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INTRODUCTION: Basically digital filter is a linear-time-invariant discrete time system. we have two kinds FIR (finite impulse response) and IIR (infinite impulse response). to realize the digital filter transfer function we have to take the help of z-transforms.

The z-transform is a powerful mathematical tool for the analysis of LTI-discrete-time system in frequency domain.

In z-domain the convolution of two time domain signals is equivalent to multiplication of their corresponding z-transforms.

This property simplifies the analysis of response of an LTI-system to various signals and also frequency response of a discrete-time system can be determined by evaluating Tof on unit circle of z-plane.

z-transform:

The z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \quad \text{--- (1)}$$

where z-Complex variable

In polar form z can be expressed as $z = re^{j\omega}$.

where r-radius of circle. mostly we consider

$r=1$, i.e. we consider unit circle.

By substituting 'z' value in Eq ① we may get

$$X(ze^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \cdot z^{-n} \quad (z=1 \text{ for unit circle})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

= Fourier Transform of $x(n)$.

Eq ① represents the Two-sided z-Transform.

one-sided z-Transform means it is having either causal or anticausal values.

Right-hand Sequence:- It is one for which $x(n) = 0$ for all $n < n_0$ where n_0 is +ve or -ve but finite. If $n_0 \geq 0$, then resulting sequence is causal and a +ve time sequence. ex: $x(n) = \{ \underset{\uparrow}{1}, 0, 2, 3, 4 \}$

Left-hand Sequence:- It is one for which $x(n) = 0$ for all $n \geq n_0$, if $n_0 \leq 0$ the resulting sequence is ~~Anticlock~~ anticausal sequence.

ex: $x(n) = \{ 1, 0, 2, 3, \underset{\uparrow}{-4} \}$

Two-Sided Sequence: A signal that has finite duration on both the left & right sides is known as Two-sided sequence.

REGION OF CONVERGENCE: The z-Transform is an infinite power series, it exists only for those values of 'z' for which this series converges. The ROC of $X(z)$ is the set of all values of 'z' for which $X(z)$ attains a finite value.

The one-sided seq can be represented as $X_+(z)$
converges - come together to meet at a point.
divergent - move in different directions
then - reach achieve.

ROC of finite duration

- * The ROC for Right hand sequence is Entire z-plane except $z=0$
- * The ROC for Left hand sequence is Entire z-plane except $z=\infty$
- * The ROC for Two-sided sequence is Entire z-plane except at $z=0$ and $z=\infty$

ROC OF INFINITE DURATION:

The concept of ROC of infinite duration can be understood with the following.

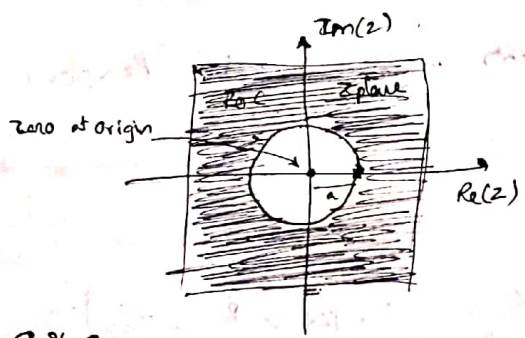
ex: determine Z-Transform and ROC of $x(n) = a^n u(n)$

We know $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ ↓ causal

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

It is infinite series as $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$ if $|x| < 1$

$$\Rightarrow X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a} \quad \text{so ROC is } |z| > a$$



ex 2

Find Z-Transform & ROC of $x(n) = -b^n \cdot u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

↙ Anticausal

$$= \sum_{n=-\infty}^{-1} -b^n \cdot z^{-n} = - \sum_{n=-\infty}^{-1} (b^n z^{-n}) - 1$$

$$= - \sum_{n=-\infty}^{-1} (b z^{-1})^n - 1 = - \sum_{n=0}^{\infty} (b^{-1} z)^n - 1$$



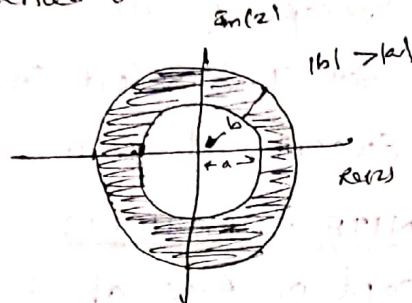
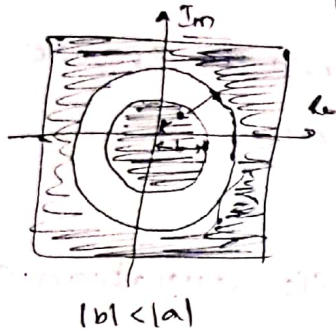
The above series is Converges for $|b^{-1} z| < 1 \Rightarrow |z| < b$

$$\therefore X(z) = - \left(\frac{1}{1 - b z^{-1}} - 1 \right) = \frac{z}{z - b} \quad \text{ROC } |z| < b$$

The
let given sequence is a Two-sided Sequence then
of infinite duration having ROC of $x(z)$ as
 $|a| < z < |b|$

i.e $x(n) = a^n u(n) + b^n u(-n-1)$

The ROC is represented as



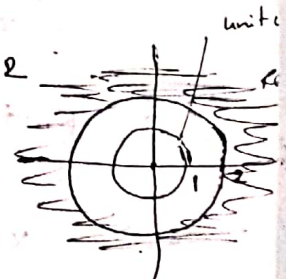
STABILITY & ROC: let $h(n)$ is the impulse response of a causal or non-causal linear time invariant system & $H(z)$ be z-Transform of $h(n)$ then stability can be found using following statement

→ A linear time-invariant system with system function $H(z)$ is stable if & only if ROC for $H(z)$ contains unit circle.

Ex: find stability of system whose impulse response is $h(n) = z^n u(n)$

$$\begin{aligned} \therefore H(z) &= \sum_{n=-\infty}^{\infty} z^n u(n) z^{-n} \\ &= \sum_{n=0}^{\infty} z^n z^{-n} = \frac{z}{z-1} \quad \because |z| > 1 \end{aligned}$$

$\therefore$ it does not contain unit circle.
So system is unstable



Converge - Approach from different ways to meet at a

PROPERTIES OF ROC:

1. The ROC is a ring or disk in z -plane - centered at origin
2. ROC cannot contain any poles.
3. If $x(n)$ is causal sequence then ROC is Entire z -plane except at $z=0$
4. If $x(n)$ is Anti-causal sequence then ROC is Entire z -plane except at $z=\infty$
5. If $x(n)$ is finite duration Two-sided sequence then ROC will consist of a ring Entire z -plane except at $z=0, z=\infty$
6. If $x(n)$ is infinite duration Two-sided seq then ROC will consist of a ring in z -plane bounded on interior & exterior by a pole not containing any pole
7. The ROC of LTI stable system contains unit circle
8. The ROC must be a connected region.

PROPERTIES OF Z-TRANSFORMS:

LINEARITY: If $X_1(z) = Z[x_1(n)]$ and $X_2(z) = Z[x_2(n)]$

Then $Z\{ax_1(n) + bx_2(n)\} = aX_1(z) + bX_2(z)$

TIME-SHIFTING (OR) TRANSLATION:

Q. If $X(z) = Z[x(n)]$ and initial conditions for $x(n)$ are zeros. Then $Z\{x(n-m)\} = z^{-m}X(z)$

$$\begin{aligned}
 \text{Proof: } Z\{x(n-m)\} &= \sum_{n=-\infty}^{\infty} x(n-m) z^{-n} \quad \text{let } l = n-m \\
 &= \sum_{l=-\infty}^{\infty} x(l) z^{-(l+m)} \quad n = l+m \\
 &= z^{-m} X(z)
 \end{aligned}$$

(b) If $x_1(z) = z\{x(n)\}$ then

$$(i) z\{x(n-m)\} = z^{-m} \left\{ X_1(z) + \sum_{k=1}^m x(-k) \cdot z^k \right\}$$

$$(ii) z\{x(n+m)\} = z^m \left\{ X_1(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right\}$$

Proof:

$$z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) \cdot z^{-n}$$

Multiply & Divide given sequence with z^m

$$= z^{-m} \sum_{n=-\infty}^{\infty} x(n-m) \cdot z^{-(n-m)}$$

let $l = n-m$

$$= z^{-m} \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-l}$$

$$= z^{-m} \left\{ \sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-\infty}^{-1} x(l) z^{-l} \right\}$$

$$= z^{-m} \left\{ X_1(z) + \sum_{k=1}^m x(-k) z^k \right\}$$

$$= z^{-m} \left\{ X_1(z) + \sum_{k=1}^m x(-k) \cdot z^k \right\}$$

when $l = -k$

$$z\{x(n+m)\} = \sum_{n=-\infty}^{\infty} x(n+m) \cdot z^{-n}$$

$$= z^m \sum_{n=-\infty}^{\infty} x(n+m) \cdot z^{-(n+m)}$$

let $l = n+m$

$$= z^m \sum_{l=-\infty}^{\infty} x(l) \cdot z^{-l}$$

$$= z^m \left\{ \sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-\infty}^{-1} x(l) z^{-l} \right\}$$

$$= z^m \left\{ X_1(z) - \sum_{k=0}^{m-1} x(k) z^{-k} \right\}$$

(iii) TIME REVERSAL:

If $X(z) = z\{x(n)\}$ then $z\{x(-n)\} = X(z^{-1})$

$$\text{Proof: } z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^l$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot (z^{-1})^{-l} = X(z^{-1})$$

$$\text{If } R_1 < |X(z)| < R_2 \Rightarrow \frac{1}{R_2} < |z| < \frac{1}{R_1} \text{ for } X(z^{-1})$$

(iv) DIFFERENTIATION OF $X(z)$:

If $X(z) = z \{x(n)\}$ Then $z \{n x(n)\} = -z \frac{d}{dz} X(z)$.

Proof: $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

differentiating z -Transform we have

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{d}{dz} (z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n) (-n) z^{-n-1}$$

$$= -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

Multiplying on both sides with $-z$

$$\Rightarrow -z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$= z \{n x(n)\}$$

(v) CONVOLUTION THEOREM: If $X(z) = z \{x(n)\}$ and

$H(z) = z \{h(n)\}$ Then $z \{x(n) * h(n)\} = X(z) \cdot H(z)$.

