UNIT - 3

Discrete fourier Transform [DFT]

$x(n) \xleftrightarrow{DTFT} X(\omega) \xrightarrow{DFT} X(k)$

$x(n) \longrightarrow X(jf)$

$x(n) \longrightarrow X(j\omega)$

$x(n) \longrightarrow X(f)$

$x(n) = \delta(n)$

$\delta(n) \longrightarrow 1$

$\delta(n-n_0) \longrightarrow e^{-j\left(\frac{2\pi}{N}\right)kn_0}$

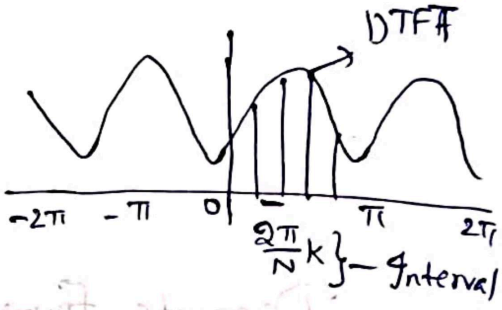
$\delta(n+n_0) \longrightarrow e^{j\left(\frac{2\pi}{N}\right)kn_0}$

$$FT[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$FT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$\omega = \frac{2\pi}{N}K$



① $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$ find DFT

$$F.T[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$= \sum_{n=0} x(2n)\, e^{-j\left(\frac{2\pi}{N}\right)2nk} + \sum_{n=0} x(2n+1)\, e^{-j\left(\frac{2\pi}{N}\right)(2n+1)k}$$

(1) (0)

② find 4-point DFT $x(n) = \{1, -1, 2, -2\}$ (k = 0 1 2 3)

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\left(\frac{2\pi}{4}\right)kn}$$

$\left\{ e^{-j\theta} = \cos\theta - j\sin\theta \right\}$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0}$$

$$= x(0)e^0 + x(1)e^0 + x(2)e^0 + x(3)e^0$$

$$= 1 + (-1) + 2 + (-2) = 0$$

$(x(1) = 1 + 2j)$

$\Rightarrow 1 + j + 2 - 2j$

$\Rightarrow 3 - j$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\left(\frac{2\pi}{4}\right)n}$$

$$= x(0)e^0 + x(1)e^{-j\frac{2\pi}{4}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$

$$= 1.(1) + (-1)e^{-j\frac{\pi}{2}} + (2)e^{-j\pi} + (-2)e^{-j\frac{3\pi}{2}}$$

$$= 1 + (1)\left(\cos\tfrac{\pi}{2} - j\sin\tfrac{\pi}{2}\right) + 2\left(\cos\pi - j\sin\pi\right) + (-2)\left(\cos\tfrac{3\pi}{2} - j\sin\tfrac{3\pi}{2}\right)$$

$$= 1 + (-1)(0 - j) + 2(-1 - 0) + (-2)(0 - j(-1))$$

Inverse Discrete fourier Transform :

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(k) e^{j\frac{2\pi}{N}Kn}$$

Properties of DFT :

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}$$

① Periodicity :

$$x(n+N) = x(n) \text{ for all values of } n$$

$$X(k+N) = X(k)$$

$$DFT[x(n)] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}$$

$$= \sum_{n=0}^{N-1} x(n)\left\{e^{-j\frac{2\pi}{N}(k+N)n}\right\}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn}\, e^{-j\frac{2\pi}{N}Nn} \qquad \because e^{-j2\pi n} = 1$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}Kn} = X(K)$$

② Linearity property :-

$$DFT[ax_1(n)] = aX_1(k)$$

$$DFT[bx_2(n)] = bX_2(k)$$

$$DFT[ax_1(n)+bx_2(n)] = \sum_{n=0}^{N-1}[ax_1(n)+bx_2(n)]\, e^{-j\frac{2\pi}{N}Kn}$$

$$= a\sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}Kn} + b\sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}Kn}$$

$$= aX_1(k) + bX_2(k)$$

③ For even sequence,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi Kn}{N}\right)$$

For odd sequence,

$$X(k) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi Kn}{N}\right)$$

④ Circular frequency Shift

$$DFT\left[x(n) e^{j\frac{2\pi \ell n}{N}}\right] = X(k-\ell)$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{+j\frac{2\pi \ell}{N}n}\, e^{-j\frac{2\pi K}{N}n}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi n}{N}(\ell-K)}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(k-\ell)}$$

$$= X(k-\ell)$$

⑤ Time Reversal of the sequence

$$[x(-n), \text{mod } N] = x(N-n)$$

$$DFT[x(-n), \text{Mod } N] = X(N-k)$$

Proof: $DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) e^{-j\frac{2\pi nK}{N}}$

Changing n to m & Let $m = N-n$ (or) $n = N-m$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}(N-m)K}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}NK}\, e^{j\frac{2\pi}{N}mK}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi K}\, e^{-j\frac{2\pi}{N}mK}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}(N-n)K}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}mK}\, e^{+j2\pi m}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m\left(\frac{K}{N}-1\right)}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi m}{N}(K-N)}$$

⑥ Complex conjugate property

$$= \left(\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \right)^*$$

$$= \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N}Kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N}(K-N)n}$$

$$= \sum_{m=0}^{N-1} x^*(n) e^{-j\frac{2\pi}{N}(N-K)n}$$

$$= x^*(N-K)$$

⑤ Time Reversal of the Sequence

$$DTFT[x(N-n)] = X(N-K)$$

Let $N-n = m$; $n = N-m$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn}$$

By changing n to m, $N-n=m$,

$n = N-m$

$$= \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}K(N-m)}$$

$$= \sum_{m=0}^{N-1} x(m) \left\{ \underbrace{e^{-j\frac{2\pi}{N}KN}}_{=1} e^{+j\frac{2\pi}{N}Km} \right\}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j\frac{2\pi}{N}Km} e^{-j2\pi m}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j2\pi m\left(\frac{K}{N}-1\right)}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi m\left(\frac{K}{N}+1\right)}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}m(-K+N)}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}(N-K)m}$$

$$= X(N-K)$$

⑥ Complex conjugate property

$$DFT[x(n)] = X(K)$$

$$= \left\{ \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn} \right\}^*$$

$$DFT[x^*(n)] = X^*(n-K)$$

$$= \left\{ \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(n-k)n} \right\}^*$$

$$= X^*(n-K)$$

⑦ DFT of delayed sequence

(Circular time shift of a sequence)

$$DFT[x(n-n_0)] = X(k) e^{-j\frac{2\pi n_0 k}{N}}$$

$$IDFT[X(k)] = \frac{1}{N}\sum_{n=0}^{N-1} X(k) e^{+j\frac{2\pi}{N}Kn} = x(n)$$

$$x(n-n_0) = \frac{1}{N}\sum_{n=0}^{N-1} X(k) e^{j\frac{2\pi}{N}K(n-n_0)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \left[X(k) e^{-j\frac{2\pi}{N}Kn_0} \right] e^{j\frac{2\pi}{N}Kn}$$

$$= IDFT\left[X(k) e^{-j\frac{2\pi}{N}Kn_0} \right]$$

$$DFT[x(n-n_0)] = X(k) e^{-j\frac{2\pi}{N}Kn_0}$$

$$DFT[x(n)] = X(k)$$

$$x(n) = IDFT[X(k)]$$

⑧ DFT of Real value of sequence

Real part,

$$DFT[x(n)] = X(k) = X_R + jX_I$$

$$X_R(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi}{N} nk\right) \quad \text{for } 0 \le K \le N-1$$

$$X_I(k) = -\sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi}{N} nk\right) \quad \text{for } 0 \le K \le N-1$$

① $X(k)$ has a complex conjugate symmetry, $X(k) = X^*(N-k)$

② Real component of even function, i.e., $X_R(k) = X_R(N-k)$

③ Imaginary component of odd function, $X_I(k) = -X_I(N-k)$

④ Magnitude function is even function, i.e., $|X(k)| = |X(N-k)|$

⑤ Phase function of odd function, $\underline{/X(k)} = -\underline{/X(N-k)}$

⑥ If $x(n) = x(-n)$ [even sequence] then $X(k)$ is purely real

⑦ If $x(n) = -x(-n)$ [odd sequence] then $x(k)$ is purely imaginary.

⑨ Multiplication of two sequences

$$DFT[x_1(n)] = X_1(k) \; ; \; DFT[x_2(n)] = X_2(k)$$

$$DFT[x_1(n) \cdot x_2(n)] = \frac{1}{N} \left[X_1(k) \textcircled{N} X_2(k)\right]$$

(from IDFT)

By Duality → Multiplication

⑩ Circular convolution of two sequences

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}Kn} \qquad K = 0, 1, 2, \ldots N-1$$

$$X_2(k) = \sum_{k=0}^{N-1} x_2(k) e^{-j\frac{2\pi}{N}Kk} \qquad K = 0, 1, 2, \ldots N-1$$

$$\therefore X_1(k) \cdot X_2(k) = X_3(k)$$

$$IDFT[X_3(k)] = x_3(m)$$

$$x_3(m) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(k) e^{j\frac{2\pi}{N}Km}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{j\frac{2\pi}{N}mk}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}kn} \sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi}{N}kl}\, e^{j\frac{2\pi}{N}mk}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-l-n)}$$

Let $m-n-l = pN$, where p is an integer

$$e^{j\frac{2\pi}{N}k(m-n-l)} = e^{j\frac{2\pi}{N}kpN} = e^{j2\pi kp} = (e^{j2\pi p})^k$$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n-l)} = \sum_{k=0}^{N-1} (\underbrace{e^{j2\pi p}}_{1})^k = \sum_{k=0}^{N-1} 1^k = \sum_{k=0}^{N-1} 1 = N$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)$$

If $x_2(l)$ is periodic sequence,

$$m-n-l = pN$$

$$l = m-n-pN$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{n=0}^{N-1} x_2(m-n) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n)$$

(11) Parseval's theorem :-

$$DFT[x_1(n)] = X_1(k)$$

$$DFT[x_2(n)] = X_2(k)$$

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$$

(12) Circular correlation :-

$$DFT[x(n)] = X(k)$$

$$DFT[y(n)] = Y(k)$$

$$DFT[\gamma_{xy}(l)] = DFT\left[\sum_{n=0}^{N-1} x(n)\, y^*((n-l), \text{mod } N)\right]$$

$$= X(k)\, Y^*(k)$$

Property	Time domain	frequency domain
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Time reversal	$(x(-n), \text{mod } N) = x(N-n)$	$X(N-k)$
Circular time shift	$x(n-l), \text{mod } N$	$X(k)\, e^{-j\frac{2\pi lk}{N}}$
Circular frequency shift	$x(n)\, e^{+j\frac{2\pi ln}{N}}$	$X((k-l), \text{mod } N)$
Circular convolution	$x_1(n) \oplus x_2(n)$	$X_1(k) \cdot X_2(k)$
Multiplication of two signals	$x_1(n) \cdot x_2(n)$	$\frac{1}{N}[X_1(k) \oplus X_2(k)]$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular correlation	$x_1(n) \oplus y^*(-n)$	$X(k)\, Y^*(k)$
Parseval's Theorem	$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ; \quad k = 0, 1, 2, \ldots N-1$$

$$x(n) = IDFT[X(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \quad ; \quad k = 0, 1, 2, \ldots N-1$$

Matrix

$$X = W_n x$$

$$\underset{N\times 1}{\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Q) Find the DFT of $x(n) = \{1, 0, 0, 1\}$ $N = 4$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{matrix} \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Method-1: $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_4 = e^{-j\frac{2\pi}{4}}$

$$W_4^0 = e^{-j\frac{2\pi}{4}\times 0} = 1$$

$$W_4^1 = e^{-j\frac{2\pi}{4}\times 1} = \cos\left(\frac{2\pi}{4}\right) - j\sin\left(\frac{2\pi}{4}\right) = -j$$

$$W_4^2 = e^{-j\frac{2\pi}{4}\times 2} = \cos(\pi) - j\sin(\pi) = 1$$

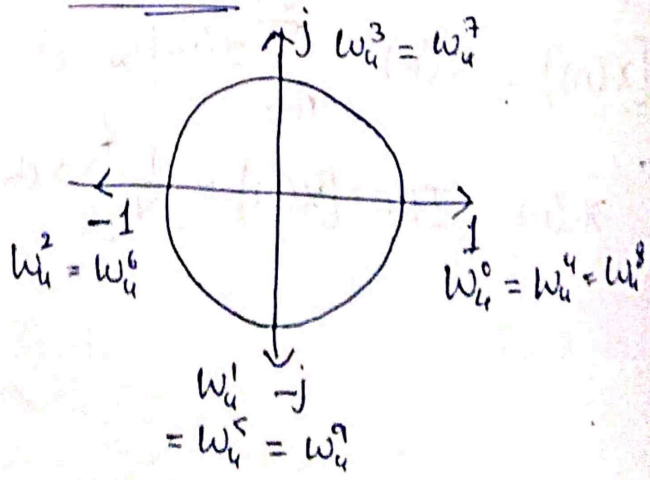
$$W_4^3 = e^{-j\frac{2\pi}{4}\times 3} = \cos\left(\frac{6\pi}{4}\right) - j\sin\left(\frac{6\pi}{4}\right) = +j$$

$$W_4^4 = e^{-j\frac{2\pi}{4}\times 4} = 1$$

$$W_4^6 = e^{-j\frac{2\pi}{4}\times 6} = -1$$

$$W_4^9 = e^{-j\frac{2\pi}{4}\times 9} = -j$$

Method-2: unit circle



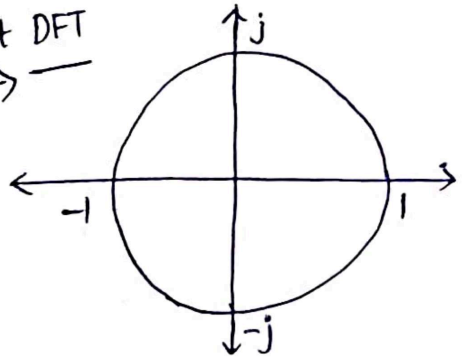
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$DFT[x(n)] = X(k) = \{2, 1+j, 0, 1-j\}$$

Q) Find the DFT of $x(n) = \{1, 0, -1, 2\}$; $N = 4$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

4 point DFT →



$W_4^0 = 1$ $\quad W_4^4 = 1$

$W_4^1 = -j$ $\quad W_4^6 = -1$

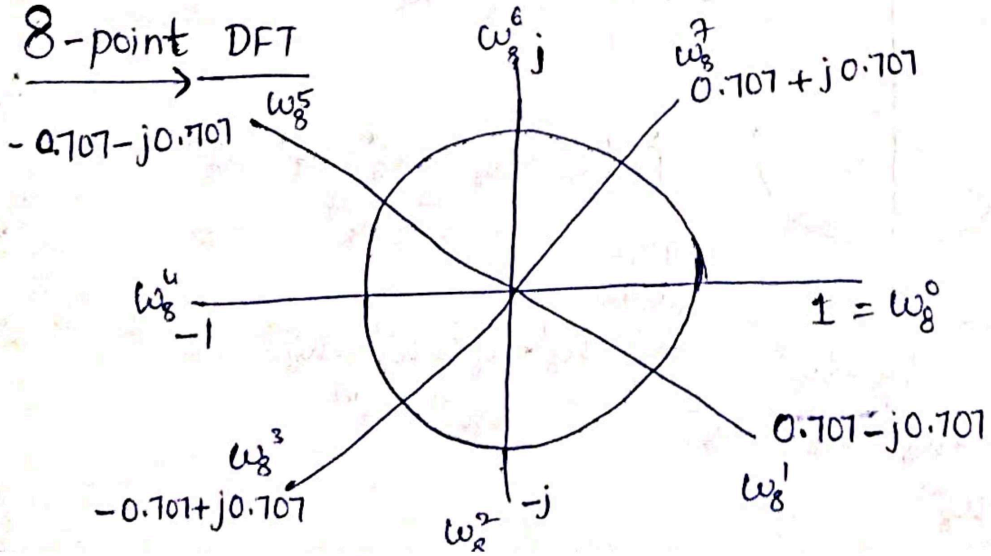
$W_4^2 = -1$ $\quad W_4^9 = -j$

$W_4^3 = j$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2+2j \\ -2 \\ 2-2j \end{bmatrix}$$

$$X(k) = \{2, 2+2j, -2, 2-2j\}$$

8-point DFT →



Q) find the 8 point DFT of $x(n) = \{1,1,0,0,0,0,0,0\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Method :-1 $\quad W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}}$

