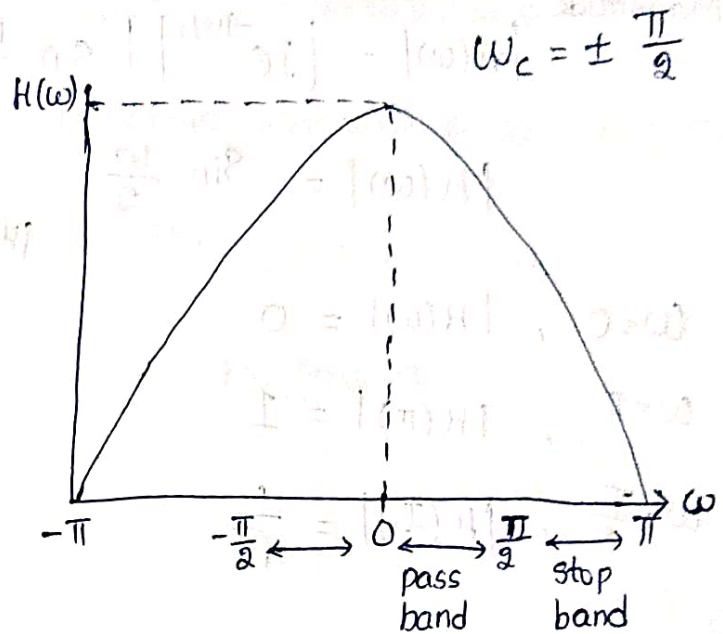
$$|H(\omega_c)| = \frac{1}{\sqrt{2}} \cdot |H(0)|$$

$$|H(\omega_c)| = \frac{1}{\sqrt{2}} \cdot 1$$

$$\cos \frac{\omega_c}{2} = \frac{1}{\sqrt{2}}$$

$$\frac{\omega_c}{2} = \cos^{-1} \frac{1}{\sqrt{2}}$$

$$\frac{\omega_c}{2} = \pm \frac{\pi}{4} \Rightarrow \omega_c = \pm \frac{\pi}{2}$$



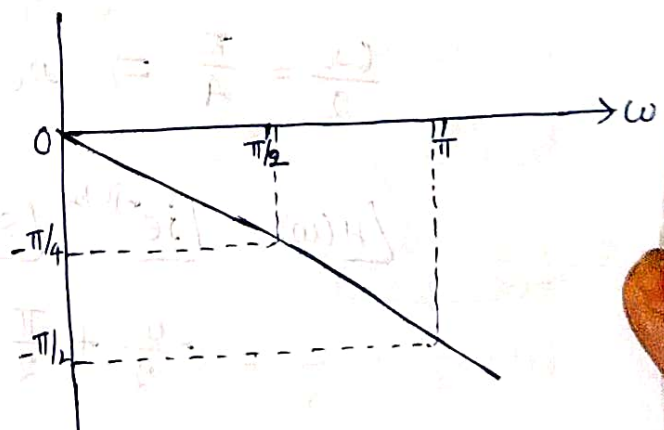
Phase response

$$H(\omega) = e^{-j\omega/2} \cos \frac{\omega}{2}$$

$$\angle H(\omega) = \angle e^{-j\omega/2} + \angle \cos \frac{\omega}{2}$$

$$= -\frac{\omega}{2} + \angle \cos \frac{\omega}{2}$$

$$\angle H(\omega) = -\frac{\omega}{2} + \tan^{-1} \left(\frac{0}{\cos \omega/2} \right)$$



Phase spectrum.

