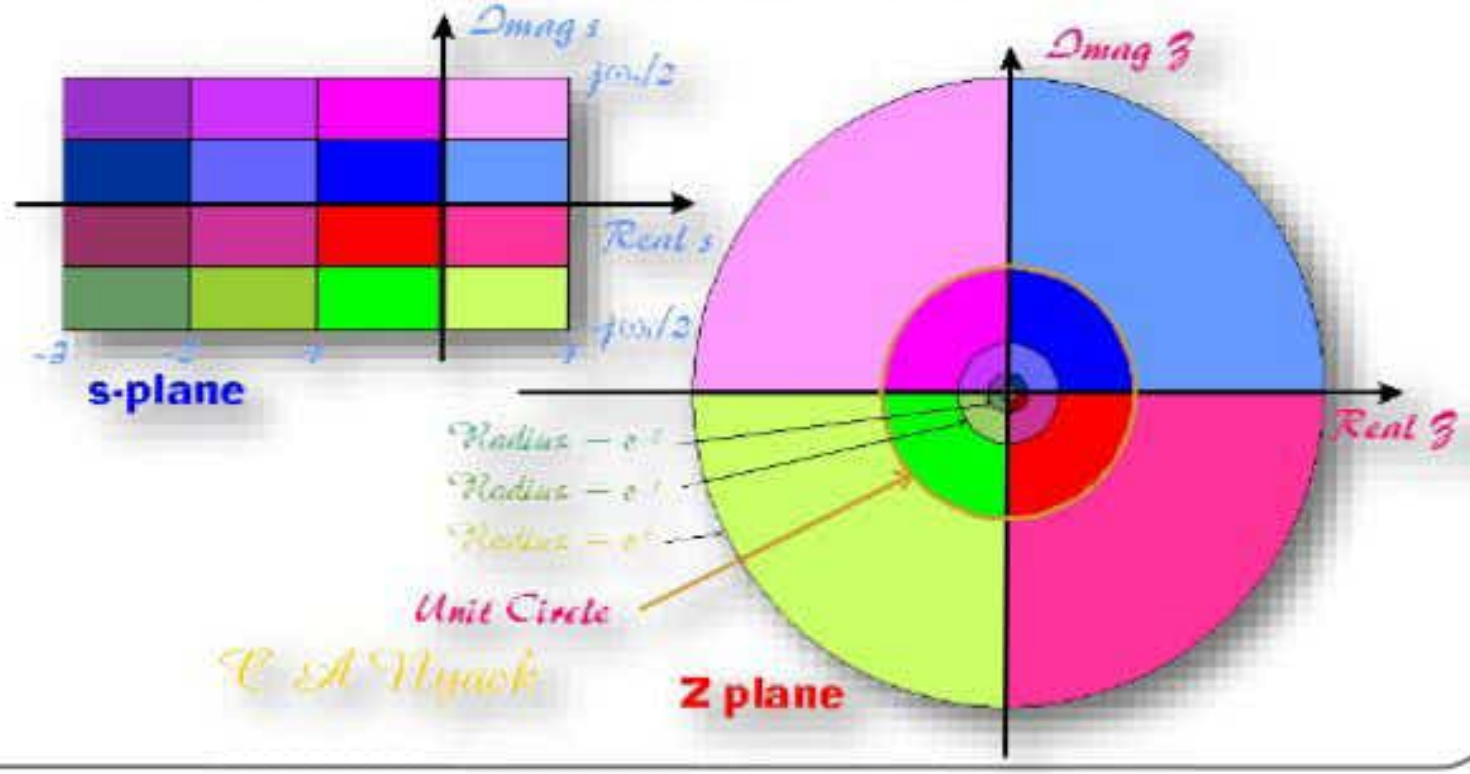
Z-Transform


Jmag s
jω/2
Real s
-jω/2
s-plane
Jmag Z
Real Z
Radius = e^-2
Radius = e^-1
Radius = e^0
Unit Circle
C A Nyack
Z plane

chapter 4

Z- Transform

Introduction :-

continuous time fourier transform of a continuous time signal $x(t)$ is given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

The above integration is valid only when the signal $x(t)$ is absolutely integrable.

i.e; $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

continuous time fourier transform is evaluated on $j\omega$-axis.

By adding real constant σ to $j\omega$-axis we can obtain laplace transform.

#Laplace transform of a continuous time signal $x(t)$ is given by

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} \underbrace{x(t)\, e^{+\sigma t}}_{x_1(t)}\, e^{-j\omega t}\, dt$$

Laplace transform of $x(t)$ = Fourier transform of $x_1(t)$ $\left[x(t)\, e^{-\sigma t}\right]$.

The discrete time fourier transform of discrete time, aperiodic, finite energy signal $x(n)$ is given by

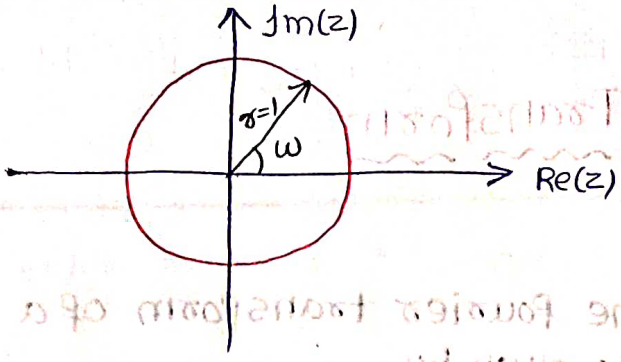
$$\mathcal{F}[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The above sum is valid only when the signal $x(n)$ is absolutely summable.

condition for validity of D.T.F.T summation:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

magnitude of $e^{-j\omega n} = |e^{-j\omega n}| = 1$ and ω indicates phase angle.



D·T·F·T is evaluated for absolutely summable signals.

D·T·F·T is evaluated on unit circle.

For NENP signals, D·T·F·T cannot be evaluated. In continuous time signals, we switch for laplace transform for the signals which are not absolutely integrable.

Similarly in discrete time signals, we switch for z-transform for the signals which are not summable.

By adding real part to jw-axis we can make the transform converge for more values.

Similarly by making certain modifications in magnitude in x(n), we make the transform converge if the signal x(n) is not absolutely summable.

That modifications can be done by z-transform.

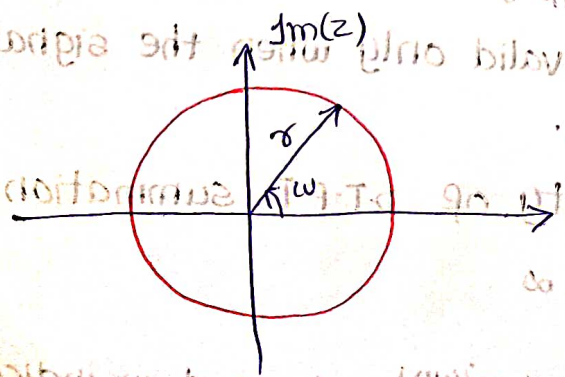
Let X(z) be the z-transform of x(n).

$$x(n) \xleftrightarrow[\text{z-Transform}]{} X(z)$$

where $z = re^{jw}$

Evaluating the discrete time fourier transform about a circle of radius 'r' instead of unit circle gives z-transform.

$$|z| = |re^{jw}|$$ radius $= r$, phase $= \omega$



Therefore the z-transform of the signal $x(n)$ is given by

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(\underbrace{re^{j\omega}}_{z})^{-n}$$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

We knew that

D.T.F.T of the signal $x(n)$ is

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Z-Transform of the signal $x(n)$ is.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \underbrace{x(n) r^{-n}}_{x_1(n)} e^{-j\omega n}$$

Let $x_1(n) = x(n) r^{-n}$

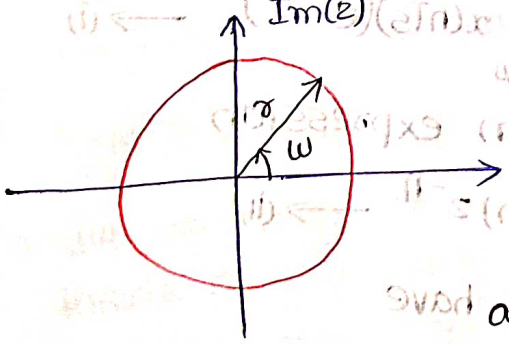
$$X(z) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n}$$

Z-Transform of the discrete time signal $x(n)$ = D.T.F.T of the signal, $x_1(n) [x(n) r^{-n}]$

$Z[x(n)] = D.T.F.T[x(n).r^{-n}]$

$D.T.F.T[x(n)] = Z[x(n)]_{r=1}$

Z-plane is also a complex plane similar to s-plane.



where

r is the radius of circle.

ω is the angle made w.r.t +ve real axis and measured in anti-clockwise direction.

condition for existance of z-Transform :-

We knew that

$$Z[x(n)] = D.T.F.T[x(n) r^{-n}]$$

For existance of D.T.F.T $x(n) r^{-n}$ should be absolutely summable.

For existance of z-Transform of $x(n)$, $x(n) r^{-n}$ should be absolutely summable.

$$\Rightarrow \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

Relation b/w laplace transform and z-Transform :-

Laplace transform : continuous-time domain

z- transform : Discrete time domain

$x(t) \longrightarrow x(nT_s) = x(n)$

$x(t)$: continuous time variable

$x(nT_s)$: Discrete time variable

Laplace transform of $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

After sampling signal is no more continuous and hence for sampled signal, the laplace transform contains summation instead of integration.

$$\mathcal{L}[x(nT_s)] = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-snT_s}$$

$$X(z) = \mathcal{L}[x(nT_s)] = \sum_{n=-\infty}^{\infty} x(nT_s) (e^{+sT_s})^{-n} \longrightarrow (i)$$

we have z- transform expression

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow (ii)$$

comparing (i) & (ii) we have

$$z = e^{sT_s}$$

$$z = e^{(\sigma + j\omega)T_s}$$

$$re^{j\omega} = e^{\sigma T_s} e^{j\omega T_s}$$

Taking magnitude on both sides.

$$|re^{j\omega}| = |e^{\sigma T_s} e^{j\omega T_s}|$$

$$r = e^{\sigma T_s}$$

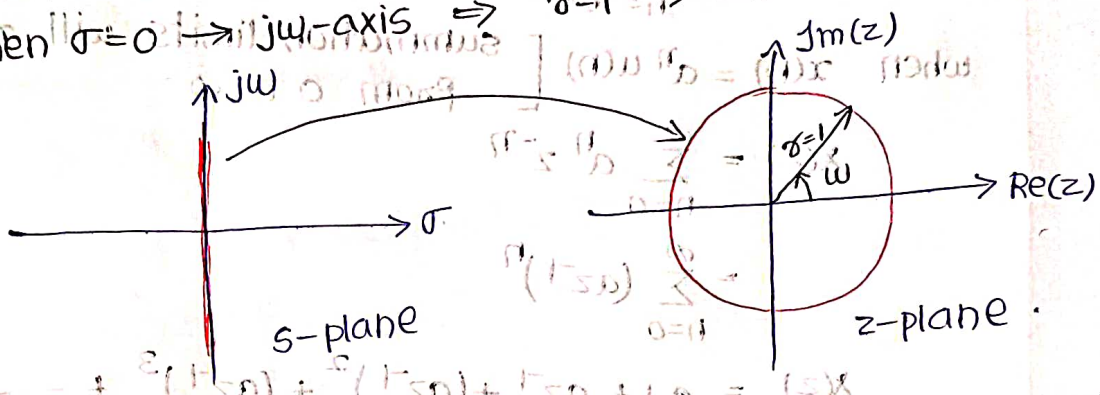
From the above eqn we can obtain laplace Transform and z-Transform mapping for different values of r & σ.

case (i) $\sigma = 0$:-

$$r = e^{\sigma T_s}$$

When $\sigma = 0 \Rightarrow r = 1$ since T_s is constant

When $\sigma = 0 \rightarrow j\omega$-axis $\Rightarrow$ $r = 1 \rightarrow$ unit circle.



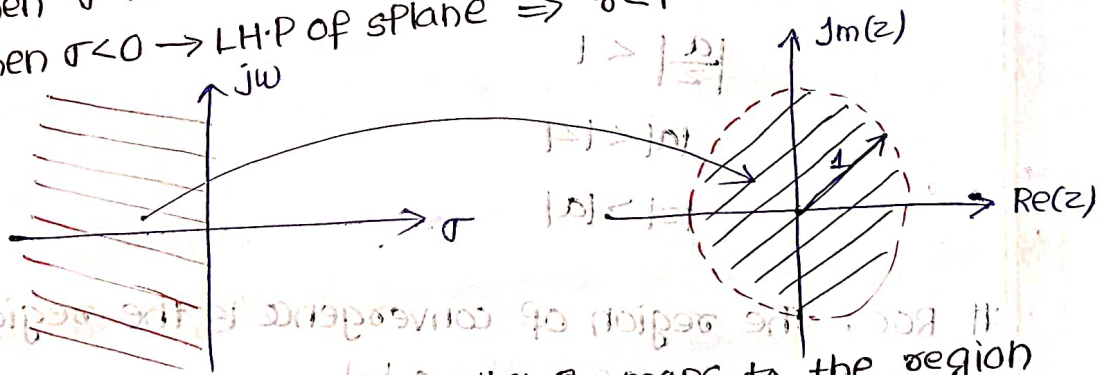
jω-axis in s-plane maps to unit circle in z-plane.

case (ii) $\sigma < 0$:-

$$r = e^{\sigma T_s}$$

When $\sigma < 0 \Rightarrow r < 1$

When $\sigma < 0 \rightarrow$ LH·P of s-plane $\Rightarrow$ $r < 1 \rightarrow$ inside unit circle.

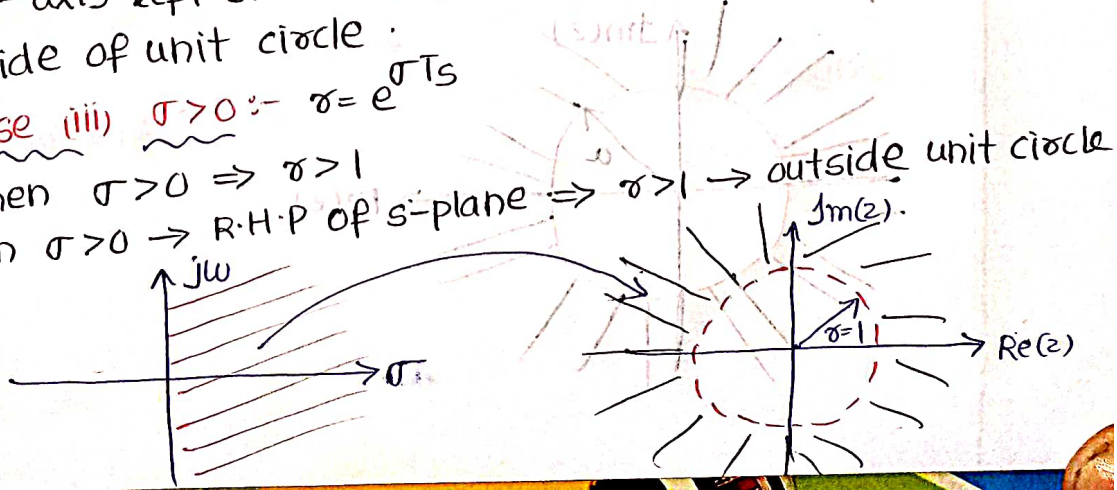


jω-axis left side in s-plane maps to the region inside of unit circle.

case (iii) $\sigma > 0$:- $r = e^{\sigma T_s}$

When $\sigma > 0 \Rightarrow r > 1$

When $\sigma > 0 \rightarrow$ R·H·P of s-plane $\Rightarrow$ $r > 1 \rightarrow$ outside unit circle



When $\sigma > 0$ i.e, RHP on s-plane maps to outside region of unit circle.

questions :

#.q1 ; obtain the z-transform for $x(n) = a^n u(n)$ and hence generalise your results

soln Given signal $x(n) = a^n u(n)$ [Right sided signal]

z-Transform of signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

when $x(n) = a^n u(n)$ [summation limits will range from 0 to ∞]

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots$$

$$X(z) = \frac{1}{1 - az^{-1}}$$

For existance of that z-transform $X(z) = \frac{1}{1-az^{-1}}$

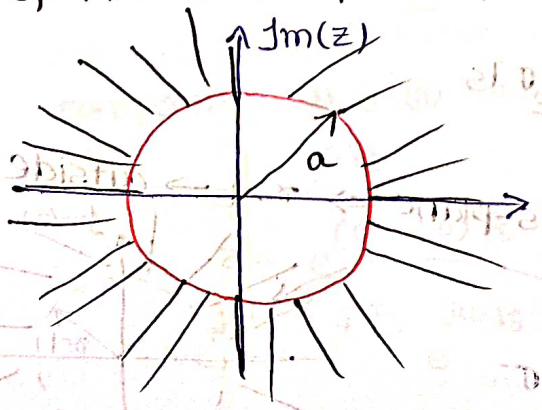
$$|az^{-1}| < 1$$

$$\left|\frac{a}{z}\right| < 1$$

$$|a| < |z|$$

$$|z| > |a|$$

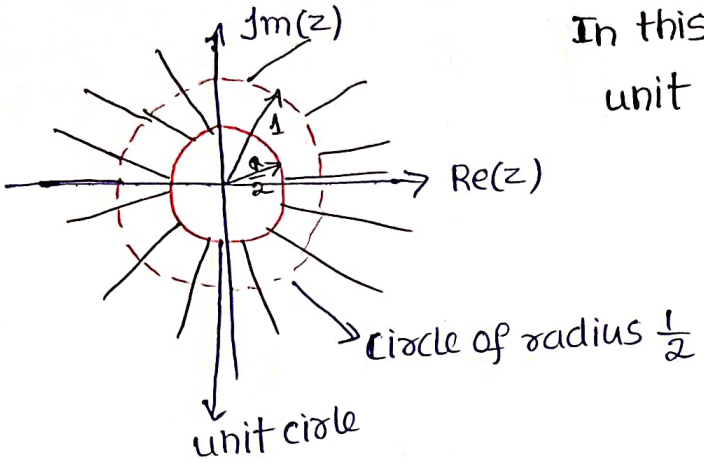
ROC : The region of convergence is the region outside of the circle of radius 'a'.