Proof:

we have $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$\text{if } Y(z) = z \{y(n)\} = \sum_{n=-\infty}^{\infty} (x(n) * h(n)) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \right] z^{-n+k-k}$$

Interchanging the order of Summation

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k} \cdot \sum_{n=-\infty}^{\infty} h(n-k) \cdot z^{-(n-k)}$$

Replace $(n-k)$ by l . i.e $l = n-k$

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \cdot \sum_{l=-\infty}^{\infty} h(l) \cdot z^{-l}$$

$$Y(z) = X(z) \cdot H(z)$$

(vi) PARSEVAL'S THEOREM:

(vi) INITIAL VALUE THEOREM:

If $X_+(z) = z \sum x(n)$ Then $x(0) = \lim_{z \rightarrow \infty} X_+(z)$

Proof:

$$X_+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

Apply The $\lim_{z \rightarrow \infty}$

$$\begin{aligned} \Rightarrow \lim_{z \rightarrow \infty} X_+(z) &= \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x(n) z^{-n} \\ &= \lim_{z \rightarrow \infty} \{ x(0) + x(1)z^{-1} + \dots \} \\ &= \lim_{z \rightarrow \infty} x(0) + \lim_{z \rightarrow \infty} x(1)z^{-1} + \dots \\ &= x(0) \quad [\because \text{remaining all are vanished}] \end{aligned}$$

(vii) FINAL VALUE THEOREM :-

If $X_+(z) = z \sum x(n)$, where ROC for $X_+(z)$ include But is not necessarily Confined to $|z| > 1$ and $(z-1)X_+(z)$ has no poles on or outside the unit circle, Then $x(\infty) = \lim_{z \rightarrow 1} (z-1)X_+(z)$ [OR]

Proof:

If $X_+(z) = z \sum x(n)$, and $(z-1)X_+(z)$ has no Poles on or outside the unit circle Then $x(\infty) = \lim_{z \rightarrow 1} (z-1)X_+(z)$ [it is steady state value]

Proof:

Let

$$z[x(n+1) - x(n)] = \sum_{n=0}^{\infty} [x(n+1) - x(n)] z^{-n}$$

$$z[x(n+1)] - z[x(n)] = \sum_{n=0}^{\infty} \{x(n+1) - x(n)\} z^{-n} \quad (\because \text{distributive})$$

$$\Rightarrow z \left\{ X_+(z) - \sum_{k=0}^{\infty} x(k) z^{-k} \right\} = \sum_{n=0}^{\infty} \{x(n+1) - x(n)\} z^{-n} \quad (\because \text{Time shifting property})$$
$$-X_+(z)$$

$$\Rightarrow z \{X_+(z)\} - x(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)] \cdot z^{-n}$$

Apply $\lim_{z \rightarrow 1}$ on both sides.

$$\begin{aligned} \Rightarrow \lim_{z \rightarrow 1} (z-1)X_+(z) - x(0) &= \lim_{z \rightarrow 1} \left\{ \sum_{n=0}^{\infty} [x(n+1) - x(n)] \cdot z^{-n} \right\} \\ &= \sum_{n=0}^{\infty} x(n+1) - x(n) \\ &= \lim_{n \rightarrow \infty} \left[\{x(1) - x(0)\} + \{x(2) - x(1)\} + \dots \right. \\ &\quad \left. + \{x(n+1) - x(n)\} \right] \\ &= \lim_{n \rightarrow \infty} [x(n+1) - x(0)] \end{aligned}$$

$$\Rightarrow \lim_{z \rightarrow 1} (z-1)X_+(z) - x(0) = \lim_{n \rightarrow \infty} x(n+1) - x(0)$$

$$\Rightarrow x(\infty) = \lim_{z \rightarrow 1} (z-1)X_+(z)$$

Ex: If $Y(z) = \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})(1-z^{-1})}$ find steady state

value of $y(n)$ if it exists

We know final value theorem.

$$y(\infty) = \lim_{z \rightarrow 1} (z-1)Y(z)$$

Similarly $y(\infty) = \lim_{z \rightarrow 1} (1-z^{-1})Y(z)$

$$= \lim_{z \rightarrow 1} (1-z^{-1}) \cdot \frac{0.5(1-0.5z^{-1})}{(1-0.25z^{-1})(1-0.75z^{-1})(1-z^{-1})}$$

By inspection there is no pole on unit circle, so

$$= \frac{0.5(1-0.5)}{(1-0.25)(1-0.75)} = 1.33$$

Ex: if $X(z) = \frac{3z}{(z-1)(z+1)}$ find $x(\infty)$.

$$x(\infty) = \lim_{z \rightarrow 1} (z-1)X(z)$$

$$= \lim_{z \rightarrow 1} (z-1) \cdot \frac{3z}{(z-1)(z+1)}$$

$$= \lim_{z \rightarrow 1} \frac{3z}{(z+1)}$$

It has a pole on unit circle so it is not defined.

Ex: find z-Transform of signal $x(n) = [3(3)^n - 4(2)^n]u(n)$

we know $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{n=0}^{\infty} [3(3)^n - 4(2)^n] u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 3(3)^n z^{-n} - \sum_{n=0}^{\infty} 4(2)^n z^{-n}$$

$$= 3 \left[\sum_{n=0}^{\infty} (3z^{-1})^n \right] - 4 \left[\sum_{n=0}^{\infty} (2z^{-1})^n \right]$$

$|z| > 3$ $|z| > 2$

$$= \frac{3}{1-3z^{-1}} - \frac{4}{1-2z^{-1}}$$

$$= \frac{3z}{z-3} - \frac{4z}{z-2}$$

ROC: $|z| > 3$

find z-Transform of sequence $x(n) = na^n u(n)$

we know $z[a^n u(n)] = \frac{z}{z-a}$ ROC: $|z| > |a|$

using differentiation property we have

$$z[n \cdot x(n)] = -z \frac{d}{dz} X(z)$$

$$z[n \cdot a^n u(n)] = -z \frac{d}{dz} \left(\frac{z}{z-a} \right)$$

$$= -z \left[\frac{(z-a)(1) - z}{(z-a)^2} \right]$$

$$= +z \left[\frac{a}{(z-a)^2} \right] \quad |z| > |a|$$

find z-Transform of $x(n) = u(n-2)$

we know

$$z[x(n-m)] = z^{-m} X(z)$$

$$\Rightarrow z[u(n-2)] = z^{-2} \cdot \frac{z}{z-1} \quad \text{ROC } |z| > 1$$

THE SYSTEM FUNCTION:

In General a system is described by a linear constant coefficient difference Equation of the form $y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$

Let Apply z-Transform on both sides, we get

$$Z[y(n)] = -Z\left[\sum_{k=1}^N a_k y(n-k)\right] + Z\left[\sum_{k=0}^M b_k x(n-k)\right]$$

$$Y(z) = - \sum_{k=1}^N a_k Y(z) \cdot z^{-k} + \sum_{k=0}^M b_k X(z) z^{-k}$$

($\because$ from shifting property)

$$\Rightarrow Y(z) \left[1 + \sum_{k=1}^N a_k z^{-k} \right] = X(z) \left[\sum_{k=0}^M b_k z^{-k} \right]$$

$$\Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}} \quad \text{--- (1)}$$

The Eq(1) is known as Transfer function or system function.

where $X(z) \rightarrow Z[x(n)] \rightarrow Z[\text{ip sequence}]$

$Y(z) \rightarrow Z[y(n)] \rightarrow Z[\text{op sequence}]$

* If $x(n) = \delta(n)$ Then $X(z) = 1$, Then Transfer function $H(z) = Y(z)$.

Ex: find system function and impulse response of system described by difference Eq $y(n) = \frac{1}{5} y(n-1) + x(n)$

Applying z-Transform on both sides.

$$Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$$

$$\Rightarrow Y(z) \left[1 - \frac{1}{5} z^{-1} \right] = X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5}z^{-1}}$$

But we know $\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}}$

$\therefore$ By applying inverse we can get

$$h(n) = \left(\frac{1}{5}\right)^n u(n).$$

POLES AND ZEROS OF A SYSTEM FUNCTION:

We know system function is $H(z)$

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

* The zeros of the system function $H(z)$ are the values of 'z' for which $H(z) = 0$.

* The poles of the system function $H(z)$ are the values of 'z' for which $H(z) = \infty$.

We can also represent a system function having both poles and zeros as

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{z^M \sum_{k=0}^M b_k z^{M-k}} = \frac{1}{z^M} \sum_{k=0}^M b_k z^{M-k}$$

it having M zeros and Mth order pole at $z=0$

These types of systems are called as

Pole-zero system.

Ex: Determine the Pole-zero plot for a system given by difference Equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Sol: Taking z-Transforms on both sides.

(7)

$$\Rightarrow Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$\Rightarrow Y(z) \left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2} \right] = X(z) [1 - z^{-1}]$$

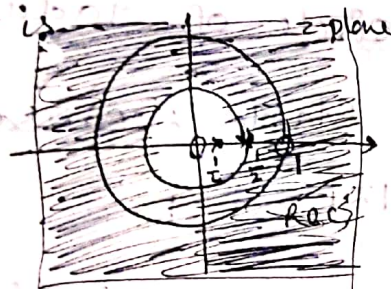
$$\begin{aligned} \Rightarrow \frac{Y(z)}{X(z)} = H(z) &= \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} \\ &= \frac{(z-1)/z}{(8z^2 - 6z + 1)/z^2} \\ &= \frac{8z(z-1)}{8z^2 - 6z + 1} = \frac{8z(z-1)}{(2z-1)(4z-1)} \\ &= \frac{4z(z-1)}{(z-\frac{1}{2})(2z-\frac{1}{2})} \end{aligned}$$

$$\text{ROC: } |z| > \frac{1}{2}$$

$$|z| > \frac{1}{4}$$

$$\therefore \text{ROC is } |z| > \frac{1}{2}$$

Pole-zero plot is



Two poles lies inside unit circle so system is stable.

from above figure we can write

- 1) ROC of System function, includes unit circle
- 2) ROC of System function, cannot contain any pole.

$$\underline{\text{Ex:}} \quad y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$$

STABILITY CRITERIAN:

A system function is said to be stable if and only if ~~the~~ all the poles of $H(z)$ are lies inside the unit circle.

But in first unit we have derived that the sufficient & necessary condition for Causal LTI system is BIBO (Bounded input bounded output) stable is

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

for Causal System $H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$

$$|H(z)| = \sum_{n=0}^{\infty} |h(n) \cdot z^{-n}| \leq \sum_{n=0}^{\infty} |h(n)| |z^{-n}|$$

When it evaluated in unit circle $|z| = 1$

$$\therefore |H(z)| \leq \sum_{n=0}^{\infty} |h(n)| < \infty$$

So we can conclude that

- * for a stable system The ROC of system function includes unit circle
- * for a causal system The ROC is exterior of some circle of radius 'r'.
- * So for a stable & causal system The ROC is $|z| > r < 1$.

