

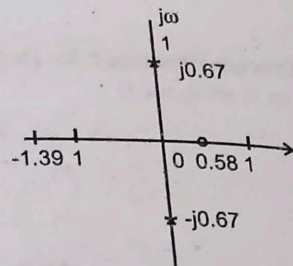
2.66 Digital Signal Processing

Solution

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.81z^{-1} - 0.81z^{-2}}{1 + 0.45z^{-2}} \\ = \frac{z^2 + 0.81z - 0.81}{z^2 + 0.45}$$

The zeros are at 0.58 and -1.39

The poles are $\pm j0.67$



Practice Problem 2.32 Find the impulse and step responses for the given system

$$y(n) + y(n-1) = x(n) - 2x(n-1)$$

May' 05 (set 4), May' 06

Ans: $y(n) = \frac{3}{2}(-1)^n u(n) - \frac{1}{2}u(n)$

Practice Problem 2.33 Determine the impulse response of the system described by the difference equation $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using z-transform
NOV'05 (set 3)

Ans: $h(n) = \frac{6}{5}(4)^n u(n) - \frac{1}{5}(-2)^n u(n)$

2.17 Simple Digital Filters

The design of digital filters are covered in Chapters 5 and 6. In the following sections we will study about simple IIR and FIR digital filters.

2.17.1 Simple IIR digital Filters

Lowpass IIR digital filters

The transfer function of a first order lowpass IIR digital filter is given by

$$H(z) = \frac{1-a}{2} \frac{1+z^{-1}}{1-az^{-1}} \quad (2.85)$$

where $|a| < 1$ for stability. The only zero of the transfer function is at $z = -1$ and pole is at $z = a$. As $|z| < a$ the frequency response of the filter can be obtained by substituting $z = e^{j\omega}$ in Eq. (2.85). That is,

$$H(e^{j\omega}) = \frac{1-a}{2} \frac{1+e^{-j\omega}}{1-ae^{-j\omega}} \quad (2.86)$$

From Eq. (2.86) we can find that

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$$H(e^{j0}) = \frac{1-a}{2} \frac{1+e^{-j0}}{1-ae^{-j0}} = 1$$

and

$$H(e^{j\pi}) = \frac{1-a}{2} \frac{1+e^{-j\pi}}{1-ae^{-j\pi}} = \frac{1-a}{2} \frac{(1-1)}{1+1} = 0$$

Therefore the frequency response of the filter $|H(e^{j\omega})|$ starts from 1 at $\omega = 0$ and

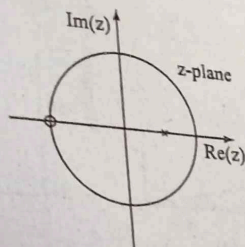


Fig. 2.13 Pole-zero plot of a lowpass filter

monotonically decreases and reaches '0' at $\omega = \pi$.

The magnitude response of the filter is

$$|H(e^{j\omega})| = \frac{1-a}{2} \frac{[(1+\cos\omega)^2 + \sin^2\omega]^{1/2}}{[(1-a\cos\omega)^2 + a^2\sin^2\omega]^{1/2}} \quad (2.87)$$

Squaring on both sides

$$|H(e^{j\omega})|^2 = \frac{(1-a)^2}{4} \frac{2+2\cos\omega}{1+a^2-2a\cos\omega} = \frac{(1-a)^2}{2} \frac{(1+\cos\omega)}{(1+a^2-2a\cos\omega)} \quad (2.88)$$

The 3dB cutoff frequency can be obtained by equating Eq. (2.88) to $\frac{1}{2}$.

That is,

$$\begin{aligned} \frac{(1-a)^2(1+\cos\omega_c)}{2(1+a^2-2a\cos\omega_c)} &= \frac{1}{2} \\ (1-a)^2(1+\cos\omega_c) &= 1+a^2-2a\cos\omega_c \\ (1+a^2-2a)(1+\cos\omega_c) &= 1+a^2-2a\cos\omega_c \\ -2a + (1+a^2-2a)\cos\omega_c &= -2a\cos\omega_c \\ \cos\omega_c &= \frac{2a}{1+a^2} \end{aligned}$$

from which $a^2\cos\omega_c - 2a + \cos\omega_c = 0$

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The solution is

$$a = \frac{1 \pm \sin \omega_c}{\cos \omega_c} \quad (2.89)$$

For a stable filter

$$a = \frac{1 - \sin \omega_c}{\cos \omega_c} \quad (2.90)$$

The magnitude response of lowpass filter for different values of a is shown in Fig. 2.14. (See MATLAB program 2.5)