$W_8^0 = e^{-j\frac{\pi}{4}\times 0} = 1$

$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right) = 0.707 - j0.707$

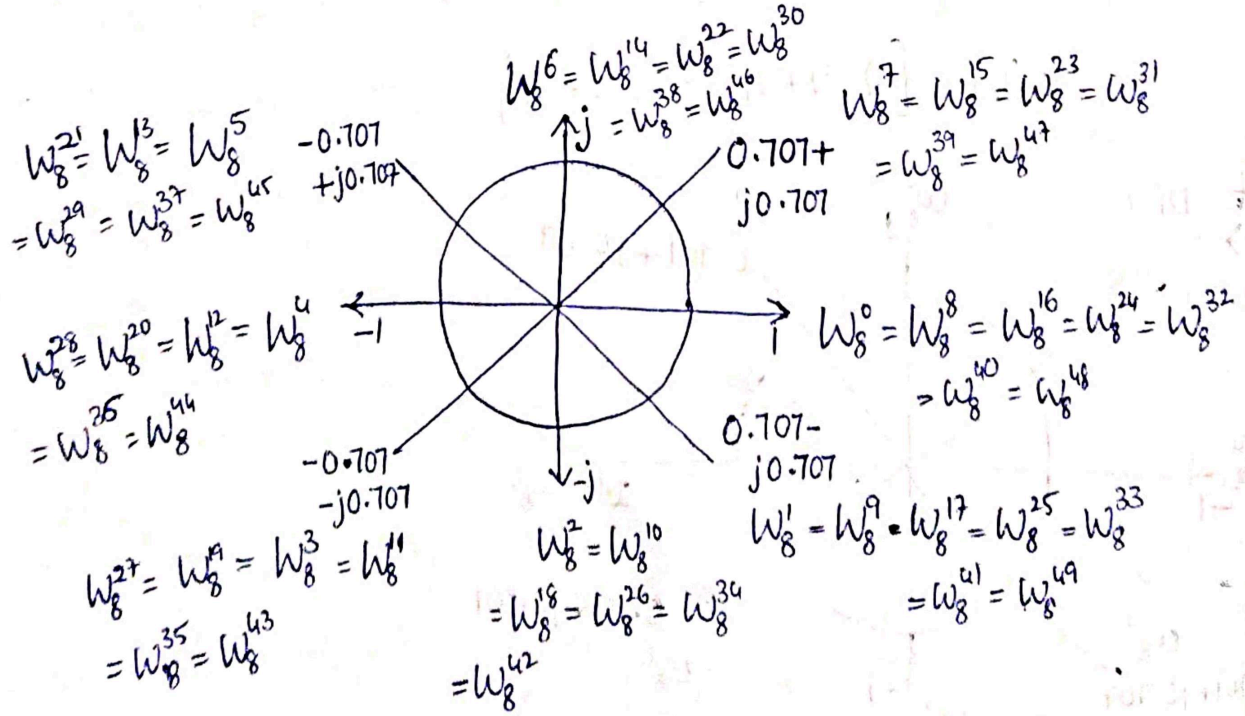
$W_8^2 = e^{-j\frac{2\pi}{4}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$

$W_8^3 = e^{-j\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right) = -0.707 - j0.707$

$W_8^4 = e^{-j\pi} = \cos(\pi) - j\sin(\pi) = -1$

$W_8^5 = e^{-j\frac{5\pi}{4}} = \cos\left(\frac{5\pi}{4}\right) - j\sin\left(\frac{5\pi}{4}\right) = -0.707 + j0.707$

$W_8^6 = e^{-j\frac{6\pi}{4}} = \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right) = +j$



		0	1	2	3	4	5	6	7	
X(0)	0	1	1	1	1	1	1	1	1	1
X(1)	1	1	0.707 − j0.707	−j	−0.707 − j0.707	−1	−0.707 + j0.707	j	0.707 + j0.707	1
X(2)	2	1	−j	−1	j	1	−j	−1	j	0
X(3)	3	1	−0.707 − j0.707	j	0.707 − j0.707	−1	0.707 + j0.707	−j	−0.707 + j0.707	0
X(4)	= 4	1	−1	1	−1	1	−1	1	−1	0
X(5)	5	1	−0.707 + j0.707	−j	0.707 + j0.707	−1	0.707 − j0.707	j	−0.707 − j0.707	0
X(6)	6	1	j	−1	−j	1	j	−1	−j	0
X(7)	7	1	0.707 + j0.707	j	−0.707 + j0.707	−1	−0.707 − j0.707	−j	0.707 − j0.707	0

$$DFT[x(n)] = X(k) = \{2,\ 1.707 - j0.707,\ 1 - j,\ 0.293 - j0.707,\ 0,\ 0.293 + j0.707,\ 1 + j,\ 1.707 + j0.707\}$$

① The IDFT from the Matrix form

$$X = W_N x$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$x = W_N^{-1} X$$

$$x = \frac{1}{N}\left[W_N^{-1}\right]^T X$$

$$x = \frac{1}{N}\left[W_N^{*}\right]^T X$$

② Using the DFT to find IDFT

$$DFT[x(n)] = X(k)$$

$$x(n) = IDFT[X(k)] = \frac{1}{N}\left[DFT[x^*(k)]\right]^*$$

Q) find IDFT of $X(k) = \{4, -2j, 0, 2j\}$ using DFT

$$x^*(k) = \{4, 2j, 0, -2j\}$$

$$DFT[x^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2j \\ 0 \\ -2j \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}[4, 8, 4, 0]^*$$

$$= \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

Q) $X(K) = \{4, 2, 0, 4\}$ using DFT

$$X^*(k) = \{4, 2, 0, 4\}$$

$$DFT[x^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+2j \\ -2 \\ 4-2j \end{bmatrix}$$

$$= \frac{1}{4}[10, 4+2j, -2, 4-2j]^*$$

$$= \frac{1}{4}[10, 4-2j, -2, 4+2j] = [2.5, 1-0.5j, -0.5, 1+0.5j]$$

Q) Let $X(k)$ be a 12-point DFT of a real sequence $x(n)$

The first 7 samples are given

$X(0) = 8$

$X(1) = -1+2j$

$X(2) = 2+3j$

$X(3) = 1-4j$

$X(4) = -2+2j$

$X(5) = 3+j$

$X(6) = -1-3j$

Determine the remaining samples of $X(k)$

A: $X(k) = X^*(N-k)$; $N = 12$

$X(7) = X^*(12-7) = X^*(5) = 3-j$

$X(8) = X^*(12-8) = X^*(4) = -2-2j$

$X(9) = X^*(12-9) = X^*(3) = 1+4j$

$X(10) = X^*(12-10) = X^*(2) = 2-3j$

$X(11) = X^*(12-11) = X^*(1) = -1-2j$

Q) The DFT of a real signal $\{1, A, -1, B, 0, -2j, C, -1+j\}$ find the values of A, B, C. $N = 8$

A: $X(1) = X^*(8-1)$ $\quad X(6) = X^*(8-6)$

$A = X^*(7) = -1-j$ $\quad C = X^*(2) = -1$

$X(3) = X^*(8-3)$

$B = X^*(5) = 2j$

Q) DFT of a real signal $X(k) = \{1, A, -1, B, -7, -2j, C, -1+j\}$ what is the signal energy. (8.25)

A: $X(1) = X^*(8-1)$ $\quad X(3) = X^*(8-3)$ $\quad X(6) = X^*(8-6)$

$A = X^*(7) = -1-j$ $\quad B = X^*(5) = +2j$ $\quad C = X^*(2) = -1$

$X(k) = \{1, -1-j, -1, 2j, -7, -2j, -1, -1+j\}$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(k)|^2$$

$$\Rightarrow \frac{1}{8} \sum_{K=0}^{7} |X(k)|^2$$

$$= \frac{1}{8}\left[|1|^2 + |(-1-j)|^2 + |(-1)|^2 + |2j|^2 + |-7|^2 + |-2j|^2 + |(1)|^2 + |(-1+j)|^2\right]$$

$$= \frac{1}{8}[$$

Q) Let $x(n)$ be a real valued sequence of length N, and $X(k)$ be its DFT with real and imaginary parts $X_R(k)$ and $X_I(k)$ respectively. Show that if $x(n)$ is Real $X_R(k) = X_R(N-K)$ and $X_I(k) = -X_I(N-k)$ for $K = 0, 1, 2, \ldots, N-1$

A: $$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}Kn}$$

$$DFT[x^*(n)] = X(k) = \sum_{n=0}^{N-1} x^*(n) e^{j\frac{2\pi}{N}Kn} \; e^{-j\frac{2\pi}{N}Nn}$$

$$X_R(k) + jX_I(k) = \sum_{n=0}^{N-1} x^*(n) \; e^{-j\frac{2\pi}{N}(N-K)n}$$

$$\therefore X^*(N-K)$$

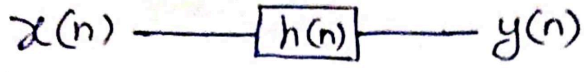
$$X_R(k) + j X_I(k) = [X_R(N-K) + jX_I(N-k)]^*$$

$$= X_R^*(N-K) - j \cdot X_I(N-K)$$

$$X_R(k) = X_R^*(N-K)$$

$$X_I(k) = -X_I^*(N-K)$$

Linear Convolution using DFT



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

If the length of the two sequences are N_1 and N_2, the total outcome, $N_1 + N_2 - 1$

① find the linear convolution of the sequence $x(n)$ and $h(n)$ using DFT $x(n) = \{1, 2\}$ $h(n) = \{2, 1\}$

$N_1 = 2$ $N_2 = 2$ → aliasing effect

$N_1 + N_2 - 1 = 2 + 2 - 1 = 3$

$x(n) = \{1, 2, 0\}$ 0 → to avoid aliasing effect

$h(n) = \{2, 1, 0\}$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$= X(k) = \sum_{n=0}^{2} x(n) e^{-j\frac{2\pi}{3}kn}$$

$$X(k) = x(0)e^{0} + x(1)e^{-\frac{2\pi}{3}jk} + x(2)e^{-j\frac{4\pi}{3}k}$$

$$X(k) = 1e^{0} + 2e^{-j\frac{2\pi}{3}k} + 0\, e^{-j\frac{4\pi}{3}k}$$

When $k=0$ $X(0) = 1 + 2 = 3$

When $k=1$ $X(1) = 1 + 2e^{-j\frac{2\pi}{3}} = 1 + 2\left[\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right]$

$= 1 + 2(-0.5 - j0.8666) = -j1.732$

When $k=2$ $X(2) = 1e^{0} + 2e^{-j\frac{4\pi}{3}} + 0$

$= +j1.732$

$$X(k) = \{3, -j1.732, +j1.732\}$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{2} h(n) e^{-j\frac{2\pi}{3}kn}$$

$$H(k) = h(0)e^{0} + h(1)e^{-j\frac{2\pi}{3}k} + h(2)e^{-j\frac{4\pi}{3}k}$$

$$H(k) = 2e^{0} + 1e^{-j\frac{2\pi}{3}k} + 0e^{-j}$$

When, $H(0) = 2+1 = 3$

$k=0$;

$k=1$; $H(1) = 1.5 - j0.866$

$k=2$; $H(2) = 1.5 + j0.866$

$$Y(k) = X(k)H(k)$$

$$Y(k) = \{3, -j1.732, +j1.732\}\{3, 1.5-j0.866, 1.5+j0.866\}$$

$$Y(k) = \{9, -1.5-j2.598, -1.5+j2.598\}$$

$$x(n) = IDFT[X(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}$$

$$y(n) = IDFT[Y(k)] = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{3}\sum_{k=0}^{2} Y(k)\, e^{+j\frac{2\pi}{3}kn}$$

$$= \frac{1}{3}\left[Y(0)e^{0} + Y(1)e^{+j\frac{2\pi n}{3}} + Y(2)e^{+j\frac{4\pi}{3}n}\right]$$

$$y(n) = \frac{1}{3}\left[9e^{0} + (-1.5-j2.598)e^{j\frac{2\pi}{3}n} + (1.5+j2.598)e^{j\frac{4\pi}{3}n}\right]$$

When, $n=0$; $y(0) = \frac{1}{3}[9 - 1.5 - j2.598 - 1.5 + j2.598] = \frac{1}{3}(6) = 2$

When, $n=1$; $y(1) = \frac{1}{3}[9$

$y(1) = 5$

When, $n=2$; $y(2) = 2$

$x(n) = \{1, 2\}$; $h(n) = \{2, 1\}$

Apply the tabular method,

$h(n)$		$x(n)$: 1	2
2		2	4
1		1	2

$y(n) = \{2, 5, 2\}$

② find the linear convolution of $x(n)$ and $h(n)$ using DFT

$x(n) = \{1, 0, 2\}$ $\quad h(n) = \{1, 1\}$

$\Rightarrow N_1 + N_2 - 1 = 3 + 2 - 1 = 4$

To avoid aliasing effect,

$x(n) = \{1, 0, 2, 0\}$ $\quad h(n) = \{1, 1, 0, 0\}$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{for } k = 0, 1, \ldots N-1$$

$$= \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn}$$

$$= x(0)e^{0} + x(1)e^{-j\frac{2\pi}{4}k} + x(2)e^{-\frac{4\pi}{4}kn} + x(3)e^{-j\frac{6\pi}{4}k}$$

$$= 1e^{0} + 0 \cdot e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + 0 \cdot e^{-j\frac{3\pi}{2}k}$$

$$X(k) = 1e^{0} + 2e^{-j\pi k}$$

When, $k = 0$; $X(0) = 1 + 2 = 3$

$k = 1$; $X(1) = 1 + 2e^{-\pi j} = 1 + 2(\cos(\pi) - j\sin(\pi)) = 1 + 2(-1) + j(0)$

$= 1 - 2 = -1$

$k = 2$; $X(2) = 1 + 2e^{-2\pi j} = 1 + 2(\cos(2\pi) - j\sin(2\pi)) = 1 + 2(1) - j(0)$

$= 3$

$k = 3$; $X(3) = 1 + 2e^{-3\pi j} = 1 + 2[\cos(3\pi) - j\sin(3\pi)] = 1 + 2[-1) - j(0)]$

$= 1 + (-2) = -1$

$X(k) = \{3, -1, 3, -1\}$

$$IFT[h(n)] = H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{3} h(n)\, e^{-j\frac{2\pi}{4}kn}$$

$$H(k) = h(0)e^{-0} + h(1)e^{-j\frac{2\pi}{4}k} + h(2)e^{-j\frac{4\pi}{4}k} + h(3)e^{-j\frac{6\pi}{4}k}$$

$$= 1e^{0} + 1e^{-j\frac{\pi}{2}k}$$

en, $k = 0$; $H(0) = 1 + 1 = 2$

$k = 1$; $H(1) = 1 + e^{-j\frac{\pi}{2}} = 1 + \left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] = 1 + 0 - j = 1 - j$

$k=2;\ H(2) = 1+e^{-j\pi} = 1+[\cos\pi - j\sin\pi] = 1+(-1)-0 = 0$

$k=3;\ H(3) = 1+e^{-j\frac{3\pi}{2}} = 1+\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] = 1+0+j = 1+j$

$H(k) = \{2, 1-j, 0, 1+j\}$

$Y(k) = X(k)H(k)$

$= \{3,-1,3,-1\}\{2,1-j,0,1+j\}$

$Y(k) = \{6, -1+j, 0, -1-j\}$

$\{e^{j\theta} = \cos\theta + j\sin\theta\}$

$\{e^{-j\theta} = \cos\theta - j\sin\theta\}$

$$y(n) = IDFT\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j\frac{2\pi}{N}kn} \quad ; n=0,1,2,\ldots,N-1$$

$$= \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\frac{2\pi}{4}kn}$$

$$= \frac{1}{4}\left[Y(0)e^{0} + Y(1)e^{j\frac{2\pi}{4}n} + Y(2)e^{j\frac{4\pi}{4}n} + Y(3)e^{j\frac{6\pi}{4}n}\right]$$

$$= \frac{1}{4}\left[6e^{0} + (-1+j)e^{j\frac{2\pi}{4}n} + 0\cdot e^{j\frac{4\pi}{4}n} + (-1-j)e^{j\frac{3\pi}{2}n}\right]$$

$$= \frac{1}{4}\left[6 + (-1+j)e^{j\frac{\pi}{2}n} + (-1-j)e^{j\frac{3\pi}{2}n}\right]$$

when, $n=0$; $y(0) = \frac{1}{4}[6+(-1+j)+(-1-j)]$

$= \frac{1}{4}[6-1+j-1-j] = \frac{1}{4}[4] = 1$

when, $n=1$; $y(1) = \frac{1}{4}[6+(-1-j)e^{j\frac{\pi}{2}} + (-1-j)e^{j\frac{3\pi}{2}}]$

$= \frac{1}{4}[6+(-1-j)(0-j)+(-1+j)(0+j)]$

$= \frac{1}{4}[6+(j-1)+(-j+1)] = \frac{1}{4}[4] = 1$

when; $n=2$; $y(2) = \frac{1}{4}[6+(-1+j)e^{j\pi} + (-1-j)e^{+j3\pi}]$

$= \frac{1}{4}[6+(-1+j)(-1)+(-1-j)(-1)]$

$= \frac{1}{4}[6+(1-j)+(1+j)] = \frac{1}{4}(8) = 2$

when, $n=3$; $y(3) = \frac{1}{4}\left[6 + (-1+j)e^{j\frac{3\pi}{2}} + (-1-j)e^{j\frac{9\pi}{2}n}\right]$