Ex: find frequency response of system $H(z) = \frac{1}{1-0.5z}$

given $H(z) = \frac{1}{1-0.5z} = \frac{z}{z-0.5}$ ROC: $|z| > \frac{1}{2}$

ROC includes unit circle.

$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 0.5} \quad \text{we know } e^{j\omega} = \cos\omega + j\sin\omega$$

$$= \frac{\cos\omega + j\sin\omega}{\cos\omega - 0.5 + j\sin\omega}$$

$$|H(e^{j\omega})| = \left| \frac{\cos\omega + j\sin\omega}{(\cos\omega - 0.5) + j\sin\omega} \right|$$

$$= \frac{1}{\sqrt{(\cos\omega - 0.5)^2 + \sin^2\omega}} = \frac{1}{\sqrt{1 - 2\cos\omega + 0.25}}$$

and phase Response $\angle H(e^{j\omega}) = \omega - \tan^{-1} \frac{\sin\omega}{(\cos\omega - 0.5)}$

INVERSE Z-TRANSFORM:

We have 4 methods to Evaluate Inverse Z-Transform. They are

1. Long division Method
2. Partial fraction Expansion Method
3. Residue Method
4. Convolution Method.

All These methods are Explained with the help of following Examples.

1. Long division Method:-

Ex: find inverse Z-Transform of following

(i) $X(z) = \frac{z}{(z-3)(z-4)} ; |z| < 3$

Sol:

i.e given That given Sequence is Anticausal

$$X(z) = \frac{z}{(z-3)(z-4)} = \frac{z}{z^2 - 7z + 12}$$

Dividing Numerator by Denominator

$$\begin{array}{r} 12 - 7z + z^2 \overline{) z} \\ \underline{z} \\ 7z^2 - z^3 \\ \underline{7z^2 - 49z^3} \\ 48z^3 - 7z^4 \\ \underline{48z^3 - 336z^4} \\ 328z^4 - 7z^5 \\ \underline{328z^4 - 2304z^5} \\ 1976z^5 - 7z^6 \\ \underline{1976z^5 - 13832z^6} \\ 11856z^6 - 7z^7 \\ \vdots \end{array}$$

$$\therefore X(z) = \frac{z}{12} + \frac{7z^2}{144} + \frac{37z^3}{1728} + \dots$$

$$= \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$$x(0) = 0, x(-1) = \frac{1}{12}, x(-2) = \frac{7}{144}, x(-3) = \frac{37}{1728}$$

$$(ii) X(z) = \frac{z+0.2}{(z+0.5)(z-1)}, |z| > 1$$

So given sequence is causal sequence

divide Numerator with denominator

$$\begin{array}{r} z^2 - 0.5z - 0.5 \overline{) z + 0.2} \quad \left(\frac{1}{z} + \frac{0.7}{z^2} + \frac{0.85}{z^3} + \frac{0.775}{z^4} + \dots \right) \\ \underline{z - 0.5 - \frac{0.5}{z}} \\ 0.7 - 0.5z^{-1} \\ \underline{0.7 - \frac{0.35}{z} - 0.35z^{-2}} \\ 0.85z^{-1} - 0.35z^{-2} \\ \underline{0.85z^{-1} - 0.425z^{-2} - 0.425z^{-3}} \\ 0.775z^{-2} + 0.425z^{-3} \\ \underline{0.775z^{-2} - 0.387z^{-3} - 0.387z^{-4}} \end{array}$$

$$\begin{aligned} \therefore X(z) &= \frac{1}{z} + \frac{0.7}{z^2} + \frac{0.85}{z^3} + \frac{0.775}{z^4} + \dots \\ &= \sum_{n=0}^{\infty} x(n)z^{-n} \end{aligned}$$

where $x(0)=0, x(1)=1, x(2)=0.7, x(3)=0.85$

2. Partial Fraction Expansion Method :

Ex: find inverse z transform of $X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}, |z| > 2$

$$\Rightarrow X(z) = \frac{z(z+3)}{z^2+3z+2}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{z+3}{z^2+3z+2} \quad (M < N)$$

$$= \frac{A}{z+1} + \frac{B}{z+2}$$

$$\Rightarrow z+3 = A(z+2) + B(z+1)$$

$$\Rightarrow +B = -1, A = 2$$

$$\therefore \frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+2}$$

(9)

$$X(z) = 2 \cdot \frac{z}{z+1} - \frac{z}{z+2}$$

$$\Rightarrow x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

Ex: find inverse z-Transform of $X(z) = \frac{z^3}{(z-2)(z-1)^2}$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2}{(z-2)(z-1)^2}$$

$$= \frac{A}{(z-2)} + \frac{B}{(z-1)} + \frac{C}{(z-1)^2}$$

† (M < N)
having simple factor

$$z^2 = A(z-1)^2 + B(z-2)(z-1) + C(z-2)$$

$$\Rightarrow z=1 \Rightarrow C = -1$$

$$z=2 \Rightarrow A = 4$$

$$B = -3$$

$$\therefore X(z) = \frac{4z}{z-2} - \frac{3z}{z-1} - \frac{z}{(z-1)^2}$$

$$\therefore x(n) = 4(2)^n u(n) - 3(1)^n u(n) - n u(n)$$

Ex: find inverse z-Transform of $X(z) = \frac{z^3}{(z-2)(z-1)}$

$$\Rightarrow \frac{X(z)}{z} = \frac{z^2}{(z-2)(z-1)} = \frac{z^2}{z^2 - 3z + 2}$$

(M = N)

$$\therefore \frac{X(z)}{z} = 1 + \frac{3z-2}{(z-1)(z-2)}$$

$$\frac{z^2 - 3z + 2}{z^2 - 3z + 2} \div \frac{z^2 - 3z + 2}{z^2 + 2}$$

$$\text{Let } 3z-2 = A(z-2) + B(z-1)$$

$$\Rightarrow z=1 \Rightarrow A = -1$$

$$\Rightarrow z=2 \Rightarrow B = 4$$

$$\therefore \frac{X(z)}{z} = 1 + \frac{4}{z-2} + \frac{1}{z-1}$$

$$\Rightarrow X(z) = z + 4 \frac{z}{z-2} - \frac{z}{z-1}$$

$$x(n) = \delta(n+1) + 4(2)^n u(n) - u(n)$$

Similarly if $X(z) = \frac{z}{(z^2-2)(z-1)}$

Then we can take as $\frac{Az+B}{z^2-2} + \frac{C}{z-1} = \frac{z}{(z^2-2)(z-1)}$

RESIDUE METHOD:

So general inverse z transforms used in P.F as

$$\frac{X(z)}{\text{inverse}} = x(n)$$

$$\frac{z}{z-1}$$

$$u(n)$$

$$\frac{z}{z-a}$$

$$a^n u(n)$$

$$\frac{z}{(z-a)^2}$$

$$\frac{na^{n-1}u(n)}{1!}$$

$$\frac{z}{(z-a)^3}$$

$$\frac{n(n-1)a^{n-2}u(n)}{2!}$$

$$\frac{z}{(z-a)^4}$$

$$\frac{n(n-2)(n-1) \cdot a^{n-3}u(n)}{3!}$$

$$\vdots$$

$$\delta(n+1)$$

$$\frac{z}{z^{-m}}$$

$$\delta(n-m)$$

$$1$$

$$\delta(n)$$

CONVOLUTION METHOD :

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In this Method The given $X(z)$ is splitted into $X_1(z)$ and $X_2(z)$ such that $X(z) = X_1(z) \cdot X_2(z)$
Next we find $x_1(n)$ & $x_2(n)$ by taking inverse z-Transform of $X_1(z)$, $X_2(z)$ resp^t. from Convolution property of z-Transform.

$$Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z) = X(z).$$

Then $x(n)$ will be obtained by Convoluting $x_1(n)$, $x_2(n)$.

Ex: find inverse z-Transform of $X(z) = \frac{1}{1-3z^{-1}+2z^{-2}}$

$$X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})} = \frac{z}{(z-1)(z-2)}$$

$$\text{Let } X(z) = X_1(z) \cdot X_2(z)$$

$$\Rightarrow x_1(n) = u(n), \quad x_2(n) = 2^n u(n).$$

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$= \sum_{k=0}^{\infty} u(k) \cdot 2^{n-k} u(n-k)$$

$$= 2^n \left[\sum_{k=0}^n 2^{-k} \right]$$

$$= 2^n \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \right] u(n)$$

$$= (2^{n+1} - 1) u(n)$$

ex:

$$X(z) = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{9}z^{-2}}$$

RESIDUE METHOD:- (Not required)

Solution of difference eq:

Ex: find response of the system $y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$ to input $x(n) = n u(n)$.
is system is stable or not

Sol: given difference eq is

$$y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$$

Taking z-transform on both sides

$$Y(z) = 0.7z^{-1}Y(z) - 0.12z^{-2}Y(z) + X(z)z^{-1} + z^{-2}X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}(1+z^{-1})}{1-0.7z^{-1}-0.12z^{-2}} \Rightarrow \frac{z+1}{z^2-0.7z+0.12}$$

$$\Rightarrow H(z) = \frac{z+1}{(z-0.4)(z-0.3)} \quad \text{--- (1)}$$

So poles are $z = 0.4, 0.3$, so ROC $|z| > 0.4$

Poles are lying inside unit circle so system is stable

But given $x(n) = n u(n)$

$$X(z) = \frac{z}{(z-1)^2} \quad \text{--- (2)}$$

from (1) & (2)

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)^2(z-0.4)(z-0.3)}$$

So by using partial fraction method we can

$$Y(z) = \frac{38.89z}{z-0.4} - \frac{26.53z}{z-0.3} - \frac{12.36z}{(z-1)^2} + \frac{4.76z}{(z-1)^2}$$

Taking inverse

$$\therefore y(n) = 38.89(0.4)^n u(n) - 26.53(0.3)^n u(n) - 12.36 u(n) + 4.76 n u(n)$$

Ex: find unit step response of the system

$$y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$$

if $y(-1) = y(-2) = 1$

Sol:

taking z-transforms

$$Y(z) - 0.7 \left\{ z^{-1} [Y(z) + \sum_{k=1}^1 y(-1) \cdot z^k] \right\} + 0.12 \left\{ z^{-2} [Y(z) + \sum_{k=1}^2 y(-k) z^k] \right\} = z^{-1} \{ X(z) + x(-1)z \} + z^{-2} \{ X(z) + \sum_{k=1}^2 x(-k)z^k \}$$

$$\Rightarrow Y(z) - 0.7 \{ z^{-1} Y(z) + y(-1) \} + 0.12 \{ z^{-2} Y(z) + z^{-1} y(-1) + y(-2) \} \\ = z^{-1} X(z) + x(-1) + z^{-2} X(z) + z^{-1} x(-1) + x(-2)$$