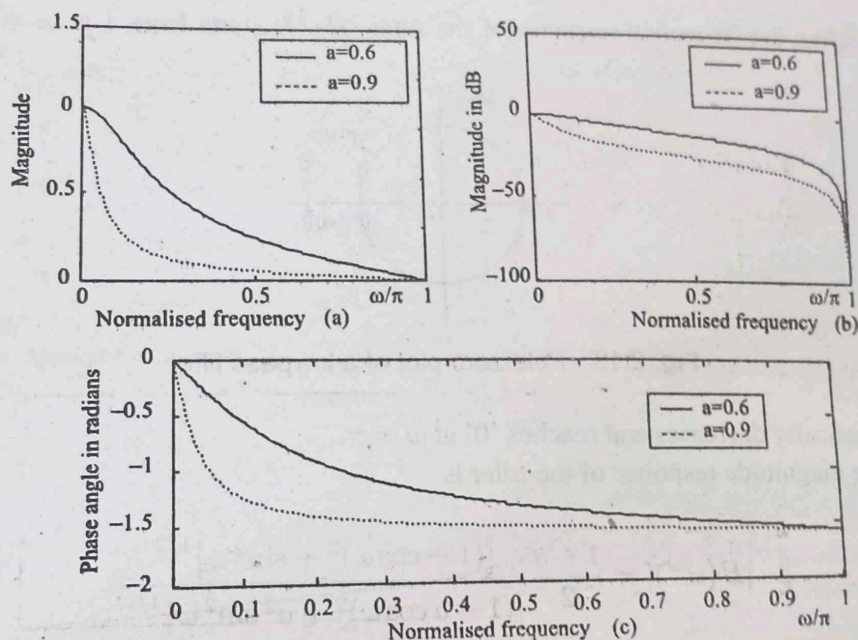


Fig. 2.14 Magnitude and phase response of the first order lowpass filter for $a = 0.6$ and $a = 0.9$.

2.17.2 Highpass IIR digital filter

The transfer function of a first - order highpass digital filter is given by

$$H(z) = \frac{1+a}{2} \left(\frac{1-z^{-1}}{1-az^{-1}} \right) \quad (2.91)$$

where $|a| < 1$ for stability. The above transfer function has a zero at $z = 1$ and a pole at $z = a$. The 3dB cutoff frequency of the filter is same as that of lowpass filter. That is,