$= \frac{1}{4}\left[6 + (-1+j)(0-j) + (-1-j)(0+j)\right]$

$= \frac{1}{4}\left[6 + j + 1 - j + 1\right] = \frac{1}{4}[8] = 2.$

$\therefore \boxed{y(n) = \{1, 1, 2, 2\}}$

Apply Tabular Method :-

$x(n) = \{1, 0, 2\}$

$h(n) = \{1, 1\}.$

h(n) \ x(n)	1	0	2
1	1	0	2
1	1	0	2

$\boxed{y(n) = \{1, 1, 2, 2\}}$

Circular convolution using DFT and IDFT

$DFT[x_1(n)] = X_1(k)$

$DFT[x_2(n)] = X_2(k)$

$X_3(k) = X_1(k)\, X_2(k)$

Let $x_1(n)$ and $x_2(n)$ are the given sequence. Let $x_3(n)$ be sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. Then

Step ① :- Take N-point DFT $x_1(n)$ and $x_2(n)$

Let $X_1(k) = DFT[x_1(n)]$

$X_2(k) = DFT[x_2(n)]$

Step ② :- Determine the product of $X_1(k)$ and $X_2(k)$

Let the product be $X_3(k) = X_1(k)\, X_2(k)$

Step ③ :- By convolution of DFT

$DFT[x_1(n) * x_2(n)] = X_1(k)\, X_2(k)$

then, $x_1(n) * x_2(n) = x_3(n)$ and

$$X_3(k) = X_1(k)\,X_2(k)$$

$$x_3(n) = \text{IDFT}[X_3(k)] = \text{IDFT}[X_1(k)\,X_2(k)]$$

① find the circular convolution of the following sequence using DFT and IDFT

$$x_1(n) = \{1, 2, 1, 2\}$$

$$x_2(n) = \{4, 3, 2, 1\}$$

A: $\text{DFT}[x_1(n)] = X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}kn}$

$$= X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j\frac{2\pi}{4}kn}$$

$$\Rightarrow X_1(k) = x_1(0)e^{0} + x_1(1)e^{-j\frac{2\pi}{4}k} + x_1(2)e^{-j\pi k} + x_1(3)e^{-j\frac{3\pi}{2}k}$$

$$X_1(k) = 1 + 2e^{-j\frac{\pi}{2}k} + 1e^{-j\pi k} + 2e^{-j\frac{3\pi}{2}k}$$

$k=0$; $X_1(0) = 1+2+1+2 = 6$

$k=1$; $X_1(1) = 1 + 2\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + (\cos\pi - j\sin\pi) + 2\left(\cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)\right)$

$= 1 + 2(0-j) + (-1) + 2(0+j) = 1 - 2j - 1 + 2j = 0$

$k=2$; $X_1(2) = 1 + 2e^{-j\frac{\pi}{2}} + 1e^{-j2\pi} + 2e^{-j3\pi}$

$= 1 + 2[(-1)] + 1[1-0] + 2[-1+0] = 1 - 2 + 1 - 2 = 2 - 4 = -2$

$k=3$; $X_1(3) = 1 + 2e^{-j\frac{3\pi}{2}} + 1e^{-j3\pi} + 2e^{-j\frac{9\pi}{2}}$

$= 1 + 2[0+j] + 1[-1+0] + 2[0-j] = 1 + 2j - 1 - 2j = 0$

$$X_1(k) = \{6, 0, -2, 0\}$$

$$DFT[x_2(n)] = X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j\frac{2\pi}{N}Kn} = \sum_{n=0}^{3} x_2(n)\, e^{-j\frac{2\pi}{4}Kn}$$

$$X_2(K) = x_2(0)e^{0} + x_2(1)e^{-j\frac{2\pi}{4}K} + x_2(2)e^{-j\pi K} + x_2(3)e^{-j\frac{6\pi}{4}K}$$

$$X_2(K) = 4 + 3e^{-j\frac{\pi}{2}K} + 2e^{-j\pi K} + 1e^{-j\frac{3\pi}{2}K}$$

$K=0$; $X_2(0) = 4+3+2+1 = 10$

$K=1$; $X_2(1) = 4 + 3\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 2\left[\cos\pi - j\sin\pi\right] + \left[\cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right)\right]$

$= 4 + 3(0-j) + 2(-1) + (0+j) = 4 - 3j - 2 + j = 2 - 2j$

$K=2$; $X_2(2) = 4 + 3e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi}$

$= 4 + 3(-1) + 2(1-0) + (-1) = 4 - 3 + 2 - 1 = 2$

$K=3$; $X_2(3) = 4 + 3e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}}$

$= 4 + 3(0+j) + 2(-1) + (0-j) = 4 + 3j - 2 - j = 2 + 2j$

$$X_2(K) = \{10,\ 2-2j,\ 2,\ 2+2j\}$$

$$X_3(K) = X_1(K) \cdot X_2(K) = \{6, 0, -2, 0\}\{10,\ 2-2j,\ 2,\ 2+2j\}$$

$$X_3(k) = \{60, 0, -4, 0\}$$

$$x_3(n) = IDFT[X_3(K)] = IDFT[X_1(K)\cdot X_2(K)] = \frac{1}{N}\sum_{K=0}^{N-1} X_3(K)\, e^{j\frac{2\pi}{N}Kn} \quad ; n = 0, 1, \ldots N-1$$

$$= \frac{1}{4}\sum_{K=0}^{3} X_3(K)\, e^{j\frac{2\pi}{4}Kn}$$

$$= \frac{1}{4}\left[X_3(0)e^{0} + X_3(1)e^{j\frac{2\pi}{4}n} + X_3(2)e^{j\frac{2\pi}{4}2n} + X_3(3)e^{j\frac{2\pi}{4}3n}\right]$$

$$x_3(n) = \frac{1}{4}\left[60e^{0} + 0 + (-4)e^{j\pi n} + 0\right] = \frac{1}{4}\left[60 - 4e^{j\pi n}\right] = 15 - e^{+j\pi n}$$

$n=0$; $x_3(0) = 15 - e^{0} = 15 - 1 = 14$

$n=1$; $x_3(1) = 15 - \left[\cos\pi - j\sin\pi\right] = 15 - \left[-1 - 0\right] = 15 + 1 = 16$

$n=2$; $x_3(2) = 15 - \left[\cos 2\pi - j\sin 2\pi\right] = 15 - \left[1 - 0\right] = 15 - 1 = 14$

$n=3$; $x_3(3) = 15 - \left[\cos 3\pi - j\sin 3\pi\right] = 15 - \left[-1 - 0\right] = 15 + 1 = 16$

$$\therefore \boxed{x_3(n) = \{14, 16, 14, 16\}}$$

Convolution of long sequences (Sectioned Method)

① Perform the linear convolution of the following sequence using Overlap-add method

$L_s \leftarrow x(n) = \{1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1\}$

$M \leftarrow h(n) = \{1, 2\}$

Step-①: $L_s = 12$, $M = 2$

$L = N - M + 1$; $N = 2^M = 2^2 = 4$

$L = 4 - 2 + 1$

$L = 3$

Step-②: $x_1(n) = \{\underbrace{1, 2, -1}_{L=3}, \underbrace{0}_{M-1 \text{ padding}}\}$

$x_2(n) = \{2, 3, -2, 0\}$

$x_3(n) = \{-3, -1, 1, 0\}$

$x_4(n) = \{1, 2, -1, 0\}$

Step-③: $h(n) = \{1, 2\}$

$h(n) = \{1, 2, \underbrace{0, 0}_{(L-1) \text{ padding zeros}}\}$

Step-④: $[y(n)]_{N\times 1} = x(n)_{N\times N} * h(n)_{N\times 1}$

$$y_1(n) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -2 \end{bmatrix} \qquad y_1(n) = \{1, 4, 3, -2\}$$

$$y_2(n) = \begin{bmatrix} 2 & 0 & -2 & 3 \\ 3 & 2 & 0 & -2 \\ -2 & 3 & 2 & 0 \\ 0 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \\ -4 \end{bmatrix} \qquad y_2(n) = \{2, 7, 4, -4\}$$

$$y_3(n) = \begin{bmatrix} -3 & 0 & 1 & -1 \\ -1 & -3 & 0 & 1 \\ 1 & -1 & -3 & 0 \\ 0 & 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ -1 \\ 2 \end{bmatrix} \qquad y_3(n) = \{-3, -7, -1, 2\}$$

$$y_4(n) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \\ -2 \end{bmatrix} \qquad y_4(n) = \{1, 4, 3, -2\}$$

Step-⑤:

1 4 3 -2
2 7 4 -4
-3 -7 -1 2
1 4 3 -2

$y(n) = 1\ 4\ 3\ 0\ 7\ 4\ -7\ -7\ -1\ 3\ 4\ 3\ -2$

$y(n) = \{1, 4, 3, 0, 7, 4, -7, -7, -1, 3, 4, 3, -2\}$

② find the o/p linear convolution of

$x(n) = \{1, 2, 3, 3, 2, 1, -1, -2, -3, +5, 6, -1, 2, 0, 2, 1\}$

$h(n) = \{3, 2, 1, 1\}$ assume the block length 7.

Ans $N = 7 \quad L_s = 16 \quad M = 4$.

$L = N - M + 1$

$= 7 - 4 + 1$

$L = 4$

$x_1(n) = \{1, 2, 3, 3, 0, 0, 0\}$

$x_2(n) = \{2, 1, -1, -2, 0, 0, 0\}$

$x_3(n) = \{-3, +5, 6, -1, 0, 0, 0\}$

$x_4(n) = \{2, 0, 2, 1, 0, 0, 0\}$

$h(n) = \{3, 2, 1, 1, 0, 0, 0\}$

$$y_1(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 3 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 3 \\ 3 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad y_1(n) = \{3, 8, 14, 18, 11, 6, 3\}$$

$$y_{d_2}(n) = \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & -1 & 1 \\ 1 & 2 & 0 & 0 & 0 & -2 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 & -2 \\ -2 & -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_2(n) = \{6, 7, 1, -5, -4, -3, -2\}$

$$y_3(n) = \begin{bmatrix} -3 & 0 & 0 & 0 & -1 & 6 & 5 \\ 5 & -3 & 0 & 0 & 0 & -1 & 6 \\ 6 & 5 & -3 & 0 & 0 & 0 & -1 \\ -1 & 6 & 5 & -3 & 0 & 0 & 0 \\ 0 & -1 & 6 & 5 & -3 & 0 & 0 \\ 0 & 0 & -1 & 6 & 5 & -3 & 0 \\ 0 & 0 & 0 & -1 & 6 & 5 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_3(n) = \{-9, 9, 25, 11, 9, 5, -1\}$

$$y_4(n) = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_4(n) = \{6, 4, 8, 9, 4, 3, 1\}$

$y(n) =$ 3 8 14 18 11 6 3

6 7 1 -5 -4 -3 -2

-9 9 25 11 9 5 -1

6 4 8 9 4 3 1

---

3 8 14 18 17 13 4 -5 -13 6 23 11 15 9 7 9 4 3 1

$y(n) = \{3, 8, 14, 18, 17, 13, 4, -5, -13, 6, 23, 11, 15, 9, 7, 9, 4, 3, 1\}$

② Perform the linear convolution using overlap-add method

$x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\}$

$h(n) = \{1, -1\}$

A: $L_s = 8$, $M = 2$. $N = 2^M = 2^2 = 4$

$L = N - M + 1$

$= 4 - 2 + 1$

$= 3$

Assume, $L = 2$

$x_1(n) = \{1, -2, 0\}$

$x_2(n) = \{2, -1, 0\}$

$x_3(n) = \{3, -4, 0\}$

$x_4(n) = \{4, -3, 0\}$

$h(n) = \{1, -1, 0\}$

$$y_1(n) = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$y_1(n) = \{1, -3, 2\}$

$$y_2(n) = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$y_2(n) = \{2, -3, 1\}$

$$y_3(n) = \begin{bmatrix} 3 & 0 & -4 \\ -4 & 3 & 0 \\ 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 4 \end{bmatrix}$$

$y_3(n) = \{3, -7, 4\}$

$$y_4(n) = \begin{bmatrix} 4 & 0 & -3 \\ -3 & 4 & 0 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}$$

$y_4(n) = \{4, -7, 3\}$

$y(n) =$ 1 −3 2

2 −3 1

3 −7 4

4 −7 3

---

1 −3 4 −3 4 −7 8 −7 3

$y(n) = \{1, -3, 4, -3, 4, -7, 8, -7, 3\}$

(4)

$x(n) = \{1, -2, 2, -1, 3, -4, 4, -3\}$ $h(n) = \{1, -1\}$

$L_s = 8$, $M = 2$

$L = N - M + 1$ $N = 2^M = \underline{2^2} = 4$

$= 4 - 2 + 1$ $\underline{L = 2}$

$= 3$

$x_1(n) = \{1, -2, 0\}$ $h(n) = \{1, -1, 0\}$

$x_2(n) = \{2, -1, 0\}$ $y_1(n) = \{1, -3, 2\}$

$y_2(n) = \{2, -3, 1\}$

$y_3(n) = \{3, -7, 4\}$

$y_4(n) = \{4, -7, 3\}$

$y(n) =$ 1 −3 2

2 −3 1

3 −7 4

4 −7 3

1 −3 4 −3 4 −7 8 −7 3

$y(n) = \{1, -3, 4, -3, 4, -7, 8, -7, 3\}$

Overlap Save Method

(1) Perform the linear convolution of the following sequences using overlap-save method.

$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$

$h(n) = \{1, 2, -1\}$

A: Step ①: $L_s = 8$, $M = 3$

$L = N - M + 1$ $N = 2^M = 2^3 = 8$

$= 8 - 3 + 1$

$L = 6$

Step-②:- $x_1(n) = \{0, 0, 1, -2, 3, 2, -3, 4\}$

(M-1) zeros

$x_2(n) = \{-3, 4, 3, -4, 0, 0, 0, 0\}$

(M-1) $x_1(n)$

Step-③:- $h(n) = \{1, 2, -1, 0, 0, 0, 0, 0\}$

(L-1) zeros

Step-④:-

$y_1(n)_{N\times 1} = x_1(n)_{N\times N} * h(n)_{N\times 1}$

$$y_1(n) = \begin{bmatrix} 0 & 4 & -3 & 2 & 3 & -2 & 1 \\ 0 & 0 & 4 & -3 & 2 & 3 & -2 \\ 1 & 0 & 0 & 4 & -3 & 2 & 3 \\ -2 & 1 & 0 & 0 & 4 & -3 & 2 \\ 3 & -2 & 1 & 0 & 0 & 4 & -3 \\ 2 & 3 & -2 & 1 & 0 & 0 & 4 \\ -3 & 2 & 3 & -2 & 1 & 0 & 0 \\ 4 & -3 & 2 & 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 1 \\ 0 \\ -2 \\ 10 \\ -2 \\ -4 \end{bmatrix}$$

$y_2(n) = x_2(n) * h(n)$

$$y_2(n) = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 & -4 & 3 & 4 \\ 4 & -3 & 0 & 0 & 0 & 0 & -4 & 3 \\ 3 & 4 & -3 & 0 & 0 & 0 & 0 & -4 \\ -4 & 3 & 4 & -3 & 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 4 & -3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 3 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 3 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 & 3 & 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 14 \\ -2 \\ -11 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Step-⑤:- $y_1(n) = \{11, -4, 1, 0, -2, 10, -2, -4\}$

$y_2(n) = \{-3, -2, 14, -2, -11, 4, 0, 0\}$

$y(n) = L_S + M - 1$

$= 8 + 3 - 1$

$= 10$

Discard (M-1) from $y_1(n)$ and $y_2(n)$

$y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4, 0, 0\}$ ← Since we are padded with zeros in $x_2(n)$ so discarded

$y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$

② find the linear convolution using ... using the following sequences.

$x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$

$h(n) = \{1, 1, 1\}$ $\quad L_S = 10$

A: $L_s = N - M + 1 \quad N = 2^M = 2^3 = 8$

$= 8 - 3 + 1 \quad M = 3$

$L_s = 6$

$x_1(n) = \{0, 0, 3, -1, 0, 1, 3, 2\}$

$x_2(n) = \{3, 2, 0, 1, 2, 1, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$$y_1(n) = \begin{bmatrix} 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \\ 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

$y_1(n) = \{5, 2, 3, 2, 2, 0, 4, 6\}$

$$y_2(n) = \begin{bmatrix} 3 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 2 & 3 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 3 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 & 3 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_2(n) = \{\ ,\ , 5, 3, 3, 4, 3, 1\}$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

$y(n) = L_s + M - 1$

$= 10 + 3 - 1$

$= 12$

③ Perform the linear convolution using overlap-add method and overlap save method.