But given $y(-1) = y(-2) = 1$ and for unit step i/p $x(-1) = x(-2) = 0$

$$\Rightarrow Y(z) - 0.7 (z^{-1} Y(z) + 1) + 0.12 [z^{-2} Y(z) + z^{-1} + 1] = z^{-1} X(z) + z^{-2} X(z)$$

But unit step i/p

$$X(z) = \frac{z}{z-1}$$

$$\Rightarrow Y(z) \left[\frac{z^2 - 0.7z + 0.12}{z^2} \right] - 0.58 + \frac{0.12}{z} = \frac{z}{z(z-1)} + \frac{z}{(z-1)z^2}$$

$$\Rightarrow Y(z) \left\{ \frac{z^2 - 0.7z + 0.12}{z^2} \right\} = 0.58 - \frac{0.12}{z} + \frac{1}{z-1} + \frac{1}{z(z-1)}$$

$$= \frac{0.58z(z-1) - 0.12(z-1) + z + 1}{z(z-1)}$$

$$\Rightarrow Y(z) = \frac{0.58z(z-1) - 0.12(z-1) + z + 1}{z(z-1) \cdot (z^2 - 0.7z + 0.12)}$$

$$= \frac{0.58z(z-1) - 0.12(z-1) + z + 1}{z(z-1) \cdot (z^2 - 0.7z + 0.12)}$$

$$\frac{Y(z)}{z} = \frac{0.58z(z-1) - 0.12(z-1) + z + 1}{(z-1)(z^2 - 0.7z + 0.12)}$$

$$= \frac{0.58z(z-1) - 0.12(z-1) + z + 1}{(z-1)(z-0.4)(z-0.3)}$$

$$\text{Let } R.H.S = \frac{A}{z-1} + \frac{B}{z-0.4} + \frac{C}{z-0.3}$$

$$\Rightarrow A(z-0.4)(z-0.3) + B(z-1)(z-0.3) + C(z-0.4)(z-1)$$

$$= 0.58z(z-1) + z - 0.12(z-1) + 1$$

$$\Rightarrow A \cdot 0.42 = 2$$

$$\Rightarrow A = \frac{2}{0.42} = 4.762$$

$$\Rightarrow -B(0.06) =$$

$$B = -22.21$$

$$C = 18.03$$

$$Y(z) = \frac{4.762}{z-1} - \frac{22.21}{z-0.4} + \frac{18.03}{z-0.3}$$

$\Rightarrow$ Taking z -inverse z -transform,

$$y(n) = 4.762u(n) - 22.21(0.4)^n u(n) + 18.03(0.3)^n u(n)$$

REALIZATION OF DIGITAL FILTERS:-

A digital filter Transfer function can be realised by two types of realizations. They are 1. Recursive 2. Non-recursive

1. For Recursive realization The current output y is a function of past outputs, Past and Present inputs. These type are called as IIR digital filters
2. For Non-recursive realization current output sample $y(n)$ is a function of only past and Present inputs. These type are called as FIR digital filters.

IIR filters can be realized in many forms.

1. Direct form-I realization
2. Direct form-II realization
3. Transposed directform realization
4. Cascade form realization
5. Parallel form realization
6. Lattice-ladder structure.

DIRECT-FORM -I REALIZATION:

Let us consider an LTI system described by the difference equation $y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k)$ — (1)

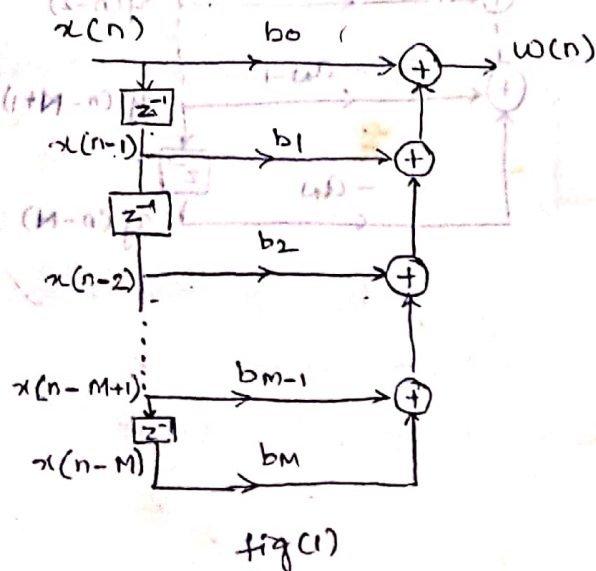
$$\Rightarrow y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$
 — (2)

Let $b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M) = w(n)$ — (3)

$\therefore$ Eq (2) can be obtained as

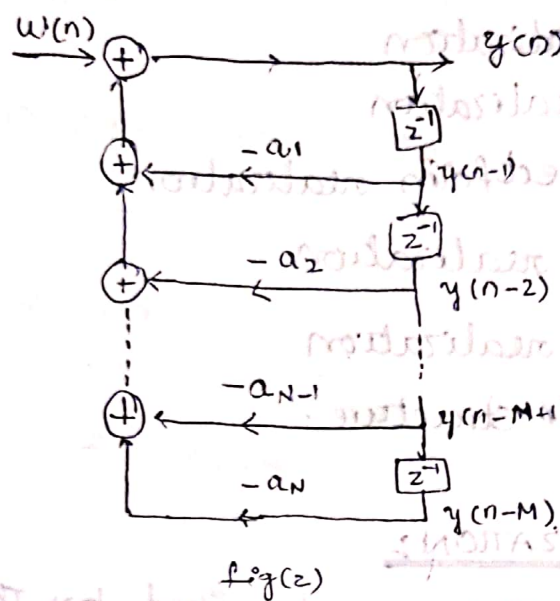
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N) + w(n)$$
 — (4)

Eq (3) can be realised as follows.



$x(z) = w(z)$
 $y(z) = w(z) - a_1 z^{-1} y(z) - a_2 z^{-2} y(z) - \dots - a_N z^{-N} y(z)$
 $y(z) (1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}) = w(z)$
 $y(z) = \frac{w(z)}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$

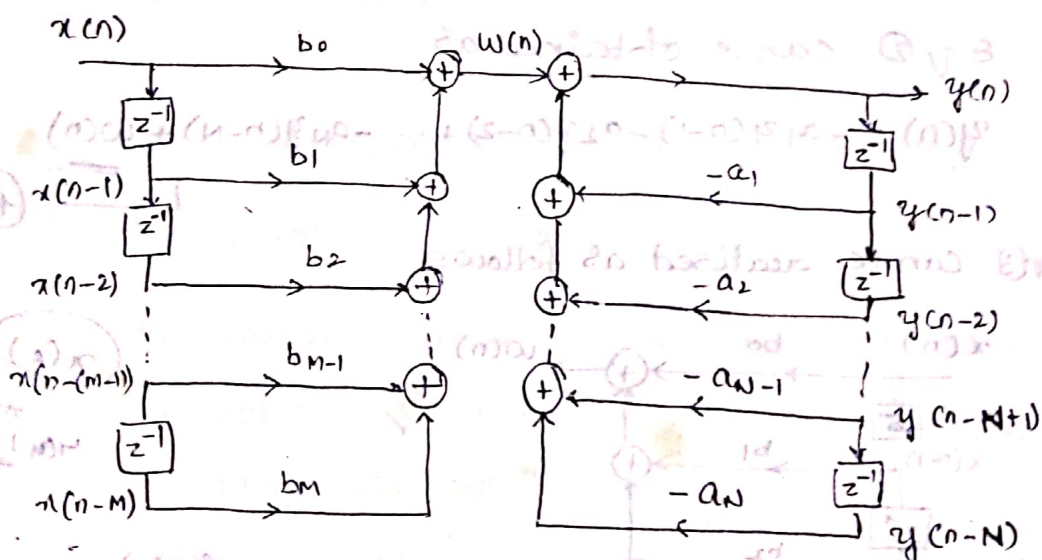
Similarly eq. (4) can be represented as



To get the realization of difference Equation
Combine Two figures. The Resulting structure is

known as Direct form - I, which is used to separate delays for both input and output. This

realization requires $M+N+1$ multiplications and $M+N$ additions and $M+N+1$ memory locations.



Ex:- Obtain the Direct form-I Realization for the System described by difference Equations $y_1(n), y_2(n)$ where

(i) $y_1(n) = 0.5 y_1(n-1) - 0.25 y_1(n-2) + x(n) + 0.4 x(n-1)$

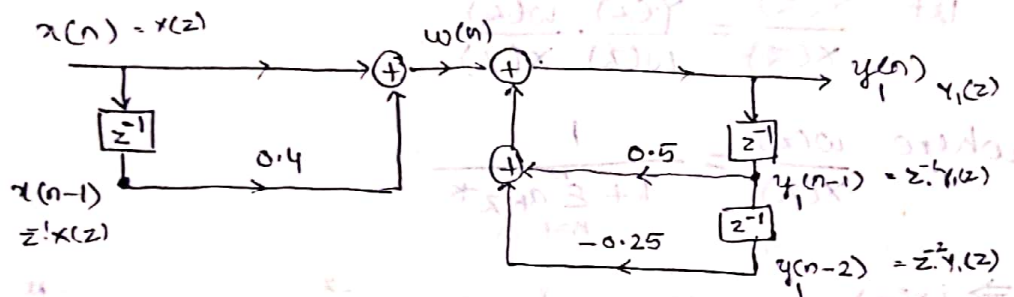
(ii) $y_2(n) = -\frac{3}{8} y_2(n-1) + \frac{3}{32} y_2(n-2) + \frac{1}{64} y_2(n-3) + x(n) + 3x(n-1) + 2x(n-2)$

Sol:-

(i) Let $x(n) + 0.4 x(n-1) = w(n)$ — (1)

Then $y_1(n) = 0.5 y_1(n-1) - 0.25 y_1(n-2) + w(n)$ — (2)