$$\cos \omega_c = \frac{2a}{1+a^2} \quad (2.92)$$

and

$$a = \frac{1 - \sin \omega_c}{\cos \omega_c} \quad (2.93)$$

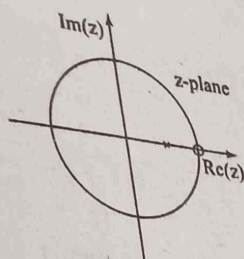


Fig. 2.15 pole zero plot of a highpass filter

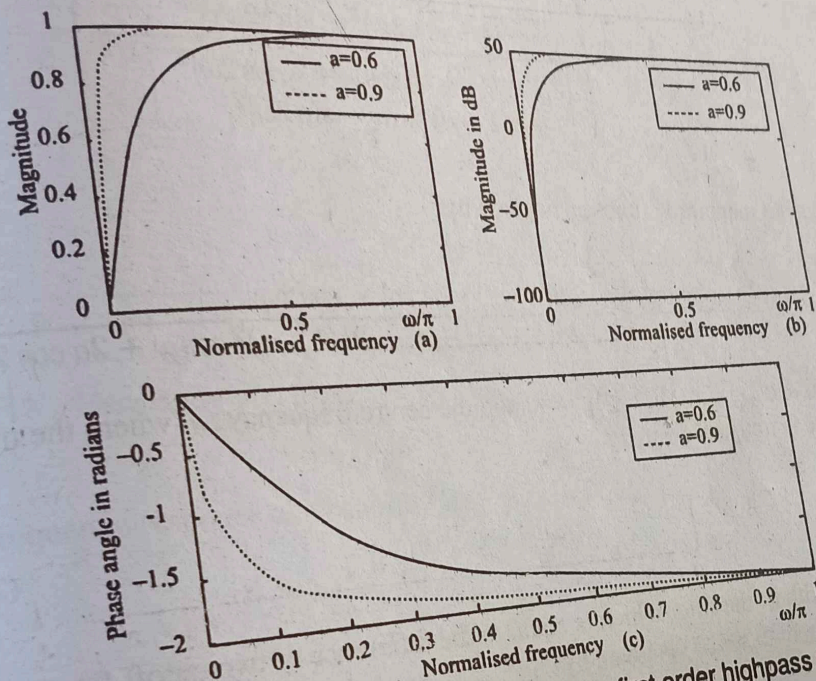


Fig. 2.16 Magnitude and phase response of the first order highpass filter for $a=0.6$ and $a=0.9$.

The magnitude response of a highpass filter for different values of 'a' is shown in Fig. 2.16. (See MATLAB program 2.6)

Bandpass IIR filters

The transfer function of a second order bandpass digital filter is given by

$$H(z) = \frac{1-a}{2} \frac{1-z^{-2}}{1-b(1+a)z^{-1}-az^{-2}} \quad (2.94)$$

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The frequency response of the filter is

$$H(e^{j\omega}) = \frac{1-a}{2} \frac{1-e^{-j2\omega}}{1-b(1+a)e^{-j\omega}-ae^{-j2\omega}} \quad (2.95)$$

The magnitude of $H(e^{j\omega})$ is given by

$$|H(e^{j\omega})| = \frac{1-a}{2} \left\{ \frac{[(1-\cos 2\omega)^2 + \sin^2 2\omega]^{1/2}}{\left[\frac{[1-b(1+a)\cos \omega + a\cos 2\omega]^2}{+[b(1+a)\sin \omega - \sin 2\omega]^2} \right]^{1/2}} \right\} \quad (2.96)$$

The square magnitude function is given by

$$|H(e^{j\omega})|^2 = \frac{(1-a)^2}{4} \left[\frac{1-\cos 2\omega}{1+b^2(1+a)^2+a^2-2b(1+a)^2\cos \omega+2a\cos 2\omega} \right] \quad (2.97)$$

At $\omega = 0$ and $\omega = \pi$; $|H(e^{j\omega})|^2 = 0$ and the centre frequency ω_0 where the magnitude value is unity, is given by

$$\omega_0 = \cos^{-1}(b) \quad (2.98)$$

The bandwidth of the filter which is equal to the difference of two cutoff frequencies ω_{c1} and ω_{c2} ($\omega_{c2} \gg \omega_{c1}$) is given by

$$\omega_{c2} - \omega_{c1} = \cos^{-1} \left(\frac{2a}{1+a^2} \right) \quad (2.99)$$

The magnitude response of bandpass filter for different values of a and b is shown in Fig. 2.17. (See MATLAB program 2.7)

Bandstop IIR digital filter

The transfer function of a second order bandstop digital filter is given by

$$H(z) = \frac{1+a}{2} \frac{1-2bz^{-1}+z^{-2}}{1-b(1+a)z^{-1}+az^{-2}} \quad (2.100)$$