$$\omega = 0 \quad H(0) = 0$$

$$\omega = \pi \quad H(\pi) = -\pi/2$$

$$\omega = \pi/2 \quad H(\pi/2) = -\pi/4$$

High pass filter FIR digital filters :

$$H(z) = \frac{1}{2}(1 - z^{-1})$$

$$H(\omega) = \frac{1}{2}(1 - e^{-j\omega})$$

$$= \frac{e^{-j\omega/2}}{2} (e^{j\omega/2} - e^{-j\omega/2}) \times \frac{j}{j}$$

$$H(\omega) = j e^{-j\omega/2} \sin \frac{\omega}{2}$$

Magnitude

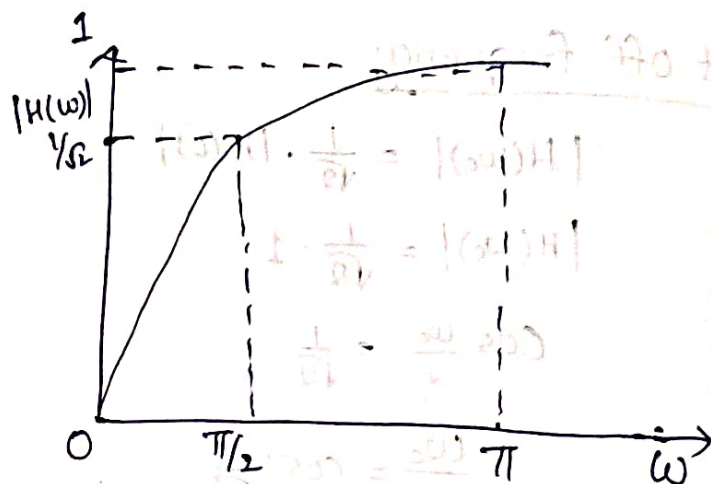
$$|H(\omega)| = |je^{-j\omega/2}| \left| \sin \frac{\omega}{2} \right|$$

$$|H(\omega)| = \sin \frac{\omega}{2}$$

$$\omega=0, |H(\omega)| = 0$$

$$\omega=\pi, |H(\pi)| = 1$$

$$\omega=\frac{\pi}{2}, |H(\frac{\pi}{2})| = \frac{1}{\sqrt{2}}$$



$$|H(\omega_c)| = \frac{1}{\sqrt{2}} |H(\pi)|$$

$$\sin \frac{\omega_c}{2} = \frac{1}{\sqrt{2}}$$

$$\frac{\omega_c}{2} = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

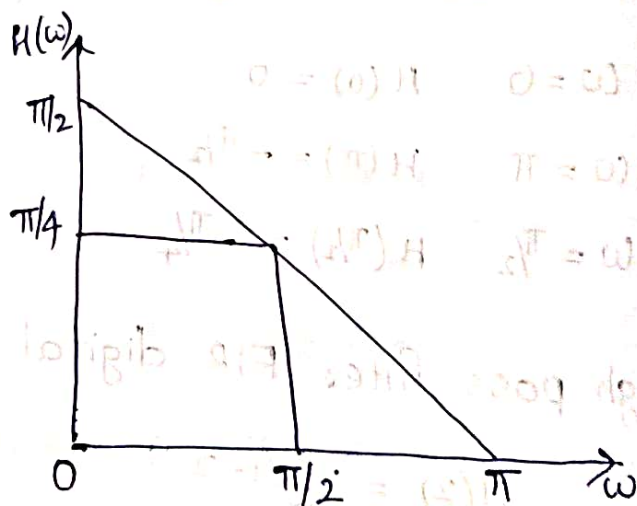
$$\frac{\omega_c}{2} = \frac{\pi}{4} \Rightarrow \omega_c = \frac{\pi}{2}$$

$$\begin{aligned} \angle H(\omega) &= \angle je^{-j\omega/2} \angle \sin \frac{\omega}{2} \\ &= -\frac{\omega}{2} + \frac{\pi}{2} \end{aligned}$$

$$\omega=0, \angle H(0) = \frac{\pi}{2}$$

$$\omega=\pi, \angle H(\pi) = 0$$

$$\begin{aligned} \omega=\frac{\pi}{2}, \angle H(\frac{\pi}{2}) &= -\pi/4 + \pi/2 \\ &= \pi/4 \end{aligned}$$