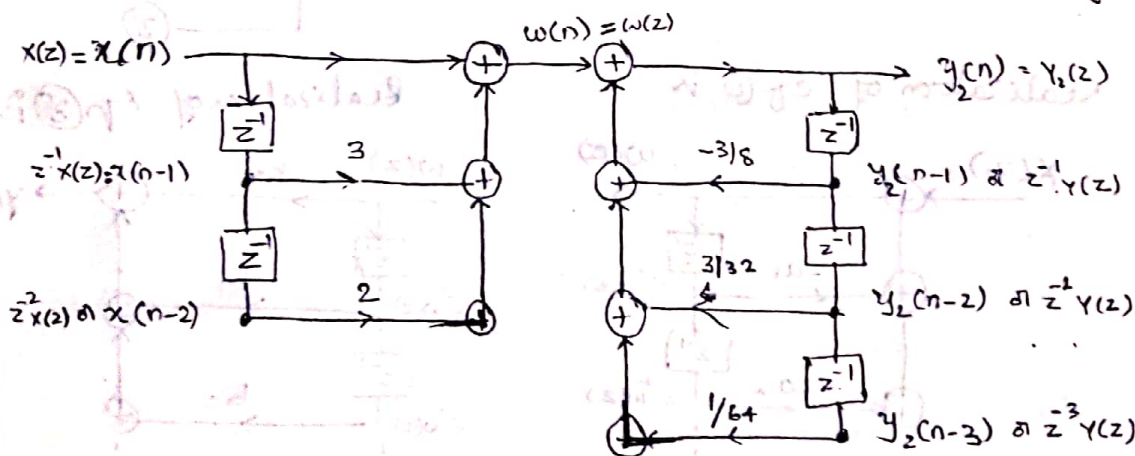
Realizing Eq (1) & (2) and Combining we get



(ii) Let $x(n) + 3x(n-1) + 2x(n-2) = w(n)$ — (1)

Then $y_2(n) = -\frac{3}{8} y_2(n-1) + \frac{3}{32} y_2(n-2) + \frac{1}{64} y_2(n-3) + w(n)$

By realizing Eq (1) & (2) and Combining we get



Z-Transform of Eq (1)

$X(z) + 3z^{-1}X(z) + 2z^{-2}X(z) = w(z)$ — (1a)

$Y_2(z) = -\frac{3}{8} z^{-1} Y_2(z) + \frac{3}{32} z^{-2} Y_2(z) + \frac{1}{64} z^{-3} Y_2(z) + w(z)$

— (2a)

DIRECTFORM-II REALIZATION :

Let the difference equation is $y(n)$ where

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k x(n-k) \quad \text{--- (1)}$$

The system function of eq (1) is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

$$\text{Let } \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

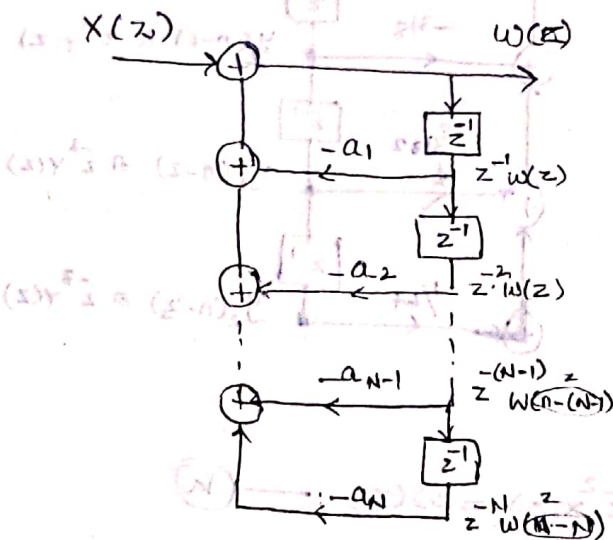
$$\text{where } \frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}}$$

$$\Rightarrow W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \dots - a_N z^{-N} W(z)$$

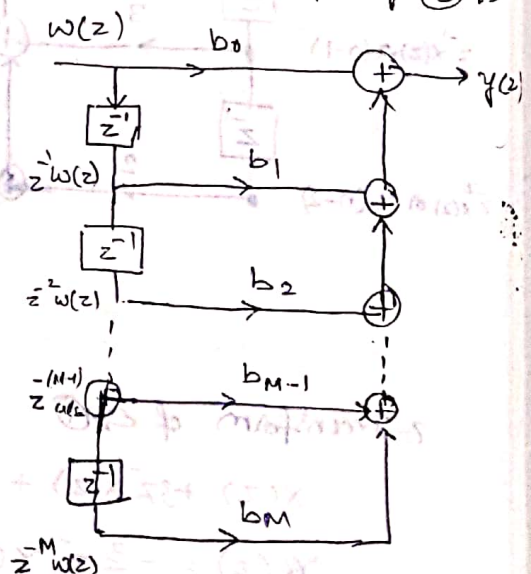
$$\text{and where } \frac{Y(z)}{W(z)} = \sum_{k=0}^M b_k z^{-k} \quad \text{--- (2)}$$

$$\Rightarrow Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \dots + b_M z^{-M} W(z)$$

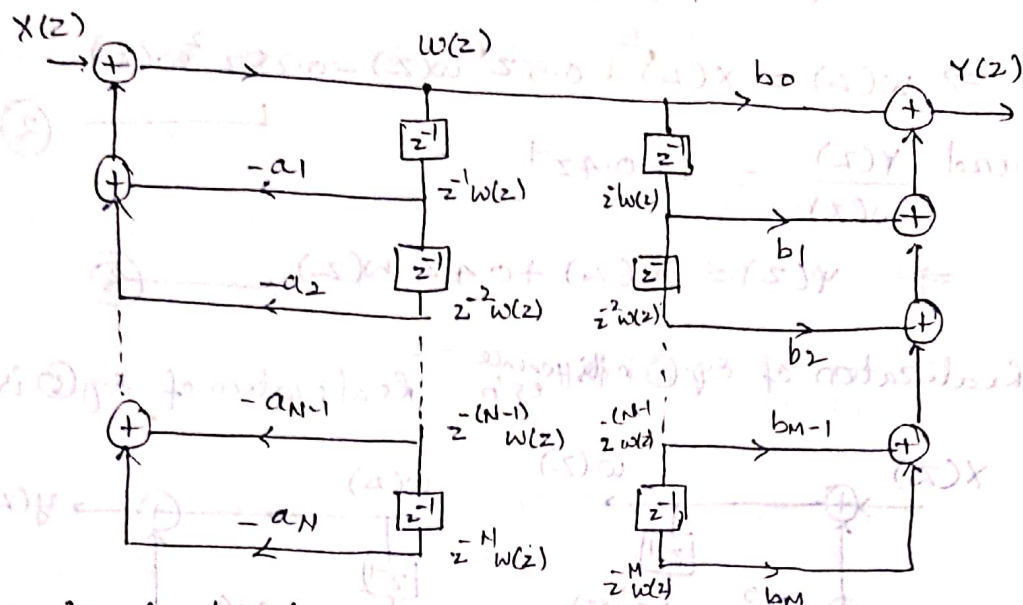
Realization of eq (2) is



Realization of eq (3) is



To get Realization of Difference Equation Combine both figures (1), (2)



This Realization is known as Direct form-II (canonic) Realization. It requires $M+N+1$ Multiplications $M+N$ Additions and maximum of $\{M, N\}$ memory locations.

Ex: Obtain the Direct form-II Realization for the system described by difference equations $y_1(n), y_2(n)$ where (i) $y_1(n) = 0.5 y(n-1) - 0.25 y(n-2) + x(n) + 0.4 x(n-1)$
(ii) $y_2(n) = -\frac{3}{8} y(n-1) + \frac{3}{32} y(n-2) + \frac{1}{64} y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$

Sol:-

(i) given diff eq is $y_1(n) = 0.5 y(n-1) - 0.25 y(n-2) + x(n) + 0.4 x(n-1)$

Let $x(n) + 0.4 x(n-1) = w(n)$ — (1)

Then $y_1(n) = 0.5 y(n-1) - 0.25 y(n-2) + w(n)$

To realize the system function of Eq (1) is

$$\Rightarrow Y(z) = 0.5 z^{-1} Y(z) - 0.25 z^{-2} Y(z) + X(z) + 0.4 z^{-1} X(z)$$

$$\Rightarrow Y(z) [1 - 0.5 z^{-1} + 0.25 z^{-2}] = X(z) [1 + z^{-1} \cdot 0.4]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} \cdot 0.4}{1 - 0.5 z^{-1} + 0.25 z^{-2}}$$

Let

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

where

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1} + 0.25z^{-2}}$$

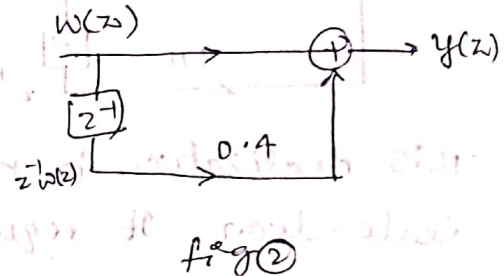
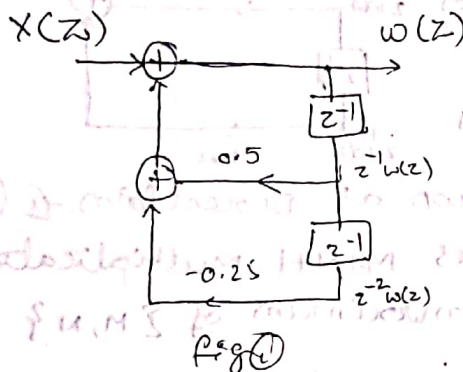
$$\Rightarrow W(z) = X(z) + 0.5z^{-1}W(z) - 0.25z^{-2}W(z)$$

$$\text{and } \frac{Y(z)}{W(z)} = 1 + 0.4z^{-1}$$

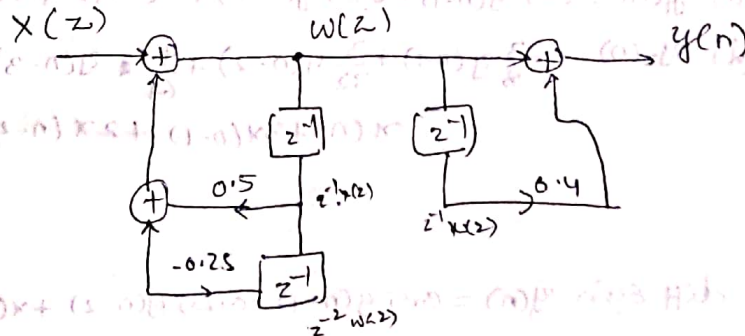
$$\Rightarrow Y(z) = W(z) + 0.4z^{-1}W(z)$$

Realization of Eq ② in difference eq'n

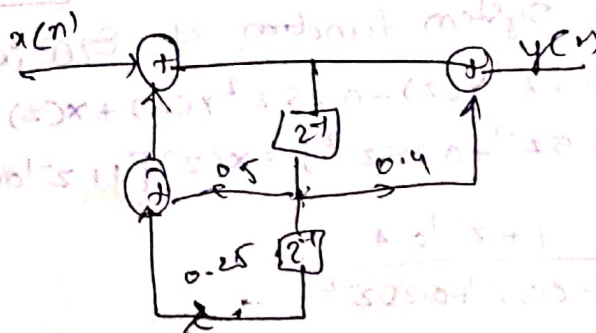
Realization of Eq ③ is



To combine Fig ① & Fig ② we get realization of difference equations.