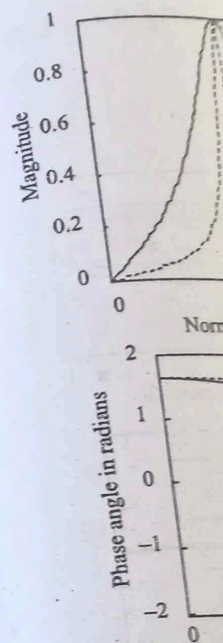


Fig. 2.17
 $b = 0.4$

The frequency

For

Similar

That
 $\omega =$
by

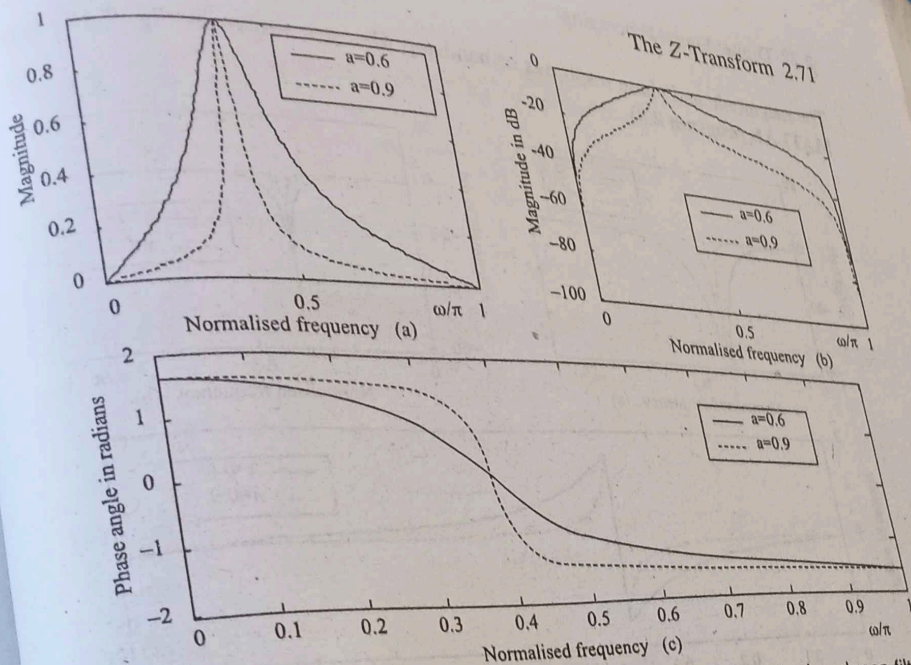


Fig. 2.17 Magnitude and phase responses of the second order bandpass filter for $b = 0.4$

The frequency response of a bandstop filter is equal to

$$H(z) = \frac{1+a}{2} \frac{1 - 2be^{-j\omega} + e^{-j2\omega}}{1 - b(1+a)e^{-j\omega} + ae^{-j2\omega}}$$

For $\omega = 0$

$$\begin{aligned} H(e^{j0}) &= \frac{1+a}{2} \left[\frac{1-2b+1}{1-b(1+a)+a} \right] = \frac{1+a}{2} \left[\frac{2(1-b)}{1-a-ab+a} \right] \\ &= \frac{1+a}{2} \left[\frac{2(1-b)}{(1+a)(1-b)} \right] = 1 \end{aligned}$$

Similarly, for $\omega = \pi$

$$H(e^{j\pi}) = \frac{1+a}{2} \left[\frac{1+2b+1}{1+b(1+a)+a} \right] = 1$$

That is, the magnitude response takes the maximum value of unity at $\omega = 0$ and $\omega = \pi$. The magnitude response goes to zero at ω_0 known as notch frequency given by

$$\omega_0 = \cos^{-1}(b)$$

The 3-dB notch bandwidth is given by

$$BW = \omega_{c2} - \omega_{c1} = \cos^{-1} \left(\frac{2a}{1+a^2} \right)$$

(2.101)

(2.102)

(2.100)

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The magnitude and phase responses of bandstop filter are shown in Fig. 2.18. (See MATLAB program 2.8)

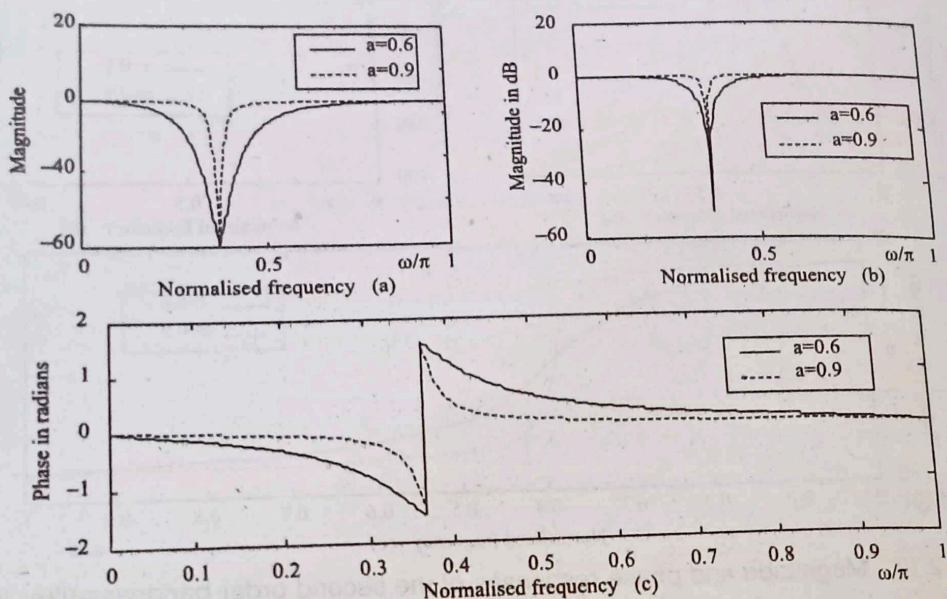


Fig. 2.18 Magnitude and phase responses of the second order bandstop filter for $b = 0.4$.

