

DISCRETE FOURIER SERIES

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DISCRETE-FOURIER SERIES : Consider a sequence $x_p(n)$ with a Period of N Samples. So That $x_p(n) = x_p(n+LN)$. Since $x_p(n)$ is periodic, it can be represented as a weighted sum of Complex Exponentials whose frequency is integer Multi-
ples of The fundamental frequency $\frac{2\pi}{N}$.

$\therefore$ These Periodic Complex Exponentials are of The form

$$e^{j2\pi kn/N} = e^{j2\pi k(n+LN)/N} \quad \text{--- (1)}$$

where L -an integer.

We know The DTFT of a sequence is Periodic it Consists of a Finite number of distinct - frequency Components. If number of frequ-
ency Components is N , Then The Harmonics are given by $\frac{2\pi k}{N}$, $k = 0, 1, 2, \dots, n-1$

$\therefore$ any periodic sequence can be represented

$$\text{as } x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_p(k) \cdot e^{j2\pi kn/N} \quad \text{where } n=0, \pm 1, \dots \quad \text{--- (2)}$$

$\rightarrow$ and $X_p(k)$, $k = 0, 1, \dots, N-1$

are called DFS Coefficients.

To obtain Fourier Coefficients Multiply both - sides of eq(2) by $e^{-j(2\pi/N)mn}$ and Sum The Product from $n=0$ to $n=N-1$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j(2\pi/N)mn} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)(k-m)n}$$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j \frac{2\pi}{N} mn} = \frac{1}{N} \sum_{k=0}^{N-1} X_p(k) \sum_{n=0}^{N-1} e^{j \frac{2\pi}{N} (k-m)n}$$

[$\because$ interchanging orders of Sum.] A

$$\text{And } \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-m)n} = N \text{ if } k-m=0, \pm N, \pm 2N, \dots \\ = 0 \text{ otherwise} \quad \text{--- (3)}$$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}mn} = X_p(m) \quad \text{--- (4)}$$

By changing the index from 'm' to 'k' Fourier Series Coefficient $X_p(k)$ in Eq (4) are obtained from $x_p(n)$ relation is

$$X_p(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \quad \text{--- (5)}$$

Eq (5) is called DFS and Eq (4) is called IDFS

By using both Equations we can observe that $x_p(n)$, $X_p(k)$ are Periodic with Period 'N' Samples.

$$\text{i.e. DFS}[x_p(n)] = X_p(k)$$

PROPERTIES OF DFS:

(i) LINEARITY: Consider 2 Periodic Sequences $x_{1p}(n)$, $x_{2p}(n)$ with Period 'N'. such that $\text{DFS}[x_{1p}(n)] = X_{1p}(k)$ and $\text{DFS}[x_{2p}(n)] = X_{2p}(k)$.

$$\text{Then } \text{DFS}[a_1 x_{1p}(n) + a_2 x_{2p}(n)] = a_1 X_{1p}(k) + a_2 X_{2p}(k).$$

(ii) TIME-SHIFTING: If $x_p(n)$ is Periodic with Period 'N' Samples and $\text{DFS}[x_p(n)] = X_p(k)$ Then $\text{DFS}[x_p(n-m)] = e^{-j(2\pi/N)mk} X_p(k)$ where $x_p(n-m)$ is a shifted version of $x_p(n)$.

(iii) PERIODIC CONVOLUTION: let $x_{1p}(n)$ and $x_{2p}(n)$ be 2 Periodic Sequences with period 'N' and $\text{DFS}[x_{1p}(n)] = X_{1p}(k)$, $\text{DFS}[x_{2p}(n)] = X_{2p}(k)$ If $X_{3p}(k) = X_{1p}(k) \cdot X_{2p}(k)$ Then Periodic Sequences $x_{3p}(n)$ with Fourier Series Coefficients $X_{3p}(k)$ is $x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) x_{2p}(n-m)$

$$\text{i.e. DFS} \left[\sum_{m=0}^{N-1} x_p(m) \cdot x_p(n-m) \right] = X_p(k) \cdot X_p(k).$$

(IV) SYMMETRY PROPERTY :

$$\begin{aligned} \rightarrow \text{If } \text{DFS}[x_p(n)] &= X_p(k) \text{ Then } \text{DFS}[x_p^*(n)] = X_p^*(-k), \\ \text{DFS}[x_p^*(-n)] &= X_p^*(k) \text{ and } \text{DFS} \left\{ \text{Re}[x_p(n)] \right\} = \text{DFS} \left[\frac{x_p(n) + x_p^*(n)}{2} \right] \\ &= \frac{1}{2} [X_p(k) + X_p^*(-k)] \\ &= X_{pe}(k) \end{aligned}$$

$$\begin{aligned} \text{DFS} \left\{ j \text{Im}[x_p(n)] \right\} &= \text{DFS} \left[\frac{x_p(n) - x_p^*(n)}{2} \right] \\ &= \frac{1}{2} [X_p(k) - X_p^*(-k)] \\ &= X_{po}(k) \end{aligned}$$

$$\begin{aligned} \rightarrow \text{we can write } x_p(n) &= x_{pe}(n) + x_{po}(n) \text{ where} \\ x_{pe}(n) &= \frac{1}{2} [x_p(n) + x_p^*(-n)] \text{ \& } \\ x_{po}(n) &= \frac{1}{2} [x_p(n) - x_p^*(-n)] \end{aligned}$$

$$\begin{aligned} \text{Then } \text{DFS}[x_{pe}(n)] &= \text{DFS} \left\{ \frac{1}{2} [x_p(n) + x_p^*(n)] \right\} \\ &= \frac{1}{2} [X_p(k) + X_p^*(-k)] \\ &= \text{Re} \{ X_p(k) \} \end{aligned}$$

$$\begin{aligned} \text{DFS}[x_{po}(n)] &= \text{DFS} \left\{ \frac{1}{2} [x_p(n) - x_p^*(-n)] \right\} \\ &= \frac{1}{2} [X_p(k) - X_p^*(-k)] \\ &= j \text{Im} \{ X_p(k) \} \end{aligned}$$

DISCRETE FOURIER TRANSFORM :

$$\text{we know DTFT}[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$

$$\text{IDTFT}[X(e^{j\omega})] = x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

There are Two drawbacks Associated with DTFT, i.e.