We knew that, from laplace transform
To make the system stable, ROC must include imaginary axis / jw axis.
similarly in z-transform,
ROC must include unit circle for stability.

case (i) $|a| < 1$:-

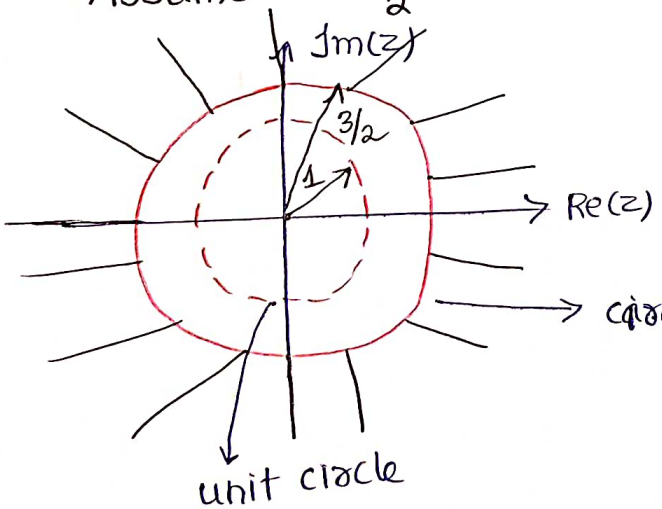
Assume $a = \frac{1}{2}$ which satisfies $|a| < 1$



In this case ROC include unit circle.

case (ii) $|a| > 1$:-

Assume $a = \frac{3}{2}$ which satisfies $|a| > 1$



In this case ROC does not include unit circle.

$\Rightarrow$ The region of convergence of $a^n u(n)$ is $|z| > |a|$

1) $x(n) = 2^n u(n)$ → Unbounded signal
→ NENP signal

2) $x(n) = (\frac{1}{2})^n u(n)$ → Absolute energy signal
→ Bounded signal.

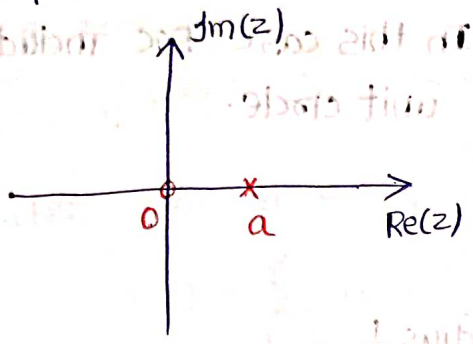
Right sided sequence :-

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

R·O·C : $|z| > |a|$

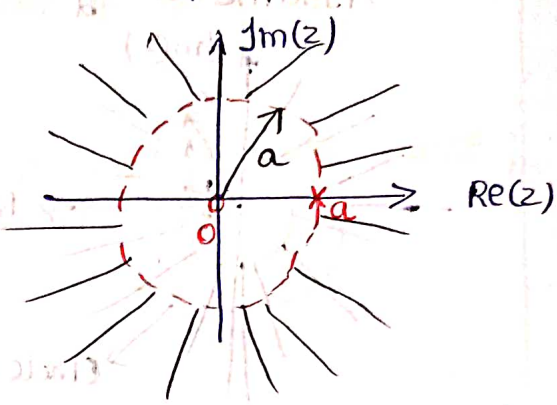
pole-zero plot :



poles at $z = a$

zeroes at $z = 0$

ROC plot :-



ROC is bounded by the pole and is exterior of a circle.

#qa : For the following signals, state ROC & Z-Transform

i) $x(n) = 2^n u(n)$ ii) $x(n) = \left(\frac{2}{3}\right)^n u(n)$

iii) $x(n) = \left(\frac{-3}{4}\right)^n u(n)$ iv) $x(n) = -(3)^{-n} u(n)$

soln i) $x(n) = 2^n u(n)$

ROC = $|z| > 2$ Z-Transform $X(z) = \frac{1}{1-az^{-1}}$

ii) $x(n) = \left(\frac{2}{3}\right)^n u(n)$

R·O·C = $|z| > \frac{2}{3}$ Z-Transform $X(z) = \frac{1}{1-\frac{2}{3}z^{-1}}$

(iii) $x(n) = \left(\frac{-3}{4}\right)^n$

R·O·C : $|z| > \frac{3}{4}$ Z-Transform $X(z) = \frac{1}{1-\left(\frac{-3}{4}\right)z^{-1}}$

$\therefore X(z) = \frac{1}{1+\frac{3}{4}z^{-1}}$

(iv) $x(n) = -(3)^{-n} u(n) = -\left(\frac{1}{3}\right)^n u(n)$

R·O·C = $|z| > \frac{1}{3}$ Z-Transform $X(z) = \frac{-1}{1-\frac{1}{3}z^{-1}}$

#Q3 : consider the following signal.

$x(n) = -(a)^n u(-n-1)$

Find Z-transform and ROC. Generalise your results.

Soln

$x(n) = -(a)^n u(-n-1)$

The signal ranges from $-\infty$ to -1

$\Rightarrow$ The Z-transform of $-a^n u(-n-1)$ is given by

$$X(z) = -\sum_{n=-\infty}^{n=-1} a^n z^{-n} = -\sum_{n=-\infty}^{\infty} (az^{-1})^n$$

Replacing n with $-n$ $\Rightarrow$ summation limits sign changes and swaps

$$X(z) = -\sum_{n=1}^{\infty} (az^{-1})^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$X(z) = -\left[a^{-1}z + (a^{-1}z)^2 + (a^{-1}z)^3 + \cdots\right]$$

$$= -(a^{-1}z)\left[1 + (a^{-1}z) + (a^{-1}z)^2 + \cdots\right]$$

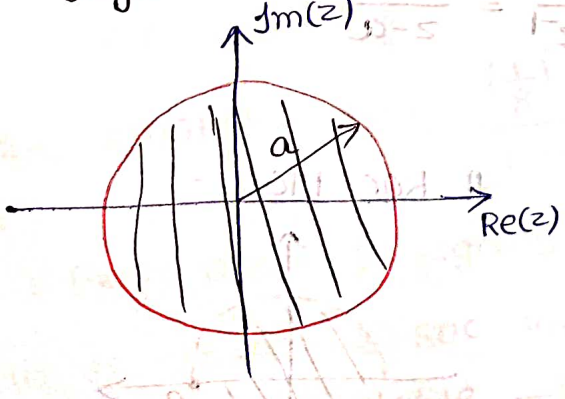
$$X(z) = \frac{-a^{-1}z}{1 - a^{-1}z}$$

Now for validity, $|a^{-1}z| < 1$

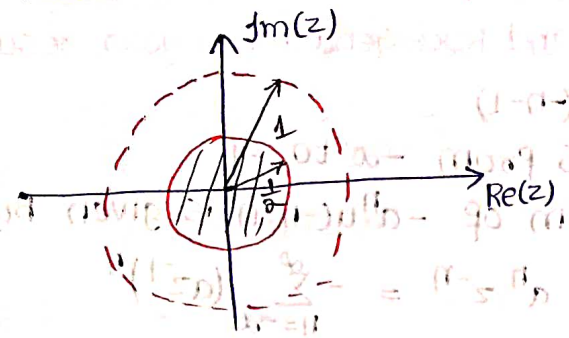
$$\left|\frac{z}{a}\right| < 1$$

$$|z| < |a|$$

$\Rightarrow$ ROC of the signal of the form $a^n u(-n-1)$ is the region inside the circle of radius a.



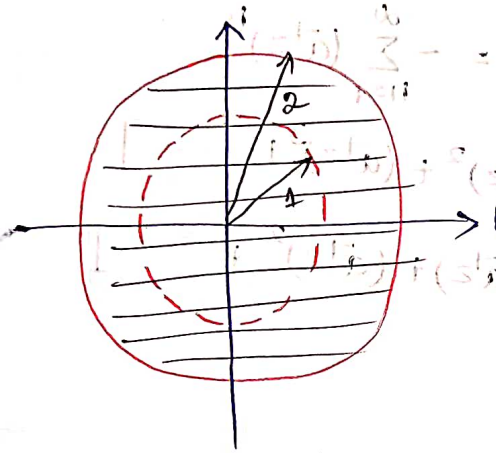
case (i) $|a|<1$:-



Assume $a=\frac{1}{2}$ which satisfies $|a|<1$

ROC: does not include unit circle.

case (ii) $|a|>1$:-



Assume $a=2$ which satiesfies $|a|>1$

ROC includes unit circle.

$$X(z) = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{-a^{-1}z}{(az^{-1}-1)a^{-1}z} = \frac{-1}{az^{-1}-1}$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

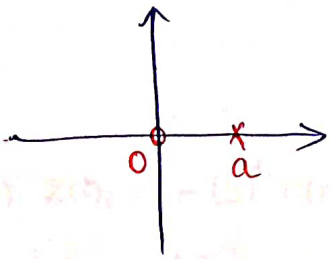
Left sided sequence :-

$$x(n) = -(a^n)\,u(-n-1)$$

$$X(z) = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

R.O.C: $|z|<|a|$

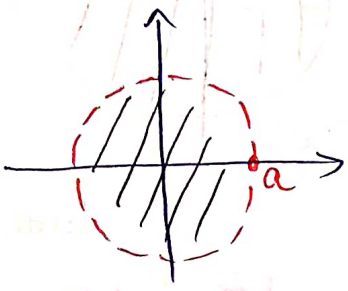
Pole-zero plot :



Poles at $z=a$

zeros at $z=0$

ROC plot :-



ROC is bounded by the pole and is interior of a circle.

Q4 : Define Region of convergence (R.O.C)

obtain ROC & z-transform for the following signals.

i) $x(n) = -(2)^n u(-n-1)$

ii) $x(n) = 3^{-n} u(-n-1)$

iii) $x(n) = \left(\frac{1}{8}\right)^n u(-n-1)$

And tell wheather the signals are bounded or not.

Soln

ROC definition :-

The set of values of z for which z-Transform converges.

$\Rightarrow |X(z)| < \infty$ is called region of convergence.

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |x(n)||z^{-n}| < \infty$$

i) $x(n) = -(2^n) u(-n-1)$

R.O.C : $|z| < 2$

z-Transform $X(z) = \frac{-2^{-1}z}{1-2^{-1}z} = \frac{1}{1-2z^{-1}}$

(ii) $x(n) = 3^{-n} u(-n-1) = \left(\frac{1}{3}\right)^n u(-n-1)$

R.O.C : $|z| < \frac{1}{3}$

z-Transform $X(z) = \frac{\left(\frac{1}{3}\right)^{-1} z}{1-\left(\frac{1}{3}\right)^{-1} z} = \frac{3z}{1-3z} = \frac{-1}{1-\frac{1}{3}z^{-1}}$

iii) $x(n) = \left(\frac{1}{8}\right)^n u(-n-1)$

R.O.C : $|z| < \frac{1}{8}$

z-Transform $X(z) = \frac{\left(\frac{1}{8}\right)^{-1} z}{1-\left(\frac{1}{8}\right)^{-1} z} = \frac{-1}{1-\frac{1}{8}z^{-1}}$

From above 3 signals :

(i) is stable $\rightarrow$ ROC includes unit circle

(ii), (iii) are un stable $\rightarrow$ ROC does not includes unit circle.

#Q5: obtain z-transform & ROC of signal

$x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{-1}{3}\right) u(n)$

Soln

Given $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{-1}{3}\right) u(n)$

$\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$ ROC: $|z| > \frac{1}{2} = R_1$

$\left(\frac{-1}{3}\right)^n u(n) \xleftrightarrow{z} \frac{1}{1-\left(\frac{-1}{3}\right)z^{-1}} = \frac{z}{z+\frac{1}{3}}$ ROC: $|z| > \frac{1}{3} = R_2$

$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$ ROC $= R_1 \cap R_2$

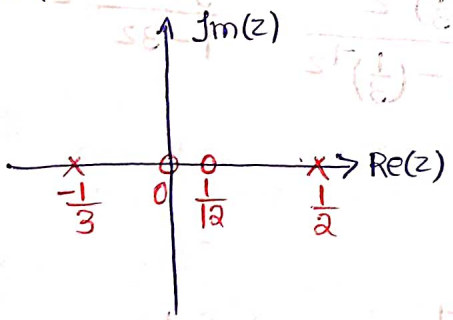
$= \frac{z\left(z+\frac{1}{3}\right) + z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$

$= \frac{z^2 + \frac{z}{3} + z^2 - \frac{z}{2}}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$

$X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$ ROC $= \left(|z| > \frac{1}{2}\right) \cap \left(|z| > \frac{1}{3}\right)$

$= |z| > \frac{1}{2}$

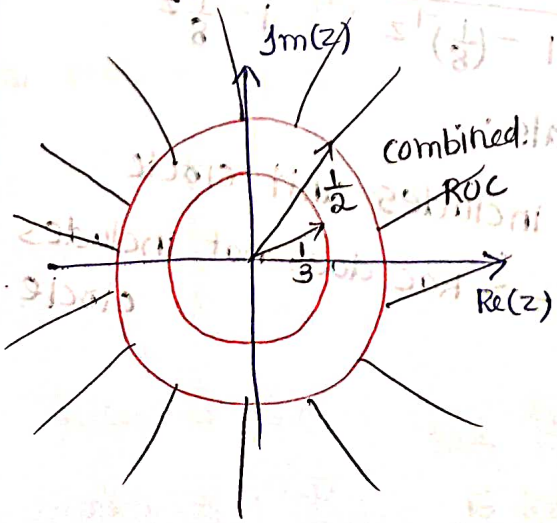
pole-zero plot :-



poles: $z = \frac{1}{2}, z = \frac{-1}{3}$

zeroes: $z = 0$ $z = \frac{1}{12}$

ROC plot :-



If both sides are right sided then common ROC will be right side of right most pole.

Q6: obtain the z-transform and ROC of $x(n)$ is given by

$$\left(\frac{-1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$$

soln

Given $x(n) = \left(\frac{-1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$

$$\left(\frac{-1}{3}\right)^n u(n) \xleftrightarrow{z} \frac{1}{1+\frac{1}{3}z^{-1}} \quad \text{R.O.C}: |z| > \frac{1}{3} = R_1$$

$$-\left(\frac{1}{2}\right)^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{R.O.C}: |z| < \frac{1}{2} = R_2$$

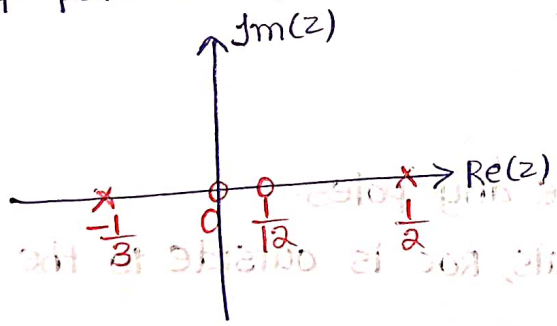
$$\Rightarrow X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \text{R.O.C}: R_1 \cap R_2$$

$$= \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} \quad \left(|z| > \frac{1}{3}\right) \cap \left(|z| < \frac{1}{2}\right)$$

$$= \frac{z^2 - \frac{z}{2} + z^2 + \frac{z}{3}}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} \quad \left[\text{R.O.C}: \frac{1}{3} < |z| < \frac{1}{2}\right]$$

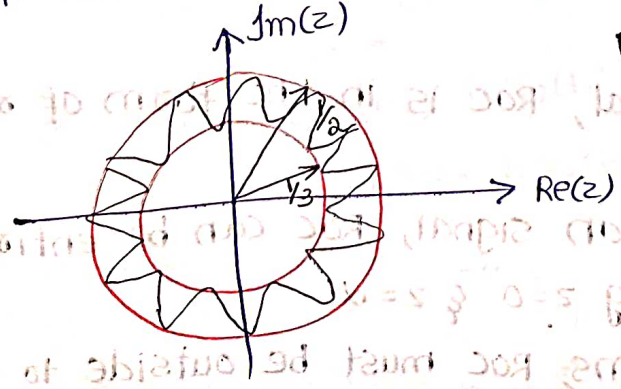
pole-zero plot:-



poles: $z = \frac{1}{2}$ and $z = \frac{-1}{3}$

zeroes: $z = 0$ $z = \frac{1}{12}$

ROC plot:-



ROC is bounded by poles and is ring.