Moving average filter FIR filter

③ Band pass filter → Band pass filter is cascading of LPF & HPF

$$H(z) = \frac{1}{2} (1 + z^{-1}) \quad (\text{low pass filter})$$

$$H(z) = \frac{1}{2} (1 - z^{-1}) \quad (\text{HPF})$$

$$H(z) = \frac{1}{2} (1 + z^{-1}) \frac{1}{2} (1 - z^{-1})$$

$$H(z) = \frac{1}{2} (1 - z^{-2})$$

$$H(\omega) = \frac{1}{2} (1 - e^{-2j\omega})$$

$$= \frac{e^{-j\omega}}{2} (e^{j\omega} - e^{-j\omega}) \times \frac{j}{j}$$

$$H(\omega) = j e^{-j\omega} \sin \omega$$

Cut off frequency

$$|H(\omega_c)| = \frac{1}{\sqrt{2}} |H(\pi/2)|$$

$$\sin \omega_c = \frac{1}{\sqrt{2}}$$

$$\omega_c = \sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

Phase response

$$\angle H(\omega) = \angle j e^{-j\omega} \angle \sin \omega$$

$$\angle H(\omega) = -\omega + \frac{\pi}{2}$$

$$\omega = 0 \quad \angle H(0) = \frac{\pi}{2}$$

$$\omega = \pi \quad \angle H(\pi) = -\pi/2$$

$$\omega = \frac{\pi}{2} \quad \angle H(\frac{\pi}{2}) = -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

Magnitude

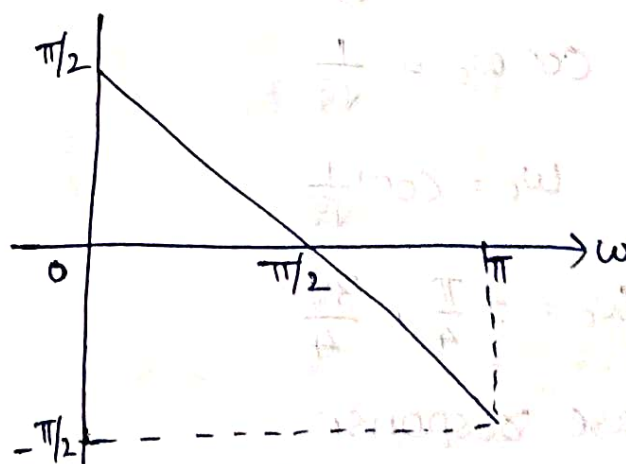
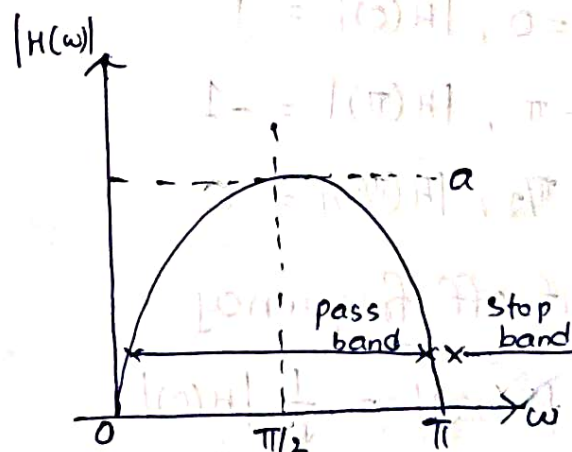
$$|H(\omega)| = |j e^{-j\omega}| |\sin \omega|$$