2.18 Simple FIR Digital filters

2.18.1 Lowpass FIR digital filters

The impulse response of a simple lowpass FIR filter (moving average filter) is given by

$$h(n) = \begin{cases} 0.5 & \text{for } n = 0, 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.103)$$

The system function of the filter is given by

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= 0.5 + 0.5z^{-1} \end{aligned} \quad (2.104)$$

$$\begin{aligned} &= \frac{1}{2}(1 + z^{-1}) \\ &= \frac{z + 1}{2z} \end{aligned} \quad (2.105)$$

The system function has a zero at $z = -1$ and a pole at $z = 0$. The transfer function of the filter can be obtained by substituting $z = e^{j\omega}$ in Eq. (2.104). That is,

$$H(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) \quad (2.106)$$

The magnitude response is given by

$$\begin{aligned} |H(e^{j\omega})| &= \frac{1}{2} [(1 + \cos \omega)^2 + \sin^2 \omega]^{1/2} \\ &= \frac{1}{2} [2(1 + \cos \omega)]^{1/2} \\ &= \cos \frac{\omega}{2} \end{aligned} \quad (2.107)$$

For $\omega = 0$

$$|H(e^{j0})| = 1$$

and for $\omega = \pi$

$$|H(e^{j\pi})| = 0$$

That is, the response starts from 1 at $\omega = 0$ and monotonically decreases to zero at $\omega = \pi$.

The phase response of the filter can be obtained by writing Eq. (2.107) as

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{2} [e^{j\omega/2} + e^{-j\omega/2}] e^{-j\omega/2} \\ &= \underbrace{e^{-j\omega/2}}_{\text{Phase response}} \underbrace{\cos \omega/2}_{\text{magnitude response}} \end{aligned} \quad (2.108)$$

$$\Rightarrow \theta(\omega) = \frac{-\omega}{2} \quad (2.109)$$

The magnitude and phase response of the filter are shown in Fig. 2.19a and 2.19b respectively. (See MATLAB program 2.9)

The cutoff frequency ω_c of the filter can be obtained by equating the magnitude response $|H(e^{j\omega})|$ to $\frac{1}{\sqrt{2}}$.

That is,

$$\cos \frac{\omega_c}{2} = \frac{1}{\sqrt{2}}$$

from which

$$\omega_c = \frac{\pi}{2}$$

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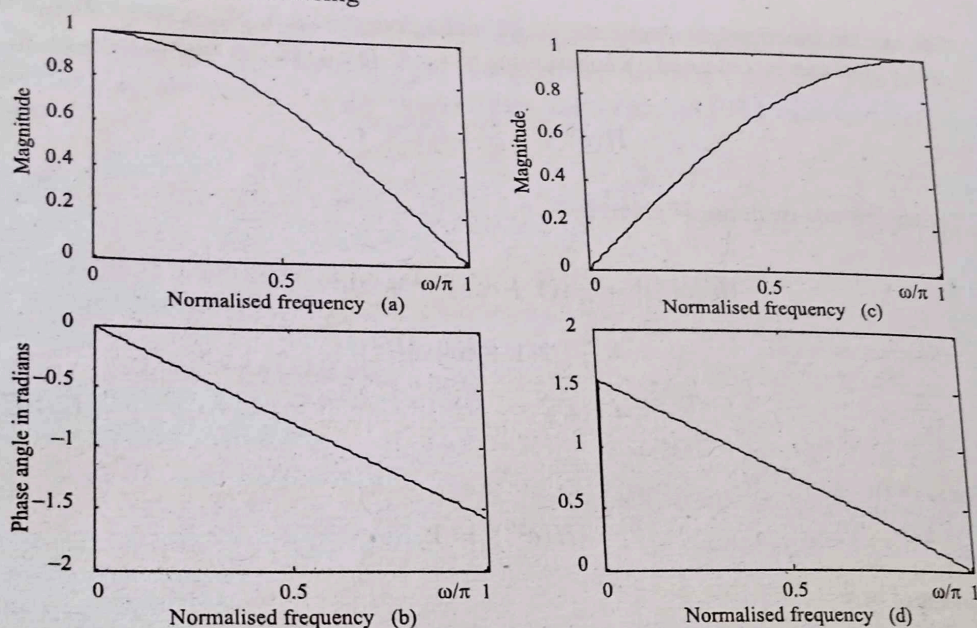