Q7 obtain the z-transform for signal

$$x(n) = a^n u(n) - b^n u(-n-1)$$

when i) $|a| > |b|$

ii) $|a| < |b|$

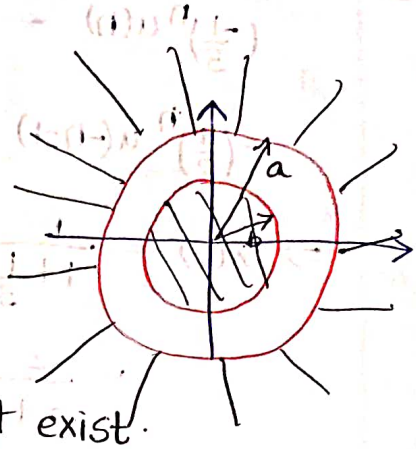
Soln

$$x(n) = a^n u(n) - b^n u(-n-1)$$

i) $$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{b|-bz^{-1}}$$

R.O.C: $(|z| > a) \cap (|z| < b)$

$\Rightarrow$ No common ROC



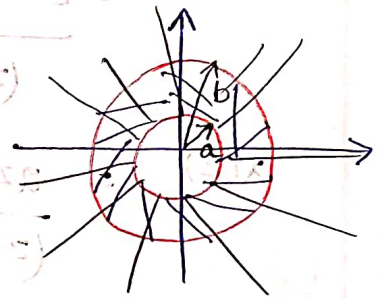
Therefore z-transform also Not exist.

ii) $$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}$$

R.O.C: $(|z| > a) \cap (|z| < b)$

ROC: $a < |z| < b$

(Ring in the z-plane)



For double sided signals, ROC is in the form of Ring in the z-plane.

Properties of ROC :-

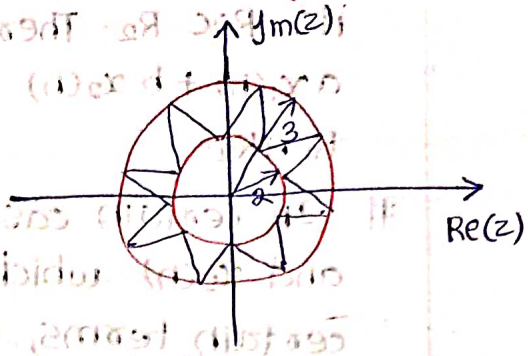
1) ROC does not include any poles.
2) For right sided signals, ROC is outside to the outer most circle.
3) For left sided signals, ROC is inside to the inner most circle.
4) For doubled sided signal, ROC is in the form of ring in z-plane.
5) For finite duration signal, ROC can be entire z-plane excluding $z=0$ & $z=\infty$
6) For causal systems, ROC must be outside to the outside circle
7) For stable systems, ROC must include unit circle.

Q8: Find X(z) and ROC for the following signals.

(i) $x(n) = 2^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$ (ii) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$

soln

i) $x(n) = 2^n u(n) - \left(\frac{1}{3}\right)^n u(-n-1)$



$x(n) = 2^n u(n) - 3^n u(-n-1)$

R.O.C: $(|z|>2) \cap (|z|<3)$

$\Rightarrow$ R.O.C: $2 < |z| < 3$

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{1}{1-3z^{-1}} \qquad 2 < |z| < 3$$

ii) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$

This problem can solved after time reversal property concept.

properties of z-Transform:

1. Linearity property

Let $x_1(n) \xleftrightarrow{z} X_1(z)$ with ROC: R_1 and

$x_2(n) \xleftrightarrow{z} X_2(z)$ with ROC: R_2

then $a\,x_1(n) + b\,x_2(n) \xleftrightarrow{z} aX_1(z) + bX_2(z)$

with ROC: at least $R_1 \cap R_2$

for both unilateral and bilateral z-transforms.

proof:-

The z transform of any signal x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

if $x(n) = a\,x_1(n) + b\,x_2(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \{a\,x_1(n) + b\,x_2(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a\,x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} b\,x_2(n) z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$X(z) = a\,X_1(n) + b\,X_2(n)$$

ROC: since the z- transform $X_1(z)$ is finite within the specified ROC, R_1. similarly $X_2(z)$ is finite within its ROC R_2. Therefore the linear combination $ax_1(n) + bx_2(n)$ should be finite within region $R_1 \cap R_2$.

In certain cases due to interaction b/w $x_1(n)$ and $x_2(n)$ which may lead to cancellation of certain terms, the overall ROC may be larger than the intersection of two regions. on other hand, if there is no common region b/w R_1 and R_2, the z- transform $ax_1(n) + bx_2(n)$ does not exist.

a) Time shifting :-

For the bilateral z- Transform

if $x(n) \xleftrightarrow{z} X(z)$ with ROC R_x

then $x(n-n_0) \xleftrightarrow{z} z^{-n_0} X(z)$ and

$x(n+n_0) \xleftrightarrow{z} z^{n_0} X(z)$ with ROC: R_x

except for the possible deletion / addition of $z=0$ or $z=\infty$.

proof :-

The bilateral z- transform of signal $x(n-n_0)$ is obtained as

$$\mathcal{Z}[x(n-n_0)] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

Let $n - n_0 = \lambda \Rightarrow n = \lambda + n_0$ — when $n \to \infty \Rightarrow \lambda \to \infty$, when $n \to -\infty \Rightarrow \lambda \to -\infty$

$$= \sum_{\lambda=-\infty}^{\infty} x(\lambda) z^{-(\lambda+n_0)}$$

$$= \sum_{\lambda=-\infty}^{\infty} x(\lambda) z^{-\lambda} z^{-n_0}$$

$$= z^{-n_0} \sum_{\lambda=-\infty}^{\infty} x(\lambda) z^{-\lambda}$$

$$\mathcal{Z}[x(n-n_0)] = z^{-n_0} X(z)$$

Similarly $\mathcal{Z}[x(n+n_0)] = z^{n_0} X(z)$

ROC: The ROC of shifted signals is altered because of the terms z^{n_0} or z^{-n_0}, which affects the roots of the denominator in $X(z)$

#3 Time reversal :-

Time reversal property states that time reflection of a discrete sequence in time domain is equivalent to replacing z by $\frac{1}{z}$ in its z- transform.

If $x(n) \xleftrightarrow{z} X(z)$ with ROC: R_x

then $x(-n) \xleftrightarrow{z} X(\frac{1}{z})$ with ROC: $\frac{1}{R_x}$

proof :-

The bilateral z- transform of $x(-n)$ is given by

$$\mathcal{Z}[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

substitute $-n = \lambda$.

$$= \sum_{\lambda=\infty}^{-\infty} x(\lambda)\, z^{\lambda}$$

$$= \sum_{\lambda=-\infty}^{\infty} x(\lambda)(z^{-1})^{-\lambda}$$

$$= X(\tfrac{1}{z})$$

$$\mathcal{Z}[x(-n)] \xleftrightarrow{z} X(\tfrac{1}{z})$$

ROC: $z^{-1} \in R_x$ or $z \in \frac{1}{R_x}$

#Q8 solution:

(ii) we knew that

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}} \quad \text{R.O.C: } |z| > a$$

By applying time reversal on $a^n u(n)$

$$\Rightarrow a^{-n} u(-n) \xleftrightarrow{z} \frac{1}{1-az} \quad \text{R.O.C: } |\tfrac{1}{z}| > a \;\; |z| < \tfrac{1}{a}$$

Given $x(n) = (\frac{1}{2})^n u(-n)$

$$= 2^{-n} u(-n)$$

$$2^{-n} u(-n) \xleftrightarrow{z} \frac{1}{1-2z} \quad |z| < \tfrac{1}{2}$$

#q9: Find x(z) and ROC of following signals

i) $x(n) = \{2, -1, 3, 2, 0, 1\}$ (↑ at 2)

ii) $x(n) = \{3, 2, -1, -4, 1\}$ (↑ at -1)

iii) $x(n) = \{1, 0, -1, 2, 3, 1\}$ (↑ at -1)

Soln We knew that

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ (Bilateral z-Transform)

i) $x(n) = \{2, -1, 3, 2, 0, 1\}$ (↑ at 2)

The range of signal is from $n=0$ to $n=5$

$$\therefore x(z) = \sum_{n=0}^{5} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$x(z) = 2 - 1z^{-1} + 3z^{-2} + 2z^{-3} + 1 \cdot z^{-5}$$

[R.O.C: entire z-plane excluding $z=0$]

ii) $x(n) = \{3, 2, -1, -4, 1\}$ (↑ at -1)

The signal is ranging from $n=-2$ to $n=2$

$$\therefore x(z) = \sum_{n=-2}^{2} x(n) z^{-n}$$

$$= x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2}$$

$$= 3z^2 + 2z - 1 + (-4)z^{-1} + z^{-2}$$

$$x(z) = 3z^2 + 2z - 1 - 4z^{-1} + z^{-2}$$

[ROC: entire z-plane excluding $z=0$ & $z=\infty$]

iii) $x(n) = \{1, 0, -1, 2, 3, 1\}$ (↑ at -1)

The signal is ranging from $n=-2$ to $n=3$

$$x(z) = \sum_{n=-2}^{3} x(n) z^{-n}$$

$$= x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= z^2 - 1 + 2z^{-1} + 3z^{-2} + z^{-3}$$

[ROC: entire z-plane excluding $z=0$ & $z=\infty$]

Q10: Find $X(z)$ & ROC for the following standard signals.

i) $x(n) = u(n) - u(n-3)$

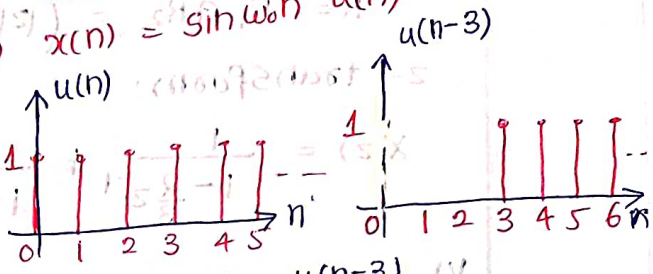
ii) $x(n) = \delta(n)$

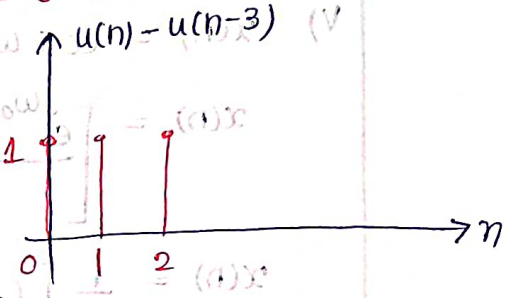
iii) $x(n) = u(n)$

iv) $x(n) = \left(\frac{2}{3}\right)^n u(n) + \left(\frac{-1}{2}\right)^n u(n)$

v) $x(n) = \cos \omega_0 n \; u(n)$

vi) $x(n) = \sin \omega_0 n \; u(n)$





Soln i) $x(n) = u(n) - u(n-3)$

$x(n) = \{1, 1, 1\}$
$\uparrow$

$$X(z) = \sum_{n=0}^{2} x(n) z^{-n}$$

$$= \sum_{n=0}^{2} 1 \cdot z^{-n}$$

$$X(z) = 1 + z^{-1} + z^{-2}$$

R.O.C: Entire z-plane excluding $z = 0$

ii) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(0) z^{-0}$$

$$X(z) = 1$$

R.O.C: Entire z-plane

iii) $x(n) = u(n)$

We knew that

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - az^{-1}} \quad |z| > a$$

If we substitute $a = 1$

$$u(n) \xleftrightarrow{z} \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad |z| > 1$$

Alternate method:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1 - z^{-1}} \quad |z| > 1$$

$$X(z) = \frac{z}{z - 1}$$

iv) $\left(\frac{2}{3}\right)^n u(n) + \left(\frac{-1}{2}\right)^n u(n)$

$R_1 = |z| > \frac{2}{3}$ $\quad R_2 : |z| > \frac{1}{2}$

$R = R_1 \cap R_2 = \left(|z| > \frac{2}{3}\right) \cap \left(|z| > \frac{1}{2}\right) = |z| > \frac{2}{3}$

z- transform:

$$X(z) = \frac{1}{1-\frac{2}{3}z^{-1}} + \frac{1}{1+\frac{1}{2}z^{-1}}$$

V) $x(n) = \cos \omega_0 n \; u(n)$

$$x(n) = \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] u(n)$$

$$x(n) = \frac{1}{2}\left[\left(e^{j\omega_0}\right)^n + \left(e^{-j\omega_0}\right)^n\right] u(n)$$

Taking z- transform on both sides

$$X(z) = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0 z^{-1}}} + \frac{1}{1-e^{-j\omega_0 z^{-1}}}\right]$$

R.O.C: $|e^{j\omega_0 z^{-1}}| > 1$

$|e^{j\omega_0}||z^{-1}| < 1$

$\Rightarrow |z| > 1$

$|e^{-j\omega_0 z^{-1}}| < 1$

$|e^{-j\omega_0}||z^{-1}| < 1$

$|z| > 1$

$|e^{-j\omega_0}| = |e^{j\omega_0}| = 1$

$$\Rightarrow X(z) = \frac{1}{2}\left[\frac{1 - e^{-j\omega_0 z^{-1}} + 1 - e^{j\omega_0 z^{-1}}}{1 - e^{-j\omega_0 z^{-1}} - e^{j\omega_0 z^{-1}} + z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - z^{-1}\left(e^{-j\omega_0} + e^{j\omega_0}\right)}{1 - z^{-1}\left(e^{-j\omega_0} + e^{j\omega_0}\right) + z^{-2}}\right]$$

$$X(z) = \frac{1}{2}\left[\frac{2 - z^{-1}(2\cos\omega_0)}{1 - z^{-1}(2\cos\omega_0) + z^{-2}}\right]$$

$$X(z) = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \qquad \text{R.O.C} : |z| < 1$$

(vi) $x(n) = \sin\omega_0 n \; u(n)$

$$x(n) = \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right) u(n)$$

$$x(n) = \frac{1}{j2}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] u(n)$$

$$x(n) = \frac{1}{j2}\left[\left(e^{j\omega_0}\right)^n u(n) - \left(e^{-j\omega_0}\right)^n u(n)\right]$$

Taking z-Transform on both sides

$$X(z) = \frac{1}{j2}\left[\frac{1}{1 - e^{j\omega_0 z^{-1}}} - \frac{1}{1 - e^{-j\omega_0 z^{-1}}}\right]$$

$$X(z) = \frac{1}{j2}\left[\frac{(1 - e^{-j\omega_0 z^{-1}}) - (1 - e^{j\omega_0 z^{-1}})}{(1 - e^{j\omega_0 z^{-1}})(1 - e^{-j\omega_0 z^{-1}})}\right]$$

$$X(z) = \frac{1}{j2}\left[\frac{1 - e^{-j\omega_0 z^{-1}} - 1 + e^{j\omega_0 z^{-1}}}{1 - e^{-j\omega_0 z^{-1}} - e^{j\omega_0 z^{-1}} + z^{-2}}\right]$$

$$X(z) = \frac{1}{j2}\left[\frac{(e^{j\omega_0} - e^{-j\omega_0}) z^{-1}}{1 - (e^{j\omega_0} + e^{-j\omega_0}) z^{-1} + z^{-2}}\right]$$

$$= \frac{1}{j2}\left[\frac{2j \sin\omega_0 z^{-1}}{1 - 2\cos\omega_0 + z^{-2}}\right]$$

$$X(z) = \frac{1}{j2}\left[\frac{2j \sin\omega_0 z^{-1}}{1 - 2\cos\omega_0 + z^{-2}}\right] \qquad \text{ROC} : |z| < 1$$

$$X(z) = \frac{\sin\omega_0 z^{-1}}{1 - 2\cos\omega_0 + z^{-2}} \qquad \text{R.O.C} : |z| < 1$$

#Q11) The region of convergence of the z-transform of the discrete time signal $x(n) = 2^n u(n)$ will be ____ (Gate - IN - 2008)

Soln $x(n) = 2^n u(n)$

R.O.C : $|z| > 2$

#Q12) If the region of convergence of $x_1(n) + x_2(n)$ is $\frac{1}{3} < |z| < \frac{2}{3}$ then region of convergence of $x_1(n) - x_2(n)$ includes (Gate - IN - 2008)

A) $\frac{1}{3} < |z| < 3$ C) $\frac{2}{3} < |z| < 3$

B) $\frac{2}{3} < |z| < 3$ ✓ D) $\frac{1}{3} < |z| < \frac{2}{3}$

Soln option : D

ROC does n't depends upon sign

#Q13) The region of convergence of the discrete time sequence $x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$ is (Gate-EC-2009)

✓ A) $\frac{1}{3} < |z| < \frac{1}{2}$ B) $|z| < \frac{1}{3}$ C) $|z| > \frac{1}{2}$ D) $2 < |z| < 3$

Soln $x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$

$\downarrow$

$R_1 : |z| > \frac{1}{3} \qquad |z| < \frac{1}{2}$

$\Rightarrow \frac{1}{3} < |z| < \frac{1}{2}$ is combined R.O.C

#4 Differentiation in the z-domain :-

This property states that multiplication of time sequence $x(n)$ with n corresponds to differentiation with respect to z and multiplication of result by $-z$ in the z-domain.