Similarly (ii) problem also we can solve.



TRANPOSED DIRECT FORM AND REALIZATION:

Before going to know about Transposed direct form we must know about a term known as - 'Signal flow graph'.

Signal flow graph:- It is a graphical representation of the relationships between the variables of set of linear difference equations.

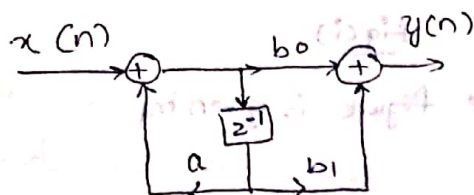
- (i) The basic elements of signal flowgraph are Branches and Nodes.
- (ii) A Node represents a system variable, which is Equal to Sum of incoming signals from all branches Connecting to The Node.

(iii) There are Two types of Nodes (a) Source (b) Sink nodes.

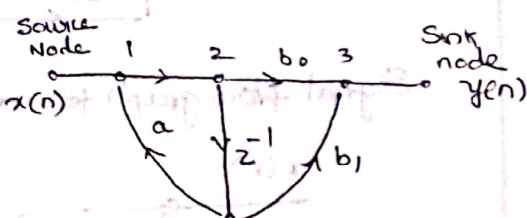
Source Node: These Nodes have no Entering Branches.

Sink Nodes: These have only Entering Nodes Branches.

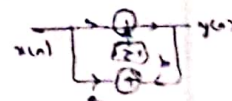
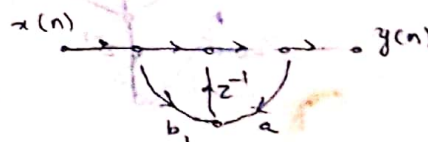
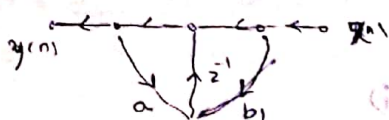
- (iv) The delay is indicated by Branch Transmittance z^{-1} . when Branch Transmittance is unity, it is left unlabeled.



Block diagram of I-order digital filter



Signal flow graph of I-order digital filter



The Transpose of a structure is defined by the following operations.

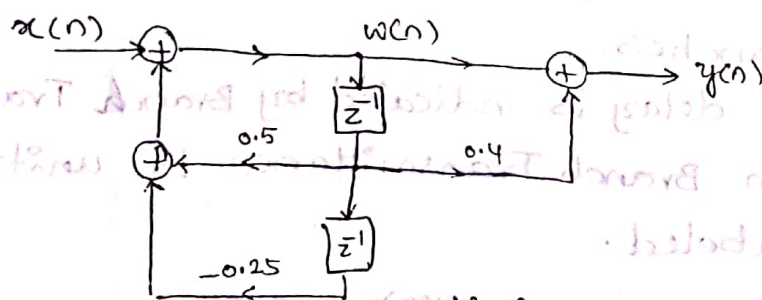
- (i) Reverse the direction of all branches in signal flow graph.
- (ii) Interchange the inputs and outputs
- (iii) Reverse the roles of all nodes in the flowgraph
- (iv) Summing points become Branching points
- (v) Branching points become Summing points.

Exo: Determine the Direct form II and Transposed direct form-II for the given system (i) $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$.
 (ii) $y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{24}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$.

Sol:

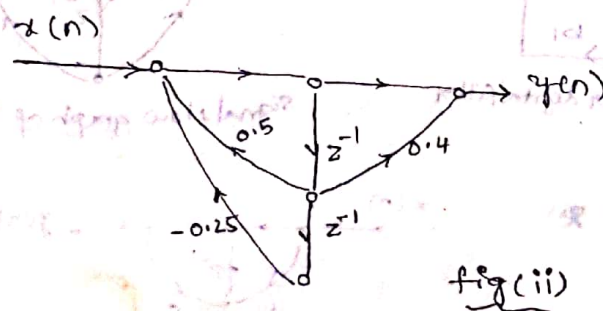
The system T.F of the given difference Equation is given by $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.4z^{-1}}{1 - 0.5z^{-1} + 0.25z^{-2}}$

By inspection, The direct form-II realization can be found as



fig(i)

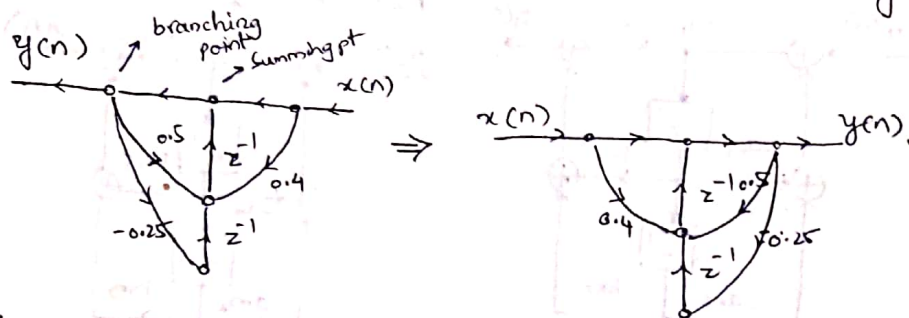
Signal flow graph for above figure is given by



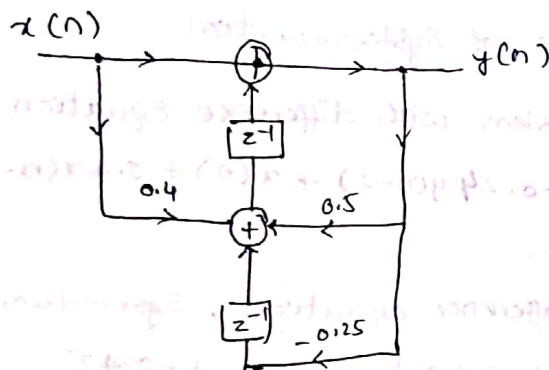
fig(ii)

To get Transposed Directform-II do The following operations. (16)

- (i) change ^{Reverse} The directions of all branches.
- (ii) Interchange input and output
- (iii) change The summing point to branching viceversa.



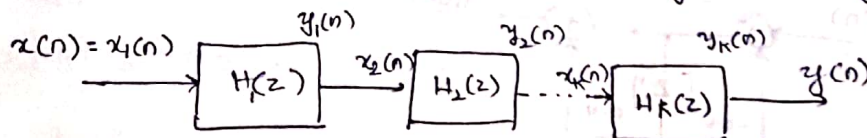
∴ Transposed structure is as follows.



CASCADE FORM :-

Let us consider an IIR system with system fun $H(z) = H_1(z) H_2(z) \dots H_K(z)$. — (1)

Eq (1) can be represented using block diagram as



Now realize Each $H_k(z)$ in direct form-II and cascade all structures.

for example let us take a system whose T.F is

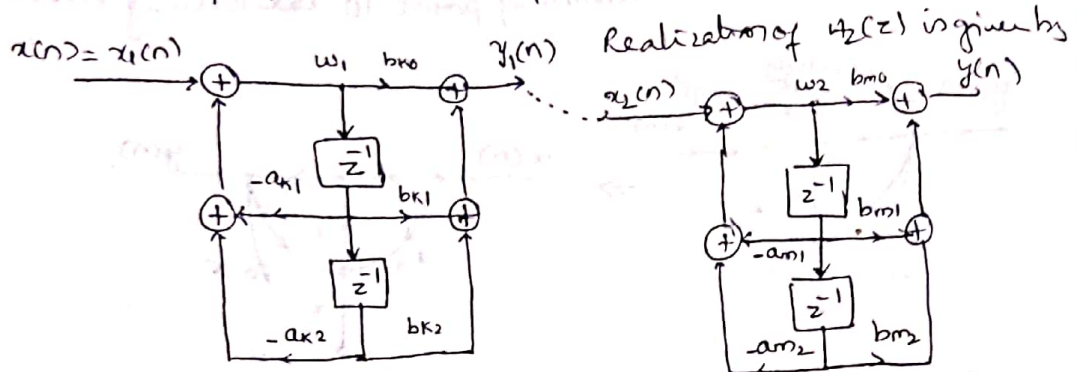
$$H(z) = \frac{(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})} \quad (2)$$

$$= H_1(z) \cdot H_2(z)$$

where $H_1(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$ and

$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$

Realization of $H_1(z)$ in direct form-II is given by



Cascading both realizations of $H_1(z)$ and $H_2(z)$ in direct form-II we can obtain cascade of system function.

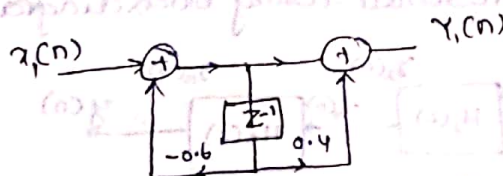
Ex: Realize the system with difference Equation —
 $y(n) = -0.2y(n-1) + 0.24y(n-2) + x(n) + 0.4x(n-1)$ in cascade form.