$x(n) = \{1, -2, 3, 2, -3, 4, 3, -4\}$

$h(n) = \{1, 2, -1\}$

A Overlap add method :-

$L_s = 8$; $M = 3$

$L = N - M + 1$ $\quad N = 2^3 = 8$

$= 8 - 3 + 1$

$L = 6$ $\quad$ ($L = 3$)

$x_1(n) = \{1, -2, 3, 0, 0\}$

$x_2(n) = \{2, -3, 4, 0, 0\}$

$x_3(n) = \{3, -4, 0, 0, 0\}$

$h(n) = \{1, 2, -1, 0, 0\}$

$$y_1(n) = \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ -2 & 1 & 0 & 0 & 3 \\ 3 & -2 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 8 \\ -3 \end{bmatrix}$$

$$y_2(n) = \begin{bmatrix} 2 & 0 & 0 & 4 & -3 \\ -3 & 2 & 0 & 0 & 4 \\ 4 & -3 & 2 & 0 & 0 \\ 0 & 4 & -3 & 2 & 0 \\ 0 & 0 & 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ +1 \\ -4 \\ 11 \\ -4 \end{bmatrix}$$

$$y_3(n) = \begin{bmatrix} 3 & 0 & 0 & 0 & -4 \\ -4 & 3 & 0 & 0 & 0 \\ 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -11 \\ 4 \\ 0 \end{bmatrix}$$

$y(n) =$ 1 0 −2 8 −3

2 +1 −4 11 −4

3 2 −11 4 0

1 0 −2 10 −2 −4 14 −2 −11 4 0

$y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4, 0\}$

$y(n) = \{1, 0, -2, 10, -2, -4, 14, -2, -11, 4\}$

$y(n) = 4+4$
$= 8+2-1$
$= 10$

Q) Let $X(k)$ be the 14-point DFT of the length 14 of real sequence $x(n)$. The first 8 samples of $X(k)$ is given

$X(0) = 8$

$X(1) = -2+j2$

$X(2) = 1+j3$

$X(3) = 3-j4$

$X(4) = -1-j5$

$X(5) = 3+j6$

$X(6) = -4-j3$

$X(7) = 5$

Q) The DFT of real sequence is given by

$\{2, 1-j3, A, 25j, 0, B, 3-j5, C\}$ find A, B, C

Q) find the linear convolution using overlap-add method and overlap save method

① $x(n) = \{4, 4, 3, 3, 2, 2, 1, 1\}$ $h(n) = \{1, 1\}$

② $x(n) = \{1, 3, 2, 4, 4, 2, 3, 1\}$ $h(n) = \{1, -1, 1\}$

Fast Fourier Transform (FFT)

$x(n) \longrightarrow x(k)$ fixed

- Decimation in time (DIT) 'n'
- Decimation in frequency (DIF) 'k'

Inverse Fast Fourier Transform (IFFT)

- DIT
- DIF

$N = 4$ → odd (1, 3), even (0, 2)

$N = 8$ → odd (1, 5, 3, 7), even (0, 4, 2, 6)

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n)$$

$$X(k) = IDFT[x(n)]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}} \quad K = 0, 1, \ldots, N-1$$

$n = 0, 1, \ldots, N-1$

Properties (Real and Imaginary)

$$X(k) = X_R(k) + X_I(k)$$

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + x_I(n)] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} [x_R(n) + x_I(n)] \left[\cos\left(\frac{2\pi nK}{N}\right) - j \sin\left(\frac{2\pi nK}{N}\right)\right]$$

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi nK}{N}\right) + x_I(n) \sin\left(\frac{2\pi nK}{N}\right)$$

$$X_I(k) = \sum_{n=0}^{N-1} \left[x_R(n) \sin\left(\frac{2\pi nK}{N}\right) - x_I(n) \cos\left(\frac{2\pi nk}{N}\right)\right]$$

Symmetry property

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

Periodicity property

$$W_N^{k+n} = W_N^k$$

Decimation in time (Radix-2) FFT

Given sequence,

$$x(n) = [x(0), x(1), x(2), x(3), \ldots, x(N-2), x(N-1)]$$

Odd sequence $f_1(n) = \{x(1), x(3), x(5), \ldots x(N-1)\}$

even sequence $f_2(n) = \{x(0), x(2), x(4), \ldots x(N-N)\}$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_n^{kn} \quad K = 0, 1, \ldots, N-1 \longrightarrow (1)$$

$$X(K) = \sum_{n=0}^{N-2} x(n) W_N^{kn} + \sum_{n=1}^{N-1} x(n) W_N^{kn} \quad K = 0, 1, 2, \ldots, N-1 \longrightarrow (2)$$

($x(n) W_N^{kn}$: even seq; $x(n) W_N^{kn}$: odd seq)

Replace $n = 2n$ in even sequence

and $n = 2n+1$ in odd sequence

$$X(K) = \sum_{n=0}^{N/2-1} x(2n) W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{k(2n+1)} \longrightarrow (3)$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} f_1 W_N^{k2n} + \sum_{n=0}^{\frac{N}{2}-1} f_2 W_N^{k(2n+1)}$$

$$W_N^{2nk} = (W_N)^{2nk} = \left(e^{-j\frac{2\pi}{N}}\right)^{2nk} = \left(e^{-j\frac{2\pi}{N/2}}\right)^{nk} = W_{N/2}^{nk}$$

$$= \sum_{n=0}^{N/2-1} f_1(n) W_{N/2}^{nk} + W_n^k \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{N/2}^{nk} \longrightarrow (4)$$

$N/2$ point of FFT

$$F_1(K) = \sum_{n=0}^{N/2-1} f_1(n)\, W_{N/2}^{kn}$$

$$F_2(K) = \sum_{n=0}^{N/2-1} f_2(n)\, W_{N/2}^{kn}$$

$$\boxed{X(k) = F_1(k) + W_n^k F_2(k)} \longrightarrow (5)$$

$K \geq \frac{N}{2}$ (Periodicity)

$$X(k) = F_1\left(K - \frac{N}{2}\right) + W_n^k F_2\left(K - \frac{N}{2}\right) \longrightarrow (6)$$

eg: $N = 8$; $K = 0, 1, \ldots, 7$

$$X(0) = F_1(0) + W_8^0 F_2(0)$$

$$X(1) = F_1(1) + W_8^1 F_2(1)$$

$$X(2) = F_1(2) + W_8^2 F_2(2)$$

$$X(3) = F_1(3) + W_8^3 F_2(3)$$

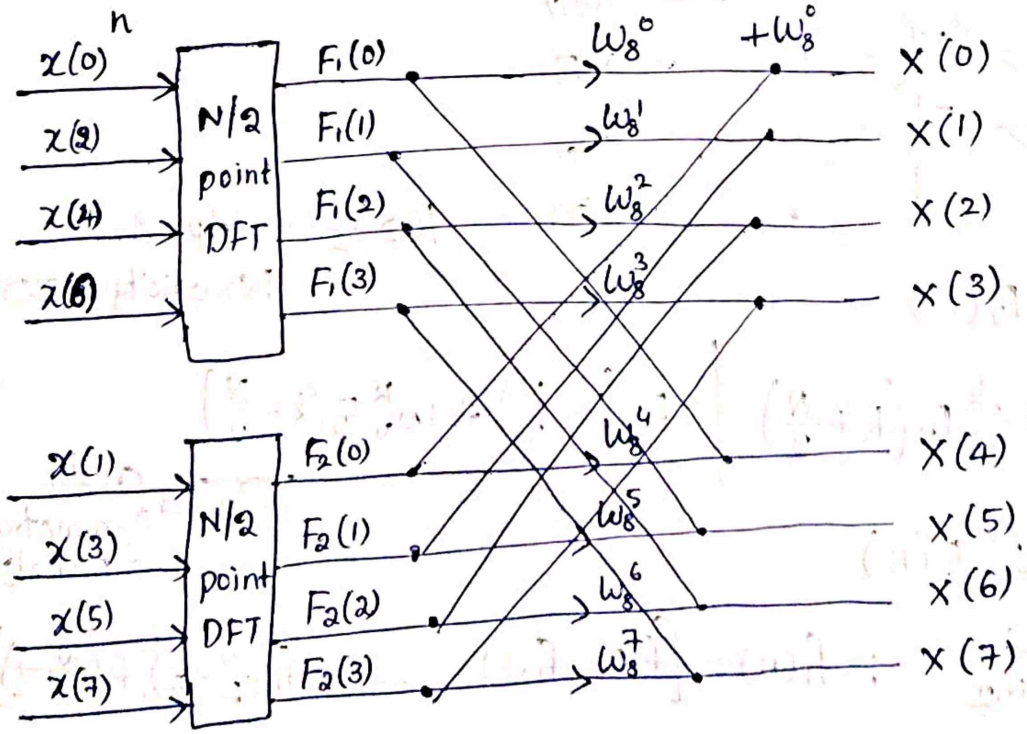
$$X(R) = F_1(K-4) + W_8^k F_2(K-4)$$

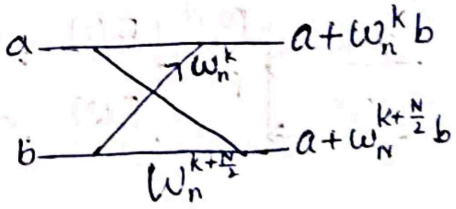
$$X(4) = F_1(0) + W_8^4 F_2(0)$$

$$X(5) = F_1(1) + W_8^5 F_2(1)$$

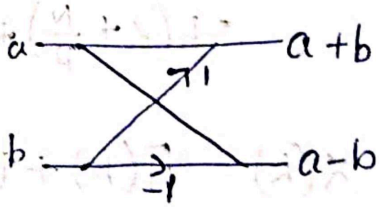
$$X(6) = F_1(2) + W_8^6 F_2(2)$$

$$X(7) = F_1(3) + W_8^7 F_2(3)$$





$$W_n^{K+\frac{N}{2}} = -W_N^K$$



$$F_1(K) = \sum_{n=0}^{N/2-1} f_1(n)\, W_{N/2}^{kn} \longrightarrow (1)$$

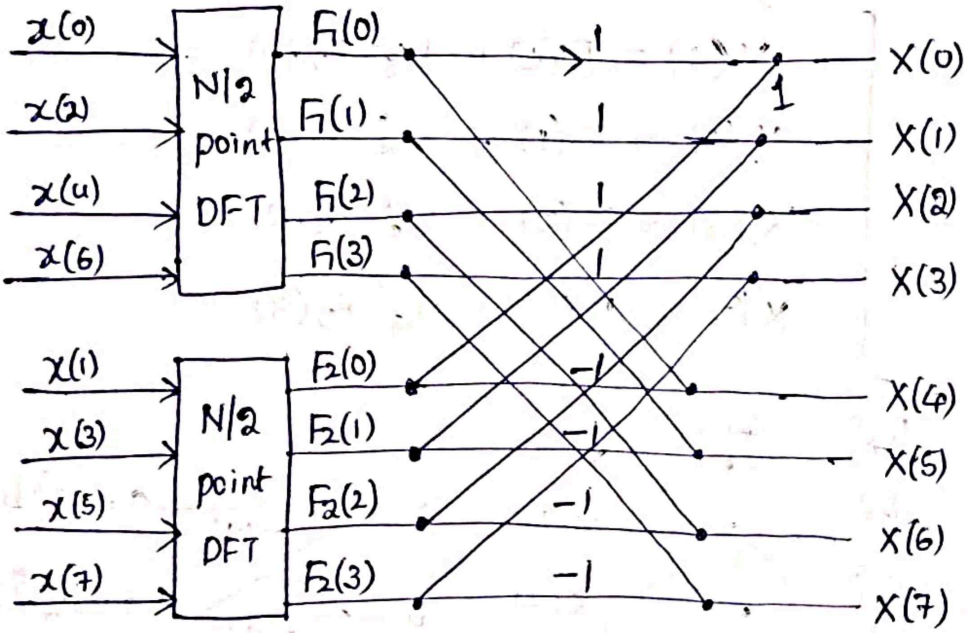
$$F_2(K) = \sum_{n=0}^{N/2-1} f_2(n)\, W_{N/2}^{kn} \longrightarrow (2)$$

$$f_1(n) = \left\{f_1(0), f_1(1), f_1(2), f_1(3), \ldots\ldots, f_1\left(\tfrac{N}{2}-2\right), f_1\left(\tfrac{N}{2}-1\right)\right\}$$

Odd sequence $\longrightarrow \left\{f_1(1), f_1(3), \ldots\ldots, f_1\left(\tfrac{N}{2}-1\right)\right\}$

even sequence $\longrightarrow \left\{f_1(0), f_1(2), \ldots\ldots, f_1\left(\tfrac{N}{2}-2\right)\right\}$

$$F_1(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-2} f_1(n)\, W_{N/2}^{kn}}_{\text{even}} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_N^{kn}}_{\text{odd}}$$



$$W_N^{k+\frac{N}{2}} = -W_n^k$$

$$W_8^{k+4} = -W_8^k$$

$K \geq \frac{N}{2} \Rightarrow$ Apply periodicity property

$$X(k) = F_1(k) + W_N^k F_2(k)$$

$$= F_1\left(K+\tfrac{N}{2}\right) + W_N^k F_2\left(K+\tfrac{N}{2}\right) \;\Big|\; F_1\left(k-\tfrac{N}{2}\right) + W_N^k F_2\left(k-\tfrac{N}{2}\right)$$

apply Symmetry property

$$\Rightarrow X(k) = F_1(k) + W_N^k F_2(k)$$

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} \quad ; \; f_1(n) = \left\{f_1(0), f_1(1), \ldots\ldots, f_1\left(\tfrac{N}{2}-2\right), f_1\left(\tfrac{N}{2}-1\right)\right\}$$

Odd $\Rightarrow \left\{f_1(1), f_1(3), f_1(5), \ldots\ldots, f_1\left(\tfrac{N}{2}-1\right)\right\}$

even $\Rightarrow \left\{f_1(0), f_1(2), f_1(4), \ldots\ldots, f_1\left(\tfrac{N}{2}-2\right)\right\}$

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-2} f_1(n)\, W_{N/2}^{kn} + \sum_{n=0}^{N/2-1} f_1(n)\, W_{N/2}^{kn}$$

$n = 2n$ (even) and $n = (2n+1)$ (odd)

$$F_1(k) = \sum_{n=0}^{N/4-1} f_1(2n)\, W_{N/2}^{k2n} + \sum_{n=0}^{N/4-1} f_1(2n+1)\, W_{N/2}^{k(2n+1)}$$

$$(W_N)^{2kn} = \left(e^{-j\frac{2\pi}{N}}\right)^{2kn} = \left(e^{-j\frac{2\pi}{N/2}}\right)^{kn} = W_{N/2}^{kn}$$

$$= \left(e^{-j\frac{2\pi}{N/4}}\right)^{kn} = W_{N/4}^{kn}$$

$$F_1(k) = \sum_{n=0}^{\frac{N}{4}-1} f_1(2n)\, W_{N/4}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} f_1(2n+1)\, W_{N/4}^{kn} \cdot W_N^k$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_{11}(n)\, W_{N/4}^{kn} + W_N^k \sum_{n=0}^{\frac{N}{4}-1} g_{12}(n)\, W_{N/4}^{kn}$$

$$F_1(k) = G_{11}(k) + G_{12}(k)$$

$$F_2(k) = G_{21}(k) + G_{22}(k)$$

$K \geq \frac{N}{4}$ $\quad \frac{8}{4} = 2$

$$F_1(k) = G_{11}\left(k - \tfrac{N}{4}\right) + W_N^k\, G_{12}\left(k - \tfrac{N}{4}\right)$$

$$F_2(k) = G_{21}\left(k - \tfrac{N}{4}\right) + W_N^k\, G_{22}\left(k - \tfrac{N}{4}\right)$$

$F_1(k) = G_{11}(k) + W_N^k\, G_{12}(k)$ $\qquad$ $F_1(k) = G_{11}(k-2) + W_N^k\, G_{12}(k-2)$

1) $F_1(0) = G_{11}(0) + W_8^0\, G_{12}(0)$

2) $F_1(1) = G_{11}(1) + W_8^1\, G_{12}(1)$

3) $F_2(0) = G_{21}(0) + W_8^0\, G_{22}(0)$

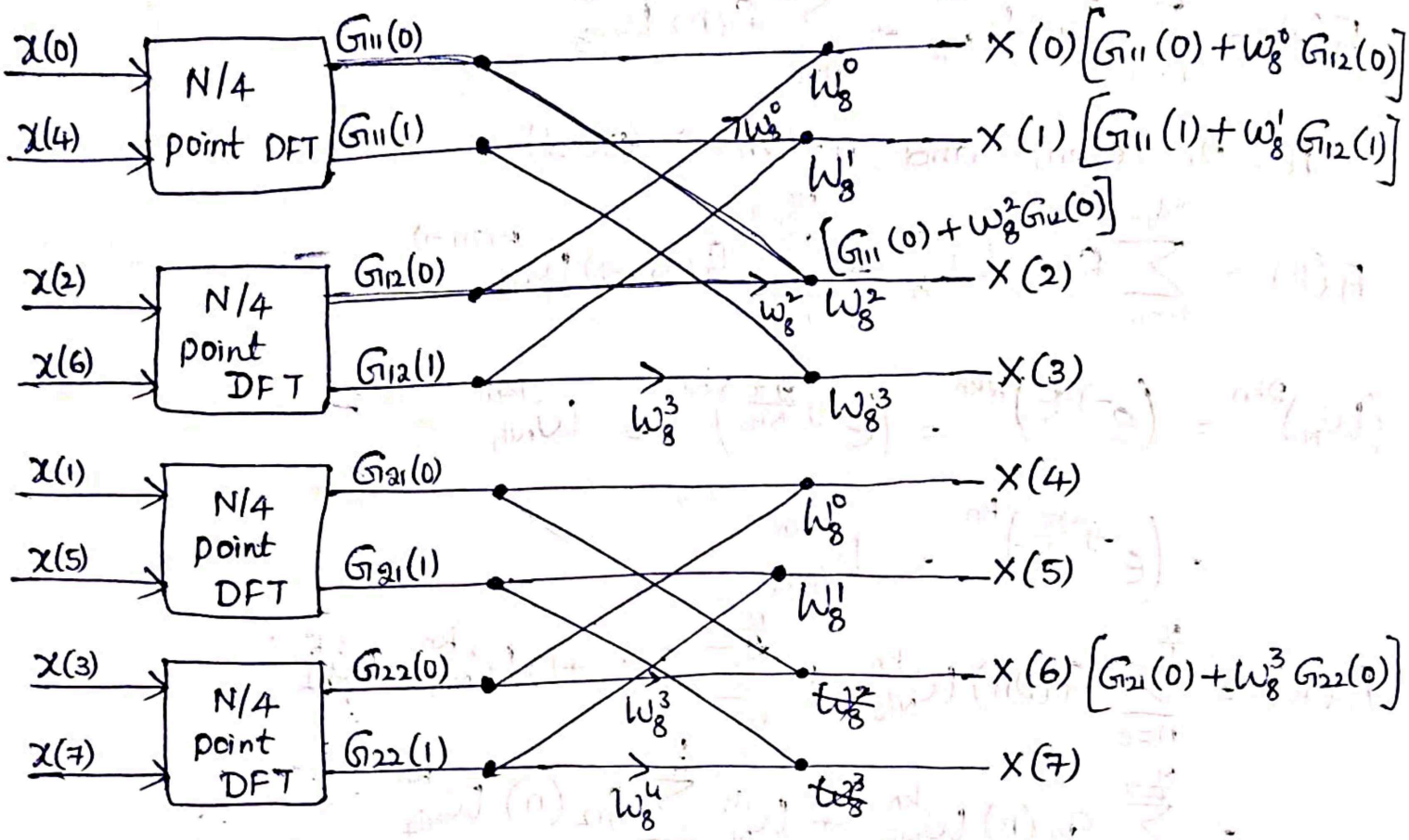
4) $F_2(1) = G_{21}(1) + W_8^1\, G_{22}(1)$