If $x(n) \xleftrightarrow{z} X(z)$ with ROC : R_x

then $n x(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz}$, with ROC : R_x

for both unilateral & bilateral z-transforms.

Proof:

The bilateral z- transform of signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

differentiating w.r.t z on both sides

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} \frac{d}{dz} \{x(n) z^{-n}\}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} \{z^{-n}\}$$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) (-n) z^{-n-1}$$

multiplying both sides by $-z$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$\Rightarrow n x(n) \xleftrightarrow{z} -z \frac{d}{dz} X(z)$$

This operation does not affect R.O.C.

5) Scaling in z- domain:

multiplication of a time sequence with an exponential sequence a^n corresponds to scaling z-domain by a factor of a.

If $x(n) \xleftrightarrow{z} X(z)$ with ROC: R_x

then $a^n x(n) \xleftrightarrow{} X\left(\frac{z}{a}\right)$ with ROC: $|a| R_x$

for both unilateral & bilateral transform.

Proof:

The bilateral z- transform of signal $x(n)$ is given by $X(z) = \mathcal{Z}(x(n)) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

for $a^n x(n)$

z-transform is defined as

$$\mathcal{Z}[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) (za^{-1})^{-n}$$

$$\mathcal{Z}[a^n x(n)] = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$\Rightarrow a^n x(n) \xleftrightarrow{z} X\left(\frac{z}{a}\right)$$

R·O·C: If z is a point in the R·O·C of $X(z)$, then the point $|a|z$ is in the R·O·C of $X(\frac{z}{a})$

#6 Time scaling :-

Time compression :-

Since the time decimation (compression) of discrete time signals is an irreversible process (because some data may lost), therefore the z-transform of $x(n)$ and the decimated sequence $y(n) = x(an)$ not related to each other.

Time expansion :-

In the discrete time domain, time expansion of sequence $x(n)$ is defined as

$$x_K(n) = \begin{cases} x(n/K) & \text{if } n \text{ is multiple of integer } K \\ 0 & \text{otherwise.} \end{cases}$$

Time scaling property of z-transform is derived only for time expansion

If $x(n) \xleftrightarrow{z} X(z)$ with R·O·C: R_x

then $x_K(n) \xleftrightarrow{z} X_K(z) = X(z^K)$ with R·O·C: R_x^K

for both the unilateral and bilateral transform.

proof :-

The unilateral z-transform of expanded sequence $x_K(n)$ is given by $\mathcal{Z}[x_K(n)] = \sum_{n=0}^{\infty} x_K(n) z^{-n}$

$$= x_K(0) + x_K(1) z^{-1} + \cdots + x_K(K) z^{-K} + x_K(K+1) z^{-(K+1)} + \cdots + x_K(2K) z^{-2K} + \cdots + x_K(3K) z^{-3K} + \cdots$$

since the expanded sequence $x_K(n)$ is zero everywhere except when n is a integer multiple of K. This reduces the above transform as follows

$$\mathcal{Z}\{x_K(n)\} = x_K(0) + x_K(K) z^{-K} + x_K(2K) z^{-2K} + x_K(3K) z^{-3K} + \cdots$$

from the interpolated of sequence definition

$x_K(n) = x(n/K)$ $\quad n = 2K \Rightarrow x_K(2K) = x(2)$

$n = 0 \Rightarrow x_K(0) = x(0)$

$n = K \Rightarrow x_K(K) = x(1)$

Thus, we can write

$$Z\{x_K(n)\} = x(0) + x(1)z^{-K} + x(2)z^{-2K} + x(3)z^{-3K} + \cdots$$

$$= \sum_{n=0}^{\infty} x(n)(z^K)^{-n}$$

$$Z\{x_K(n)\} = X(z^K)$$

Time expansion of a discrete time sequence by a factor of K, corresponds to replacing z as z^K in its z- transform.

7) Time differencing property :-

If $x(n) \xleftrightarrow{z} X(z)$ with R.O.C : R_x then

$x(n) - x(n-1) \xleftrightarrow{z} (1-z^{-1})X(z)$ with R.O.C : R_x except for the possible deletion of $z=0$ for both unilateral and bilateral transform

proof:

The z- transform of signal $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

when $x(n) = x(n) - x(n-1)$ then

$$X'(z) = \sum_{n=-\infty}^{\infty} \{x(n) - x(n-1)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} - \sum_{n=-\infty}^{\infty} x(n-1)z^{-n}$$

Substitute $n-1 = \lambda$ in and summation

$$\Rightarrow n = \lambda + 1$$

As $n \to \infty \Rightarrow \lambda \to \infty$

$n \to -\infty \Rightarrow \lambda \to -\infty$

$$X'(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} - \sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-(\lambda+1)}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} - z^{-1}\sum_{\lambda=-\infty}^{\infty} x(\lambda)z^{-\lambda}$$

$$= X(z) - z^{-1}X(z)$$

$$Z[x(n) - x(n-1)] = (1-z^{-1})X(z)$$

8) Time convolution property :-

Time convolution property states that convolution of 2 sequences in time domain corresponds to multiplication in z-transform.

Let $x_1(n) \xleftrightarrow{z} X_1(z)$ R.O.C = R_1

and $x_2(n) \xleftrightarrow{z} X_2(z)$ R.O.C = R_2

then the convolution property states that

$$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z)\, X_2(z)$$ R.O.C at least $R_1 \cap R_2$

for both unilateral and bilateral transforms.

Proof:

we knew that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Taking z transform on both sides

$$x_1(n) * x_2(n) \xleftrightarrow{z} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\, z^{-n}$$

Interchanging the order of 2 summations we get

$$x_1(n) * x_2(n) \xleftrightarrow{z} \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}$$

substituting $n-k=\lambda$ in and summation

$n = \lambda + k$

As $n \to -\infty$ $\lambda \to -\infty$

$n \to \infty$ $\lambda \to \infty$

$$x_1(n) * x_2(n) \xleftrightarrow{z} \sum_{k=-\infty}^{\infty} x_1(k) \sum_{\lambda=-\infty}^{\infty} x_2(\lambda)\, z^{-(\lambda+k)}$$

$$x_1(n) * x_2(n) \xleftrightarrow{z} \sum_{k=-\infty}^{\infty} x_1(k) \sum_{\lambda=-\infty}^{\infty} x_2(\lambda)\, z^{-\lambda} z^{-k}$$

$$x_1(n) * x_2(n) \xleftrightarrow{} \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} \sum_{\lambda=-\infty}^{\infty} x_2(\lambda)\, z^{-\lambda}$$

$$x_1(n) * x_2(n) \xleftrightarrow{} X_1(z)\, X_2(z)$$

#9) conjugation property :-

If $x(n) \xleftrightarrow{z} X(z)$ with R.O.C = Rx

then $x^*(n) \xleftrightarrow{z} X^*(z^*)$ with R.O.C = Rx

If $x(n)$ is real then $X(z) = X^*(z^*)$

proof:

The z-transform of signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The z-transform of signal $x^*(n)$ is given by equation

$$\mathcal{Z}[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} = \sum_{n=-\infty}^{\infty} \{x(n)(z^*)^{-n}\}^* \longrightarrow (i)$$

Taking complex conjugate on both sides for z-transform eqn

$$X^*(z) = \sum_{n=-\infty}^{\infty} \{x(n) z^{-n}\}^*$$

Replace z by z^*

$$X^*(z^*) = \sum_{n=-\infty}^{\infty} \{x(n) (z^*)^{-n}\}^* \longrightarrow (ii)$$

from eqn (i) & (ii)

$$\mathcal{Z}\{x^*(n)\} = X^*(z^*)$$

for real $x(n) \Rightarrow x^*(n) = x(n)$ and then

$$\mathcal{Z}\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= X(z)$$

$$\mathcal{Z}\{x^*(n)\} = \mathcal{Z}\{x(n)\}$$

$$\Rightarrow X^*(z^*) = X(z)$$

10) Initial Value Theorem :-

If $x(n) \xleftrightarrow{z} X(z)$ with R.O.C : Rx

then initial value theorem states that

$$x(0) = \lim_{z \to \infty} X(z)$$

The initial value theorem is valid only for unilateral z-transform.

Proof :-

For a causal signal $x(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + \dots$$

Taking limit as $z \to \infty$ on both sides we get

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left[x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \dots \right]$$

$$= \lim_{z \to \infty} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \dots \right]$$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} x(0)$$

$$\Rightarrow x(0) = \lim_{z \to \infty} X(z)$$

II) Final Value Theorem :-

If $x(n) \xleftrightarrow{z} X(z)$ with R.O.C = R_z

Then final value theorem states that

$$x(\infty) = \lim_{z \to 1} (z-1) X(z)$$

Final value theorem is applicable if $X(z)$ has no poles outside the unit circle. This theorem can be applied to either the unilateral or bilateral z-transform.

[Note:
unilateral time shifting property is given by

$$x(n+n_0) \xleftrightarrow{z} z^{n_0} \left(X(z) - \sum_{m=0}^{n_0-1} x(m) z^{-m} \right)$$

This property is proved later]

$$Z\{x(n+1) - x(n)\} = \lim_{K \to \infty} \left\{ \sum_{n=0}^{K} (x(n+1) - x(n)) z^{-n} \right\} \to (iii)$$

from unilateral z-transform for $n_0 = 1$

$$x(n_0+1) \xleftrightarrow{z} z \left(X(z) - \sum_{m=0}^{0} x(m) z^{-m} \right)$$

$$x(n+1) \xleftrightarrow{z} z\left(X(z) - x(0)\right) \quad \rightarrow (iv)$$

Substituting (iv) in iii

$$\mathcal{Z}\{x(n+1)\} - z\{x(n)\} = \lim_{K\to\infty}\left\{\sum_{n=0}^{K}\left(x(n+1)-x(n)\right)z^{-n}\right\}$$

$$\Rightarrow z\,X(z) - z\,x(0) - X(z) = \lim_{K\to\infty}\left\{\sum_{n=0}^{K}\left(x(n+1)-x(n)\right)z^{-n}\right\}$$

$$\Rightarrow (z-1)X(z) - z\,x(0) = \lim_{K\to\infty}\left\{\sum_{n=0}^{K}\left(x(n+1)-x(n)\right)z^{-n}\right\}$$

Taking limit as $z \to 1$ on both sides

$$\lim_{z\to 1}\left\{(z-1)X(z) - z\,x(0)\right\} = \lim_{z\to 1}\left[\lim_{K\to\infty}\left\{\sum_{n=0}^{K}\left(x(n+1)-x(n)\right)z^{-n}\right\}\right]$$

$$\lim_{z\to 1}(z-1)X(z) - x(0) = \lim_{K\to\infty}\left\{\sum_{n=0}^{K}\left(x(n+1)-x(n)\right)\right\}$$

$$\lim_{z\to 1}(z-1)X(z) - x(0) = \lim_{K\to\infty}\left\{x(1)-x(0)+x(2)-x(1)+x(3)-x(2)+\cdots+x(K+1)-x(K)\right\}$$

$$\lim_{z\to 1}(z-1)X(z) - x(0) = \lim_{K\to\infty}\left\{x(K+1)-x(0)\right\}$$

$$\lim_{z\to 1}(z-1)X(z) - x(0) = x(\infty) - x(0)$$

$$x(\infty) = \lim_{z\to 1}(z-1)X(z)$$

12 Time shifting for unilateral z-transform :-

For the unilateral z-transform

if $x(n) \xleftrightarrow{z} X(z)$ with R.O.C R_x

then $x(n+n_0) \xleftrightarrow{z} z^{n_0}\left(X(z) - \sum_{m=0}^{n_0-1} x(m)\,z^{-m}\right)$

and $x(n-n_0) \xleftrightarrow{z} z^{-n_0}\left(X(z) + \sum_{m=0}^{n_0-1} x(m)\,z^{+m}\right)$

with R.O.C R_x except for possible deletion or addition of $z=0$ or $z=\infty$

Proof :-

The unilateral z-transform of signal $x(n-n_0)$ is given by

$$Z\{x(n-n_0)\} = \sum_{n=0}^{\infty} x(n-n_0) z^{-n}$$

multiplying R.H.S by z^{n_0} and z^{-n_0}

$$Z\{x(n-n_0)\} = \sum_{n=0}^{\infty} x(n-n_0) z^{-n} \cdot z^{n_0} z^{-n_0}$$

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{n=0}^{\infty} x(n-n_0) z^{-(n-n_0)}$$

Substitute $n-n_0 = \lambda$

$\Rightarrow n = \lambda + n_0$

As $n \to 0 \Rightarrow \lambda \to -n_0$

$n \to \infty \Rightarrow \lambda \to \infty$

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{\lambda=-n_0}^{\infty} x(\lambda) z^{-\lambda}$$

$$= z^{-n_0} \sum_{\lambda=-n_0}^{-1} x(\lambda) z^{-\lambda} + z^{-n_0} \sum_{\lambda=0}^{\infty} x(\lambda) z^{-\lambda}$$

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{\lambda=0}^{\infty} x(\lambda) z^{-\lambda} + z^{-n_0} \sum_{\lambda=-n_0}^{-1} x(\lambda) z^{-\lambda}$$

Replace $\lambda = -\lambda$ in 2nd summation

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{\lambda=0}^{\infty} x(\lambda) z^{-\lambda} + z^{-n_0} \sum_{\lambda=1}^{n_0} x(-\lambda) z^{\lambda}$$

change of variable as

$\lambda \to n$ in 1st summation

$\lambda \to m$ in 2nd summation

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{n=0}^{\infty} x(n) z^{-\lambda} + z^{-n_0} \sum_{m=1}^{n_0} x(-m_0) z^{+m}$$

$$Z\{x(n-n_0)\} = z^{-n_0} \left(X(z) - \sum_{m=0}^{n_0-1} x(-m) z^{m} \right).$$

Q14) If $x(n) = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^{n} u[n]$, then the region of convergence (R.O.C) of its z-transform in the z-plane will be

(A) $\frac{1}{3} < |z| < 3$ (B) $\frac{1}{3} < |z| < \frac{1}{2}$ ✓ (C) $\frac{1}{2} < |z| < 3$ D) $\frac{1}{3} < |z|$

Soln

$$x(n) = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^{n} u[n]$$

Let $x_1[n] = \left(\frac{1}{3}\right)^{|n|}$ $\qquad |n| = \begin{cases} n & n \geq 0 \\ -n & n < 0 \end{cases}$

$$x_1[n] = \left(\frac{1}{3}\right)^{n} u[n] + \left(\frac{1}{3}\right)^{-n} u(-n-1)$$

$$x_1[n] = \left(\frac{1}{3}\right)^{n} u(+n) + 3^{n} u(-n-1)$$

$$x_2[n] = -\left(\frac{1}{2}\right)^{n} u[n]$$

$$x(n) = \underset{\text{R.O.C } |z| > \frac{1}{3}}{\left(\frac{1}{3}\right)^{n} u(n)} + \underset{|z| < 3}{3^{n} u(-n-1)} - \underset{|z| > \frac{1}{2}}{\left(\frac{1}{2}\right)^{n} u(n)}$$

R.O.C for combined signal $= \left(|z| > \frac{1}{3}\right) \cap \left(|z| < 3\right) \cap \left(|z| > \frac{1}{2}\right)$

$$\Rightarrow \text{R.O.C} = \frac{1}{2} < |z| < 3$$

Q15) Consider the discrete time signal given by

$x(n) = (-0.25)^{n} u(n) + (0.5)^{n} u(-n-1)$.

The region of convergence of its z-transform would be

Soln

$$x(n) = \underset{|z| > 0.25}{(-0.25)^{n} u(n)} + \underset{|z| < 0.5}{(0.5)^{n} u(-n-1)}$$

$$\text{R.O.C} = 0.25 < |z| < 0.5$$

The annular region b/w the two circles, both centered at origin and having radii 0.25 & 0.5

Q16) If the signal $x[n] = (2.0)^{|n|}, \; -\infty < n < \infty$ then the R.O.C of its transform will be ——

Soln

$$x(n) = 2^{|n|}$$

$$x(n) = 2^{n} u(n) + 2^{-n} u(-n-1)$$

$\Rightarrow x(n) = 2^n u(n) + \left(\frac{1}{2}\right)^n u(-n-1)$

$\downarrow \qquad \downarrow$

$|z| > 2 \qquad |z| < \frac{1}{2}$

combined R.O.C $= (|z|>2) \cap (|z|<\frac{1}{2})$

No common R.O.C exist.