$$|H(\omega)| = 1 \cdot \sin \omega$$

$$\omega = 0, |H(0)| = 0$$

$$\omega = \pi, |H(\pi)| = 0$$

$$\omega = \frac{\pi}{2}, |H(\frac{\pi}{2})| = 1$$



$$H(z) = \frac{1}{2} (1 - z^2)$$

Band stop filter

$$H(z) = \frac{1}{2}(1 + z^{-2})$$

$$H(\omega) = \frac{1}{2}(1 + e^{-j\omega 2})$$

$$= \frac{e^{-j\omega}}{2}(e^{j\omega} + e^{-j\omega})$$

$$H(\omega) = e^{-j\omega} \cos \omega$$

Magnitude response

$$|H(\omega)| = |e^{-j\omega}| |\cos \omega|$$

$$|H(\omega)| = \cos \omega$$

$$\omega = 0, |H(0)| = 1$$

$$\omega = \pi, |H(\pi)| = -1$$

$$\omega = \pi/2, |H(\pi/2)| = 0$$

Cut off frequency

$$|H(\omega_c)| = \frac{1}{\sqrt{2}} |H(0)|$$

$$|H(\omega_c)| = \frac{1}{\sqrt{2}} \cdot 1$$

$$\cos \omega_c = \frac{1}{\sqrt{2}}$$

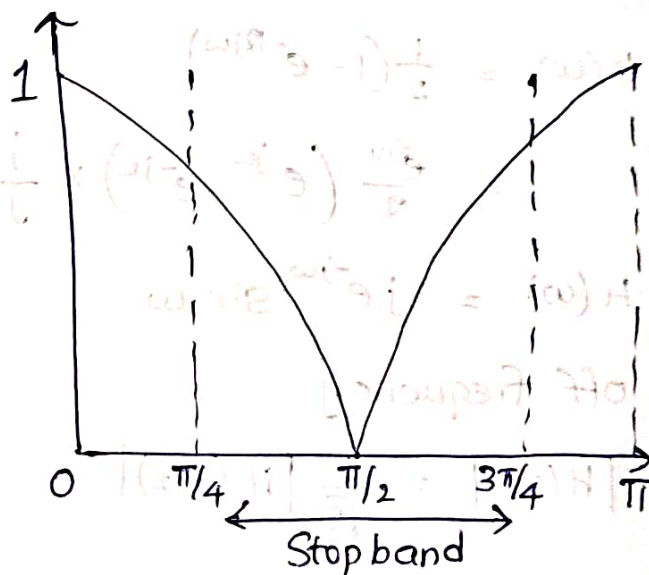
$$\omega_c = \cos^{-1} \frac{1}{\sqrt{2}}$$

$$\omega_c = \pm \frac{\pi}{4}, \frac{3\pi}{4}$$

Phase response

$$\angle H(\omega) = \angle e^{-j\omega} + \angle \cos \omega$$

$$= -\omega + \cos \omega$$



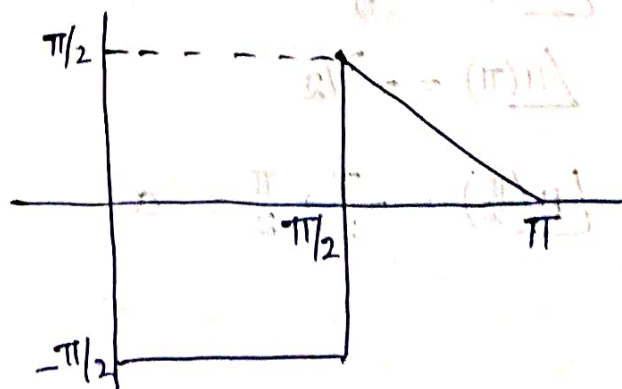
This filter also said to be as notch filter.

$$\omega = 0, \angle H(0) = 1$$

$$\omega = \pi, \angle H(\pi) = -\pi + \cos \pi$$

$$\angle H(\pi) = -\pi - \pi = -2\pi$$

$$\cos \omega = \begin{cases} 0 & 0 \text{ to } \pi/2 \\ 180 & \pi/2 \text{ to } \pi \end{cases}$$