5) $F_1(2) = G_{11}(0) + W_8^2\, G_{12}(0)$

6) $F_1(3) = G_{11}(1) + W_8^3\, G_{12}(1)$

7) $F_2(2) = G_{21}(0) + W_8^2\, G_{22}(0)$

8) $F_2(3) = G_{21}(1) + W_8^3\, G_{22}(1)$



6|9

$$F_1(k) = G_{11}(k) + W_8^k G_{12}(k)$$

$$F_2(k) = G_{21}(k) + W_8^k G_{22}(k)$$

$$G_{11}(k) = \sum_{n=0}^{\frac{N}{4}-1} g_{11}(n)\, W_{N/4}^{nk} \qquad (N=8)$$

Each $\frac{N}{4}$ point $\frac{N}{8}$ point

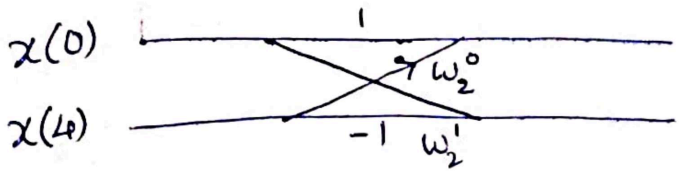
$x(0)$ and $x(4)$

$$G_{11}(k) = \sum_{n=0}^{1} x(n)\, W_{N/4}^{nk}$$

When, $k=0$ $\quad G_{11}(0) = x(0) + W_2^0 x(4)$

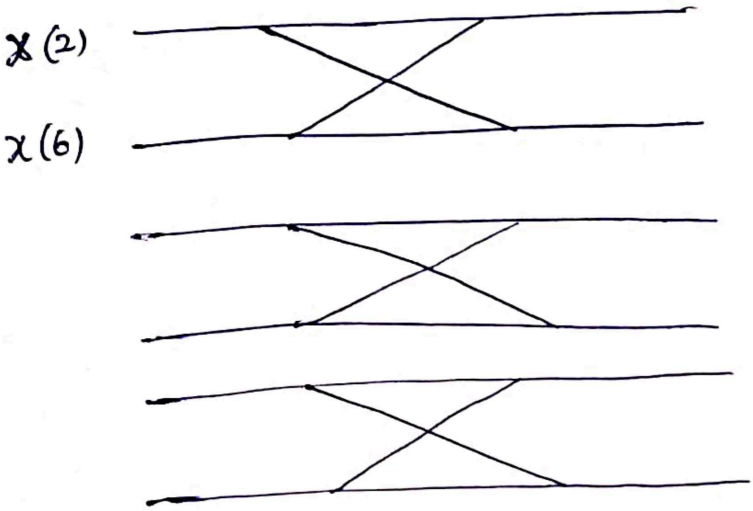
$$G_{11}(1) = x(0) + W_2^1 x(4)$$

$x(2)$ $\quad x(6)$



$x(0) + \omega_2^0 x(4) = X(0)$ $G_{11}(0)$

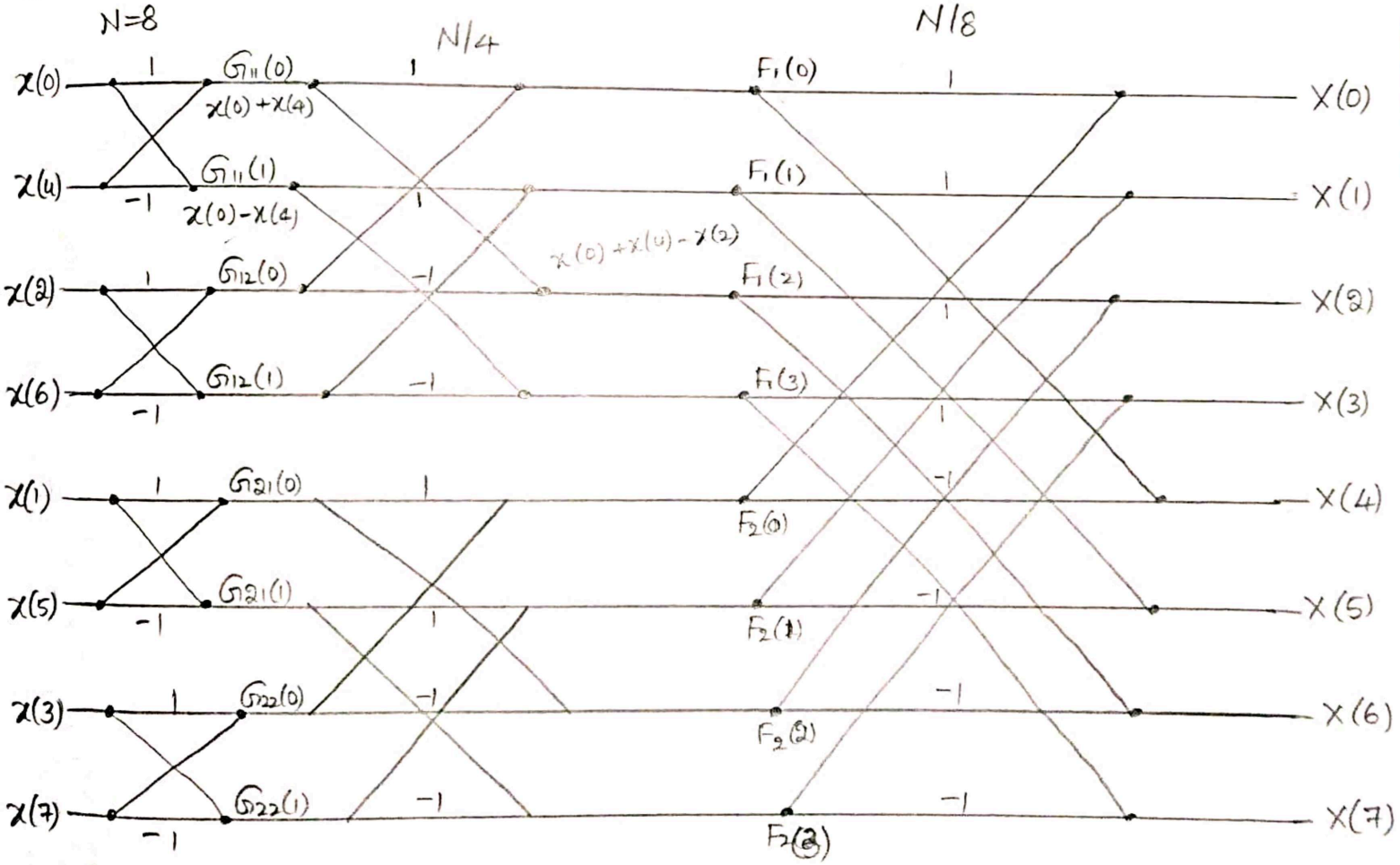
$X(0) + \omega_2^1 x(4)$ $G_{11}(1)$



DIT-FFT Algorithm (N=8)

$G_{11}(0) = x(0) + x(4)$

$G_{11}(1) = x(0) - x(4)$



$$W_N^{K+\frac{N}{2}} = -W_N^K$$

$$= -1$$

11/9 $X(k) = \sum_{n=0}^{N-1} x(n) W_n^{kn}$; $W_N = e^{-j\frac{2\pi}{N}}$

① Twiddle factor of W_N^k

$$X(k) = F_1(k) + W_N^k F_2(k)$$

$$F_1(k) = \sum_{n=0}^{N-1} f_1(n) W_{N/2}^{kn} \quad ; \quad F_2(k) = \sum_{n=0}^{N-1} f_2(n) W_{N/2}^{kn}$$

$k \geq N/2$

$N = 8$

$k = 0$ to 3 $\quad X(k) = F_1\left(k - \frac{N}{2}\right) + W_N^k F_2\left(k - \frac{N}{2}\right)$ periodicity.

When, $k=0$ $\quad X(0) = F_1(0) + W_8^0 F_2(0)$ $\quad X(4) = F_1(0) + W_8^4 F_2(0)$

$k=1$ $\quad X(1) = F_1(1) + W_8^1 F_2(1)$ $\quad X(5) = F_1(1) + W_8^5 F_2(1)$

$k=2$ $\quad X(2) = F_1(2) + W_8^2 F_2(2)$ $\quad X(6) = F_1(2) + W_8^6 F_2(2)$

$k=3$ $\quad X(3) = F_1(3) + W_8^3 F_2(3)$ $\quad X(7) = F_1(3) + W_8^7 F_2(3)$

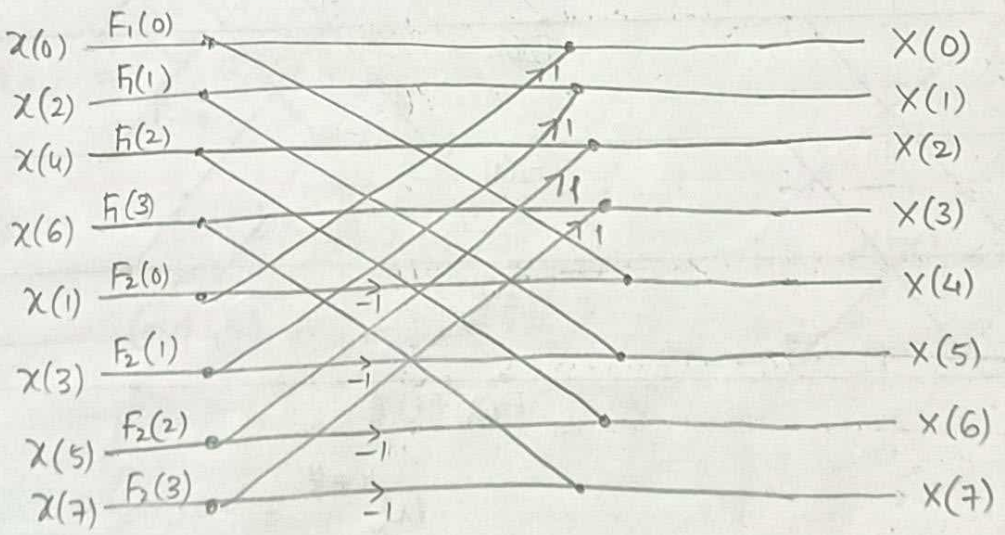
$$W_N^{k+N/2} = -W_N^k$$

$X(4) = F_1(0) + (-1) W_8^4 F_2(0) \Rightarrow X(4) = F_1(0) + (-1) W_8^0 F_2(0)$

$X(5) = F_1(1) + (-1) W_8^5 F_2(1) \Rightarrow X(5) = F_1(1) + (-1) W_8^1 F_2(1)$

$X(6) = F_1(2) + (-1) W_8^6 F_2(2) \Rightarrow X(6) = F_1(2) + (-1) W_8^2 F_2(2)$

$X(7) = F_1(3) + (-1) W_8^7 F_2(3) \Rightarrow X(7) = F_1(3) + (-1) W_8^3 F_2(3)$



$W_8^5 = -0.707 + j0.707$

$= 0.707 - j0.707 = W_8^1$

$$F_1(k) = G_{11}(k) + W_{N/2}^{k} \cdot G_{12}(k)$$

$$F_2(k) = G_{21}(k) + W_{N/2}^{k} G_{22}(k)$$

$k=0,1$

$$F_1(0) = G_{11}(0) + W_4^0 G_{12}(0) \quad | \quad F_2(0) = G_{21}(0) + W_4^0 G_{22}(0)$$

$$F_1(1) = G_{11}(1) + W_4^1 G_{12}(1) \quad | \quad F_2(1) = G_{21}(1) + W_4^1 G_{22}(1)$$

$k=2,3$

$$F_1(k) = G_{11}\left(k-\frac{N}{4}\right) + W_{N/2}^{k} G_{12}\left(k-\frac{N}{4}\right)$$

$$F_2(k) = G_{21}\left(k-\frac{N}{4}\right) + W_{N/2}^{k} G_{22}\left(k-\frac{N}{4}\right)$$

$k=2,3$

$$F_1(2) = G_{11}(0) + W_4^2 G_{12}(0) \quad | \quad F_2(2) = G_{21}(0) + W_4^2 G_{22}(0)$$

$$F_1(3) = G_{21}(1) + W_4^3 G_{12}(1) \quad | \quad F_2(3) = G_{21}(1) + W_4^3 G_{22}(1)$$

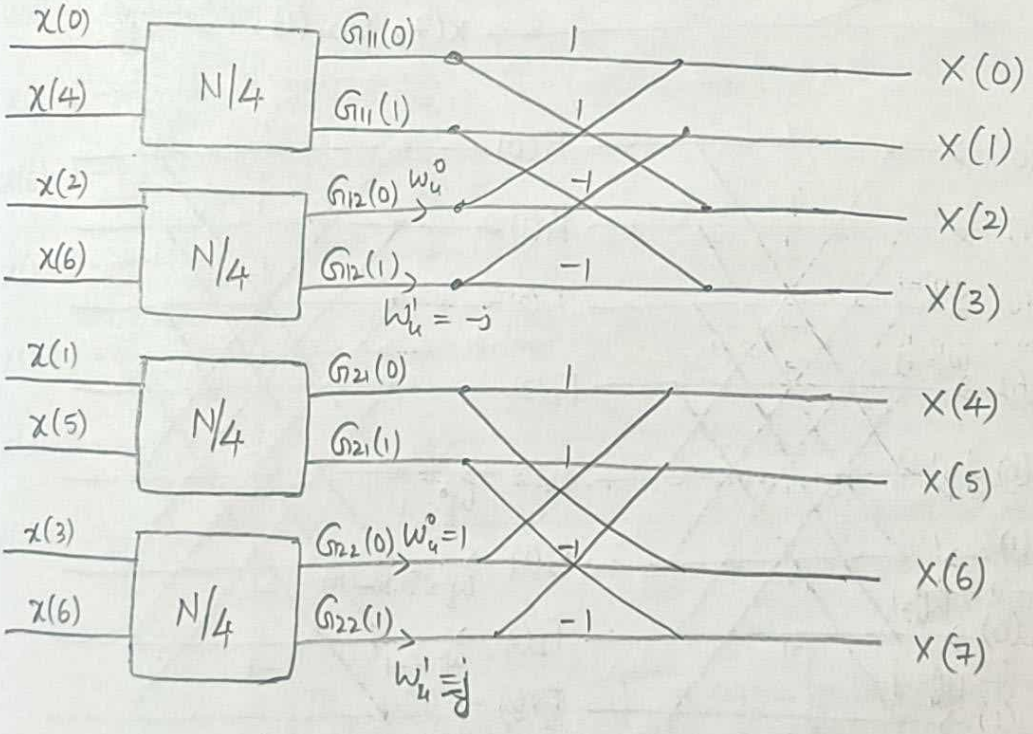
$$W_N^{k+\frac{N}{4}} = -W_N^{k}$$

$$F_1(2) = G_{11}(0) + (-1) W_4^2 G_{12}(0) \Longrightarrow F_1(2) = G_{11}(0) + (-1) W_4^0 G_{12}(0)$$

$$F_1(3) = G_{11}(1) + (-1) W_4^3 G_{12}(1)$$

$$F_2(2) = G_{21}(0) + (-1) W_4^2 G_{22}(0)$$

$$F_2(3) = G_{21}(1) + (-1) W_4^3 G_{22}(1)$$



Second DIT twiddle factor W_N^k

$$G_{11}(k) = \sum_{n=0}^{N/4-1} g_{11}(n)\, W_{N/4}^{nk} \qquad 0 \le k \le \frac{N}{4}-1$$