Q17) consider the sequence $x(n) = a^n u(n) + b^n u(n)$ where $u(n)$ denotes the unit step sequence and $0<|a|<|b|<1$. The region of convergence of z-transform of $x(n)$ is ——

soln

$x(n) = a^n u(n) + b^n u(n)$

$\downarrow \qquad \downarrow$

$|z| > |a| \qquad |z| > |b|$

since, $|b| > |a|$

combined R.O.C is $|z| > b$

Q18) Determine the z-transform of $x(n)$.

$$x(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else where} \end{cases}$$

soln

$$x(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{else where} \end{cases}$$

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ (Bilateral z- Transform definition)

$X(z) = \sum_{n=0}^{N-1} 1 \cdot z^{-n}$

$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots + z^{-(N-1)}$

$= \sum_{n=0}^{N-1} (z^{-1})^n$

$= \frac{1 - (z^{-1})^N}{1 - z^{-1}}$

$X(z) = \frac{1 - z^{-N}}{1 - z^{-1}}$ if $z \neq 1$

$$X(z) = \begin{cases} \frac{1-z^{-N}}{1-z} & z \neq 1 \\ N & z = 1 \end{cases}$$

#Q19: Determine z-transform of following signals

i) $x(n) = a^n \cos\omega_0 n\, u(n)$ (ii) $x(n) = a^n \sin\omega_0 n\, u(n)$

Soln: We knew that scaling property in z-domain

$$a^n x(n) \longleftrightarrow X(z)$$

i) $x(n) = a^n \cos\omega_0 n\, u(n)$

we have already

$$\cos\omega_0 n\, u(n) \xleftrightarrow{z} \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$$

$$\& \quad \cos\omega_0 n\, u(n) \xleftrightarrow{z} \frac{1 - \frac{1}{z}\cos\omega_0}{1 - \frac{2}{z}\cos\omega_0 + \frac{1}{z^2}}$$

$$a^n \cos\omega_0 n\, u(n) \longleftrightarrow \frac{1 - \frac{1}{(z/a)}\cos\omega_0}{1 - \frac{2}{(z/a)}\cos\omega_0 + \frac{1}{(z/a)^2}}$$

$$a^n \cos\omega_0 n\, u(n) \longleftrightarrow \frac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$$

ii) $x(n) = a^n \sin\omega_0 n\, u(n)$

We already have

$$\sin\omega_0 n\, u(n) \xleftrightarrow{z} \frac{z^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + z^{-2}}$$

$$\sin\omega_0 n\, u(n) \xleftrightarrow{z} \frac{\frac{\sin\omega_0}{z}}{1 - \frac{2}{z}\cos\omega_0 + \frac{1}{z^2}}$$

By z-transform scaling

$$a^n \sin\omega_0 n\, u(n) = \frac{\frac{1}{z/a}\sin\omega_0}{1 - \frac{2}{z/a}\cos\omega_0 + \frac{1}{(z/a)^2}}$$

$$a^n \sin\omega_0 n\, u(n) = \frac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$$

Analysis of LTI systems using z-transform :-

The z-transform is very useful tool in analysis of discrete LTI systems. As laplace transform is used in solving differential equations which describe continuous LTI system, the z-transform is used to solve difference equation which describe the discrete LTI system.

similar to laplace transform, for continuous time domain, the z-transform gives transfer function of discrete LTI systems which is the ratio of the z-transform of the output variable to the z-transform of the input variable when initial condition are set to zero.

Response of LTI discrete time systems :-

A discrete-time LTI system is always described by a linear constant coefficient difference equation given as follows

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$a_N y(n-N) + a_{N-1} y(n-(N-1)) + a_{N-2} y(n-(N-2)) + \cdots + a_0 y(n)$$
$$= b_M x(n-M) + b_{M-1} x(n-(M-1)) + b_{M-2} x(n-(M-2)) + \cdots + b_0 x(n)$$

where N is the order of system.

From time shifting property of z-transform

$$x(n-n_0) \xleftrightarrow{z} z^{-n_0} X(z)$$

$\Rightarrow$ The above difference eqn is changed into algebraic eqn

$$a_N z^{-N} Y(z) + a_{N-1} z^{-(N-1)} Y(z) + a_{N-2} z^{-(N-2)} Y(z) + \cdots + a_0 Y(z) =$$
$$b_M z^{-M} X(z) + b_{M-1} z^{-(M-1)} X(z) + b_{M-2} z^{-(M-2)} X(z) + \cdots + b_0 X(z)$$

$$\Rightarrow Y(z) \{ a_N z^{-N} + a_{N-1} z^{-(N-1)} + a_{N-2} z^{-(N-2)} + \cdots + a_0 \}$$
$$= X(z) \{ b_M z^{-M} + b_{M-1} z^{-(M-1)} + b_{M-2} z^{-(M-2)} + \cdots + b_0 \}$$

$\Rightarrow$ Transfer function

$$\frac{Y(z)}{X(z)} = \frac{a_N z^{-N} + a_{N-1} z^{-(N-1)} + a_{N-2} z^{-(N-2)} + \cdots + a_0}{b_M z^{-N} + b_{M-1} z^{-(N-1)} + b_{N-2} z^{-(N-2)} + \cdots + b_0}$$

This eqn can be solved for $Y(z)$ to find the response $y(n)$

The solution or total response $y(n)$ consists of 2 parts: 1) zero input response / Free Response / Natural Response.

2) zero state response / Forced response

1) zero input response / free response / Natural response :-

The zero input response $y_{zi}(n)$ is mainly due to initial conditions of the system.

The zero input response is obtained from system difference equation when $x(n) = 0$

By substituting $x(n) = 0$ and $y(n) = y_{zi}(n)$ in system difference eqn we get

$$a_N y_{zi}(n-N) + a_{N-1} y_{zi}(n-(N-1)) + a_{N-2} y_{zi}(n-(N-2)) + \cdots + a_1 y_{zi}(n-1) + a_0 y_{zi}(n) = 0$$

on taking z-transform of above eqn with given initial conditions we can form an eqn for $Y_{zi}(z)$. The zero-input response $y_{zi}(n)$ is given by inverse z-transform of $Y_{zi}(z)$.

Note :-

The zero input response is also called the natural response of the system and it is denoted by $y_{zi}(n)$

2) zero state Response / Forced Response :-

The zero-state response $y_{zs}(n)$ is the response of the system due to the input signal and with zero initial conditions. The zero-state response is obtained from the difference eqn governing the system for specific input $x(n)$ for $n \geq 0$ with zero initial conditions

substituting $y(n) = y_{zs}(n)$ we get

$$a_N y_{zs}(n-N) + a_{N-1} y_{zs}(n-(N-1)) + a_{N-2} y_{zs}(n-(N-2)) + \cdots + a_1 y_{zs}(n-1) + a_0 y_{zs}(n) = b_m x(n-m) + b_{m-1} x(n-(m-1)) + b_{m-2} x(n-(m-2)) + \cdots + b_1 x(n-1) + b_0 x(n)$$

Taking z-transform of the above equation with zero initial conditions for output (i.e, $y(-1) = y(-2) = \cdots = 0$) we can form an equation for $Y_{ss}(z)$

The zero state response $y_{ss}(n)$ is given by inverse z-transform of $Y_{ss}(z)$

Note :-

The zero state response is also called the forced response of the system and is denoted as $y_f(n)$.

Total response :-

The total response $y(n)$ is the response of the system due to input signal and initial conditions. The total response is given by sum of zero input response $y_{zi}(n)$ and zero state response $y_{zs}(n)$

$$y(n) = y_{zi}(n) + y_{zs}(n)$$

Impulse Response & Transfer function :

System function or transfer function is defined as ratio of the z-transform of the output $y(n)$ and the input $x(n)$ with zero initial conditions.

Let $x(n) \xleftrightarrow{z} X(z)$ is the input and $y(n) \xleftrightarrow{z} Y(z)$ is the output of an LTI discrete-time system having impulse response $h(n) \xleftrightarrow{z} H(z)$.

The response $y(n)$ of the discrete time system is given by convolution sum of input and impulse response as

$$y(n) = x(n) * h(n)$$

By applying convolution property of z-transform we obtain

$$Y(z) = X(z)\,H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

where $H(z)$ is defined as the transfer function of the system. It is the z-transform of the impulse response.

Alternatively we can say that the inverse z-transform of the transfer function is the impulse response of the system

Impulse response

$$h(n) = z^{-1}\{H(z)\} = z^{-1}\left\{\frac{Y(z)}{X(z)}\right\}$$

Stability and causality of LTI discrete systems using z-Transform :-

z- transform is also used in characterization of discrete LTI systems.

1) causality :-

A linear time-invarient discrete time system is said to be causal if the impulse response $h(n) = 0$ for $n<0$ and it is therefore right sided. The R.O.C of such a system $H(z)$ is a exterior of a circle.

** If $H(z)$ is rational then the systems are said to be causal if

i) The R.O.C is the exterior of a circle outside the outermost pole and

ii) The degree of the numerator polynomial of $H(z)$ should be less than or equal to degree of denominator polynomial.

2) Stability :-

An LTI discrete-time system is said to be BIBO stable if the impulse response $h(n)$ is absolutely summable.

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

z- transform of $h(n)$ is given as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

Let $z = e^{j\omega}$ (which describes a unit circle in z-plane)

Then $|H(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \right|$

$$\leq \sum_{n=-\infty}^{\infty} \left| h(n) e^{-j\omega n} \right|$$

$$= \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

which is the condition for stability.

*** An LTI system is stable if the R.O.C of its system ~~resp~~ function H(z) contains unit circle $|z| = 1$

** An LTI discrete time system with rational system function H(z) is said to be both causal and stable if all the poles of H(z) lies inside the unit circle.

Block diagram representation :-

In z-domain, the input and output relation of an LTI discrete time system is represented by transfer function which is a rational function of z, is given by

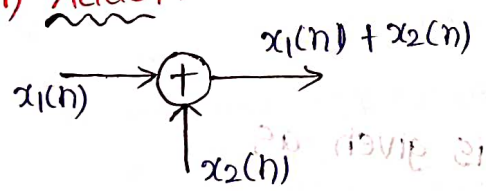
$$H(z) = \frac{Y(z)}{X(z)}$$

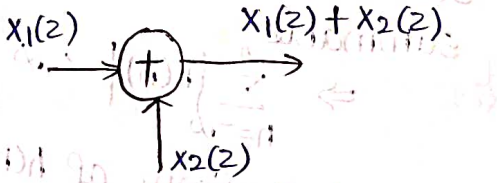
$$H(z) = \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m}{a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_{N-1} z + a_N}$$

where N = order of the system $m \leq N$ and $a_0 = 1$

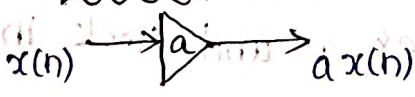
The above transfer function is realized using unit delay elements, unit advancements, adders, and multipliers.

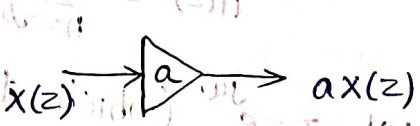
1) Adder :



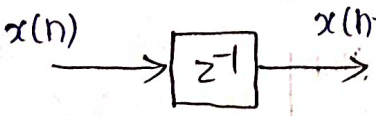


2) constant multiplier :



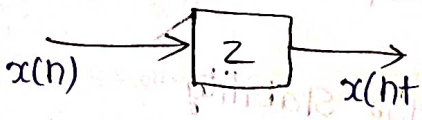


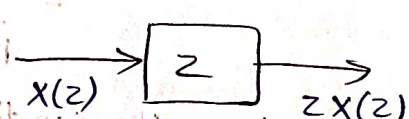
3) unit delay element :





4) unit advance element :-





q20 : The transfer function of a discrete time LTI system is given by

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

consider the following statements

S1 : The system is causal & stable for R.O.C $|z| > \frac{1}{2}$

S2 : The system is stable but not causal for R.O.C $|z| < \frac{1}{4}$

S3 : The system is neither stable nor causal for R.O.C $\frac{1}{4} < |z| < \frac{1}{2}$

Which of the following statement(s) is/ are valid?

Soln $H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$

$-3/4$

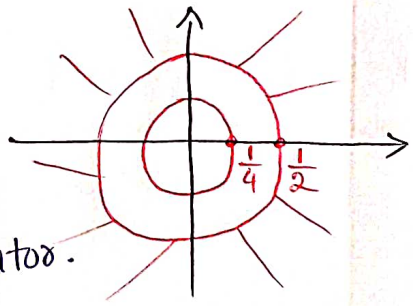
$-\frac{1}{2} \times -\frac{1}{4} = \frac{1}{8}$

$$= \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{2 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) - \frac{1}{4}z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$H(z) = \frac{2 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

poles : $z = \frac{1}{2}$ $z = \frac{1}{4}$



The degree of numerator is less than degree of denominator.

$\Rightarrow$ system is causal

And all poles are lies inside the circle.

$\Rightarrow$ system is stable { when $|z| > \frac{1}{2}$ }

And when $\frac{1}{4} < |z| < \frac{1}{2}$ $\Rightarrow$ system is neither causal neither stable.

#qa0 Find $x(z)$ if $x(n) = (n+1)u(n)$

Soln $x(n) = (n+1)u(n)$

$x(n) = n u(n) + u(n)$

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

$n u(n) \xleftrightarrow{z} -z \frac{d}{dz}\{z(u(n))\}$ [differentiation in z- domain

$= -z \frac{d}{dz}\left\{\frac{z}{z-1}\right\}$

$= -z \left[\frac{(z-1)-z}{(z-1)^2}\right]$

$n u(n) \xleftrightarrow{z} \frac{z}{(z-1)^2}$

$x(n) = n u(n) + u(n)$

$X(z) = \frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z + z(z-1)}{(z-1)^2}$

$x(z) = \frac{z+z^2-z}{(z-1)^2} = \left(\frac{z}{z-1}\right)^2$ with R.O.C $|z|>1$

#qa1) Find z- transform for following signals.

i) $x(n) = a^{n-1}u(n-1)$ ii) $x(n) = nu(n)$

iii) if $x(n) = x_1(n) * x_2(n)$

where $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$ $x_2(n) = \left(\frac{1}{5}\right)^n u(n)$

Soln i) $x(n) = a^{n-1}u(n-1)$

we knew that

$a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$

After time shifting

$a^{n-1}u(n-1) \xleftrightarrow{} z^{-1}\frac{1}{1-az^{-1}}$

ii) $x(n) = nu(n)$

we knew that

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

By differentiation property

$$nu(n) \xleftrightarrow{z} -z\frac{d}{dz}\{z(x(n))\}$$

$$nu(n) \xleftrightarrow{z} -z\frac{d}{dz}\left\{\frac{z}{z-1}\right\}$$

$$nu(n) \xleftrightarrow{z} -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$nu(n) \xleftrightarrow{z} \frac{z}{(z-1)^2}$$

iii) $x(n) = x_1(n) * x_2(n)$.

By convolution property

$$X(z) = X_1(z)\, X_2(z)$$

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \qquad x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$X_1(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \qquad X_1(z) = \frac{1}{1-\frac{1}{5}z^{-1}}$$

$$X(z) = X_1(z)\, X_2(z)$$

$$= \left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{5}z^{-1}}\right)$$

$$= \frac{1}{1-\frac{1}{3}z^{-1}-\frac{1}{5}z^{-1}+\frac{1}{15}z^{-2}}$$

$$X(z) = \frac{1}{1-\left(\frac{1}{3}+\frac{1}{5}\right)z^{-1}+\frac{1}{15}z^{-2}}$$

$$X(z) = \frac{1}{1-\frac{8}{15}z^{-1}+\frac{1}{15}z^{-2}}$$

#q22) consider two discrete time signals $x_1(n) = \{1, 1\}$ and $x_2(n) = \{1, 2\}$ for $n = 0, 1$. The z-transform of the convoluted sequence $x(n) = x_1(n) * x_2(n)$ is

Soln

$$x_1(n) = \underset{\uparrow}{\{1, 1\}} = \delta(n) + \delta(n-1)$$

$$x_2(n) = \underset{\uparrow}{\{1, 2\}} = \delta(n) + \delta(n-2)$$

By time shifting property & Linearity property

$x_1(n) = \delta(n) + \delta(n-1) \xleftrightarrow{z} 1+z^{-1} = X_1(z)$

$x_2(n) = \delta(n) + 2\delta(n-1) \xleftrightarrow{z} 1+2z^{-1} = X_2(z)$

$x(n) = x_1(n) * x_2(n)$

$\Rightarrow X(z) = X_1(z)\, X_2(z)$

$= (1+z^{-1})(1+2z^{-1})$

$= 1+2z^{-1}+z^{-1}+2z^{-2}$

$X(z) = 1+3z^{-1}+2z^{-2}$

q23) Find $f(\infty)$

i) $F(z) = \dfrac{z+2}{(z-0.8)^2}$

ii) $F(z) = \dfrac{z+1}{3(z-1)(z+0.9)}$

iii) $X(z) = \dfrac{3z+4}{(z-1)(z+1)}$

iv) $X(z) = \dfrac{z(8z-7)}{4z^2-7z+3}$

Soln From final value theorem we can say

$$f(\infty) = \lim_{z\to 1} (z-1)\, F(z)$$

if all poles lies inside the unit circle.

i) $F(z) = \dfrac{z+2}{(z-0.8)^2}$ $z = 0.8$ (repeated twice)

⇓

Both poles lies inside the unit circle.

$$f(\infty) = \lim_{z\to 1} (z-1)\left(\frac{z+2}{(z-0.8)^2}\right) = 0$$

~~(ii) $F(z) = \dfrac{3z+4}{(z-1)(z+1)}$~~

ii) $F(z) = \dfrac{z+1}{3(z-1)(z+0.9)}$

poles $z = -0.9$ → Lie inside the unit circle

$z = 1$ → lies on the unit circle but it will cancel with $(z-1)$ term in final value theorem

$$f(\infty) = \lim_{z\to 1} (z-1)\left(\frac{z+1}{3(z-1)(z+0.9)}\right) = \frac{2}{3(1+0.9)} = 0.3508$$

iii) $F(z) = \dfrac{3z+4}{(z-1)(z+1)}$

poles: $z=-1$ $z=1$

$z=1$ lies on the unit circle but get cancelled with $(z-1)$ term in final value theorem

$z=-1$ lies on the unit circle but does not get cancelled with $(z-1)$ term in final value theorem.