(i) Limits of summation which extends b/w $-\infty$ to ∞ implying The length of The signal must be infinitely long.

Note: DTFT is used for finite or infinite sequences.
DFT is used for finite sequences.

(ii) The frequency variable ' ω ' which is continuous implying there is an infinite number of frequency points to be computed.

* As a Result of first drawback DTFT Requires infinite time. To overcome these problems the limits of the summation are reduced by truncating an infinite signal to a finite signal. This process is called "windowing", because only a portion of the actual discrete signal is available for transform operation

* To overcome 2nd drawback the number of frequency points to be computed is - Restricted to a finite number. DFT is the Mathematical Procedure to Determine the harmonic or frequency content of a discrete signal.

The Fourier Transform of discrete-time signal is $X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

When Fourier Transform is calculated at only discrete points, it is called Discrete Fourier Transform. It is denoted by $X(K)$. It is given as

DFT :
$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \quad \text{--- (1)}$$

where K - sample of $X(\omega)$ in frequency Domain

We know Fourier Series Coefficient

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{--- (1a)}$$

Comparing these two Equations (1) & (1a) we get

$$X(K) = N \cdot C_K$$

The inverse DFT is given as

IDFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, \quad n=0, 1, \dots, N-1 \quad (2)$$

we know Discrete Fourier Series

$$x(n) = \sum_{k=0}^{N-1} c(k) e^{j2\pi kn/N}$$

we know discrete Fourier Series

$$x(n) = \sum_{k=0}^{N-1} c(k) e^{j2\pi kn/N} \quad (2a)$$

$$\left[\because c(k) = \frac{X(k)}{N} \right]$$

$$\therefore \boxed{\text{IDFT} = \frac{1}{N} x(n)} \quad \text{from Eqs (2) and (2a)}$$

Here 'N' is number of samples in $x(n)$. Hence DFT must be at least N samples. If DFT $[x(n)]$ has $< N$ -point then Aliasing will be takes place when IDFT is taken.

* Let us we defined a term $W_N = e^{-j\frac{2\pi}{N}}$ known as Twiddle factor

Hence we can write DFT and IDFT Equations as follows.

$$\underline{\text{DFT:}} \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad \text{where } k=0, 1, \dots, N-1$$

$$\underline{\text{IDFT:}} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad \text{where } n=0, 1, \dots, N-1.$$

Let the sequence $x(n)$ as a vector of 'N' samples.

Then That sequence can be represented using Column matrix as

$$x_N(n) = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N \times 1} \quad \text{where } n=0, 1, \dots, N-1$$

Similarly its DFT is represented as

$$X_N(k) = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N \times 1} \quad \text{where } k=0, 1, \dots, N-1$$

So the twiddle factors values can be represented as a matrix $[W_N]$ of size $N \times N$ as:

$$[W_N] = \begin{bmatrix} W_N^{kn} |_{k=0, n=0} & W_N^{kn} |_{k=0, n=1} & \dots & W_N^{kn} |_{k=0, n=N-1} \\ W_N^{kn} |_{k=1, n=0} & W_N^{kn} |_{k=1, n=1} & \dots & W_N^{kn} |_{k=1, n=N-1} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{kn} |_{k=N-1, n=0} & W_N^{kn} |_{k=N-1, n=1} & \dots & W_N^{kn} |_{k=N-1, n=N-1} \end{bmatrix}$$

$$= \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \dots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

i.e. $n \rightarrow$ row wise, $k \rightarrow$ column wise changes.

$\therefore$ The N -point DFT $[x(n)] = X(k)$ can be represented in matrix form as $X_N(k) = [W_N] x_N(n)$

$$\text{IDFT}[X(k)] = x(n) \text{ as } x_N(n) = \frac{1}{N} [W_N^*] X_N(k)$$

$$\text{where } W_N^* = W_N^{-kn}$$

PERIODICITY OF W_N^{kn} :

Let $W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$, we can calculate the values of W_N for $N=8$.

$$\text{we know } W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_8 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

Let $kn = a$ where $a = 0, 1, \dots, N-1$, then

$kn = a$	$W_8^a = e^{-j\frac{\pi}{4}a}$	Magnitude	Phase
0	$W_8^0 = 1$	1	0
1	$W_8^1 = e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$	1	$-\pi/4$
2	$W_8^2 = e^{-j\frac{\pi}{2}} = -j$	1	$-\pi/2$
3	$W_8^3 = e^{-j\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$	1	$-\frac{3\pi}{4}$

contd.

$kn = a$	$W_8^a = e^{-j\pi/4 a}$	Magnitude	phase
4	$W_8^4 = e^{-j\pi} = -1$	1	$-\pi$
5	$W_8^5 = e^{-j\frac{5\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-\frac{5\pi}{4}$
6	$W_8^6 = e^{-j\frac{3\pi}{2}} = j$	1	$-\frac{3\pi}{2}$
7	$W_8^7 = e^{-j\frac{7\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-\frac{7\pi}{4}$
8	$W_8^8 = e^{-j2\pi} = 1 = W_8^0$	1	$-2\pi = 0$
9	$W_8^9 = e^{-j\frac{9\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} = W_8^1$	1	$-\frac{9\pi}{4} = -\frac{\pi}{4}$
10	$W_8^{10} = e^{-j\frac{5\pi}{2}} = -j = W_8^2$	1	$-\frac{5\pi}{2} = -\frac{\pi}{2}$
11	$W_8^{11} = e^{-j\frac{11\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} = W_8^3$	1	$-\frac{11\pi}{4} = -\frac{3\pi}{4}$
12	$W_8^{12} = e^{-j3\pi} = -1 = W_8^4$	1	$-3\pi = -\pi$
13	$W_8^{13} = e^{-j\frac{13\pi}{4}} = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = W_8^5$	1	$-\frac{13\pi}{4} = -\frac{5\pi}{4}$
14	$W_8^{14} = e^{-j\frac{7\pi}{2}} = j = W_8^6$	1	$-\frac{7\pi}{2} = -\frac{3\pi}{2}$
15	$W_8^{15} = e^{-j\frac{15\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = W_8^7$	1	$-\frac{15\pi}{4} = -\frac{\pi}{4}$
16	$W_8^{16} = e^{-j4\pi} = 1 = W_8^8$	1	$-4\pi = -2\pi$