Fig. 2.19 (a) Magnitude response of FIR lowpass filter (b) Phase response of FIR lowpass filter (c) Magnitude response of FIR highpass filter (d) Phase response of FIR highpass filter

2.18.2 Highpass FIR digital filter

The system function of a simple highpass FIR filter is given by

$$H(z) = \frac{1}{2}[1 - z^{-1}] \quad (2.110)$$

The frequency response is given by

$$\begin{aligned} H(e^{j\omega}) &= H(z) \Big|_{z=e^{j\omega}} \\ &= \frac{1}{2}[1 - e^{-j\omega}] \\ &= \frac{1}{2}e^{-j\omega/2}[e^{j\omega/2} - e^{-j\omega/2}] \\ &= je^{-j\omega/2} \sin\left(\frac{\omega}{2}\right) \end{aligned} \quad (2.111)$$

The magnitude response is given by

$$|H(e^{j\omega})| = \sin \frac{\omega}{2} \quad (2.112)$$

For $\omega = 0$; $|H(e^{j0})| = 0$
and for $\omega = \pi$; $|H(e^{j\pi})| = 1$

That is, the magnitude response starts at 0 at $\omega = 0$ and increases to 1 at $\omega = \pi$. The cutoff frequency at which the magnitude is $\frac{1}{\sqrt{2}}$ is equal to $\frac{\pi}{2}$. The magnitude and phase response of FIR highpass filter are shown in Fig. 2.19c and 2.19d respectively.

2.19 Comb Filter

A filter which has multiple passband and stopbands is known as comb filter. It can be viewed as a notch filter in which the peaks and notches occur periodically across the frequency band.

Let us consider a moving average FIR filter described by difference equation

$$y(n) = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k) \quad (2.113)$$

The system function of the filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} \quad (2.114)$$

$$= \frac{1}{M} \left[\frac{1 - z^{-M}}{1 - z^{-1}} \right] \quad (2.115)$$

The frequency response is given by

$$\begin{aligned} H(e^{j\omega}) &= H(z) \Big|_{z=e^{j\omega}} \\ &= \frac{1}{M} \left[\frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} \right] \end{aligned} \quad (2.116)$$