$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}$$

To illustrate the comb filter, consider the moving average FIR filter

$$y(n) = \frac{1}{M+1} \sum_{k=0}^M x(n-k)$$

$$Y(z) = \frac{1}{M+1} \sum_{n=-\infty}^{\infty} \sum_{k=0}^M x(n-k) z^{-n}$$

$$= \frac{1}{M+1} \sum_{k=0}^M X(z) z^{-k}$$

$$= \frac{1}{M+1} X(z) \sum_{k=0}^M z^{-k}$$

$$H(z) = \frac{1}{M+1} \sum_{k=0}^M z^{-k}$$

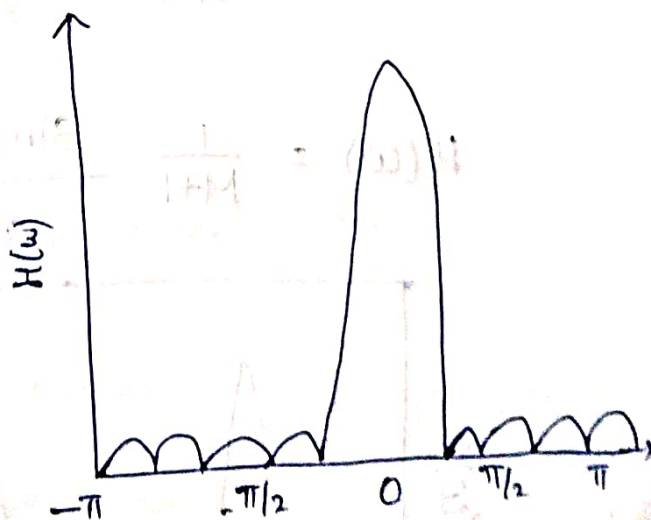
$$H(z) = \frac{1}{M+1} \frac{[1 - z^{-(M+1)}]}{[1 - z^{-1}]}$$

$$H(\omega) = \frac{e^{-j\omega M/2}}{M+1} \cdot \frac{\sin \omega \left(\frac{M+1}{2}\right)}{\sin \frac{\omega}{2}} \Rightarrow \text{frequency response}$$

Magnitude response if $|H(\omega)|$ for $M=10$

$$\frac{2\pi k}{M+1}$$

$$k=0, 1, 2, \dots$$



FIR filter system function

$$H(z) = \sum_{k=0}^M G(k) z^{-k}$$

replace the z by z^L where L is the +ve integer

$$H(z) = \sum_{k=0}^M h(k) z^{-kL} \rightarrow \textcircled{1}$$

If the frequency response of the original FIR filter is $H(\omega)$, the frequency response of eqn ① is

$$H_L(\omega) = \sum_{k=0}^M h(k) e^{-j\omega kL}$$

$$= H(L\omega)$$

$$H(z) = \frac{1}{M+1} \frac{[1 - z^{-(M+1)}]}{[1 - z^{-1}]}$$

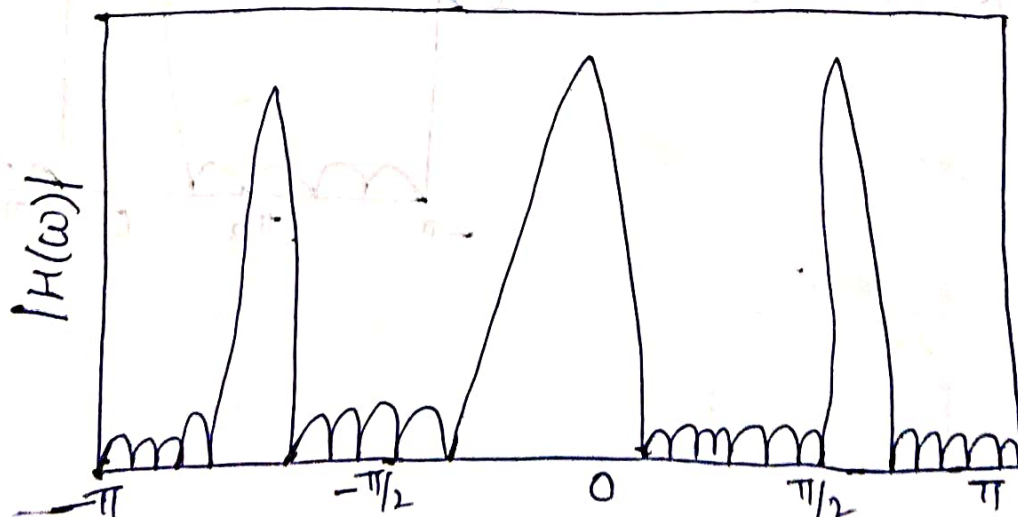
$$H(z) = \frac{1}{M+1} \frac{[1 - z^{-L(M+1)}]}{1 - z^{-L}}$$