$$G_{11}(k) = \sum_{n=0}^{N/4-1} x(n)\, W_{N/4}^{nk}$$

$N/4$

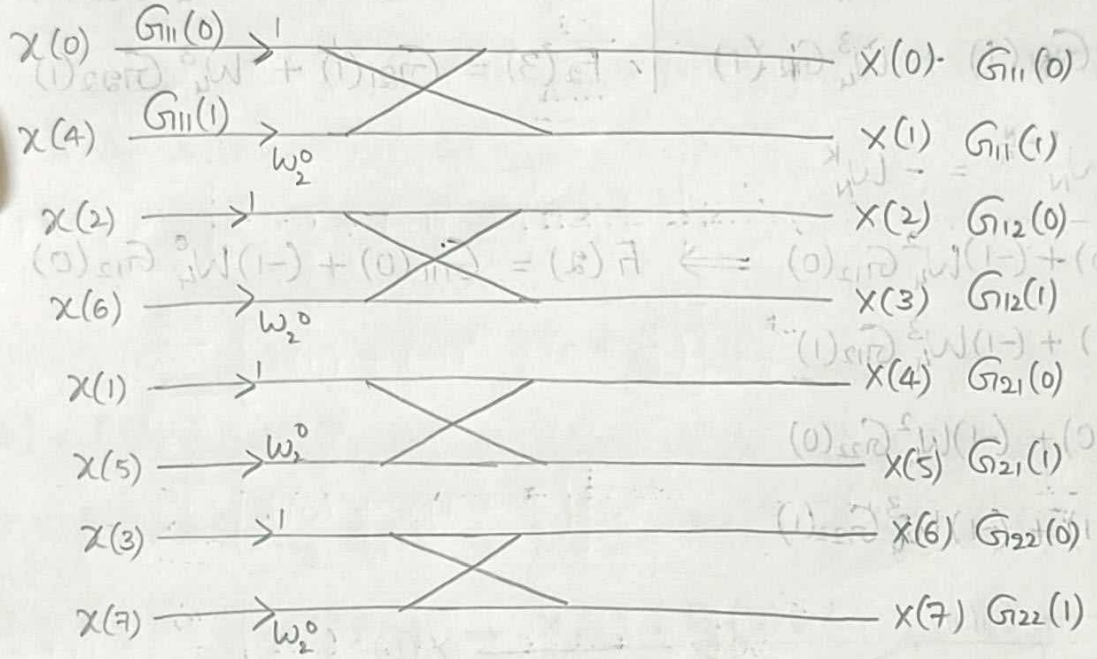
$x(n) \Rightarrow x(0)$ and $x(4)$

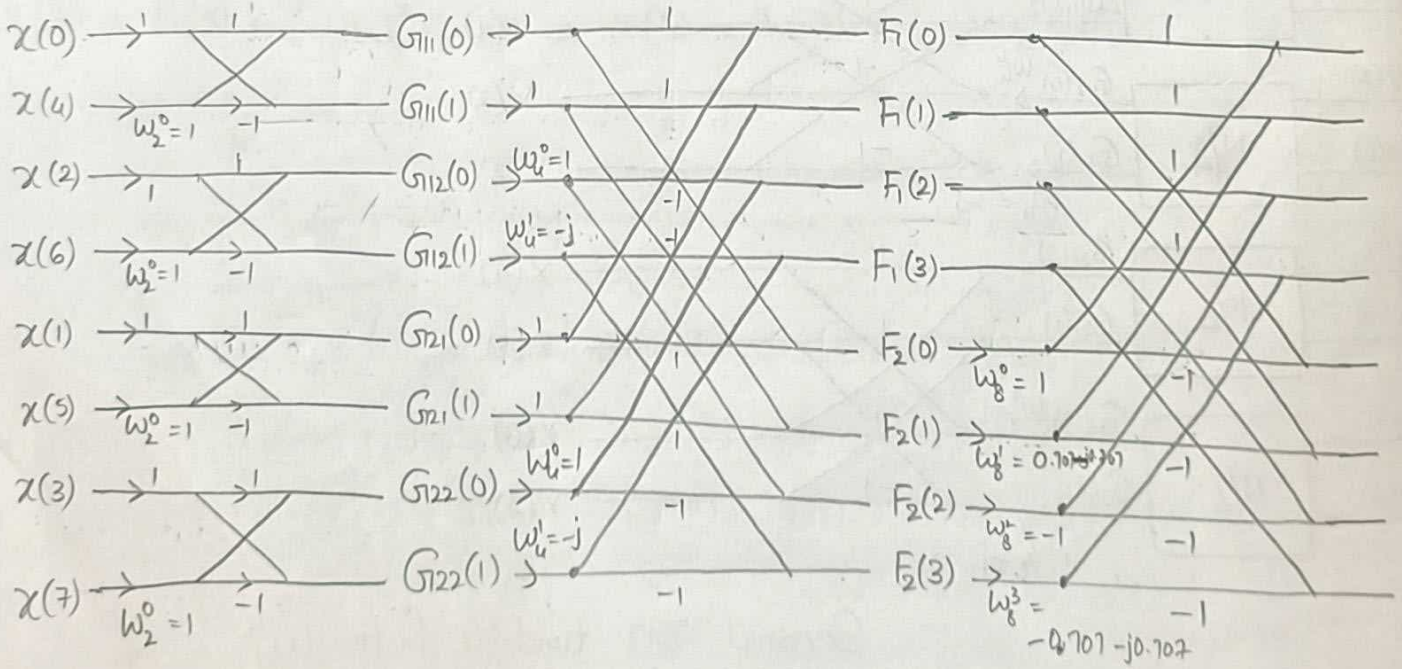
$k=0 \quad G_{11}(0) = x(0) + W_2^0\, x(4)$

$k=1 \quad G_{11}(1) = x(0) + W_2^1\, x(4)$

$= x(0) + (-1) W_2^1 x(4)$

$= x(0) + (-1) W_2^0 x(4)$

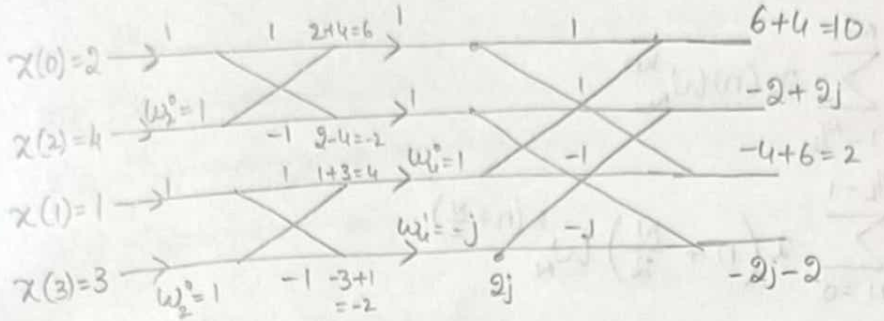




① find 4-point DFT of the sequence $x(n) = \{2, 1, 4, 3\}$ using DIT-FFT algorithm.

Ans

$$x(n) = \underset{0}{2}, \underset{1}{1}, \underset{2}{4}, \underset{3}{3}$$



$X(0) = 10 = 10$

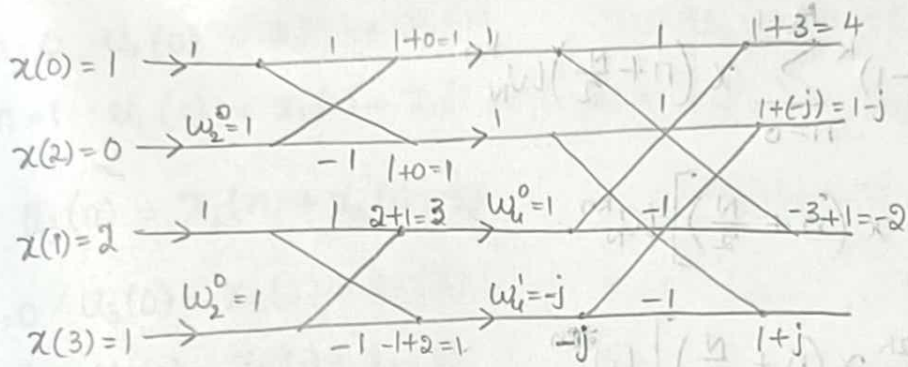
$X(1) = -2+2j = -2-2j$

$X(2) = 2 = 2$

$X(3) = -2j-2 = +2j-2$

② $x(n) = \{1, 2, 0, 1\}$

$$\underset{0}{1}, \underset{1}{2}, \underset{2}{0}, \underset{3}{1}$$



$X(0) = 4$

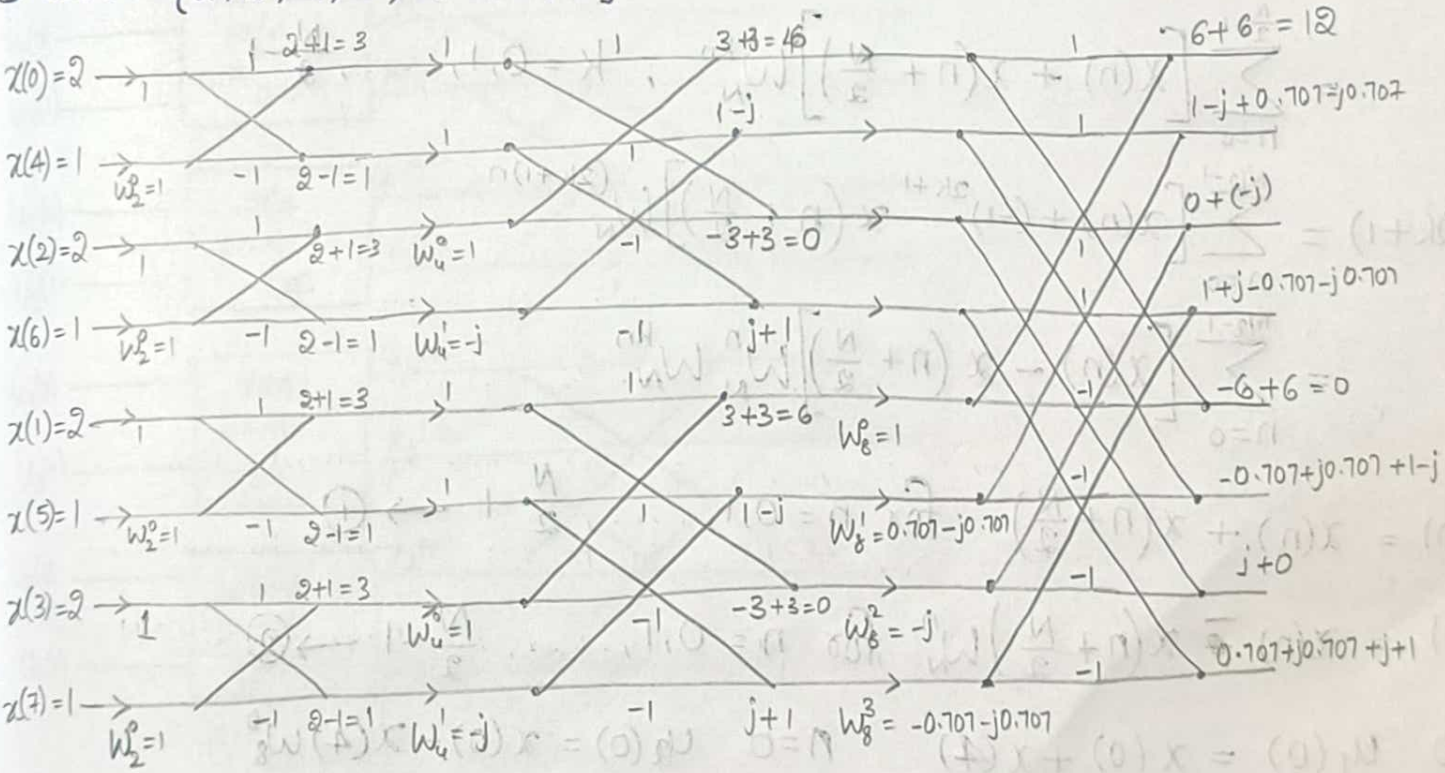
$X(1) = 1-j$

$X(2) = -2$

$X(3) = 1+j$

③ $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$

$$\underset{0}{2}, \underset{1}{2}, \underset{2}{2}, \underset{3}{2}, \underset{4}{1}, \underset{5}{1}, \underset{6}{1}, \underset{7}{1}$$



$$= \{12, 1.707-1.707j, -j, 0.293+j0.293, 0, 0.293-j0.293, j, 1.707+1.707j\}$$

$X(0)$ $X(1)$ $X(2)$ $X(3)$ $X(4)$ $X(5)$ $X(6)$ $X(7)$

Decimation in frequency (Radix-2) at N = 8 alogrithm

X(k) → Decimation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad k = 0, 1, \ldots, N-1$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{k(n+\frac{N}{2})}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + W_N^{\frac{N}{2}k} \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} x\left(n+\frac{N}{2}\right)\right] W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_N^{kn} \; ; \; k = 0, 1, \ldots, \frac{N}{2}-1$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2k+1} x\left(n+\frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n} W_N^{kn}$$

$$U_1(n) = x(n) + x\left(n+\frac{N}{2}\right) \quad \text{for } n = 0, 1, \ldots, \frac{N}{2}-1 \rightarrow (1)$$

$$U_2(n) = x(n) - x\left(n+\frac{N}{2}\right) W_N^n \quad \text{for } n = 0, 1, \ldots, \frac{N}{2}-1 \rightarrow (2)$$

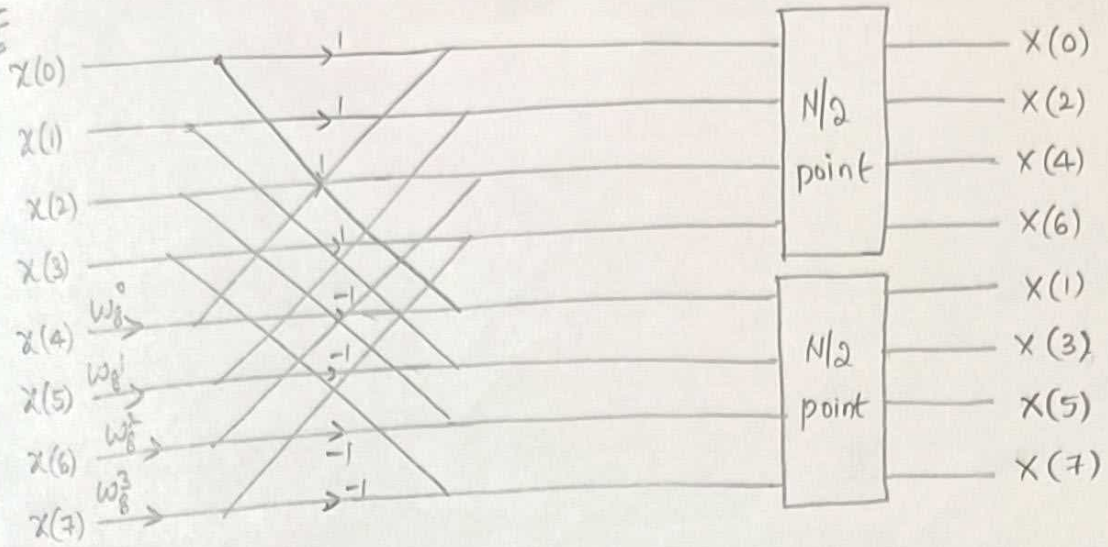
$n=0$ $U_1(0) = x(0) + x(4)$ $\quad$ $n=0$ $U_2(0) = x(0) - x(4) W_8^0$