So we can write W_N^a is a periodic with period 'N'. That is $W_8^0 = W_8^8 = W_8^{16} = \dots$ and $W_8^1 = W_8^9 = W_8^{17} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

So In General we can write

$$W_N^a = W_N^{a \pm N} = W_N^{a \pm 2N} \dots$$

This is known as periodic Property of TWIDDLE FACTOR.

and also we can write from the table

$$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}, \quad W_8^5 = -\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

$$W_8^2 = -1 - j, \quad W_8^6 = +j$$

In General we can write

$$W_N^a = -W_N^{a \pm \frac{N}{2}}$$

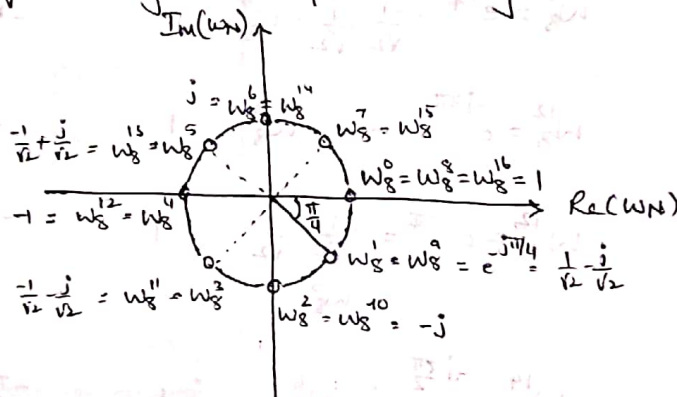
This is known as symmetry property of the Twiddle factor.

* Magnitude of Twiddle factor W_N^{nk} is $|W_N|$

$$|W_N| = |e^{-j\frac{2\pi}{N}}| = |\cos \frac{2\pi}{N} - j \sin \frac{2\pi}{N}| \text{ and}$$

* phase angle of Twiddle factor is $\angle e^{-j\frac{2\pi}{N}} = -\frac{2\pi}{N}$.

The Symmetry and periodicity can be shown as



ZERO PADDING:-

Let a signal with length 'L' is $x(n) = \{x(0), x(1), \dots, x(L-1)\}$
 i.e. it has L samples, minimum number of Equally-spaced frequency points can be calculated b/w $0 \rightarrow 2\pi$ without time domain aliasing, so frequency Resolution is $\frac{2\pi}{L}$. This frequency Resolution is said to be poor if straight line interpolation of frequency samples are different from Actual magnitude and phase response. If we increase number of points b/w $0 \rightarrow 2\pi$ we can get better Representation of Magnitude and phase Response i.e. The frequency Resolution is Improved by Adding

(6)

zero's to signal as $x(n) = \{x(0), x(1), \dots, x(L-1), 0, 0, \dots, 0\}$

* In order to have 'N' frequency points b/w 0 and 2π , we add $N-L$ zeros to the sequence. Then the sequence length is = to 'N'. Then the frequency Resolution of N-point DFT is $\frac{2\pi}{N}$.

Example problems:

1. Find the DFT of sequence $x(n) = \{1, 1, 0, 0\}$ and IDFT of $Y(K) = \{1, 0, 1, 0\}$

Sol: let $N = L = 4$

$$\text{We know } X(K) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi K n}{N}}, K=0, 1, \dots, N-1$$

$$\begin{aligned} \text{Then } X(0) &= \sum_{n=0}^3 x(n) = x(0) + x(1) + x(2) + x(3) \\ &= 1 + 1 + 0 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} X(1) &= \sum_{n=0}^3 x(n) e^{-j \frac{2\pi n}{4}} = x(0) + x(1) e^{-j \frac{\pi}{2}} + x(2) e^{-j \pi} + x(3) e^{-j \frac{3\pi}{2}} \\ &= 1 + \cos \frac{\pi}{2} - j \sin \frac{\pi}{2} \\ &= 1 - j \end{aligned}$$

$$\begin{aligned} X(2) &= \sum_{n=0}^3 x(n) e^{-j \pi n} = x(0) + x(1) e^{-j \pi} + x(2) e^{-j 2\pi} + x(3) e^{-j 3\pi} \\ &= 1 + \cos \pi - j \sin \pi \\ &= 0 \end{aligned}$$

$$\begin{aligned} X(3) &= \sum_{n=0}^3 x(n) e^{-j \frac{3\pi n}{2}} = x(0) + x(1) e^{-j \frac{3\pi}{2}} + x(2) e^{-j 3\pi} + x(3) e^{-j \frac{9\pi}{2}} \\ &= 1 + \cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} \\ &= 1 + j \end{aligned}$$

$$\therefore \text{DFT } \{x(n)\} = X(K) = \{2, 1-j, 0, 1+j\}$$

$$\text{We know IDFT } [Y(K)] = y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K) e^{j \frac{2\pi K n}{N}}$$

$$\begin{aligned} \Rightarrow y(0) &= \frac{1}{4} \sum_{K=0}^3 Y(K), n=0, 1, 2, 3 \\ &= \frac{1}{4} \{Y(0) + Y(1) + Y(2) + Y(3)\} \\ &= \frac{1}{4} (1 + 0 + 0 + 0) = 0.5 \end{aligned}$$

$$\begin{aligned}
 y(1) &= \frac{1}{N} \sum_{k=0}^3 Y(k) \cdot e^{j\pi k/2} \\
 &= \frac{1}{4} \left[Y(0) + Y(1)e^{j\pi/2} + Y(2)e^{j\pi} + Y(3)e^{j3\pi/2} \right] \\
 &= \frac{1}{4} [1 + 0 + \cos\pi + j\sin\pi + 0] \\
 &= \frac{1}{4} [1 + 0 - 1 + 0] = 0
 \end{aligned}$$

$$\begin{aligned}
 y(2) &= \frac{1}{4} \sum_{k=0}^3 Y(k) \cdot e^{j2\pi k} \\
 &= \frac{1}{4} \{ Y(0) + Y(1)e^{j2\pi} + Y(2)e^{j4\pi} + Y(3)e^{j6\pi} \} \\
 &= \frac{1}{4} \{ 1 + 0 + \cos 2\pi + j\sin 2\pi + 0 \} \\
 &= \frac{1}{4} \{ 1 + 0 + 1 + 0 \} = 0.5
 \end{aligned}$$

$$\begin{aligned}
 y(3) &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j3\pi k/2} \\
 &= \frac{1}{4} \{ Y(0) + Y(1)e^{j3\pi/2} + Y(2)e^{j3\pi} + Y(3)e^{j9\pi/2} \} \\
 &= \frac{1}{4} \{ 1 + 0 + \cos 3\pi + j\sin 3\pi + 0 \} \\
 &= \frac{1}{4} \{ 1 + 0 + (-1) + 0 \} = 0
 \end{aligned}$$

$$\therefore y(n) = \{0.5, 0, 0.5, 0\}$$

2) Find DFT of a sequence $x(n)=1$ for $0 \leq n \leq 2$
 $= 0$ otherwise

for (i) $N=4$ (ii) $N=8$ plot $|x(k)|$ & $\angle x(k)$.