frequency response

$$H_L(\omega) = \frac{1}{M+1} \frac{\sin(\omega L(M+1)/2)}{\sin(\frac{\omega L}{2})}$$

$$H(\omega) = \frac{1}{M+1} \frac{\sin \omega (\frac{M+1}{2})}{\sin(\frac{\omega}{2})}$$

$$L=3, M=10$$



All pass filter

$$|H(\omega)| = 1 \quad 0 \leq \omega < \pi$$

$$\boxed{H(z) = z^{-k}} \rightarrow \text{All pass filter}$$

This is also said to be trivial-all pass filter. that has linear response characteristics. All pass filter must be describe in terms of system function

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \dots + a_1z^{-N+1} + z^{-N}}{1 + a_1z^{-1} + a_2z^{-2} + \dots + a_Nz^{-N}}$$

$$H(z) = \frac{\sum_{k=0}^N a_k z^{-N+k}}{\sum_{k=0}^N a_k z^{-k}} \quad a_0 = 1$$

The coefficients $|a_k|$ are real values

$$A(z) = \sum_{k=0}^N a_k z^{-k} \quad a_0 = 1$$

$$\text{All pass filter } \boxed{H(z) = z^{-N} \frac{A(z^{-1})}{A(z)}} \text{ --- Polynomial function}$$

$$|H(\omega)|^2 = H(z)H(z^{-1}) \big|_{z=e^{j\omega}}$$

Minimum phase, Maximum phase and Mixed phase

System function,

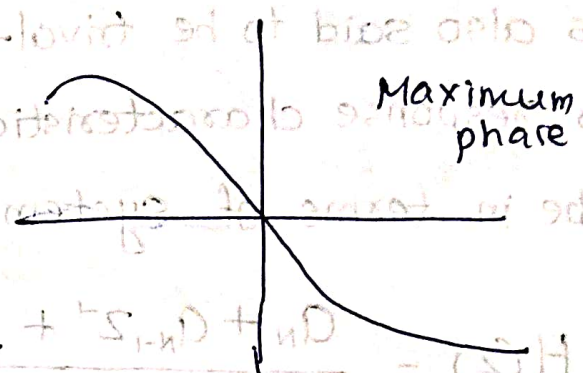
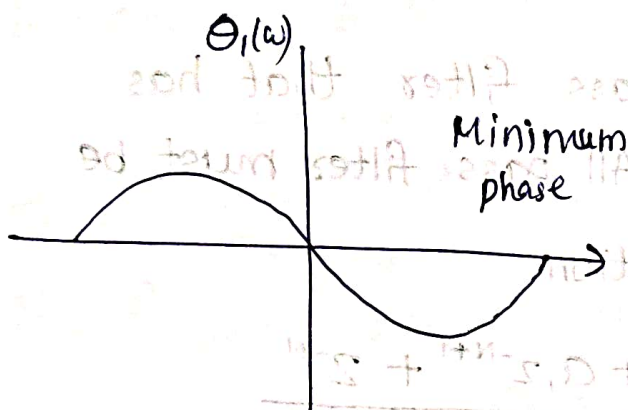
$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2})$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1)$$

$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos \omega}$$

$$\theta_1(\omega) = -\omega + \tan^{-1} \frac{\sin \omega}{\frac{1}{2} + \cos \omega}$$

$$\theta_2(\omega) = -\omega + \tan^{-1} \frac{\sin \omega}{2 + \cos \omega}$$



Q) $H_1(z) = 6 + z^{-1} - z^{-2}$ $(2 + \frac{1}{2})(2 - \frac{1}{3})$ $\Rightarrow \frac{1}{2}, \frac{1}{3}$ minimum phase

$H_2(z) = 1 + z^{-1} - 6z^{-2}$ $(2+2)(2-3)$ maximum phase $\Rightarrow 2, 3$

$z = \frac{1}{2}, 3 \rightarrow$ mixed phase

$$A(z) = \sum_{k=0}^n a_k z^{-k}$$

Polynomial
function

$$H(z) = \frac{(1-z^{-1})^n A(z)}{(z^{-1})^n A(z)}$$