$\Rightarrow$ Final value theorem is not applied.

iv) $F(z) = \dfrac{z(8z-7)}{4z^2-7z+3}$

$4z^2-7z+3 = 0$

$4z^2-4z-3z+3=0$

$4z(z-1)-3(z-1)=0$

$(4z-3)(z-1)=0$

poles: $z=1$ $z=\frac{3}{4}$

$z=1$ lies on unit circle but get cancelled with $(z-1)$ term in final value theorem

$z=\frac{3}{4}$ lies inside the unit circle.

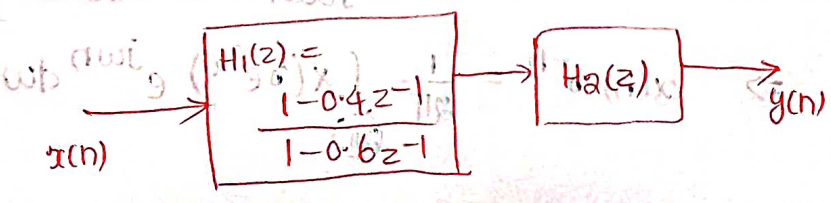
$\Rightarrow F(\infty) = \lim_{z\to 1} (z-1)\left(\dfrac{z(8z-7)}{(z-1)(4z-3)}\right)$

$= \lim_{z\to 1} \dfrac{z(8z-7)}{4z-3}$

$= \dfrac{1(8-7)}{4-3}$

$F(\infty) = 1$

Q24) Two systems $H_1(z)$ & $H_2(z)$ are connected in cascade as shown below. The overall output $y(n)$ is same as the input $x(n)$ with a one unit delay. The transfer function of 2nd system $H_2(z)$ is:



Soln Given that

$$y(n) = x(n-1)$$

$$\Rightarrow Y(z) = z^{-1} X(z) \longrightarrow (i)$$

And From the block diagram we have

$$\frac{Y(z)}{X(z)} = H_1(z)\, H_2(z)$$

$$\frac{Y(z)}{X(z)} = \left(\frac{1-0.4z^{-1}}{1-0.6z^{-1}}\right) H_2(z) \longrightarrow (ii)$$

from eqn (i)

$$\frac{Y(z)}{X(z)} = z^{-1} \longrightarrow (iii)$$

comparing (ii) & (iii)

$$\left(\frac{1-0.4z^{-1}}{1-0.6z^{-1}}\right) H_2(z) = z^{-1}$$

$$H_2(z) = z^{-1}\left(\frac{1-0.6z^{-1}}{1-0.4z^{-1}}\right)$$

Inverse z-transform :-

We knew that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

$$\Rightarrow X(z) = D.T.F.T\left(x(n)\, r^{-n}\right)$$

$$\Rightarrow D.T.F.T\left(x(n)\, r^{-n}\right) = X(z)$$

$$\Rightarrow D.T.F.T\left(x(n)\, r^{-n}\right) = X(re^{j\omega})$$

$$x(n)\, r^{-n} = D.T.F.T^{-1}\left(X(re^{j\omega})\right)$$

[By applying inverse discrete time fourier transform]

$$\Rightarrow x(n)\, r^{-n} = \frac{1}{2\pi} \int_{(2\pi)} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{r^n}{2\pi} \int_{2\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi} \int X(re^{j\omega}) (re^{j\omega})^n d\omega$$

Let we knew that $z = re^{j\omega}$

$$\frac{dz}{d\omega} = jre^{j\omega}$$

$$\frac{dz}{d\omega} = jz$$

$$d\omega = \frac{1}{jz} dz$$

$$\Rightarrow \quad x(n) = \frac{1}{2\pi} \int X(z) z^n \frac{1}{jz} dz$$

$$\Rightarrow \quad x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz \longrightarrow \text{Inverse z- Transform eq}^n$$

#Q25 Find the signal $x(n)$ for which z- transform is

$$X(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$$

Soln Given that

$$X(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$$

$$X(z) = 4z^4 - z^3 + 0 \cdot z^2 - 3z + 0 \cdot z^0 + 4z^{-1} + 3z^{-2}$$

from time shifting property

$$x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$$

$$x(n+n_0) \longleftrightarrow z^{n_0} X(z)$$

$$4z^4 \longleftrightarrow 4\, x(n+4)$$

$$-z^3 \longleftrightarrow -1\, x(n+3)$$

$$-3z \longleftrightarrow -3\, x(n+1)$$

$$4z^{-1} \longleftrightarrow 4\, x(n-1)$$

$$3z^{-2} \longleftrightarrow 3\, x(n-2)$$

where $x(n) = \delta(n)$

$$\Rightarrow x(n) = 4\delta(n+4) - \delta(n+3) - 3\delta(n+1) + 4\delta(n-1) + 3\delta(n-2)$$

$$x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}$$

qa6 Find the inverse z- transform of

$z^{-2} \log_e (1-az^{-1})$

Soln Before solving the problem

let us consider

$$x(z) = \log_e (1+az^{-1})$$

$$\frac{dx(z)}{dz} = \frac{1}{1+az^{-1}}(-az^{-2})$$

we knew that

$$n x(n) \longleftrightarrow -z \frac{dx(z)}{dz}$$

$$= -z\left[\frac{-az^{-2}}{1+az^{-1}}\right]$$

$$= \frac{az^{-1}}{1+az^{-1}}$$

$$n x(n) = (-a)^n u(+n-1)$$

$$x(n) = \frac{(-a)^n}{n} u(+n-1)$$

we knew that

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

$$x(n-n_0) \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$$

$$\Rightarrow z^{-2} \log_e (1-az^{-1}) \longleftrightarrow \frac{(-a)^{n-2}}{n-2} u(+(n-2)-1)$$

$$z^{-2} \log_e (1-az^{-1}) \longleftrightarrow \frac{(-2)^{n-2}}{n-2} u(n-3)$$

qa7) compute the convolution $x(n)$ of the signal

$x_1(n) = \{1, 3, 3, 1\}$ (↑ at 1)

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 2 & 3 < n \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Soln

$x_1(n) = \{1, 3, 3, 1\}$
$\uparrow$

$X_1(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$

$x_2(n) = \{1, 1, 1, 1, 2, 2\}$
$\uparrow$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + 2z^{-4} + 2z^{-5}$

$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z)\, X_2(z)$

$X_1(z)\, X_2(z) = (1 + 3z^{-1} + 3z^{-2} + z^{-3})(1 + z^{-1} + z^{-2} + z^{-3} + 2z^{-4} + 2z^{-5})$

$= 1 + z^{-1} + z^{-2} + z^{-3} + 2z^{-4} + 2z^{-5} + 3z^{-1} + 3z^{-2} + 3z^{-3} + 3z^{-4}$
$+ 6z^{-5} + 6z^{-6} + 3z^{-2} + 3z^{-3} + 3z^{-4} + 3z^{-5} + 6z^{-6} + 6z^{-7} +$
$z^{-3} + z^{-4} + z^{-5} + z^{-6} + 2z^{-7} + 2z^{-8}$

$= 1 + 4z^{-1} + 7z^{-2} + 8z^{-3} + 9z^{-4} + 12z^{-5} + 13z^{-6} + 8z^{-7} + 2z^{-8}$

$X_1(z)\, X_2(z) = 1 + 4z^{-1} + 7z^{-2} + 8z^{-3} + 9z^{-4} + 12z^{-5} + 13z^{-6} + 8z^{-7} + 2z^{-8}$

$x_1(n) * x_2(n) = \{1, 4, 7, 8, 9, 12, 13, 8, 2\}$
$\uparrow$

Normal convolution:

$x_1(n) * x_2(n)$

	①	1	1	1	2	2
①	1	1	1	1	2	2
3	3	3	3	3	6	6
3	3	3	3	3	6	6
1	1	1	1	1	2	2

$x_1(n) * x_2(n) = \{1, 4, 7, 8, 9, 12, 13, 8, 2\}$
$\uparrow$

928) The z-transform of a anticausal ~~sys~~ signal is $X(z) = \frac{3 - 4z}{1 - 2z + 5z^2}$. The value of $x(0)$ is ____.

Soln

The signal is anticausal $\Rightarrow$ unilateral signal.

$\Rightarrow$ Initial value theorem is valid.

$x(0) = \lim_{z \to \infty} X(z)$

$= \lim_{z \to \infty} \frac{3 - 4z}{1 - 2z + 5z^2}$

$$= \lim_{z\to\infty} \frac{z\left(\frac{3}{z}-4\right)}{z^2\left(\frac{1}{z^2}-\frac{2}{z}+5\right)}$$

$$x(0) = \lim_{z\to\infty} \frac{\left(\frac{3}{z}-4\right)}{z\left(\frac{1}{z^2}-\frac{2}{z}+5\right)}$$

As $\lim_{z\to 0}$, $x(0) = 0$

#q29) Determine the system function and the unit sample response of the system described by difference eqn $y(n) = \frac{1}{2}y(n-1) + 2x(n)$

Soln Given difference eqn

$$y(n) = \frac{1}{2}y(n-1) + 2x(n)$$

$$y(n) - \frac{1}{2}y(n-1) = 2x(n)$$

Taking z-transform on both sides:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = 2X(z)$$

$$Y(z)\left(1-\frac{1}{2}z^{-1}\right) = 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

By taking inverse z-transform:

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) = \frac{2}{1-\frac{1}{2}z^{-1}}X(z)$$

for unit sample input

$$x(n) = \delta(n) \qquad X(z) = 1$$

$$\Rightarrow Y(z) = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$\Rightarrow y(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Note :-

1) $\delta(n) \xleftrightarrow{z} 1$

R·O·C : Entire z-plane including $z=0$ and $z=\infty$

2) $\delta(n+1) \xleftrightarrow{} z$

R·O·C : Entire z-plane including $z=0$ excluding $z=\infty$

3) $\delta(n-1) \xleftrightarrow{} z^{-1}$

R·O·C : Entire z-plane excluding $z=0$ including $z=\infty$

930) check for stability & causality of a system with impulse response $h(n) = \left(\frac{1}{3}\right)^n u(n) + 2^n u(-n-1)$

Soln

$$h(n) = \left(\frac{1}{3}\right)^n u(n) + 2^n u(-n-1)$$

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{2^{-1}z}{1-2^{-1}z}$$

$$= \frac{z}{z-\frac{1}{3}} - \frac{z}{z-2}$$

$$= \frac{z^2-2z-z^2+\frac{z}{3}}{(z-2)(z-1/3)}$$

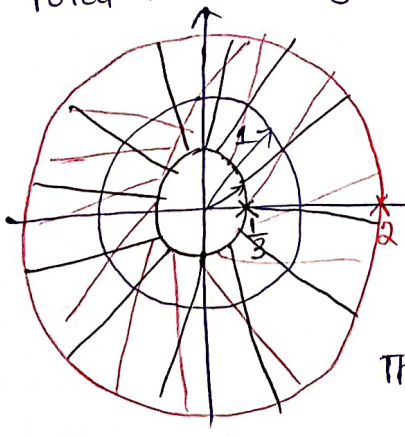
$$H(z) = \frac{-2z+\frac{z}{3}}{(z-2)(z-\frac{1}{3})} = \frac{-\frac{5}{3}z}{(z-2)(z-\frac{1}{3})}$$

poles: $z=2$ $z=\frac{1}{3}$ zeroes : $z=0$

R·O·C: $\left(\frac{1}{3}\right)^n u(n) \longleftrightarrow |z| > \frac{1}{3}$

$2^n u(-n-1) \longleftrightarrow |z| < 2$

Total R·O·C : $\frac{1}{3} < |z| < 2$



The system should be causal when R·O·C is outside the outermost pole. But here the R·OC lies insider the outermost pole
⇒ system is non causal.

The R·O·C include unit circle
⇒ system is stable

⇒ The system is stable but not causal.

Q31) A linear discrete time system has the characteristic eqⁿ $z^3 - 0.81z = 0$. comment on stability.

Soln characteristic eqⁿ of system

$$C.E = z^3 - 0.81z = 0$$

$$z(z^2 - 0.81) = 0$$

$$z(z+0.9)(z-0.9) = 0$$

$z = 0$
$z = 0.9$
$z = -0.9$

} All poles lie inside the unit circle and we can say the system is stable.

Q32) The z-transform of the signal is given by $C(z) = \frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}$. Determine final value.

Soln

$$C(\infty) = \lim_{z\to 1} (z-1)\left(\frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}\right)$$

$$= \lim_{z\to 1} (z-1)\left(\frac{z^{-5}(z^4-1)}{4z^{-2}(z-1)^2}\right)$$

$$= \lim_{z\to 1} (z-1)\left(\frac{z^{-3}(z^4-1)}{4(z-1)^2}\right)$$

$$= \lim_{z\to 1}\left[\frac{z^{-3}(z^4-1)}{4(z-1)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^{-3}(z^2+1)(z^2-1)}{4(z-1)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^{-3}(z^2+1)(z+1)(z-1)}{4(z-1)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^{-3}(z^2+1)(z+1)}{4}\right]$$

$$= \frac{(1)(1+1)(1+1)}{4}$$

$$C(\infty) = \frac{1}{}$$

Methods for finding inverse z-Transform:-

M1: Inverse z-transform by contour Integration:-

cauchy Residue Theorem:-

Let $f(z)$ be a function of complex variable z and c be a closed path in a z-plane. If the derivative $\frac{df}{dz}$ exists on and inside the contour c and if $f(z)$ has no poles at $z=z_0$ then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0) & \text{if } z_0 \text{ is inside } c \\ 0 & \text{if } z_0 \text{ is outside } c \end{cases} \longrightarrow ①$$

more generally if the $(k+1)^{th}$ order derivative of $f(z)$ exist and $f(z)$ has no poles at $z=z_0$ then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}} f(z)\Big|_{z=z_0} & \text{if } z_0 \text{ is inside } c \\ 0 & \text{if } z_0 \text{ is outside } c \end{cases} \longrightarrow ②$$

The values on the right hand side of eqn ① & ② are called residues at $z=z_0$.

The $x(n)$ is given as

$$x(n) = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz$$

$$= \sum_{\text{all poles } |z_i| \text{ inside } c} [\text{residue of } X(z) z^{n-1} \text{ at } z=z_i]$$

$$x(n) = \sum (z-z_i) X(z) z^{n-1}\Big|_{z=z_i}$$

9.33) $X(z) = \frac{z^{18}}{(z-0.5)(z-1)(z-4)}$ $X(z)$ converges for $|z|=1$

find $x(-16)$

Soln: Poles: $z=0.5 \quad z=1 \quad z=4$

since it is given that $x(z)$ converges to $|z|=1$ so the poles $z=1$ is only significant.

Residue at $z=1$

$$= (z-1)\left(\frac{z^{18}}{(z-1)(z-\frac{1}{2})(z-4)}\right) z^{n-1}\Big|_{z=1}$$

$$= \frac{1^{18}}{(1-\frac{1}{2})(1-4)} 1^{n-1}$$

$$= \frac{1}{(\frac{-1}{2})(-3)}$$

$$= \frac{2}{3}$$

$$x(n) = \frac{2}{3} \quad \forall\ n$$

#Ma; Inverse z-transform by partial fractions:-

case (i) Distinct poles :-

#934) Determine partial fraction expansion for the given function.

$$x(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

soln Procedure :-

Suppose that the poles $P_1, P_2, P_3, \ldots P_N$ are all different (distinct).

Thus we seek an expansion of the form

$$\frac{X(z)}{z} = \frac{A1}{z-P_1} + \frac{A2}{z-P_2} + \frac{A3}{z-P_3} + \ldots + \frac{A_N}{z-P_N}$$

Above expansion can be obtained as

1) Eliminate -ve power
2) Take common on R.H.S and brought that z to L.H.S
3) Decompose the rest of the expression on R.H.S into partial fractions.

Now we need to find coefficients $A_1, A_2, A_3, \ldots A_N$

With $z=P_K$ yields K^{th} coefficient as

$$A_K = (z-P_K)\ \frac{X(z)}{z}\Big|_{z=P_K}$$

Given z-transform

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

multiply numerator & denominator by z^2 to eliminate -ve powers.