Sol: from the given Difference Equation, System function

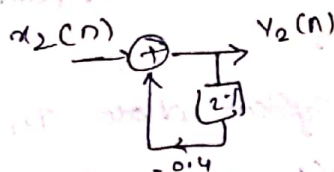
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.4z^{-1}}{1 + 0.2z^{-1} - 0.24z^{-2}} = \frac{1 + 0.4z^{-1}}{(1 + 0.6z^{-1})(1 - 0.4z^{-1})} = H_1(z)H_2(z)$$

where $H_1(z) = \frac{1 + 0.4z^{-1}}{1 + 0.6z^{-1}}$, $H_2(z) = \frac{1}{1 - 0.4z^{-1}}$

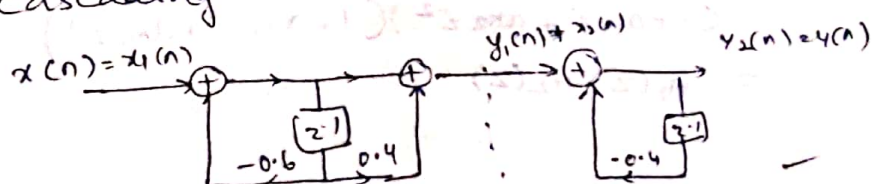
$H_1(z)$ can be realized in direct form-II as



$H_2(z)$ can be realized in direct form-II as



$\therefore$ Cascading both Realization we get



PARALLEL FORM STRUCTURE :-

A parallel form Realization ^(IIR Syst) Can be obtained by Performing a Partial Expansion of $H(z)$, where

$$H(z) = c + \sum_{k=1}^N \frac{C_k}{1 - p_k z^{-1}} \quad \text{where } \{p_k\} \text{ are poles.}$$

①

This Eqn ① can be written as

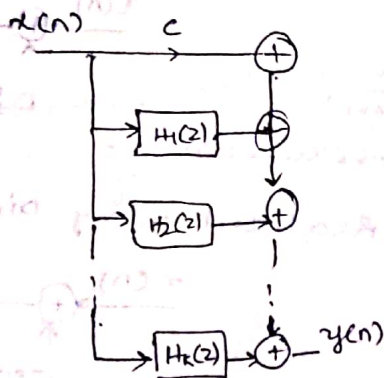
$$H(z) = c + \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + \dots + \frac{C_N}{1 - p_N z^{-1}}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = c + H_1(z) + H_2(z) + \dots + H_N(z)$$

$$\text{Now } Y(z) = cX(z) + H_1(z) \cdot X(z) + H_2(z) \cdot X(z) + \dots + H_N(z) \cdot X(z)$$

②

$\therefore$ Eqn ② can be realised in parallel form as



Ex: Realize The system given by Difference Equation

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-2] \text{ by Parallel form.}$$

Sol: Applying z-Transform we get

$$Y(z) = 0.5z^{-1}Y(z) - 0.25z^{-2}Y(z) + X(z) + 0.4z^{-2}X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + 0.4z^{-2}}{1 - 0.5z^{-1} + 0.25z^{-2}} \quad Y(z) = 0.5z^{-1}Y(z) + 0.25z^{-2}Y(z) + X(z) + 0.4z^{-2}X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + 0.4z^{-2}}{1 - 0.5z^{-1} + 0.25z^{-2}}$$

$$= -\frac{1}{0.6} + \frac{0.8}{(1 - 0.6z^{-1})(1 + 0.4z^{-1})}$$

$$\begin{aligned} & \frac{1 + 0.4z^{-2}}{1 - 0.5z^{-1} + 0.25z^{-2}} = \frac{1 + 0.4z^{-2}}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})} \\ & = \frac{1 + 0.4z^{-2}}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})} = \frac{A}{(1 - 0.6z^{-1})} + \frac{B}{(1 + 0.4z^{-1})} \end{aligned}$$

$$\text{Let } \frac{0.8 - z^{-1}}{(1 - 0.6z^{-1})(1 + 0.4z^{-1})} = \frac{A}{(1 - 0.6z^{-1})} + \frac{B}{(1 + 0.4z^{-1})}$$

$$\Rightarrow A(1+0.4z^{-1}) + B(1-0.6z^{-1}) = 0.8 - z^{-1}$$

$$\text{At } z = -\frac{1}{0.4} \Rightarrow B\left(1 + \frac{0.3}{0.2}\right) = 0.8 + \frac{1}{0.4}$$

$$\Rightarrow B(1.6) = 0.32$$

$$\Rightarrow B = 0.2$$

$$\text{At } z = 0.6 \Rightarrow A\left(1 + \frac{0.4}{0.6}\right) = 0.8 - \frac{1}{0.6}$$

$$\Rightarrow A = -0.52$$

LATTICE STRUCTURE OF IIR SYSTEM:-

$$\therefore \frac{Y(z)}{X(z)} = -\frac{1}{0.6} + \frac{1}{0.3} \left[\frac{-0.52}{1-0.6z^{-1}} + \frac{0.2}{1+0.4z^{-1}} \right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = -\frac{1}{0.6} + \frac{-0.52}{0.3(1-0.6z^{-1})} + \frac{0.2}{0.3(1+0.4z^{-1})}$$

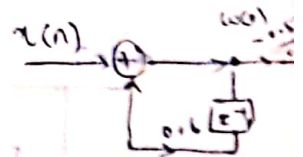
$$= C_1 + H_1(z) + H_2(z)$$

$H_1(z)$ Realization using Direct form-II is

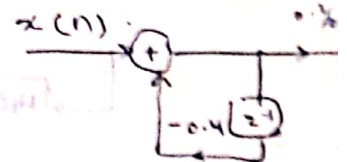
$$\frac{Y_1(z)}{X_1(z)} = \frac{-0.52}{1-0.6z^{-1}} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X_1(z)}$$

$$\frac{Y_1(z)}{W_1(z)} = -0.52/0.3 \quad \frac{W_1(z)}{X_1(z)} = \frac{1}{1-0.6z^{-1}}$$

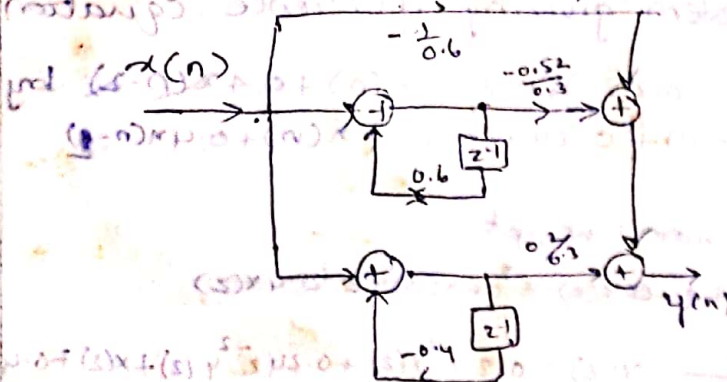
$$\Rightarrow Y_1(z) = \frac{-0.52}{0.3} W_1(z) \Rightarrow X(z) + 0.6z^{-1}W_1(z) = W_1(z)$$



Similarly $H_2(z)$ can be Realized using Direct form-II



Now Realization of $H(z)$ is



(Non-trivial)

examples:

- ① $y(n) = 2\pi \cos(\omega_0) y(n-1) - \pi^2 y(n-2) + x(n) - \pi \cos(\omega_0) x(n-1)$.
- ② $y(n) = 0.5 y(n-1) - 0.25 y(n-2) + x(n) + 0.4 x(n-1)$.
- ③ $y(n) = 2 y(n-1) + 3 y(n-2) + x(n) + 2 x(n-1) + 3 x(n-2)$.
- ④ $y(n) = 0.5 y(n-1) + 0.06 y(n-2) + 0.3 x(n) + 0.5 x(n-1)$.
- * ⑤ $y(n) = -0.1 y(n-1) + 0.72 y(n-2) + 0.7 x(n) - 0.252 x(n-2)$ (Parallel example)
- ⑥ $y(n) + y(n-1) - 4 y(n-3) = x(n) + 3 x(n-2)$.
- ⑦ $y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{2} x(n-1)$ (Cascade ex)
- * ⑧ $y(n) = -0.2 y(n-1) + 0.24 y(n-2) + x(n) + 0.4 x(n-1)$
- ⑨ $y(n) = \frac{1}{2} y(n-1) + \frac{1}{4} y(n-2) + x(n) + x(n-1)$
- * ⑩ $y(n) = -0.1 y(n-1) + 0.2 y(n-2) + 3 x(n) + 3.6 x(n-1) + 0.6 x(n-2)$.
- ⑪ $y(n) = \frac{1}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) + x(n-1)$ (Transposer ex)
- * ⑫ $y(n) = 0.2 y(n-1) + 0.24 y(n-2) + x(n) + 0.4 x(n-1)$ [Parallel form]

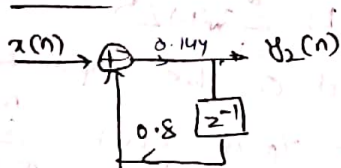
Parallel form:

$$H(z) = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}} = 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}} = 0.35 + \frac{0.206}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}}$$

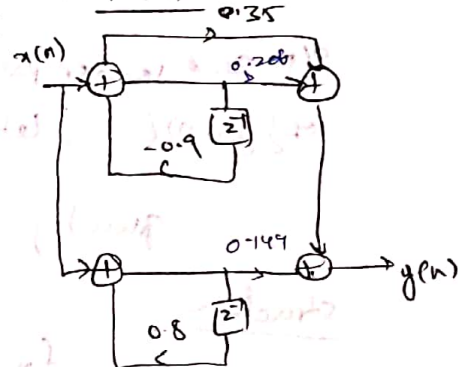
$H_1(z) = D-I$



$H_2(z) = D-II$



$H(z)$



lattice-ladder structure

Let all pole system is $H(z) = \frac{1}{1 + \sum_{k=1}^N a_N(k) z^{-k}} = \frac{1}{A_N(z)}$

difference

Ex of IIR $\rightarrow y(n) = - \sum_{k=1}^N a_N(k) y(n-k) + x(n)$ — (1)

$\Rightarrow x(n) = y(n) + \sum_{k=1}^N a_N(k) y(n-k)$ — (2)

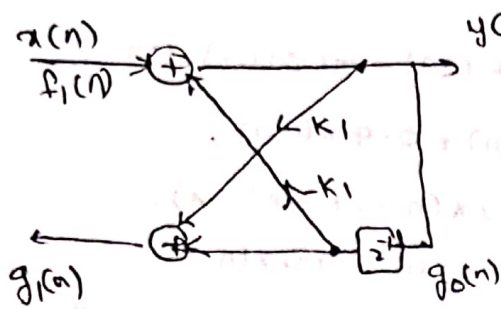
for $N=1$

$\Rightarrow y(n) = -a_1(1) y(n-1) + x(n)$ — (3)

$x(n) = y(n) + a_1(1) y(n-1)$ — (4)

$x(n) = f_1(n)$

Lattice structure is (a frame structure made of crossed wood & metal strips)



Comparing Eq

$$x(n) = f_1(n)$$

$$\text{Then } y(n) = f_0(n) = f_1(n) - K_1 g_0(n-1) \\ = x(n) - K_1 y(n-1)$$

Comparing Eq (4) & (6)

$$K_1 = a_1(1)$$

∴ In general converting Lattice to Direct form

$$a_m(m) = K_m$$

Converting Direct form to Lattice

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m) \cdot a_m(m-k)}{1 - a_m^2(m)}$$

$$1 \leq m \leq N$$

Lattice-ladder structure:

$$\text{Let } H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^M b_M(k) z^{-k}}{1 + \sum_{k=1}^N a_N(k) z^{-k}} \quad \text{where } N \geq M$$