Sol: given length of sequence $L=3$ ($0 \leq n \leq 2$)

But $N=4$ so we can add a zero to given sequence

$$N - L \text{ zeros} = 4 - 3 = 1$$

$$\therefore \text{sequence is } x(n) = \{1, 1, 1, 0\}$$

$$\text{we know DFT}[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, k=0, \dots, N-1$$

$$\begin{aligned}
 \therefore X(0) &= \sum_{n=0}^3 x(n) \\
 &= x(0) + x(1) + x(2) + x(3) \\
 &= 1 + 1 + 1 + 0 = 3
 \end{aligned}$$

$\therefore$ Magnitude $|X(0)| = 3$

phase $\angle X(0) = 0$

for $k=1$, $X(1) = \sum_{n=0}^3 x(n) e^{-j\pi n/2}$

$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$

$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + 0$

$= 1 - j - 1 = -j$

$\therefore$ Magnitude $|X(1)| = 1$

phase $\angle X(1) = -\pi/2$

$X(2) = \sum_{n=0}^3 x(n) e^{-j\pi n}$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$

$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$

$= 1 - 1 + 1 = 1$

$\therefore$ Magnitude $|X(2)| = 1$

phase $\angle X(2) = -\pi = 0$

$X(3) = \sum_{n=0}^3 x(n) e^{-j3\pi n/2}$

$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$

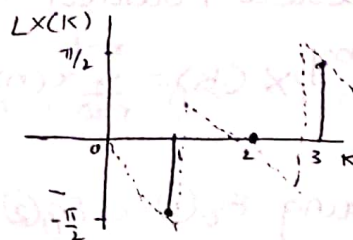
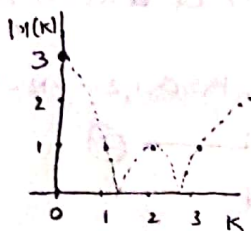
$= 1 + j - 1 = j$

$\therefore$ Magnitude $|X(3)| = 1$

phase $\angle X(3) = -\frac{3\pi}{2} = \frac{\pi}{2}$

$\therefore x(k) = \{3, -j, 1, j\}$

Plots of $|x(k)|$, $\angle x(k)$

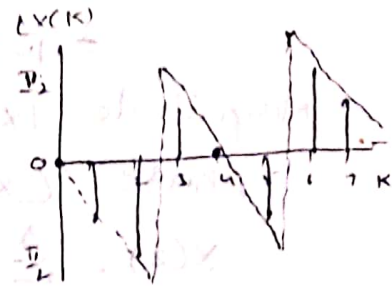
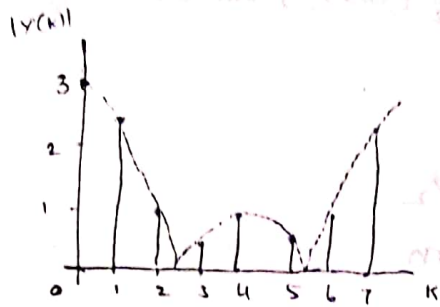


Similarly for $N=8$, we get

$$X(K) = \left\{ 3, 1.707 - j1.707, -j, 0.293 + j0.293, 1, 0.293 - j0.293, j, 1.707 + j1.707 \right\}$$

$$|X(K)| = \{ 3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414 \}$$

$$\angle X(K) = \left\{ 0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4} \right\}$$



3. Determine 8-point DFT of sequence $x(n)$, where

$$x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$$

Sol: $X(K) = \{6, -0.707 - j1.707, 1 - j, 0.707 + j0.293, 0, 0.707 - j0.293, 1 + j, -0.707 + j1.707\}$

4. Find IDFT of $X(K) = \{4, 1, 0, 1 - j, 1, 0, 1 + j, 1\}$

Sol: $x(n) = \{ \quad \quad \quad \}$

RELATION BETWEEN DFT AND OTHER TRANSFORMS:

(i) RELATION TO THE FOURIER TRANSFORM:

The Fourier Transform of a finite duration sequence $x(n)$ having length N is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \quad \text{--- (1)}$$

The Discrete Fourier Transform of $x(n)$ is

$$\text{given by } X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}, \quad K=0, 1, 2, \dots, N-1 \quad \text{--- (2)}$$

Comparing Eq (1) & Eq (2) we get

$$X(K) = X(e^{j\omega}) \Big|_{\omega = \frac{2\pi K}{N}}, \quad K=0, 1, \dots, N-1$$

(ii) RELATION TO THE Z-TRANSFORM

Let the Z-Transform of a sequence having finite duration 'N' as $X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$ — (1), and

$$\text{DFT}[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} \quad \text{--- (2)}$$

Comparing Eq (1) & Eq (2) we get

$$X(k) = X(e^{j\omega}) \Big|_{\substack{\omega = \frac{2\pi k}{N} \\ z = e^{j\omega}}}, \quad k = 0, 1, 2, \dots, N-1.$$

PROPERTIES OF DISCRETE FOURIER TRANSFORM :-

(i) PERIODICITY :-

If $X(k)$ is N-point DFT of a finite duration sequence $x(n)$ then $x(n+N) = x(n)$ for all 'n'.
 $X(k+N) = X(k)$ for all 'k'.

(ii) LINEARITY :-

If two finite duration sequences $x_1(n)$ & $x_2(n)$ are linearly combined as $x_3(n) = ax_1(n) + bx_2(n)$ then $\text{DFT}[x_3(n)] = X_3(k) = aX_1(k) + bX_2(k)$.