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

0.5 → -1, -0.5

$$X(z) = \frac{z^2}{z^2-z-0.5z+0.5}$$

$$X(z) = \frac{z^2}{z(z-1)-0.5(z-1)}$$

$$X(z) = \frac{z^2}{(z-1)(z-0.5)}$$

Taking z from R.H.S to L.H.S

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1)\left(\frac{z}{(z-1)(z-0.5)}\right)\Big|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_2 = (z-0.5)\left(\frac{z}{(z-1)(z-0.5)}\right)\Big|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\Rightarrow \frac{X(z)}{z} = \frac{2}{z-1} + \frac{(-1)}{z-0.5}$$

complex conjugate poles:

q35) Determine the partial fraction expansion of

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Soln Given z-transform $X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$

multiply numerator & denominator by z^2 to eliminate -ve power

$$X(z) = \frac{z^2+z}{z^2-z+0.5}$$

$$\frac{X(z)}{z} = \frac{z+1}{z^2-z+0.5}$$

poles : $z^2 - z + 0.5 = 0$

$$z = \frac{1}{2} \pm j\frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{z+1}{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-\left(\frac{1}{2}+j\frac{1}{2}\right)} + \frac{A_2}{z-\left(\frac{1}{2}-j\frac{1}{2}\right)}$$

$$A_1 = \left[z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[\frac{z+1}{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}\right]_{z=\frac{1}{2}+j\frac{1}{2}}$$

$$A_1 = \frac{\frac{1}{2}+j\frac{1}{2}+1}{\frac{1}{2}+j\frac{1}{2}-\frac{1}{2}+j\frac{1}{2}} = \frac{\frac{3}{2}+j\frac{1}{2}}{j} = \frac{-\frac{1}{2}+j\frac{3}{2}}{-1}$$

$$A_1 = \frac{1}{2} - j\frac{3}{2}$$

$$A_2 = \left[z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]\left[\frac{(z+1)}{\left(z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right)\left(z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right)}\right]_{z=\frac{1}{2}-j\frac{1}{2}}$$

$$= \frac{\frac{1}{2}-j\frac{1}{2}+1}{\frac{1}{2}-j\frac{1}{2}-\frac{1}{2}-j\frac{1}{2}} = \frac{\frac{3}{2}-j\frac{1}{2}}{-j} = \frac{1}{2}+j\frac{3}{2}$$

$$A_1^* = A_2$$

$$\therefore \frac{X(z)}{z} = \frac{\frac{1}{2}-j\frac{3}{2}}{z-\left(\frac{1}{2}+j\frac{1}{2}\right)} + \frac{\frac{1}{2}+j\frac{3}{2}}{z-\left(\frac{1}{2}-j\frac{3}{2}\right)}$$

complex conjugate poles results in complex conjugate coefficients in partial fraction expansion.

case (ii) pole of multiple pole :-

If $X(z)$ has a pole of multiplicity l, that is it contains in its denominator the factor $(z-p_k)^l$, in this case a different expansion is needed.

#936 Determine the partial ~~ext~~ fraction expansion of

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

soln Given z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Eliminating -ve powers by multiplying numerator & denominator by z^3

$$X(z) = \frac{z^3}{(1+z)(1-z)^2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

The above z-transform can be decomposed into partial fractions as

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_{21}}{z-1} + \frac{A_{22}}{(z-1)^2}$$

Now we need to determine these coefficients A_1, A_{21} and A_{22}.

Let us findout coefficient A_1 (distinct root)

$$A_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1}$$

$$= (z+1)\left(\frac{z^2}{(z+1)(z-1)^2}\right)\Big|_{z=-1}$$

$$A_1 = \frac{z^2}{(z-1)^2}\Big|_{z=-1} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

Now let's find coefficients A_{21} & A_{22} related to repeated poles:

$$A_{22} = (z-1)^2\frac{X(z)}{z}\Big|_{z=1}$$

$$= (z-1)^2\left[\frac{z^2}{(z+1)(z-1)^2}\right]_{z=1} = \frac{1}{1+1} = \frac{1}{2}$$

Now

$$A_{21} = \frac{d}{dz}\left\{(z-1)^2 \frac{z^2}{(z+1)(z-1)^2}\right\}\Big|_{z=1}$$

$$= \frac{2z(z+1) - z^2}{(z+1)^2}$$

$$= \frac{2z^2 + 2z - z^2}{(z+1)^2}$$

$$A_{21} = \frac{z^2+2z}{(z+1)^2}\Big|_{z=1}$$

$$A_{21} = \frac{1+2}{2^2} = \frac{3}{4}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{4(z+1)} + \frac{3}{4(z-1)} + \frac{1}{2(z-1)^2}$$

Procedure for finding inverse z-transform from partial fraction expansion:

We already take common 'z' on R.H.S and brought it to L.H.S at initial stage. After finding partial fraction expansion coefficients resend the 'z' on L.H.S to R.H.S.

case (i) Real & distinct poles :-

$$\frac{X(z)}{z} = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2} + \frac{A_3}{z-P_3} + \cdots + \frac{A_n}{z-P_n}$$

$$\Rightarrow X(z) = \frac{A_1 z}{z-P_1} + \frac{A_2 z}{z-P_2} + \frac{A_3}{z-P_3} + \cdots + \frac{A_n}{z-P_n}$$

$$\Rightarrow X(z) = A_1\left(\frac{1}{1-z^{-1}P_1}\right) + A_2\left(\frac{1}{1-z^{-1}P_2}\right) + A_3\left(\frac{1}{1-z^{-1}P_3}\right) + \cdots + A_n\left(\frac{1}{1-z^{-1}P_n}\right)$$

The inverse z-transform $x(n) = z^{-1}\{X(z)\}$ can be obtained by inverting each term in above expansion and taking the corresponding linear combination.

$$z^{-1}\left\{\frac{1}{1-P_K z^{-1}}\right\} = \begin{cases} P_K^n u(n) & \text{when R.O.C } |z|>|P_K| \text{ causal signals} \\ -(P_K)^n u(-n-1) & \text{when R.O.C } |z|<|P_K| \text{ anticausal signals.} \end{cases}$$

If $x(n)$ is causal then R.O.C is $|z| > P_{max}$.

where $P_{max} = \max\{|P_1|, |P_2|, \ldots |P_n|\}$.

$$\Rightarrow x(n) = \{A_1 P_1^n + A_2 P_2^n + A_3 P_3^n + \ldots + A_n P_n^n\} u(n)$$

A causal signal having a z-transform that contains real and distinct poles, is a linear combination of real exponential signals.

Case (ii) Distinct & complex poles:

We knew that poles are complex conjugate then their corresponding partial fraction coefficients are also complex conjugate to each other.

Therefore the signal $x(n)$ is of the form

$$x(n) = \{A_K (P_K)^n + A_K^* (P_K^*)^n\} u(n)$$

$$A_K = |A_K| e^{j\alpha_K} \text{ and } P_K = |P_K| e^{j\beta_K}$$

$|A_K|$ & $|P_K|$ are amplitudes & α_K & β_K are phase components of A_K and P_K

$$\Rightarrow x(n) = \{|A_K| e^{j\alpha_K} (|P_K| e^{j\beta_K})^n + |A_K| e^{-j\alpha_K} (|P_K| e^{-j\beta_K})^n\} u(n)$$

$$\Rightarrow x(n) = \{|A_K||P_K|^n e^{j\alpha_K} e^{j\beta_K n} + |A_K||P_K|^n e^{-j\alpha_K} e^{-j\beta_K n}\} u(n)$$

$$\Rightarrow x(n) = |A_K||P_K|^n \{e^{j(n\beta_K+\alpha_K)} + e^{-j(n\beta_K+\alpha_K)}\} u(n)$$

$$\Rightarrow x(n) = \{2|A_K||P_K|^n \cos(n\beta_K+\alpha_K)\} u(n)$$

$$\Rightarrow z^{-1}\left\{\frac{A_K}{1-P_K z^{-1}} + \frac{A_K^*}{1-P_K^* z^{-1}}\right\} = 2|A_K||P_K|^n \cos(n\beta_K+\alpha_K) u(n)$$

if the R.O.C is $|z| > |P_K|$.

⇒ Each pair of complex-conjugate poles in z-domain results in a causal sinusoidal signal component with an exponential envelope.

⇒ The distance $|P_k|$ of the pole from the origin determines the exponential weighting

case (i) if $|P_k|>1$ ⇒ Growing exponential envelope

case (ii) if $|P_k|<1$ ⇒ Decaying exponential envelope

case (iii) if $|P_k|=1$ ⇒ constant exponential envelope.

⇒ The angle of poles B_k with respect to positive real axis provides the frequency of the sinusoidal signal.

case (iii) multiple poles :-

In case of multiple poles either real or complex, the inverse transform of the terms of the form $\frac{A}{(z-P_k)^n}$ is required.

For double poles :-

$$z^{-1}\left\{\frac{Pz^{-1}}{(1-Pz^{-1})^2}\right\} = nP^n u(n)$$

provided that R.O.C $|z|>|P|$

Q87) Determine the inverse z-transform of

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

if a) R.O.C $|z|>1$

b) R.O.C $|z|<0.5$

c) R.O.C $0.5<|z|<1$

Soln Given z-transform

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

Eliminate -ve power by multiplying numerator & denominator by z^2

$$X(z) = \frac{z^2}{z^2-1.5z+0.5}$$

0.5

-1 -0.5

$$X(z) = \frac{z^2}{z^2-z-0.5z+0.5}$$

$$X(z) = \frac{z^2}{z(z-1)-0.5(z-1)}$$

$$X(z) = \frac{z^2}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5} \quad \text{(partial fraction decomposition)}$$

$$A_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1}$$

$$A_1 = (z-1)\left(\frac{z}{(z-1)(z-0.5)}\right)_{z=1} = \frac{1}{1-0.5} = \frac{1}{0.5} = 2$$

$$A_2 = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5}$$

$$A_2 = (z-0.5)\left(\frac{z}{(z-1)(z-0.5)}\right)_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\Rightarrow \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\Rightarrow X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

$$\Rightarrow X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

poles : 1, 0.5

$\min(1, 0.5) = 0.5$ $\quad \max(1, 0.5) = 1$

a) R.O.C $|z|>1$

The R.O.C is the outside of outer most pole.

$\Rightarrow$ It is right sided signal

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

$$x(n) = 2u(n) - (0.5)^n u(n)$$

$$x(n) = \{2-(0.5)^n\}u(n)$$

b) R·O·C : $|z| < 0.5$)

⇓

In this case R·O·C is inside the most inner pole. Therefore it is left sided sequence (Anti causal)

$$\Rightarrow \quad x(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

$$\Rightarrow \quad x(n) = 2u(-n-1) + (0.5)^n u(-n-1)$$

$$\Rightarrow \quad x(n) = \{-2 + (0.5)^n\} u(-n-1)$$

c) R·O·C = $0.5 < |z| < 1$

⇓

In this case R·O·C is an annular region / ring.

$\Rightarrow |z| < 1$ $|z| \geq 0.5$

Left sided → Right sided

$$\Rightarrow \quad x(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

$$x(n) = -2u(-n-1) - (0.5)^n u(n)$$

$$x(n) = -\left[2u(-n-1) + (0.5)^n u(n)\right]$$

q38) Determine the causal signal $x(n)$ whose z-transform is given by

$$x(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Soln

z- Transform given

$$x(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Eliminate -ve by multiplying numerator & denominator by z^2

$$x(z) = \frac{z^2+z}{z^2-z+0.5}$$

$$x(z) = \frac{z(z+1)}{z^2-z+0.5}$$

$$\frac{x(z)}{z} = \frac{z+1}{z^2-z+0.5}$$

poles: $z^2 - z + 0.5 = 0$

$$z = 0.5 \pm j0.5$$

$$P = 0.5 \pm j0.5$$

$$|P| = \sqrt{0.5^2 + 0.5^2} = \frac{1}{\sqrt{2}}$$

$$\beta_K = \tan^{-1}\left(\frac{\text{Im}(P)}{\text{Real}(P)}\right) = \tan^{-1}\left(\frac{0.5}{0.5}\right) = 45^\circ = \frac{\pi}{4}$$

Partial fraction expansion:

$$\frac{X(z)}{z} = \frac{z+1}{z^2 - z + 0.5}$$

$$\frac{X(z)}{z} = \frac{z+1}{\left(z - \left(\frac{1}{2} + j\frac{1}{2}\right)\right)\left(z - \left(\frac{1}{2} - j\frac{1}{2}\right)\right)}$$

$$\frac{X(z)}{z} = \frac{A_1}{\left[z - \left(\frac{1}{2} + j\frac{1}{2}\right)\right]} + \frac{A_2}{\left[z - \left(\frac{1}{2} - j\frac{1}{2}\right)\right]}$$

$$A_1 = \left[z - \left(\frac{1}{2} + j\frac{1}{2}\right)\right]\left[\frac{z+1}{z^2 - z + 0.5}\right]$$

$$A_1 = \left[z - \left(\frac{1}{2} + j\frac{1}{2}\right)\right]\left[\frac{z+1}{[z - (0.5 + j0.5)][z - (0.5 - j0.5)]}\right]_{z = 0.5 + j0.5}$$

$$A_1 = \frac{z+1}{z - (0.5 - j0.5)} = \frac{0.5 + j0.5 + 1}{0.5 + j0.5 - 0.5 + j0.5}$$

$$A_1 = \frac{1.5 + j0.5}{j} = \frac{j1.5 + j^2 0.5}{j^2} = \frac{-0.5 + j1.5}{-1} = 0.5 - j1.5$$

$$A_2 = A_1^* = (0.5 - j1.5)^* = 0.5 + j1.5$$

$$|A_1| = \sqrt{0.5^2 + (-1.5)^2} = \frac{\sqrt{10}}{2}$$

$$\alpha_1 = \tan^{-1}\left(\frac{-1.5}{0.5}\right) = -71.565$$

$$x(n) = 2|A_1||P_1|^n \cos(n\beta_1 + \alpha_1)\, u(n)$$

$$= 2 \times \frac{\sqrt{10}}{2} \times \left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4}n - 71.565\right) u(n)$$

$$x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi n}{4} - 71.565\right) u(n)$$

#939 Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Soln Given z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

multiplying numerator & denominator by z^3 to eliminate -ve powers of z

$$X(z) = \frac{z^3}{(z^3+1)(z^2-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_{21}}{z-1} + \frac{A_{22}}{(z-1)^2}$$

$$A_1 = (z+1)\left(\frac{z^2}{(z+1)(z-1)^2}\right)_{z=-1} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$A_{21} = \frac{d}{dz}\left\{(z-1)^2 \frac{z^2}{(z+1)(z-1)^2}\right\}_{z=1} = \frac{1}{1!}\frac{d}{dz}\left\{\frac{z^2}{z+1}\right\}_{z=1}$$

$$= \frac{2z(1) - z^2}{(z+1)^2}\bigg|_{z=1} = \frac{2z(z+1) - z^2}{(z+1)^2}\bigg|_{z=1}$$

$$= \frac{z^2+2z}{(z+1)^2}\bigg|_{z=1} = \frac{3}{4}$$

$$A_{22} = \left\{(z-1)^2 \frac{z^2}{(z+1)(z-1)^2}\right\}_{z=1} = \frac{z^2}{z+1} = \frac{1}{2}$$

$$A_1 = \frac{1}{4} \quad A_{21} = \frac{3}{4} \quad A_{22} = \frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{1}{4(z+1)} + \frac{3}{4(z-1)} + \frac{1}{2(z-1)^2}$$

$$X(z) = \frac{z}{4(z+1)} + \frac{3z}{4(z-1)} + \frac{z}{2(z-1)^2}$$

$$X(z) = \frac{1}{4(1+z^{-1})} + \frac{3}{4(1-z^{-1})} + \frac{z^{-1}}{2(1-z^{-1})^2}$$

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}n\,u(n)$$

$$x(n) = \left\{\frac{(-1)^n}{4} + \frac{3}{4} + \frac{n}{2}\right\} u(n)$$

Q40) Find inverse z-transform using long division method

$$X(z) = \frac{1+az^{-1}}{1-az^{-1}+z^{-2}}$$

(i) if $x(n)$ is causal

(ii) if $x(n)$ is anti causal / non causal

Soln i) $x(n)$ is causal → n ranges from 0 to ∞

$$1-az^{-1}+z^{-2}\;\overline{)\;1+az^{-1}}\;\;(1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}$$

$1+az^{-1}-z^{-2}$

$4z^{-1}-z^{-2}$

$4z^{-1}-8z^{-2}+4z^{-3}$

$7z^{-2}-4z^{-3}$

$7z^{-2}-14z^{-3}+7z^{-4}$

$10z^{-3}-7z^{-4}$

$10z^{-3}-20z^{-4}+10z^{-5}$

$13z^{-4}-10z^{-5}$

$13z^{-4}-26z^{-5}+13z^{-6}$

$16z^{-5}-13z^{-6}$

$$X(z) = \frac{1+az^{-1}}{1-az^{-1}+z^{-2}} = 1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+16z^{-5}+\cdots$$

$\uparrow$ $n=0$

$$x(n) = \{1,\ 4,\ 7,\ 10,\ 13,\ 16,\ \cdots\}$$

$\uparrow$

ii) $x(n)$ is non-causal / non causal

We have to write the numerator as dividened & denominator as divisor in reverse order

$$x(z) = \frac{az^{-1}+1}{z^{-2}-az^{-1}+1}$$

$$z^{-2}-2z^{-1}+1 \;\Big)\; 2z^{-1}+1 \;\Big(\; 2z+5z^2+8z^3+11z^4+14z^5$$

$\frac{2z^{-1}}{z^{-2}} = 2z$

$\frac{5}{z^{-2}} = 5z^2$

$2z^{-1}-4+2z$

$5-2z$

$5-10z+5z^2$

$8z-5z^2$

$8z-16z^2+8z^3$

$11z^2-8z^3$

$11z^2-22z^3+11z^4$

$14z^3-11z^4$

$14z^3-28z^4+14z^5$

$17z^4-14z^5$

$$X(z) = \frac{2z^{-1}+1}{z^{-2}-2z^{-1}+1} = 2z+5z^2+8z^3+11z^4+14z^5+\cdots$$

$n=-1 \quad n=-2 \quad n=-3 \quad n=-4 \quad n=-5$

$$x(n) = \{\underset{\uparrow}{14}, 11, 8, 5, 2, 0\}$$

one-sided z-Transform :-

The one sided or unilateral z-transform of a signal $x(n)$ is defined by

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$x(n) \overset{z^+}{\longleftrightarrow} x^+(n)$$

The one sided z-transform differs from the two sided transform in the low limit of the summation, which is always zero, whether or not the signal $x(n)$ is zero for $n<0$.

characteristics of one-sided z-Transform :-

1) It does not contain information about the signal $x(n)$ for negative instants of time (i.e, $n<0$)

2) It is unique only for causal signals, because only these signals are zero for $n<0$.