$n=1$ $U_1(1) = x(1) + x(5)$ $\quad$ $n=1$ $U_2(1) = x(1) - x(5) W_8^1$

$n=2$ $U_1(2) = x(2) + x(6)$ $\quad$ $n=2$ $U_2(2) = x(2) - x(6) W_8^2$

$n=3$ $U_1(3) = x(3) + x(7)$ $\quad$ $n=3$ $U_2(3) = x(3) - x(7) W_8^3$

DIF



$$u_1(n) = x(n) + x\left(n + \frac{N}{4}\right)$$

$$u_1(n) = x(n) + x(n+2)$$

$$u_2(n) = x(n) - x\left(n + \frac{N}{4}\right) W_{N/2}^{n}$$

$n=0 \quad u_1(0) = x_1(0) + x_1(2)$

$n=1 \quad u_1(1) = x_1(1) + x_1(3)$

$n=0 \quad u_2(0) = x_1(0) - x_1(2) W_4^0$

$n=1 \quad u_2(1) = x_1(1) - x_1(3) W_8^1$

$$u_3(n) = x_2(n) + x_2(n+2)$$

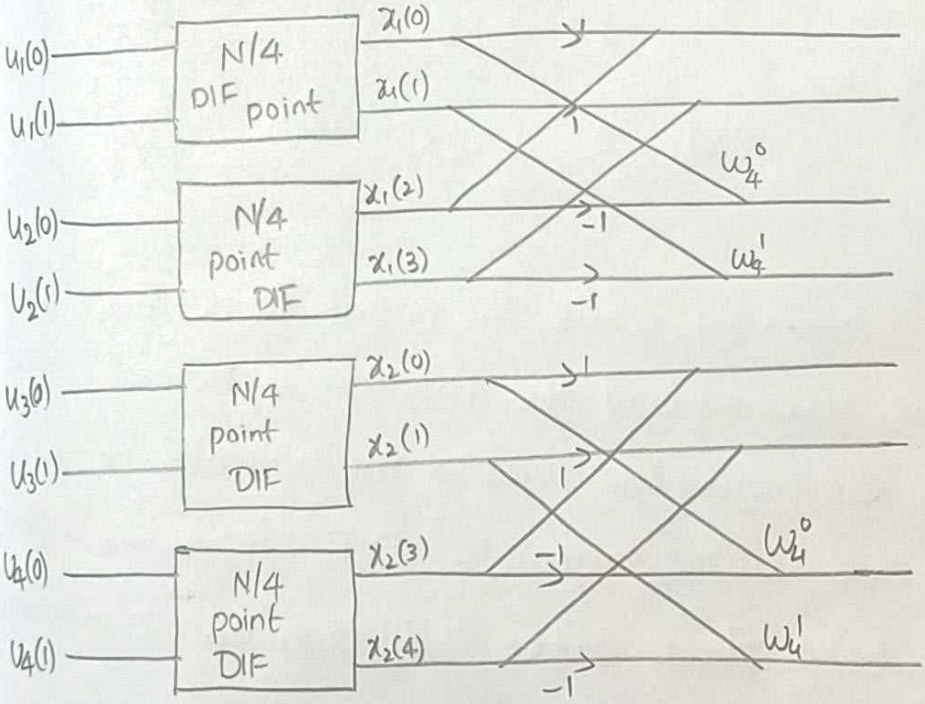
$$u_4(n) = x_2(n) - x_2\left(n + \frac{N}{4}\right) W_{N/2}^{n}$$

$n=0 \quad u_3(0) = x_2(0) + x_2(2)$

$n=1 \quad u_3(1) = x_2(1) + x_2(3)$

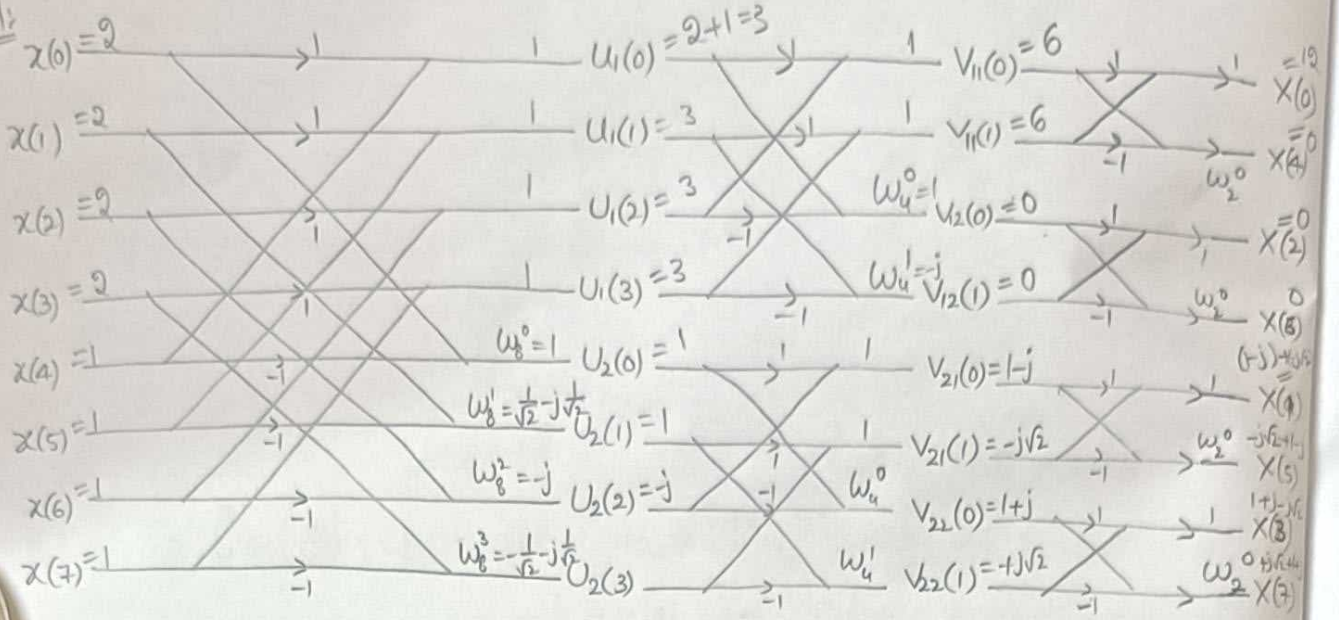
$n=0 \quad u_4(0) = x_2(0) - x_2(2) W_4^0$

$n=1 \quad u_4(1) = x_2(1) - x_2(3) W_4^1$



(2) $x(n) = \{2,2,2,2,1,1,1,1\}$

Sol:



$X(k) = \{12, 1-j(\sqrt{2}-1), 0, 1+j-\sqrt{2}, 0,$

⇒ Calculate the no.of addition and multiplication needed in the calculation of DFT using FFT algorithm.

* Multiplication,

$\frac{N}{2}\log_2 N$ If $N = 32$

$\Rightarrow \frac{32}{2}\log_2 2^5 \Rightarrow 16 \times 5 = 80$

Addition, $N \log_2 N$ If $N = 32$

$\Rightarrow 32 \log_2 2^5 \Rightarrow 32 \times 5 = 160$

* Differences :-

1) In DIT the input bit is reversed while the output is normal order. In DIF the output bit is reversed and the input is in normal order.
2) Consider the butterfly diagram, the complex multiplication takes place before, addition and subtraction of DIF. However the complex multiplication takes place after addition and subtraction in DIF.

* Similarities :-

1) Both the algorithms requires the same no.of operations to compute DIT
2) Both algorithms require bit reversal at the same place during operation.

IDFT using FFT algorithm:-

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$x(n) = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}\right]^* = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{kn}$$

$$X(k) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

$$X^*(k) = \{4,\ 1+j2414,\ 0,\ 1+j0.414,\ 0,\ 1-j0.414,\ 0,\ 1-j2.414\}$$

$$DIF = \{x(0),\ x(4),\ x(2),\ x(6),\ x(1),\ x(5),\ x(3),\ x(7)\}$$

$$DIT = \{x(0),\ x(1),\ x(2),\ x(3),\ x(4),\ x(5),\ x(6),\ x(7)\}.$$

18/09

Z- Transform

① Extension of DTFT

$$DTFT[x(n)] = X(\omega) = X(e^{j\omega})$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

DTFT → stable system

Z-transform → stable and unstable

→ digital filter design and synthesis.

Types of Z-transform :-

1) Bilateral or two sided Z-transform

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

2) Unilateral or one sided Z-transform

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

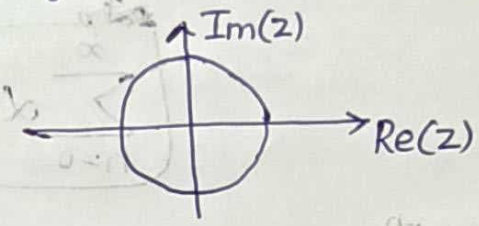
Region of Convergence (ROC) :-

① Z-transform gets converges

② Values of z can be calculated.

Properties of ROC :-

① The ROC is a ring or disk in the z-plane centered to the origin.

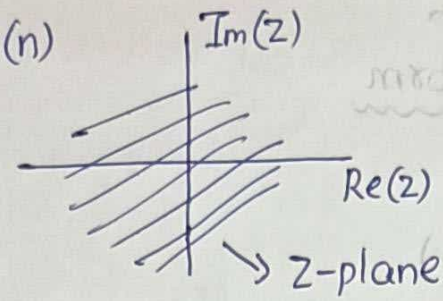


② The ROC does not contain any poles, $X(z) \rightarrow$ finite values

③ i) If $x(n)$ is finite duration (right-sided sequence), the ROC is entire z-plane except at $z=0$.

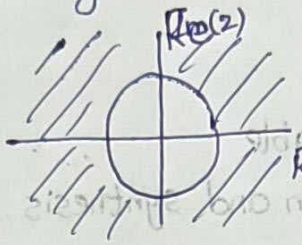
ii) If $x(n)$ is infinite duration (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.

$x(n) = \delta(n)$



④ If the ROC is right sided sequence is outside the circle

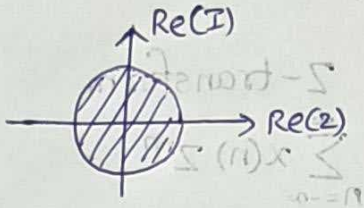
$x(n) = \alpha^n u(n)$



$u(n) - 1 = \begin{cases} 1 & n \geq 0 \\ 0 & n \leq 0 \end{cases}$

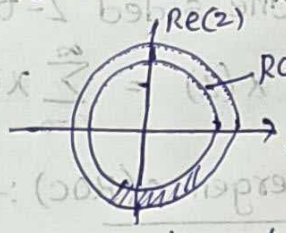
⑤ If the ROC is left sided sequence is inside the circle

$x(n) = -\alpha^n u(-n-1)$

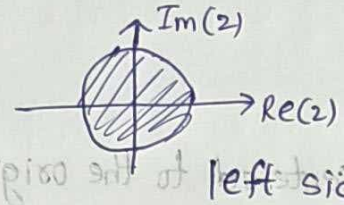


⑥ If ROC is two sided sequence is concentric ring

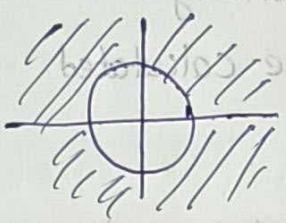
$x(n) = \alpha^n u(n) - \alpha^n u(-n-1)$



⑦ If $x(n)$ is rational number, the ROC is bounded by poles or extend to infinity.



left sided sequence.



right sided sequence.

Prove the sequence,

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \; ; |\alpha| < 1$$

ⓐ $x(n) = a^n u(n)$

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

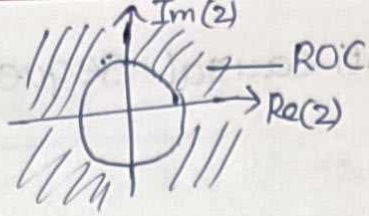
$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + az^{-1} + (az^{-1})^2 + \ldots$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$\begin{cases} |az^{-1}| < 1 \\ |a| < z \\ z > |a| \end{cases}$$



$u(-n-1) = \begin{cases} 1 & \text{for } n \le -1 \\ 0 & \text{for } n \ge 0 \end{cases}$

$$\begin{cases} \sum_{n_1}^{n_2} a^n = \dfrac{a^{n_1} - a^{n_2+1}}{1-a} \\ \sum_{1}^{\infty} a^n = \dfrac{a^1 - a^{\infty+1}}{1-a} \\ \sum_{1}^{\infty} a^n = \dfrac{a}{1-a} \end{cases}$$

(b) $x(n) = -a^n u(-n-1)$

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n z^{-n} = -\sum_{n=1}^{\infty} (az^{-1})^n$$

$$\sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-a}$$

$$\sum_{n=1}^{\infty} (az^{-1})^n = \frac{a}{1-a}$$

$$= \frac{az^{-1}}{1-az^{-1}} = \frac{a\frac{1}{z}}{1-a\frac{1}{z}} = \frac{\frac{a}{z}}{\frac{z-a}{z}}$$

$$= \frac{a}{z^2-a}$$

$|az^{-1}| < 1$

$|a| < z$

$z > |a|$

③ find the ROC and Z-transform of the causal sequence

$x(n) = \{1, 0, -2, 3, 5, 4\}$ (↑ at 1)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + 0z^{-1} + (-2)z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

④ If $x(n) = \{5, 3, -2, 0, 4, -3\}$ find $X(z)$ and ROC

(↑ at 5)

A finite sequence

$$= 5\delta(n) + 3\delta(n-1) + (-2)\delta(n-2) + 0\delta(n-3) + 4\delta(n-4) - 3\delta(n-5)$$

$$= 5 + 3z^{-1} - 2z^{-2} + 0 + 4z^{-4} - 3z^{-5}$$

⑤ find Z- transform and ROC of anti causal sequence

$$x(n) = \{4, 2, 3, -1, -2, 1\}$$
(1 at the arrow)

⑥ $x(n) = -a^n u(-n-1)$

$$u(-n-1) = \begin{cases} 1 & \text{for } n \le -1 \\ 0 & \text{for } n > 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= -\sum_{n=-\infty}^{1} a^{-n} z^{n}$$

$$= \sum_{n=1}^{\infty} a^{-n} z^{n}$$

$$= \sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= \frac{a^{-1}z}{1 - a^{-1}z}$$

$|a^{-1}z| < 1$

$|z| < a$

$z < a$

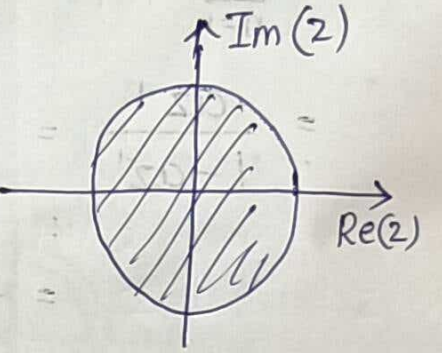
$a > z$

$$= \frac{\frac{1}{a}z}{1 - \frac{z}{a}}$$

$$= \frac{-z}{a - z}$$

$$= \frac{z}{z - a}$$

$$X(z) = \frac{z}{z-a}$$



$$\sum_{n=n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a} = \frac{a}{1-a}$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

⑦ $x(n) = n\, 2^n \sin\left(\frac{\pi}{2} n\right) u(n)$ find z transform and ROC.

⑧ find convolution of $x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^n u(n)$

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z)$$

$$a^n u(n) \longrightarrow x_1(n) \; ; \; a^n u(n) \longrightarrow x_2(n)$$

$$X_1(z) \cdot X_2(z)$$

Property	Z-transform	ROC
	$x_1(n) \xleftrightarrow{} X_1(z)$ $x_2(n) \xleftrightarrow{} X_2(z)$	R_1 R_2
① Linearity	$ax_1(n) + bx_2(n) \xleftrightarrow{ZT}$ $aX_1(z) + bX_2(z)$	$R_1 \cap R_2$
② Time Shifting → Time delay	$x(n-m) \xleftrightarrow{ZT} z^{-m} X(z)$	entire z-plane except $z=0$
② Time Shifting → Time advance	$x(n+m) \xleftrightarrow{ZT} z^{m} x(z)$	entire z-plane except $z=\infty$
③ Multiplication by exponential sequence on scaling in z-domain	$a^n u(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$	$\lvert a\rvert R_1 < z < \lvert a\rvert R_2$
④ Time reversal	$x(-n) \xleftrightarrow{ZT} X(z^{-1})$	$\frac{1}{R_2} < \lvert z\rvert < \frac{1}{R_1}$
⑤ Time expansion	$x\left(\frac{n}{k}\right) \xleftrightarrow{ZT} X(z^k)$	
⑥ Differentiation in z-domain	$n\,x(n) \xleftrightarrow{ZT} -z \frac{d}{dz} X(z)$	$R_1 < \lvert z\rvert < R_2$
⑦ Conjugation	$x^*(n) \xleftrightarrow{ZT} X^*(z^*)$	$R_1 < z < R_2$
⑧ Accumulation	$\sum_{k=-\infty}^{\infty} x(k) \xleftrightarrow{ZT} \frac{1}{1-z^{-1}} X(z)$	
⑨ Convolution	$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z)$	At least the intersection of R_1 & R_2
⑩ Multiplication	$x_1(n).x_2(n) \xleftrightarrow{ZT} \frac{1}{2\pi j}\oint_c X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1} dv$	At least the $R_{1\ell} R_{2\ell} < \lvert z\rvert < R_{1u} R_{2u}$
⑪ Correlation	$R_{x_1x_2}(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{ZT}$ $X_1(z)\, X_2(z^{-1})$	At least intersection of the ROC of $X_1(z)$ and $X_2(z^{-1})$
⑫ Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$	
⑬ Initial value theorem	$x(0) = \lim_{n\to 0} x(n) = \lim_{n\to\infty} X(z)$	
⑭ Final value theorem	$x(\infty) = \lim_{n\to\infty} x(n) = \lim_{n\to 0} X(z)$	