If $x_1(n)$ has length N_1 and $x_2(n)$ has length N_2 then Maximum length of $x_3(n)$ will be $N_3 = \text{Max}(N_1, N_2)$.

Let N_1 -point DFT is $X_1(k) = \sum_{n=0}^{N_1-1} x_1(n) e^{-j\frac{2\pi kn}{N_1}}$

where $0 \leq k \leq N_1-1$

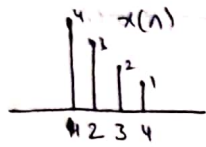
Let N_2 -point DFT of $x_2(n)$ is $X_2(k) = \sum_{n=0}^{N_2-1} x_2(n) e^{-j\frac{2\pi kn}{N_2}}$

where $0 \leq k \leq N_2-1$

$\therefore \text{DFT}[ax_1(n) + bx_2(n)] = aX_1(k) + bX_2(k)$

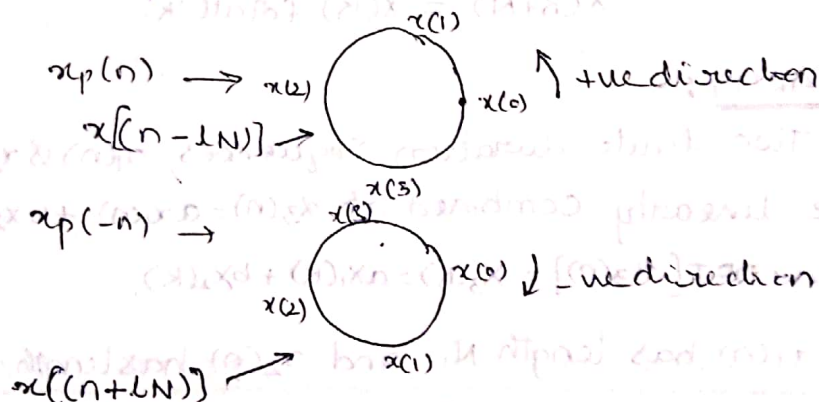
(iii) circular shift of a sequence.

The Periodic Extension of a sequence can be represented as $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$



If we consider periodic sequence $x_p(n)$ is wrapping the finite duration sequence $x(n)$. This can be expressed using a circle. representation will be of modulo-N

$$x_p(n) = x[(n \text{ modulo } N)]$$



In General shifted version $x(n)$ can be shown as

$$x(n) = [x(0), x(1), x(2), \dots, x(N-2), x(N-1)]$$

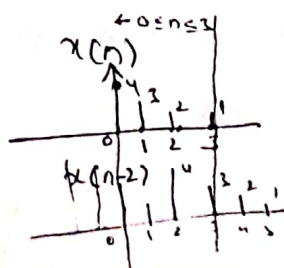
$$x((n-1))_N = [x(N-1), x(0), x(1), \dots, x(N-3), x(N-2)]$$

$$x((n-2))_N = [x(N-2), x(N-1), x(0), \dots, x(N-4), x(N-3)]$$

$$x((n-k))_N = [x(N-k), x(N-k+1), x(N-k+2), \dots, x(N-k-1)]$$

$$x((n-N))_N = [x(0), x(1), x(2), \dots, x(N-2), x(N-1)]$$

ex:



$$i.e. x(n) = x((n-N))_N$$

$$\Rightarrow x(n-N+N) = x((n-N))_N$$

$$\text{Similarly } \underline{x((n-m))_N = x(N+n-m)}$$

Now. Come to The property

$$\text{DFT}[x(n)] = X(K) \text{ Then } \text{DFT}[x((n-m))_N] = e^{-j2\pi \frac{Km}{N}} \cdot X(K)$$

proof: Apply above concept (d) following the below.

$$\text{DFT}[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{Kn}{N}}$$

$$\begin{aligned} \text{DFT}[x((n-m))_N] &= \sum_{n=0}^{N-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} \\ &= \sum_{n=0}^{m-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} + \sum_{n=m}^{N-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} \end{aligned}$$

$$\text{But we know } x((n-m))_N = x(N+n-m)$$

$$\begin{aligned} \therefore \text{let } \sum_{n=0}^{m-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} &= \sum_{n=0}^{m-1} x(N+n-m) \cdot e^{-j2\pi \frac{Kn}{N}} \\ &= \sum_{l=N-m}^{N-1} x(l) e^{-j2\pi K \frac{(l+m-N)}{N}} \quad \text{let } l = N+n-m \\ &= \sum_{l=N-m}^{N-1} x(l) e^{-j2\pi K \frac{(l+m)}{N}} \quad \Rightarrow n = l+m-N \\ &\quad \left(\because e^{j2\pi K} = 1 \text{ for } K=0, 1, 2, \dots \right) \end{aligned}$$

Similarly

$$\begin{aligned} \text{let } \sum_{n=m}^{N-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} &= \sum_{n=m}^{N-1} x(N-m+n) e^{-j2\pi \frac{Kn}{N}} \\ &= \sum_{l=0}^{N-m-1} x(l) \cdot e^{-j2\pi K \frac{(l+m)}{N}} \quad \text{let } l = n-m+N \\ &\quad n = l+m-N \end{aligned}$$

from eq ① & eq ② we can write

$$\begin{aligned} \sum_{n=0}^{N-1} x((n-m))_N e^{-j2\pi \frac{Kn}{N}} &= \sum_{l=N-m}^{N-1} x(l) e^{-j2\pi K \frac{(l+m)}{N}} + \sum_{l=0}^{N-m-1} x(l) e^{-j2\pi K \frac{(l+m)}{N}} \\ &= \sum_{l=0}^{N-1} x(l) e^{-j2\pi K \frac{(l+m)}{N}} \\ &= e^{-j2\pi K m} \cdot X(K) \end{aligned}$$

(iv) TIME REVERSAL OF THE SEQUENCE

The time reversal of an N -point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in clockwise direction. It is denoted as $x((-n))_N$.

$$\text{i.e. } x((-n))_N = x(N-n) \quad 0 \leq n \leq N-1.$$

$$\text{If DFT}[x(n)] = X(k) \text{ Then DFT}[x((-n))_N] = \text{DFT}[x(N-n)] \\ = X((-k))_N = X(N-k).$$