3) The one sided z-transform $x^+(z)$ of $x(n)$ is identical to the two sided z-transform of the signal $x(n)u(n)$. Since $x(n)u(n)$ is causal, the R.O.C of its transform, and hence the R.O.C of $x^+(z)$ is always the exterior of a circle. Thus we deal with one sided z-transform, it is not necessary to refer to their R.O.C.

#Q41) Determine the one sided z-transform for following signals.

$x_1(n) = \{1, 2, 5, 7, 0, 1\}$ $x_2(n) = \{1, 2, 5, 7, 0, 1\}$

$x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$ $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$x_5(n) = \delta(n)$ $x_6(n) = \delta(n-k)$ $x_7(n) = \delta(n+k)$

soln

i) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

$x^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

(ii) $x_2(n) = \{1, 2, 5, 7, 0, 1\}$

$x^+(z) = 5 + 7z^{-1} + z^{-3}$

(iii) $x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$

$x^+(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$

(iv) $x_4(n) = \{2, 4, 5, 7, 0, 1\}$

$x^+(z) = 5 + 7z^{-1} + z^{-3}$

v) $x_5(n) = \delta(n) \longleftrightarrow x^+(z) = 1$

(vi) $x_6(n) = \delta(n-k) \longleftrightarrow x^+(z) = z^{-k}$

(vii) $x_7(n) = \delta(n+k) \longleftrightarrow x^+(z) = 0$

#Q42 Determine the one sided z-transform for following signals.

a) $x(n) = a^n u(n)$

b) $x_1(n) = x(n-2)$ where $x(n) = a^n u(n)$

c) $x_2(n) = x(n+2)$ where $x(n) = a^n u(n)$

soln

Given signal $x(n) = a^n u(n)$

From the definition of unilateral/one sided z-transform

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

a) when $x(n) = a^n u(n)$

$$\Rightarrow X^+(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$\Rightarrow X^+(z) = \frac{1}{1-az^{-1}}$$

b) $x(n-2) = a^{n-2} u(n-2)$

From time shifting property of one sided z-transform

$$\mathcal{Z}\{x(n-n_0)\} = z^{-n_0}\left[X^+(z) + \sum_{m=0}^{n_0-1} x(-m) z^m\right]$$

$$\mathcal{Z}\{x(n-2)\} = z^{-2}\left[X^+(z) + \sum_{m=0}^{1} x(-m) z^m\right]$$

$$\mathcal{Z}\{x(n-2)\} = z^{-2}\left[X^+(z) + x(0) + x(-1) z^{m}\right]$$

$$a^{n-2} u(n-2) = z^{-2}\left[\frac{1}{1-az^{-1}} + a^0 + a^{-1} z\right]$$

$$= z^{-2}\left[\frac{1}{1-az^{-1}} + 1 + a^{-1} z\right]$$

$$\mathcal{Z}\{a^{n-2} u(n-2)\} = \frac{z^{-2}}{1-az^{-1}} + z^{-2} + a^{-1} z^{-1}$$

c) $x(n+2) = a^{n+2} u(n+2)$

From time shifting property of one sided z- transform

$$\mathcal{Z}\{x(n+n_0)\} = z^{n_0}\left\{X^+(z) - \sum_{m=0}^{n_0-1} x(m) z^{-m}\right\}$$

$$\mathcal{Z}\{x(n+2)\} = z^2\left\{X^+(z) - \sum_{m=0}^{1} x(m) z^{-m}\right\}$$

$$= z^2\left\{X^+(z) - x(0) - x(1) z^{-1}\right\}$$

$$\mathcal{Z}\{a^{n+2} u(n+2)\} = z^2\left\{\frac{1}{1-az^{-1}} - 1 - az^{-1}\right\}$$

$$= \frac{z^2}{1-az^{-1}} - z^2 - az$$

#643 The impulse response of a relaxed linear time invarient system is $h(n) = a^n u(n)$, $|a|<1$ determine the value of step response of system as $n \to \infty$

Soln Given impulse response of the system $h(n) = a^n u(n)$

For step response $x(n) = u(n)$

The response of distrete time LTI system is given by convolution sum

$$\Rightarrow y(n) = h(n) * x(n)$$

⇒ If we excite a causal system with a causal input signal the output will also be causal.

⇒ since $h(n) = a^n u(n)$ $x(n) = u(n)$ are causal signal the response is also causal.

From convolution property

$$y(n) = h(n) * x(n)$$

$$\updownarrow$$

$$Y(z) = H(z)\,X(z)$$

$$h(n) = a^n u(n) \overset{\leftrightarrow}{} H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$x(n) = u(n) \longleftrightarrow X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\Rightarrow \quad Y(z) = \left(\frac{z}{z-a}\right)\left(\frac{z}{z-1}\right)$$

$$\Rightarrow \quad Y(z) = \frac{z}{(z-a)(z-1)}$$

We need final value of step response i.e, $y(\infty)$

From final value theorem

$$y(\infty) = \lim_{z\to 1} (z-1)\left(Y(z)\right)$$

$$= \lim_{z\to 1} (z-1)\left[\frac{z}{(z-a)(z-1)}\right]$$

$$= \lim_{z\to 1} \frac{z}{z-a}$$

$$y(\infty) = \frac{1}{1-a}$$

#944) The well known fibonacci sequence of integer numbers is obtained by computing each term as the sum of 2 previous ones. The first few terms of the sequence are

1, 1, 2, 3, 5, 8, ---

Determine a closed form expression for the n^{th} term of fibonacci sequence.

Soln Let $y(n)$ be the n^{th} term of fibonacci sequence.

n^{th} term of fibonacci sequence = sum of 2 previous terms

$$\Rightarrow y(n) = y(n-1) + y(n-2)$$

Initial conditions:

$$y(0) = y(-1) + y(-2) = 1 \longrightarrow 1$$

$$y(1) = y(0) + y(-1) = 1 \longrightarrow 2$$

we knew that $y(0) = 1$ substituting in eqn 2

we get

$$y(1) = y(0) + y(-1) = 1$$

$$1 + y(-1) = 1$$

$$y(-1) = 0$$

substituting $y(-1) = 0$ in eqn ①

$$y(0) = y(-1) + y(-2) = 1$$

$$0 + y(-2) = 1$$

$$y(-2) = 1$$

$$\Rightarrow y(0) = 1 \quad y(-1) = 0 \quad y(-2) = 1$$

Now we need to determine $y(n)$ for $n \geq 0$ using above initial conditions.

From one sided time shifting property

$$\mathcal{Z}\{x(n-n_0)\} = z^{-n_0}\left[X^+(z) + \sum_{m=-1}^{-n_0} x(-m) z^{-m}\right]$$

$$\Rightarrow \mathcal{Z}\{y(n-1)\} = z^{-1}\left[Y(z) + y(-1)z^{1}\right]$$

$$\mathcal{Z}\{y(n-1)\} = Yz^{-1} + y(-1)$$

$$z\{y(n-2)\} = z^{-n_0}\left[Y^+(z) + \sum_{m=-1}^{-n_0} y(m) z^{-m}\right]$$

$$= z^{-2}\left[Y^+(z) + \sum_{m=-1}^{-2} y(m) z^{-m}\right]$$

$$= z^{-2}\left[Y^+(z) + y(-1)z^1 + y(-2)z^{-2}\right]$$

$$z\{y(n-2)\} = Y^+(z)z^{-2} + y(-1)z + y(-2)$$

$$y(n) = y(n-1) + y(n-2)$$

obtaining z- transform on both sides

$$Y(z) = z\{y(n-1)\} + z\{y(n-2)\}$$

$$= Y(z)z^{-1} + y(-1) + Y(z)z^{-2} + y(-1)z^{-1} + y(-2)$$

$$Y(z) - Y(z)z^{-1} - Y(z)z^{-2} = y(-1) + y(-1)z^{-1} + y(-2)$$

$$Y(z)\{1 - z^{-1} - z^{-2}\} = y(-1)(1+z^{-1}) + y(-2)$$

$$Y(z) = \frac{(1+z^{-1})y(-1) + y(-2)}{1 - z^{-1} - z^{-2}}$$

substituting initial conditions $y(-1) = 0 \quad y(-2) = 1$

$$Y(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$

Eliminating -ve power by multiplying numerator & denominator by z^2

$$Y(z) = \frac{z^2}{z^2 - z - 1}$$

poles: $z^2 - z - 1 = 0$

$$z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

poles are real & distinct

$$Y(z) = \frac{z^2}{z^2 - z - 1}$$

$$Y(z) = \frac{z^2}{\left[z - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z - \left(\frac{1-\sqrt{5}}{2}\right)\right]}$$

Partial fraction expansion:

$$\frac{Y(z)}{z} = \frac{z}{\left[z-\left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z-\left(\frac{1-\sqrt{5}}{2}\right)\right]} = \frac{A}{z-\left(\frac{1+\sqrt{5}}{2}\right)} + \frac{B}{z-\left(\frac{1-\sqrt{5}}{2}\right)}$$

$$A = \left[z-\left(\frac{1+\sqrt{5}}{2}\right)\right]\cdot\left[\frac{z}{\left[z-\left(\frac{1+\sqrt{5}}{2}\right)\right]\left[z-\left(\frac{1-\sqrt{5}}{2}\right)\right]}\right]_{z=\left(\frac{1+\sqrt{5}}{2}\right)}$$

$$A = \frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}} = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{2}{2\sqrt{5}}\right) = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$B = \frac{\frac{1-\sqrt{5}}{2}}{\frac{1-\sqrt{5}}{2}-\frac{1+\sqrt{5}}{2}} = \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{2}{-2\sqrt{5}}\right) = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$$

$$\frac{Y(z)}{z} = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1}{z-\left(\frac{1+\sqrt{5}}{2}\right)}\right) - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1}{z-\left(\frac{1-\sqrt{5}}{2}\right)}\right)$$

$$Y(z) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left[\frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)z^{-1}}\right] - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left[\frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)z^{-1}}\right]$$

$$Y(z) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}u(n) - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}u(n)$$

$$Y(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)^{n+1}\left[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\right]u(n)$$

Q45) Determine the step response of the system

$y(n) = \alpha\, y(n-1) + x(n)$ $\quad -1 < \alpha < 1$

when the initial condition is $y(-1) = 1$

soln

Given difference eqn

$$y(n) = \alpha\, y(n-1) + x(n)$$

$\Rightarrow \quad y(n) - \alpha y(n-1) = x(n)$

obtaining z- transform on both sides

$$Y(z) - \alpha\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z)\left(1-z^{-1}\alpha\right) - \alpha y(-1) = X(z)$$

$$Y(z) - \frac{\alpha}{1-\alpha z^{-1}} y(-1) = \frac{X(z)}{1-\alpha z^{-1}}$$

since $x(n) = u(n) \Rightarrow X(z) = \frac{1}{1-z^{-1}}$ and $y(-1) = 1$

substituting in above eqn

$$Y(z) - \frac{\alpha}{1-\alpha z^{-1}} = \frac{1}{(1-\alpha z^{-1})(1-z^{-1})}$$

$$Y(z) = \frac{1}{(1-\alpha z^{-1})(1-z^{-1})} + \frac{\alpha}{1-\alpha z^{-1}}$$

$$Y(z) = \frac{1+\alpha(1-z^{-1})}{(1-\alpha z^{-1})(1-z^{-1})}$$

$$Y(z) = \frac{\alpha+1-\alpha z^{-1}}{(1-\alpha z^{-1})(1-z^{-1})}$$

Eliminating -ve powers by multiplying numerator & denominator by z^2

$$Y(z) = \frac{(1+\alpha)z^2 - \alpha z}{(z-\alpha)(z-1)}$$

$$\frac{Y(z)}{z} = \frac{(1+\alpha)z - \alpha}{(z-\alpha)(z-1)}$$

By partial fraction expansion

$$\frac{Y(z)}{z} = \frac{A}{z-\alpha} + \frac{B}{z-1}$$

$$A = (z-\alpha)\frac{Y(z)}{z} = \cancel{(z-\alpha)}\left[\frac{(1+\alpha)z-\alpha}{\cancel{(z-\alpha)}(z-1)}\right]_{z=\alpha} = \frac{(1+\alpha)\alpha-\alpha}{\alpha-1}$$

$$A = \frac{\alpha+\alpha^2-\alpha}{\alpha-1} = \frac{\alpha^2}{\alpha-1}$$

$$B = (z-1)\frac{Y(z)}{z} = \cancel{(z-1)}\left[\frac{(1+\alpha)z-\alpha}{(z-\alpha)\cancel{(z-1)}}\right]_{z=1} = \frac{(1+\alpha)-\alpha}{1-\alpha}$$

$$B = \frac{1}{1-\alpha}$$

$$\frac{Y(z)}{z} = \left(\frac{\alpha^2}{\alpha-1}\right)\left(\frac{1}{z-\alpha}\right) + \left(\frac{1}{1-\alpha}\right)\left(\frac{1}{z-1}\right)$$

$$Y(z) = \frac{-\alpha^2}{1-\alpha}\left(\frac{1}{1-\alpha z^{-1}}\right) + \left(\frac{1}{1-\alpha}\right)\left(\frac{1}{1-z^{-1}}\right)$$

$$Y(z) = \frac{-\alpha^2}{1-\alpha}\cdot\alpha^n u(n) + \frac{1}{1-\alpha}u(n)$$

$$Y(z) = -\frac{\alpha^{n+2}}{1-\alpha}u(n) + \frac{1}{1-\alpha}u(n)$$

$$Y(z) = \frac{1}{1-\alpha}\left(1-\alpha^{n+2}\right)u(n)$$

Analysis of LTI systems in the z-domain :-

consider a pole zero system described by general linear constant coefficient difference eqn

$$y(n)+y(n-1)+y(n-2)+\cdots+y(n-N) = x(n)$$

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_m x(n-m)$$

If $a_0 = 1$, then the eqn become

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 u(n-1) + b_2 x(n-2) + \cdots + b_m x(n-M)$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

suppose that the signal $x(n)$ is applied to the pole-zero system at $n=0$. Thus the signal $x(n)$ is assumed to be causal. The effects of all previous input signals to the system are reflected in the initial conditions $y(-1), y(-2), \ldots, y(-N)$. since the input $x(n)$ is causal and since we are interested in determining the output $y(n)$ for $n \geq 0$, we can use the one sided z-transform, which allows us to deal with the initial conditions. Thus the one-sided z-transform of above eqn becomes

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k}\left(Y^+(z) + \sum_{n=1}^{k} y(-n)z^n\right) + \sum_{k=0}^{M} b_k z^{-k} X^+(z)$$

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k} Y^+(z) - \sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n + \sum_{k=0}^{m} b_k z^{-k} x^+(z)$$

$$\Rightarrow Y^+(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{m} b_k z^{-k} x^+(z) - \sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n$$

$$\Rightarrow Y^+(z) = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} x^+(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\Rightarrow \quad Y^+(z) = H(z)X(z) + \frac{N_0(z)}{A(z)}$$

where $N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n$

Therefore the output of the system with non zero initial conditions can be subdivided into 2 parts.

1) zero-state response

The first component is the zero-state response of system defined in the z-domain as

$$Y_{zs}(z) = H(z)\,x(z)$$

2) zero-input response

The second component is corresponds to the output resulting from the non zero initial conditions. This output is the zero input response of the system which is defined in z-domain as

$$Y_{zi}(z) = \frac{N_0(z)}{A(z)}$$

Hence the total response is the sum of these output components, which can be expressed in the time domain by determing the inverse z-transforms of $Y_{zs}(z)$ and $Y_{zi}(z)$ seperately, and then adding the results

Thus $Y(z) = Y_{zs}(z) + Y_{zi}(z)$

$$\Rightarrow \quad y(n) = y_{zs}(n) + y_{zi}(n)$$