Sequence	Z-transform	ROC
① $\delta(n)$	1	All z plane except at $z=0$ and $z=\infty$
② $u(n)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
③ $u(-n)$ $x(-n) \xleftrightarrow{ZT} X(z^{-1})$	$\frac{z}{z-1}\Big\rvert_{z=\frac{1}{z}} = \frac{2\times z^{-1}}{2\times z^{-1} - z^{-1}}$ $= \frac{1}{1-z}$	$\lvert z\rvert < 1$
④ $u(-n-1)$ $x(n-m) \xleftrightarrow{ZT} z^{-m}x(z)$ $x(n+m) \xleftrightarrow{ZT} z^{m}x(z)$	$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
⑤ $u(-n-2)$	$\frac{z^2}{z-1}$	$\lvert z\rvert < 1$
⑥ $u(-n-k)$	$\frac{z^k}{z-1}$	$\lvert z\rvert < 1$
⑦ $\delta(n-k)$	z^k	All z-plane except at $z=0$ and $z=\infty$
Multiplication $a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > a$
$a^n u(-n)$	$\frac{a}{z-a}$	$\lvert z\rvert < a$
$\boxed{x(-n) \xleftrightarrow{ZT} X(z^{-1})}$	$\frac{z}{z-a}$	$\lvert z\rvert < a$
$-a^n u(-n-1)$ $nx(n) \xleftrightarrow{ZT} -z\frac{d}{dz}X(z)$	$\frac{-z}{(z-1)^2}$	$\lvert z\rvert > 1$
$n\,u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$n a^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$n a^n u(-n-1)$	$\frac{z}{z-e^{-j\omega}}$	$\lvert z\rvert > 1$
$e^{-j\omega n} u(n)$		

Sequence	Z - transform	ROC
$\cos \omega n\, u(n)$	$\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$	$\lvert z\rvert > 1$
$\sin \omega n\, u(n)$	$\frac{z\sin\omega}{z^2-2z\cos\omega+1}$	$\lvert z\rvert > 1$
$a^n \cos \omega n\, u(n)$	$\frac{z(z-a\cos\omega)}{z^2+2az\cos\omega+a^2}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin \omega n\, u(n)$	$\frac{az\sin\omega}{z^2-2az\cos\omega+1}$	$\lvert z\rvert > \lvert a\rvert$
$(n+1)\, a^n u(n)$	$\frac{z^2}{(z-a)^2}$	$\lvert z\rvert > a$
$-n\, u(-n-1)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert < 1$
$n a^n u(n)$	$\frac{z}{(z-a)^2}$	$\lvert z\rvert > a$
$\frac{[n(n-1)\, a^{n-2} u(n)]}{2!}$	$\frac{z}{(z-a)^3}$	$\lvert z\rvert > a$
$\frac{n(n-1)\ldots[n(k-2)]\, a^{n-k+1}}{(k-1)!}$	$\frac{z}{(z-a)^k}$	$\lvert z\rvert > a$
$n^k a^k\ ;\ k<0$	$-\left(-\frac{d}{dz}\right)^n \frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$
$a^{\lvert n\rvert}$	$\frac{1-a^2}{(1-az)(1-az^{-1})}$	$\lvert a\rvert < z < \lvert\frac{1}{a}\rvert$

Q) find the initial and final value theorem if $X(z)$ is given by

(a) $\frac{z+1}{(z-(0.6))^2}$ (b) $\frac{z+2}{4(z-1)(z+0.7)}$ (c) $\frac{2z+3}{(z+1)(z+3)(z-1)}$

Inverse Z-transform:-

$$X(z) = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \ldots\ldots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots\ldots + a_N}$$

(i) The roots of Numerator polynomial are those value of z for which $X(z) = 0$ and are referred as zeros of $X(z)$

(ii) The roots of denominator polynomial are those values of z for which $X(z) = \infty$ and are referred as poles of $X(z)$

(iii) In z-plane, the location of zeros are represented by (•) small circle, whereas the location of poles are represented by cross symbol (××).

There are four methods to calculate inverse Z-transform,

(i) Power series method / long division method.

(ii) Partial fraction expansion method.

(iii) Complex inversion integral method (Residue method)

(iv) Convolution Integral method.

(i) Power series method / long division method

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

(1) $Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$ (causal signal $x(n) = 0 ; n > 0$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \ldots + x(\infty)z^{-\infty}$$

ROC, $z > a$

(ii) $Z[x(n)] = X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n}$ (Anticausal signal)

$$= x(-\infty)z^{\infty} + \ldots + x(-3)z^3 + x(-2)z^2 + x(-1)z^1 + x(0)z^0$$

ROC, $z < a$

$$X(z) = \frac{z > a}{z < a} ; \frac{N(z)}{D(z)}$$

Q) find the inverse Z-transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}$$

$$X(z) = x(-3)z^{-3} + x(-2)z^{-2} + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$x(n) = \{1, 2, 1, 1, -2, -3, 4\}$$
$$\uparrow$$

Q) $X(z) = \frac{1}{z-a}$; ROC, $|z| > a$

$$= \frac{1}{z(1-az^{-1})} = \frac{z^{-1}}{(1-az^{-1})}$$

$$= z^{-1}(1-az^{-1})^{-1}$$

$$= z^{-1}\left[1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \ldots\right]$$

$$= z^{-1} + az^{-2} + a^2z^{-3} + a^3z^{-4} + \ldots$$

$$= \sum_{n=1}^{\infty} a^{n-1} z^{-n} = \sum_{n=0}^{\infty} a^{n-1} u(n-1) z^{-n} = X(k) \qquad \begin{matrix} z > a \\ z < a \end{matrix}$$

$$x(n) = a^{n-1} u(n-1)$$

Q) $\frac{1}{1-az^{-1}}$; $|z| > a$

$$= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \ldots$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$x(n) = a^n u(n)$$

Q) $X(z) = \cos(3z)$, ROC $|z| < \infty$

Q) $X(z) = \sin(2z)$, ROC $|z| < \infty$

Taylor's expansion series,

$$\Rightarrow \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(3z)^{2k}}{(2k)!}$$

$$= 1 - \frac{(3z)^2}{2!} + \frac{(3z)^4}{4!} - \frac{(3z)^6}{6!} + \dots$$

$$\sum_{n=-\infty}^{0} x(n) z^{-n} = x(-\infty) z^{\infty} + \dots + x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0$$

$$\left[\dots \frac{-81}{80} z^6 + \frac{27}{8} z^4 - \frac{9}{2} z^2 + 1 z^0 \right]$$

$$\Rightarrow \left[\dots \frac{-81}{80}, 0, \frac{27}{8}, 0, \frac{-9}{2}, 0, 1 \right]$$

$$\Rightarrow \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$$

Q) $x(z) = \ln(1 + z^{-1})$; ROC : $|z| > 0$

$$\ln(1+\theta) = \theta - \frac{\theta^2}{2} + \frac{\theta^3}{3} - \frac{\theta^4}{4} + \frac{\theta^5}{5} + \dots$$

$$X(z) = \ln(1+\theta) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta^k}{k} \quad \text{if } |\theta| < 1$$

2. Partial Fraction expansion Method :-

(i) $\frac{X(z)}{z}$

(ii) $X(z) = \frac{N(z)}{D(z)}$

$D(z) > N(z)$

case-(i) :- If $\frac{X(z)}{z}$ consists of distinct poles.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \text{----}$$

case-ii :- If $\frac{X(z)}{z}$ consists of l number of repeated poles

$$\frac{z}{(z-1)^l} = \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \text{----} \frac{1}{(z-1)^l}$$

$$C_{kl} = (z-p_k)^l \left.\frac{X(z)}{z}\right|_{z=pk}$$

(1) $X(z) = \frac{z^{-1}}{3-4z+z^{-2}}$ ROC $z > 1$

Sol $X(z) = \frac{z}{3z^2-4z^{-1}+1}$

$$= \frac{1}{3}\left[\frac{z}{z^2-\frac{4}{3}z+\frac{1}{3}}\right]$$

$$X(z) = \frac{1}{3}\frac{z}{(z-1)(z-\frac{1}{3})}$$

$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{1}{(z-1)(z-\frac{1}{3})}\right]$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{1}{(z-1)(z-\frac{1}{3})}\right] = \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$A = (z-1)\frac{X(z)}{z}$$

$$= (z-1) \times \frac{1}{3}\left[\frac{1}{(z-1)(z-\frac{1}{3})}\right]\Bigg|_{z=+1}$$

$$A = \frac{1}{3}\left[\frac{3}{2}\right] = \frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right) \times \frac{X(z)}{z}$$

$$= \left(z-\frac{1}{3}\right) \times \frac{1}{3}\left[\frac{1}{(z-1)\left(z-\frac{1}{3}\right)}\right]\Bigg|_{z=1/3}$$

$$= \frac{1}{3}\left[\frac{1}{z-1}\right]\Bigg|_{z=\frac{1}{3}} \Rightarrow 1/3 \times \frac{1}{(-2/3)} = \frac{1}{3} \times \frac{-3}{2}$$

$$B = -\frac{1}{2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{2(z-1)} - \frac{1}{2\left(z-\frac{1}{3}\right)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{2}\left[\frac{1}{(z-1)} - \frac{1}{\left(z-\frac{1}{3}\right)}\right]$$

Taking z^{-1} both the sides.

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right] \qquad \text{ROC } |z| > 1$$

Q. $X(z) = \dfrac{z(z-1)}{(z+1)^3(z+2)}$, ROC $|z| > 2$.

Sol $X(z) = \dfrac{A}{(z+1)} + \dfrac{B}{(z+1)^2} + \dfrac{C}{(z+1)^3} + \dfrac{D}{(z+2)}$

$$D = (z+2) \times \frac{(z-1)}{(z+1)^3(z+2)}\Bigg|_{z=-2}$$

$$D = \frac{-2-1}{(-2+1)^3} = \frac{-3}{(-1)^3} = 3 \quad \therefore$$

$\therefore D = 3$

$$C = (z+1)^3 \frac{X(z)}{z}\Bigg|_{z=-1}$$

$$= (z+1)^3 \cdot \frac{(z-1)}{(z+1)^3(z+2)}\Bigg|_{z=-1}$$

$$= \frac{z-1}{z+2} = \frac{-2}{1} = -2$$

$\therefore C = -2$

$$B = \frac{1}{1!}\frac{d}{dz}\left[(z+1)^3 \cdot \frac{X(z)}{z}\right]\Bigg|_{z=-1}$$

$$= \frac{1}{1!}\frac{d}{dz}\left[(z+1)^3 \times \frac{z-2}{(z+1)^3(z+2)}\right]\Bigg|_{z=-1}$$

$$= \frac{d}{dz}\left[\frac{z-1}{z+2}\right]\Bigg|_{z=-1} \Rightarrow \therefore B = 1$$

Inverse z transform :-

③ Residue method / complex inversion integral Method / Contour Method :-

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{(bilateral)}$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{(unilateral)}$$

Inverse z transform :-

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

When c represents the circle in z-plane in ROC of $X(z)$

$x(n) = \sum$ Residues of $X(z)z^{n-1}$ at the poles insides the c

distinct · $x(n) = \sum (z - z_i) X(z) z^{n-1}$ at the poles inside the c.

repeated · l-number of poles

$$x(n) = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \sum (z - z_i)^k X(z) z^{n-1})$$

at the poles inside c.

① find the inverse z-transform of $X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}}$ ROC $|z| > 3$ using Residue method.

Sol $\frac{z(z+2)}{z^2+4z+3} = \frac{z(z+2)}{(z+1)(z+3)}$

$x(n) = \sum$ Residue of $X(z)z^{n-1}$ at poles $z = -1, z = -3$

at poles $= -1$

$$x(n) = (z+1) \cdot \frac{z(z+2)}{(z+1)(z+3)} z^{n-1} \Big|_{z=-1}$$

$$x(n) = \frac{z^n (z+2)}{z+3} \Big|_{z=-1}$$

$$= \frac{(-1)^n (-1+2)}{-1+3} = \frac{(-1)^n}{2}$$

$= \frac{1}{2}(-1)^n$ at $z = -1$

$$x(n) = (z+3)\frac{z(z+2)}{(z+1)(z+3)} z^{n-1}\Big|_{z=-3}$$

$$= \frac{z^n(z+2)}{z+1}\Big|_{z=-3}$$

$$= \frac{(-3)^n(-3+2)}{-3+1} = \frac{1}{2}(-3)^n$$

$$\boxed{x(n) = \frac{1}{2}(-1)^n + \frac{1}{2}(-3)^n}$$

$x(n) = \frac{d}{2}$

② Determine the inverse z-transform of

$$X(z) = \frac{3z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]^2} \quad \text{ROC } |z| > \frac{1}{4}$$

Sol^n

$$X(z) = \frac{3z}{\left(z - \frac{1}{2}\right)^2} \quad \text{ROC } |z| > \frac{1}{4}$$

$$x(n) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}} \sum (z - z_i)^k X(z) \cdot z^{n-1} \text{ at poles } z = 1/2$$

$$= \frac{1}{(2-1)!}\frac{d}{dz}\left[\left(z - \frac{1}{2}\right)^2 \cdot \frac{3z^1}{\left(z - \frac{1}{2}\right)^2} \cdot z^{n-1}\right]_{z=1/2}$$

$$= \frac{1}{1}\frac{d}{dz} 3z^n$$

$$= 3n z^{n-1}\Big|_{z=\frac{1}{2}}$$

$$x(n) = 3n\left(\frac{1}{2}\right)^{n-1}$$

③ Determine the inverse z-transform of

$$X(z) = \frac{z}{(z+1)^2(z+2)^3}$$ using Residue method.

Sol^n

$$x(n) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}} \sum (z - z_i)^k X(z) z^{n-1} \text{ at poles } z = -1, -2, -3$$

$$= \frac{1}{1!} \cdot \frac{d}{dz}(z+1)^2 \frac{z}{(z+1)^2(z+2)^3} z^{n-1}\Big|_{z=-1}$$

$$= \frac{d}{dz} \frac{z^n}{(z+2)^3}$$

$$\Rightarrow \left.\frac{(z+2)^3\, n z^{n-1} - 2^n\, 3(z+2)^2}{(z+2)^6}\right|_{z=-1}$$

$$\Rightarrow -(n+3)(-1)^n$$

$$x(n) = \frac{1}{(3-1)!}\frac{d^2}{dz^2}\left[(z+2)^3 \frac{z}{(z+1)^2(z+2)^3} z^{n-1}\right] \text{ at pole } z=-2$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z^n}{(z+1)^2}\right]\Bigg|_{z=-2}$$

$$X(z) = \frac{z}{(z+1)^2(z+2)^3} = (-2)^n\left[0.125n^2 - 1.25n + 3\right].$$

$\therefore$

*when ROC $|z|>2$, both the residues must be positive.

$$\therefore x(n) = -(n+3)(-1)^n u(n) + (-2)^n\left[0.125n^2 - 1.25n + 3\right] u(n)$$

*when ROC $|z|<1$, both the residues will be negative.

$$\therefore x(n) = -(n+3)(-1)^n u(-n-1) + -(-2)^n\left[0.125n^2 - 1.25n + 3\right] u(-n-1)$$

*when ROC $1<|z|<2$, pole at $z=-1$ is positive and $z=-2$ is negative.

$$\therefore x(n) = -(n+3)(-1)^n u(n) - (-2)^n\left[0.125n^2 - 125n + 3\right] u(-n-1)$$

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④ Convolution Method :-

$$Z[x_1(n)] = X_1(z)$$

$$Z[x_2(n)] = X_2(z)$$

$$Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$$= \sum_{k=0}^{n} x_1(n)\, x_2(n-k)$$

Ⓟ Find the inverse z-transform of $X(z) = \frac{z^2}{(z-2)(z-3)}$ using convolution Method.

Sol $Z[x_1(n) * x_2(n)] = \frac{z}{z-2}\,\frac{z}{z-3}$

$$= X_1(z)\, X_2(z)$$

$$x_1(n) * x_2(n) = Z^{-1}\left[\frac{z}{z-2}\right] Z^{-1}\left[\frac{z}{z-3}\right]$$

$= (2)^n U(n) \cdot (3)^n U(n)$

$= \sum_{k=0}^{n} x_1(n) \cdot x_2(n-k)$

$= \sum_{k=0}^{n} (2)^k U(k) \cdot (3)^{n-k} \cdot U(n-k)$

$= 3^n \sum_{k=0}^{n} (2)^k (3)^{-k} U(k) \cdot U(n-k)$

$= 3^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$

$\therefore \sum_{n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a}$

$= \frac{a^0 - a^{n_2+1}}{1-a}$

$x_1(n) * x_2(n) = 3^n \cdot \frac{n - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}}$

$= 3^{n+1} \left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$

$Z\left[x_1(n) * x_2(n)\right] = 3^{n+1} - 2^{n+1}$

$= 3^{n+1} U(n) - 2^{n+1} U(n)$

$\therefore a^n U(n) = \frac{z}{z-a}$

(2) Find the inverse z-transform of $X(z) = \frac{z}{(z-1)(z-\frac{1}{2})}$ using convolution method.

Sol $X_1(z) = \frac{z}{z-1}$

$X_2(z) = \frac{z}{z-\frac{1}{2}}$

$Z\left[X_1(z) \cdot X_2(z)\right] = \frac{z}{z-1} \cdot \frac{z}{z-\frac{1}{2}}$

$= Z^{-1}\left[\frac{z}{z-1}\right] \cdot Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$

$= U(n)\left(\frac{1}{2}\right)^{n-1} U(n-1)$

$X(z) = \frac{z^2}{(z-2)(z-3)}$

$\therefore \left[x(n) = a^n U(n) \quad = \frac{z}{z-a}\right]$