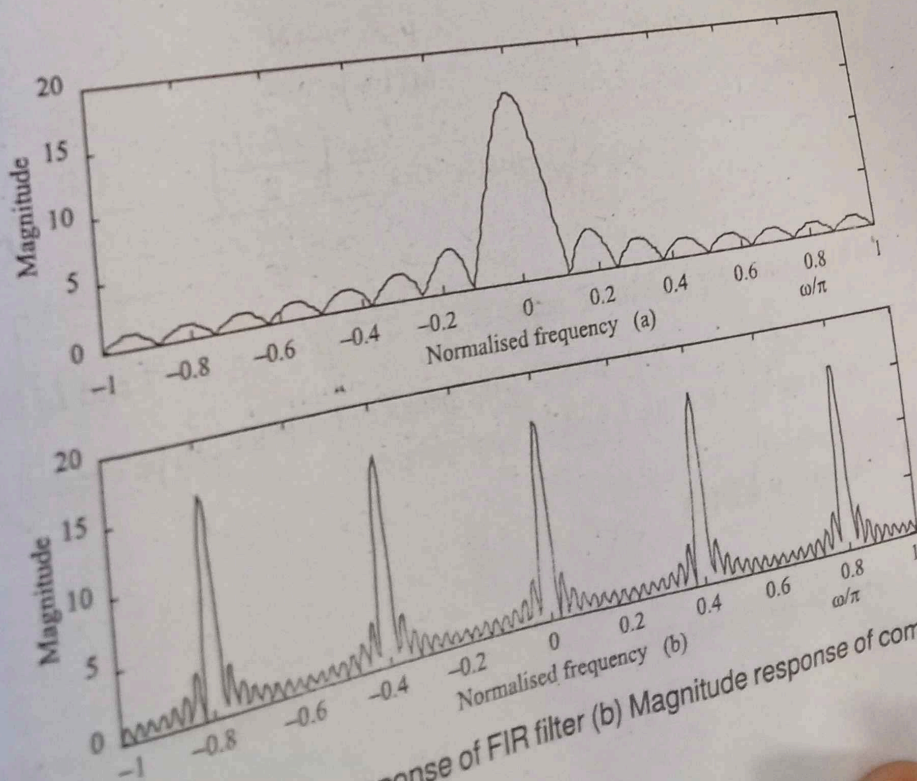


Fig. 2.20 (a) Magnitude response of FIR filter (b) Magnitude response of comb filter

$$\begin{aligned}
 &= \frac{1}{M} \frac{e^{-j\omega M/2}}{e^{-j\omega/2}} \left[\frac{e^{j\omega M/2} - e^{-j\omega M/2}}{e^{j\omega/2} - e^{-j\omega/2}} \right] \\
 &= \frac{1}{M} e^{-j\omega(M-1)/2} \left[\frac{\sin \frac{\omega M}{2}}{\sin \frac{\omega}{2}} \right] \quad (2.117)
 \end{aligned}$$

The magnitude response of the filter is shown in Fig. 2.20a. The filter has a peak magnitude at $\omega = 0$ and $M - 1$ notches at $\omega = \frac{2\pi l}{M}$; $1 \leq l \leq M - 1$. (See MATLAB program 2.10)

A comb filter can easily be generated from either an FIR or IIR filter with system function $H(z)$ by replacing z by z^L , where L is a positive integer. Now the system function of the comb filter is

$$G(z) = H(z^L) \quad (2.118)$$

and the frequency response is given by

$$G(e^{j\omega}) = H(e^{j\omega L}) \quad (2.119)$$

From Eq. (2.119) we can find that if the magnitude response $|H(e^{j\omega})|$ exhibits a peak at $\frac{\pi}{3}$ then the magnitude response of $|G(e^{j\omega})|$ exhibits L peaks at $\frac{\pi k}{3L}$, $1 \leq k \leq L$. Similarly if the magnitude response $|H(e^{j\omega})|$ exhibits notch at $\omega = \frac{\pi}{4}$ then the magnitude response $|G(e^{j\omega})|$ will have L notches at $\omega = \frac{\pi k}{4L}$; $1 \leq k \leq L$. The frequency response of a comb filter generated from a moving average FIR filter can be obtained by replacing $e^{j\omega}$ by $e^{j\omega L}$ in Eq. (2.116)

$$G(e^{j\omega}) = H(e^{j\omega L}) = \frac{1 - e^{-j\omega LM}}{M(1 - e^{-j\omega L})} \quad (2.120)$$

$$= \frac{1}{M} e^{-j\omega L(M-1)/2} \frac{\sin \left(\frac{\omega LM}{2} \right)}{\sin \frac{\omega L}{2}} \quad (2.121)$$

The magnitude response has L peaks at $\omega = \frac{2\pi k}{L}$, $0 \leq k \leq L - 1$ and $L(M - 1)$ notches at $\omega = \frac{2\pi k}{LM}$; $1 \leq k \leq LM - 1$ the peaks and notches can be created at the desired locations by selecting L and M appropriately.

2.20 Allpass Filter

A filter that has a constant magnitude response for all frequencies is known as allpass filter. That is, for an allpass filter

$$|H(e^{j\omega})| = 1 \text{ for } 0 \leq \omega \leq \pi \quad (2.122)$$

And Nth order allpass filter is described by the system function

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \dots + a_1z^{-(N-1)} + z^{-N}}{1 + a_1z^{-1} + \dots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}} \quad (2.123)$$

where the coefficients $\{a_k\}$ are real.