[changing the index from n to $N-n$ we get

$$\begin{aligned} \text{DFT}[x(N-n)] &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi k(N-n)/N} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kN/N} e^{j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi k(N-n)/N} \quad \left[\because \frac{j2\pi kN}{N} = j2\pi k \text{ for } k=0,1,2 \right] \\ &= X(N-k). \end{aligned}$$

Proof:

$$\begin{aligned} \text{DFT}[x(N-n)] &= \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} x(a) e^{-j2\pi kn/N} e^{j2\pi ka/N} \quad \left[\begin{array}{l} \text{Let } N-n=a \\ n=N-a \end{array} \right] \\ &= \sum_{n=0}^{N-1} x(a) e^{j2\pi ka/N} \quad \left[\because e^{-j2\pi kn/N} = 1 \text{ for } k=0 \right] \\ &= X((-k))_N = X(N-k) \end{aligned}$$

(v) CIRCULAR FREQUENCY SHIFT:

$$\text{If DFT}[x(n)] = X(k) \text{ Then DFT}[x(n) e^{j2\pi l n/N}] = X((k-l))_N$$

Proof:

$$\begin{aligned} \text{DFT}[x(n) e^{j2\pi l n/N}] &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} e^{j2\pi l n/N} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k-l)/N} \\ &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N+k-l)/N} \\ &= X(N+k-l) = X((k-l))_N \end{aligned}$$

vi) COMPLEX CONJUGATE PROPERTY:

If $\text{DFT}[x(n)] = X(K)$ Then $\text{DFT}[x^*(n)] = X^*(N-K) = X^*((-K))_N$

$$\begin{aligned} \text{Proof: } \text{DFT}[x^*(n)] &= \sum_{n=0}^{N-1} x^*(n) \cdot e^{-j2\pi kn/N} \\ &= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N} \right]^* \\ &= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-K)/N} \right]^* \\ &= X^*(N-K) = X^*((-K))_N \end{aligned}$$

$$\begin{aligned} \text{Similarly IDFT}[X^*(K)] &= \left(\frac{1}{N} \sum_{K=0}^{N-1} X^*(K) e^{j2\pi Kn/N} \right)^* \\ &= \frac{1}{N} \left[\sum_{K=0}^{N-1} X(K) e^{-j2\pi Kn/N} \right]^* \\ &= \frac{1}{N} \left[\sum_{K=0}^{N-1} X(K) \cdot e^{j2\pi K(N-n)/N} \right]^* \\ &= x^*(N-n). \end{aligned}$$

(vii) CIRCULAR CONVOLUTION :- let Two finite sequences $x_1(n), x_2(n)$ of Length 'N' with DFTs as $X_1(K), X_2(K)$ resp.

So we have a sequence $x_3(n)$ with DFT is $X_3(K)$

where $X_3(K) = X_1(K) \cdot X_2(K)$.

Proof:

Let The Convolution of Two sequences is

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m).$$

$$x_3(n)_N = \sum_{m=0}^{N-1} x_1(m)_N \cdot x_2(n-m)_N$$

But for $0 \leq n \leq N-1$, $x_3(n)_N = x_3(n)$, $x_1(m)_N = x_1(m)$

$$\therefore x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N \quad \text{--- (1)}$$

Eq (1) represents the circular convolution of $x_1(n), x_2(n)$

$$x_3(n) = x_1(n) \circledast x_2(n)$$

$$\therefore \text{DFT}[x_1(n) \circledast x_2(n)] = X_1(K) \cdot X_2(K)$$

iii) CIRCULAR CORRELATION:

for complex valued sequences $x(n)$, $y(n)$ if
 $\text{DFT}[x(n)] = X(K)$, $\text{DFT}[y(n)] = Y(K)$ then

$$\begin{aligned}\text{DFT}[\tilde{x}y(L)] &= \text{DFT}\left[\sum_{n=0}^{N-1} x(n) \cdot y^*((n-L)_N)\right] \\ &= X(K) \cdot Y^*(K)\end{aligned}$$

(ix) MULTIPLICATION OF TWO SEQUENCES:

If $\text{DFT}[x_1(n)] = X_1(K)$, $\text{DFT}[x_2(n)] = X_2(K)$ then

$$\text{DFT}[x_1(n) \cdot x_2(n)] = \frac{1}{N} [X_1(K) \otimes X_2(K)]$$

Proof:

$$\begin{aligned}\text{DFT}[x_1(n)x_2(n)] &= \sum_{n=0}^{N-1} x_1(n) \cdot x_2(n) \cdot e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot e^{+j2\pi ln/N} \cdot x_2(n) \cdot e^{-j2\pi kn/N} \\ &= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j2\pi n(k-l)/N} \\ &= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j2\pi n(N+k-l)/N} \\ &= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot x_2((k-l)_N) \\ &= \frac{1}{N} [X_1(K) \otimes X_2(K)]\end{aligned}$$

(x) PARSEVAL'S THEOREM:-

If $\text{DFT}[x(n)] = X(K)$ and $\text{DFT}[y(n)] = Y(K)$ then

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \cdot Y^*(K)$$

$$\begin{aligned}\text{Let } \sum_{n=0}^{N-1} x(n) \cdot y^*(n) &= \sum_{n=0}^{N-1} x(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} Y(K) \cdot e^{j2\pi kn/N} \right]^* \\ &= \sum_{k=0}^{N-1} \frac{1}{N} Y^*(K) \cdot \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} Y^*(K) \cdot X(K)\end{aligned}$$

COMPARISON BETWEEN CIRCULAR & LINEAR CONVOLUTION

The linear convolution of sequences $x(n)$ and $h(n)$ produces a result $y(n)$ which contains number of samples as $N = N_1 + N_2 - 1$ where N_1 is length of $x(n)$, N_2 is length of $h(n)$.

But in circular convolution, after the convolution the sequence having number of samples as $N = \max(N_1, N_2)$. If $N_2 < N_1$, we must add $N_1 - N_2$ number of zeros to sequence $h(n)$, so both sequences periodic with N .

linear convolution can be used to find the response of a filter. Circular convolution can't be used to find response of a linear filter without zero padding.

We have 2 methods to find circular convolution of two sequences. They are

- (1) Concentric Circle Method
- (2) Matrix Multiplication Method.

CONCENTRIC CIRCLE METHOD:-

Let two sequences $x_1(n)$ and $x_2(n)$. The circular convolution of these two sequences is $x_3(n) = x_1(n) \otimes x_2(n)$ can be found by using following steps.