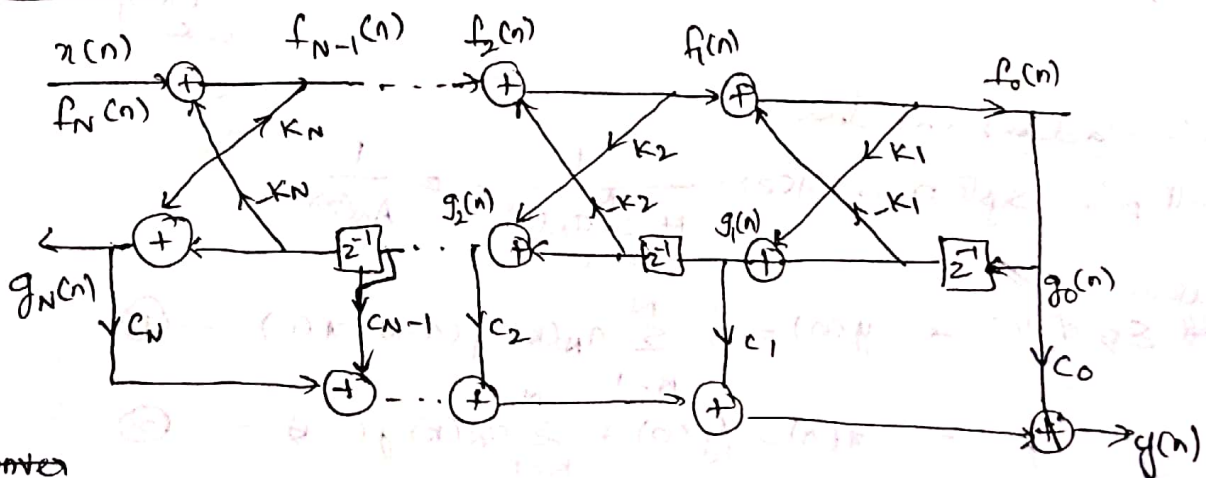
o/p of ladder part can be expressed as a weighted linear comb of $\{g_m(n)\}$

$$\text{let o/p is } y(n) = \sum_{m=0}^M c_m g_m(n)$$

ladder Coeff

$$\text{given by } c_m = b_m - \sum_{i=m+1}^M c_i a_i(i-m); \quad m = M, M-1, \dots, 0$$

Structure:



Center

Example:

convert the following pole-zero structure into ladder structure. $H(z) = \frac{1 + 2z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{8}z^{-3}}$

Sol: given $B_M(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

and $A_N(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{8}z^{-3}$

$a_3(0) = 1, a_3(1) = \frac{13}{24}, a_3(2) = \frac{5}{8}, a_3(3) = \frac{1}{8}$

$K_m = a_m(m) \rightarrow K_3 = a_3(3) = \frac{1}{8}$

we know $a_{m-1}(k) = \frac{a_m(k) - a_m(m) \cdot a_m(m-k)}{1 - a_m^2(m)}$

for $m=3$ and $k=1 \Rightarrow a_2(1) = \frac{a_3(1) - a_3(3) \cdot a_3(2)}{1 - a_3^2(3)} = \frac{\frac{13}{24} - \frac{1}{8}(\frac{5}{8})}{1 - (\frac{1}{8})^2} = \frac{3}{8}$

for $m=3, k=2 \Rightarrow K_2 = a_2(2) = \frac{a_3(2) - a_3(3) \cdot a_3(1)}{1 - a_3^2(3)} = \frac{\frac{5}{8} - \frac{1}{8}(\frac{13}{24})}{1 - (\frac{1}{8})^2} = \frac{1}{2}$

for $m=2, k=1 \Rightarrow K_1 = a_1(1) = \frac{a_2(1) - a_2(2) \cdot a_2(1)}{1 - a_2^2(2)} = \frac{\frac{3}{8} - \frac{1}{2}(\frac{3}{8})}{1 - (\frac{1}{2})^2} = \frac{1}{4}$

$\therefore \boxed{K_1 = \frac{1}{4}, K_2 = \frac{1}{2}, K_3 = \frac{1}{8}}$

for ladder structure

$C_m = b_m - \sum_{i=m+1}^M c_i a_i(i-m); \quad m = M, M-1, \dots, 0$

at $m=3 \Rightarrow C_3 = b_3 = 1$

at $m=2 \Rightarrow C_2 = b_2 - C_3 a_3(1) = 2 - 1(\frac{13}{24}) = 1.4583$

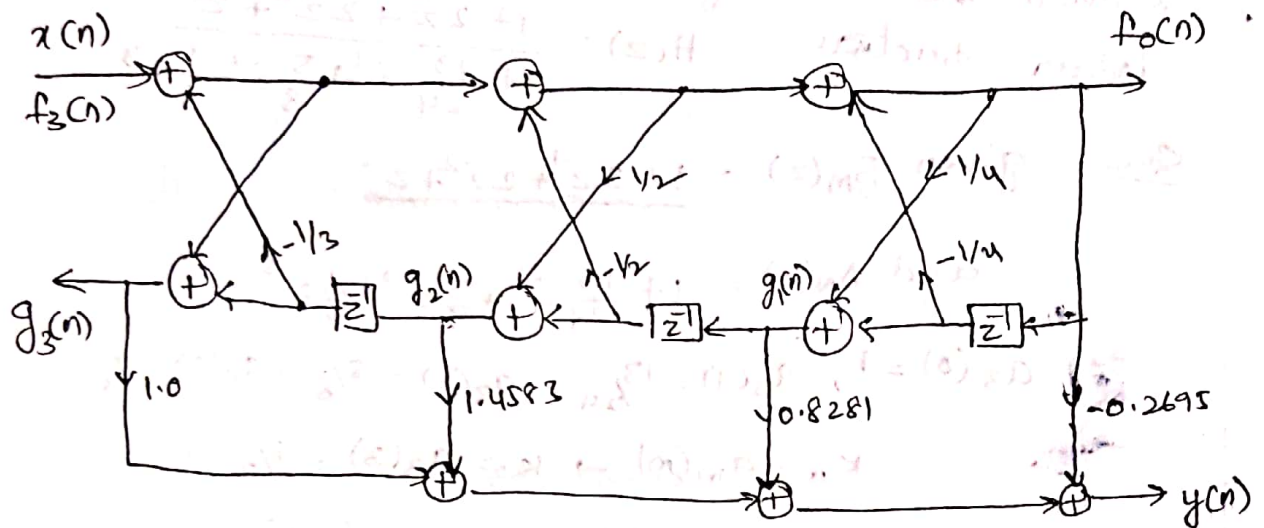
at $m=1 \Rightarrow C_1 = b_1 - \sum_{i=2}^3 c_i a_i(i-m) = b_1 - [C_2 a_2(1) + C_3 a_3(2)]$
 $= 2 - [(1.4583)(\frac{3}{8}) + 1(\frac{5}{8})] = 0.8281$

at $m=0 \Rightarrow C_0 = b_0 - \sum_{i=1}^3 c_i a_i(i-m)$
 $= b_0 - [C_1 a_1(1) + C_2 a_2(2) + C_3 a_3(3)]$
 $= 1 - [0.8281(\frac{1}{4}) + 1.4583(\frac{1}{2}) + 1(\frac{1}{8})] = -0.2695$

$\therefore \boxed{C_0 = -0.2695, C_1 = 0.8281, C_2 = 1.4583, C_3 = 1}$

coefficients.

The lattice ladder structure for pole-zero filter is



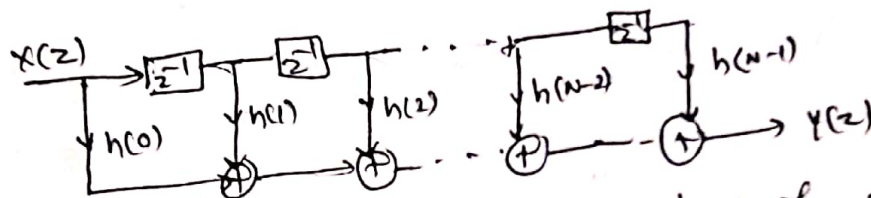
FIR FILTER REALIZATION:

we have following models under Non-recursive Method to realize FIR digital filters.

- ① Transversal structure
- ② cascade realization
- ③ Linear phase Realization
- ④ Lattice-ladder structure.

Transversal or Direct form

system function of FIR filter is $H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$
 $\Rightarrow H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(N-1)z^{-(N-1)}$



Ex: Determine Direct form realization of system function

$$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

cascade realization.

Ex: obtain cascade realization of system function

$$H(z) = (1 + 2z^{-1} - z^{-2})(1 + z^{-1} - z^{-2}) \quad \text{--- (1)}$$

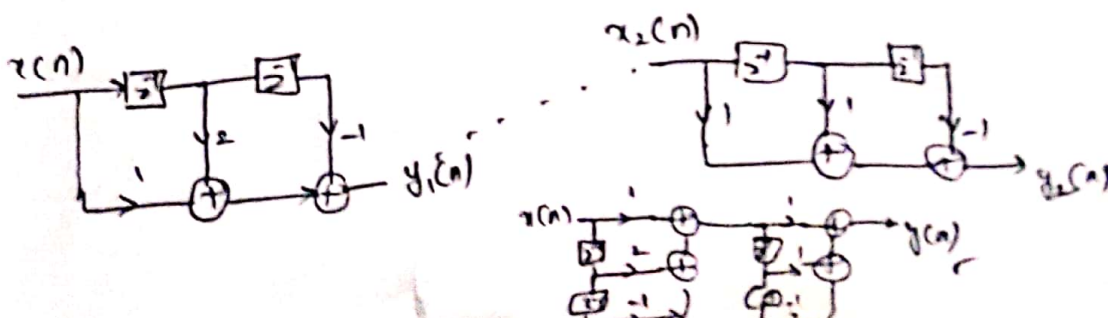
$$\Rightarrow H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = 1 + 2z^{-1} - z^{-2}$$

$$\Rightarrow Y_1(z) = X_1(z) + 2z^{-1}X_1(z) - z^{-2}X_1(z) \quad \text{--- (2)}$$

$$H_2(z) = 1 + z^{-1} - z^{-2}$$

$$\Rightarrow Y_2(z) = X_2(z) + z^{-1}X_2(z) - z^{-2}X_2(z) \quad \text{--- (3)}$$



③ Linear phase

Obtain the linear phase realization of following system

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

Note: for this find the realization of system function and consider present values as system symmetric. Then construct realization diagram.

Given

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

$$\rightarrow y(n] = \frac{1}{2}x(n) + \frac{1}{3}x(n-1) + x(n-2) + \frac{1}{4}x(n-3) + x(n-4) + \frac{1}{3}x(n-5) + \frac{1}{2}x(n-6)$$

By inspection method given system is symmetric

$$\frac{1}{2} \quad \frac{1}{3} \quad 1 \quad \frac{1}{4} \quad 1 \quad \frac{1}{3} \quad \frac{1}{2}$$