1. Graph N samples of $x_1(n)$ as Equally spaced Points around an outer circle in Anticlockwise direction
2. Start at The same point as $x_1(n)$ graph N samples of $x_2(n)$ as Equally spaced points around an inner circle in clockwise direction.
3. Multiply corresponding samples on the Two circles and sum the products to produce output.
4. Rotate The inner circle one sample at a time in Anticlockwise direction and go to step 3.
5. Repeat step 4 until The inner circle first sample lines up with the first sample of the exterior circle once again.

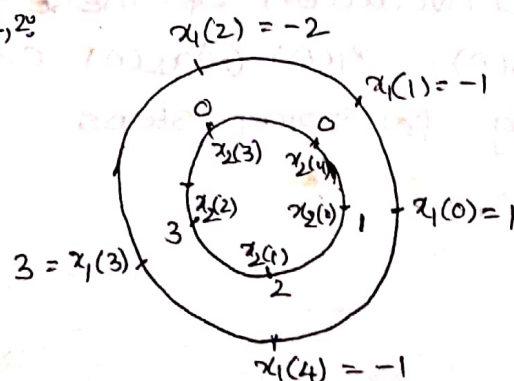
Ex: find circular convolution of Two sequences
 $x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3\}$

Sol: To find circular convolution both seq. must be of same length. so we add Two zeros to $x_2(n)$.

$$\therefore x_1(n) = \{1, -1, -2, 3, -1\}$$

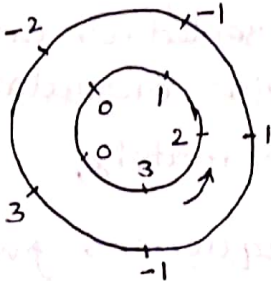
$$x_2(n) = \{1, 2, 3, 0, 0\}$$

Step 1, 2:

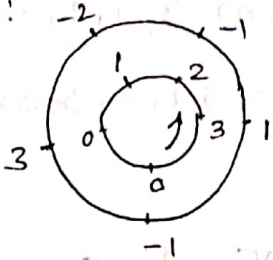


Step 3:

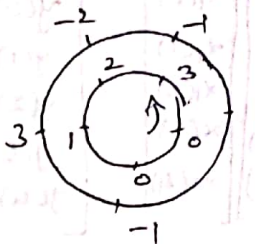
$$\begin{aligned}
 y(0) &= x_1(0) \cdot x_2(0) + x_1(1) \cdot x_2(4) + x_1(2) \cdot x_2(3) + x_1(3) \cdot x_2(2) \\
 &\quad + x_1(4) \cdot x_2(1) \\
 &= 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1) \\
 &= 8
 \end{aligned}$$

Step 4:

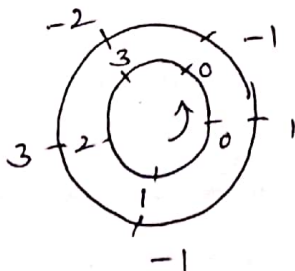
$$\begin{aligned}
 y(1) &= 1(2) + 1(-1) + 0(-2) + 0(3) + 3(1) \\
 &= -2
 \end{aligned}$$

Step 5:

$$\begin{aligned}
 y(2) &= 1(3) - 1(2) - 2(1) + 3(0) + (-1)(0) \\
 &= -1
 \end{aligned}$$



$$\begin{aligned}
 y(3) &= 1(0) + (-1)(3) - 2(2) + 3(1) - 1(0) \\
 &= -4
 \end{aligned}$$



$$\begin{aligned}
 y(4) &= 1(0) - 1(0) - 2(3) + 3(2) - 1(1) \\
 &= -1
 \end{aligned}$$

$$\therefore y(n) = \{8, -2, -1, -4, -1\}$$

MATRIX MULTIPLICATION METHOD :- In this method circular convolution of two sequences $x_1(n)$, $x_2(n)$ can be obtained by representing the sequences in Matrix form as follows

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \dots & x_2(2) & \dots & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \dots & x_2(3) & \dots & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \dots & x_2(4) & \dots & x_2(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \dots & x_2(0) & \dots & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \dots & x_2(1) & \dots & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

$N \times N$ $N \times 1$ $N \times 1$

The sequence $x_1(n)$ is represented as column-matrix. $x_2(n)$ repeated via circular shift and is represented by $N \times N$ matrix.

Ex: Find Convolution of Two sequences given by

$$x_1(n) = \{1, -1, -2, 3, -1\}, x_2(n) = \{1, 2, 3\}$$

by adding zeros to $x_2(n) = \{1, 2, 3, 0, 0\}$

given $N=5$ So matrix is

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\Rightarrow \begin{matrix} & x_2(n) & x_1(n) & \\ \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} & = & \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix} \end{matrix}$$

$$\therefore y(n) = \{8, -2, -1, -4, -1\}$$

find convolution of above using DFT & IDFT

$$\text{i.e. } X_3(K) = X_1(K) \cdot X_2(K)$$

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi \frac{Kn}{N}} \rightarrow \text{fnd}$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi \frac{Kn}{N}} \rightarrow \text{fnd}$$

$$\text{Next } X_3(K) = X_1(K) \cdot X_2(K)$$

$$\text{Then } x_3(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_3(K) e^{j2\pi \frac{Kn}{N}}$$

LINEAR CONVOLUTION FROM CIRCULAR CONVOLUTION (c)

In Signal processing applications we are interested in linear convolution of two sequences. Let two finite duration sequences $x(n)$ and $h(n)$ with samples N_1 and N_2 respectively. The linear convolution $y(n) = x(n) * h(n)$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=0}^{N_1+N_2-1} x(k)h(n-k)$$

We have here a finite sequence.

$y(n)$ has length of $N_1 + N_2 - 1$ samples.

But in circular convolution samples for $y(n)$ are $N = \max(N_1, N_2)$.

If $N_2 < N_1$, to find circular convolution we must add zeros to $N_1 - N_2$ to $h(n)$ sequence.

So to find circular convolution $x(n), h(n)$ must have $N_1 + N_2 - 1$ samples.

So we must add $N_2 - 1$ zeros to $x(n)$ and $N_1 - 1$ zeros to $h(n)$.

LINEAR CONVOLUTION USING DFT:

To perform linear convolution using DFT it is necessary to evaluate DFT at sufficiently large number of points so that we can avoid occurring of time-aliasing effect.

Let $x_1(n), h(n)$ of lengths N_1 and N_2 . Then $y_3(n) = x_1(n) * h(n)$ is obtained by $Y_3(k) = X_1(k) \cdot H(k)$ and taking IDFT of $Y_3(k)$.

Ex: perform linear convolution using DFT

$$\text{where } x(n) = \begin{cases} 1 & \text{for } n=0 \\ 0.5 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} 0.5 & \text{for } n=0 \\ 1 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$$

Sol: first find $N = N_1 + N_2 - 1$

then find $X(K)$ and $H(K)$ -

Multiply both $X(K)$ and $H(K)$ to $Y(K)$

$$Y(K) = X(K) \cdot H(K)$$

then find $y(n) = \text{IDFT}[Y(K)]$.

Ex: given 8-pt DFT of a sequence $x(n) = 1$ $0 \leq n \leq 3$
 $= 0$ $4 \leq n \leq 7$

then compute DFT of (i) $x_1(n) = \begin{cases} 1 & n=0 \\ 0 & 1 \leq n \leq 4 \\ 1 & 5 \leq n \leq 7 \end{cases}$

(ii) $x_2(n) = \begin{cases} 0 & 0 \leq n \leq 1 \\ 1 & 2 \leq n \leq 5 \\ 0 & 6 \leq n \leq 7 \end{cases}$

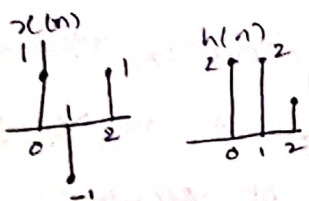
first find $X(K)$

then find $x_1(K)$ apply time shifting

$$x_1(n) = x((n-(-3))_N) \Rightarrow X_1(K) = \text{DFT}[x((n-m))_N]$$

$$\xrightarrow{\text{DFT}} [x((n-m))_N] = X(K) \cdot e^{-j2\pi km/N}$$

→ use DFT to compute linear convolution of signals chosen



FILTERING LONG DURATION SEQUENCES: Let an input sequence $x(n)$ is of long duration is to be processed with a system having impulse response of finite duration, by convolving two sequences. Because of length of input sequence it would not be practical to store it all before performing linear convolution. So the input sequence must be divided up into blocks. we have two methods for this sectioned convolution. They are

- (i) Overlap-Save Method
- (ii) Overlap-Add Method.

(i) Overlap-Save Method:-

Let input sequence of length N_1 , and an impulse sequence of length N_2 . So in this input sequence is divided into blocks of size $N = N_1 + N_2 - 1$, for each block consists of $N_2 - 1$ data points of previous block followed by N_1 new data points. Now impulse sequence length is increased with adding $N_1 - 1$ zeros, then $N = N_1 + N_2 - 1$, and $y_k(n) = x_k(n) \otimes h(n)$.

Ex: find $y(n)$ with $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ and $h(n) = \{1, 1, 1\}$

Sol: let $N_1 = 10, N_2 = 3$ $N = N_1 + N_2 - 1 = 12$

given input sequence can be divided as follows

$$\begin{aligned} x_1(n) &= \{ \underbrace{0, 0}_{N_2-1}, \underbrace{3, -1, 0}_{N_1} \} \\ x_2(n) &= \{ -1, 0, \underbrace{1, 3, 2}_{N_1} \} \\ x_3(n) &= \{ 3, 2, 0, \underbrace{1, 2}_{N_1} \} \\ x_4(n) &= \{ 1, 2, 1, 0, 0 \} \end{aligned}$$

here Each input sequence length $N_1 = 5$

$h(n)$ length $N_2 = 3$

But $N = \max(N_1, N_2)$
 $= 5$, 2 zeros added to $h(n)$

$$\therefore h(n) = \{1, 1, 1, 0, 0\}$$

$$\therefore y_1(n) = x_1(n) \otimes h(n) = \{ \underbrace{-1, 0, 3, 2, 2} \}$$

$$y_2(n) = x_2(n) \otimes h(n) = \{ \underbrace{4, 1, 0, 4, 6} \}$$

$$y_3(n) = x_3(n) \otimes h(n) = \{ \underbrace{6, 7, 5, 3, 3} \}$$

$$y_4(n) = x_4(n) \otimes h(n) = \{ \underbrace{1, 3, 4, 3, 1} \}$$

discard

$$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\} \checkmark$$

(ii) Overlap-Add Method:

Let a block contains 3 data bits.

$$\text{Then } x_1(n) = \{3, -1, 0, 0, 0\}$$

$$x_2(n) = \{1, 3, 2, 0, 0\}$$

$$x_3(n) = \{0, 1, 2, 0, 0\}$$

$$x_4(n) = \{1, 0, 0, 0, 0\}$$

$$\begin{bmatrix} h(n) \\ x_1(n) \\ x_2(n) \\ x_3(n) \\ x_4(n) \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{Then } y_1(n) = x_1(n) \otimes h(n) = \{3, 2, 2, -1, 0\}$$

$$y_2(n) = x_2(n) \otimes h(n) = \{1, 4, 6, 5, 2\}$$

$$y_3(n) = x_3(n) \otimes h(n) = \{0, 1, 3, 3, 2\}$$

$$y_4(n) = x_4(n) \otimes h(n) = \{1, 1, 1, 0, 0\}$$

$$3, 2, 2, -1, 0$$

↓ ↓ add

$$1, 4, 6, 5, 2$$

↓ ↓ add

$$0, 1, 3, 3, 2$$

↓ ↓

$$1, 1, 1, 0, 0$$

unusable

$$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\} \checkmark$$



$$\begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}$$

3, 2, 2, -1, 0

$$\begin{array}{c|cccc} & 0 & 0 & 1 & 1 & 1 \\ \hline -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 3 & 3 & 3 \\ 2 & 0 & 0 & 2 & 2 & 2 \end{array}$$

0, 0, -1, -1, 0, 4, 6, 5, 2

00 3, -1, 0

11100

$$\begin{array}{c|ccccc} & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 2 & 2 & 2 \\ -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

0 0 3, 2, 2, -1, 0

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -0.5 \\ 2 \end{bmatrix}$$

-1 + 2
-1