Let the denominator of the system function can be denoted as

$$D(z) = 1 + a_1z^{-1} + a_2z^{-2} + \dots + a_{N-1}z^{-(N-1)} + a_Nz^{-N} \quad (2.124)$$

Then the system function $H(z)$ can be expressed as

$$H(z) = \frac{z^{-N}(a_Nz^N + a_{N-1}z^{N-1} + \dots + a_1z + 1)}{1 + a_1z^{-1} + a_2z^{-2} + \dots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}} \quad (2.125)$$

$$= \frac{z^{-N}D(z^{-1})}{D(z)} \quad (2.126)$$

From Eq. (2.126) we find

$$H(z)H(z^{-1}) = z^{-N} \frac{D(z^{-1})}{D(z)} \cdot z^N \frac{D(z)}{D(z^{-1})} = 1 \quad (2.127)$$

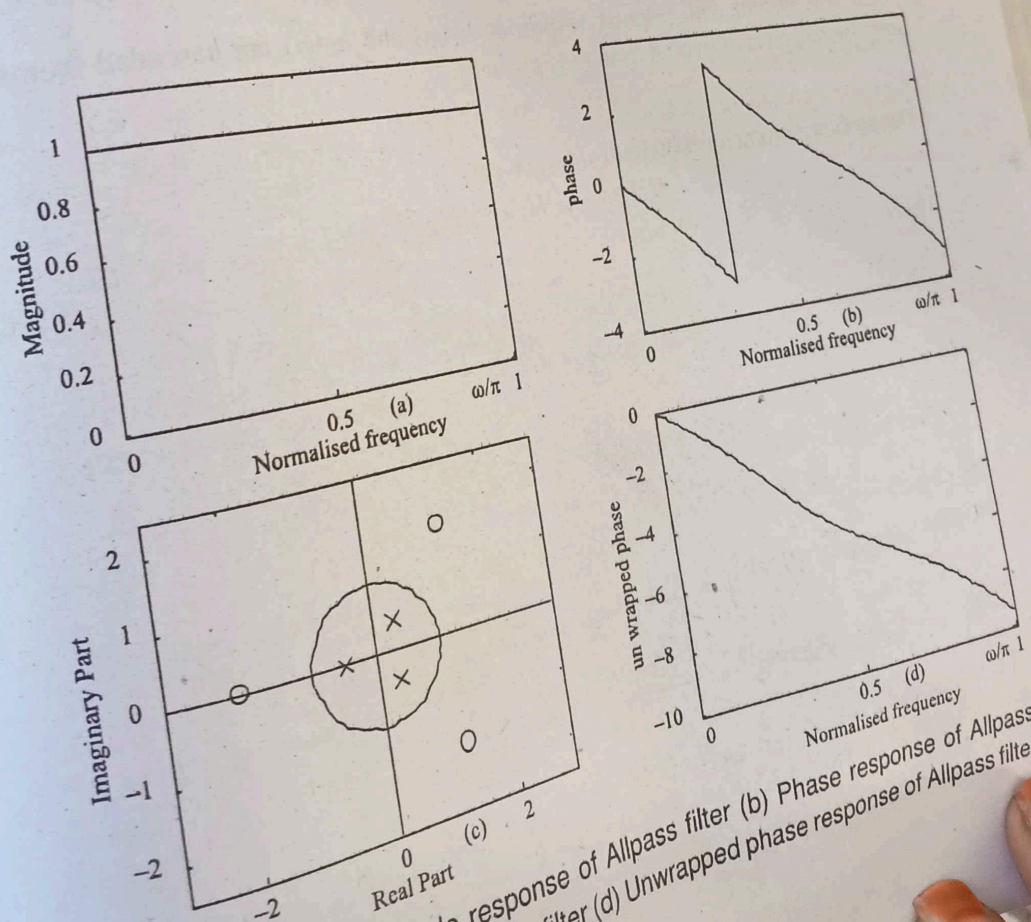


Fig. 2.21 (a) Magnitude response of Allpass filter (b) Phase response of Allpass filter (c) Pole-Zero plot of Allpass filter (d) Unwrapped phase response of Allpass filter

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We have

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1}) \Big|_{z=e^{j\omega}} \quad (2.128)$$

Therefore

$|H(e^{j\omega})|^2 = 1$ and the system is an allpass system. Note from Eq. (2.127) that if z_0 is a pole of $H(z)$ then it has a zero at $\frac{1}{z_0}$.

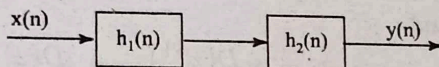
Fig. 2.21 shows the pole zero pattern of an allpass filter with system function

$$H(z) = \frac{0.125 - 0.05z^{-1} - 0.1z^{-2} + z^{-3}}{1 - 0.1z^{-1} - 0.05z^{-2} + 0.125z^{-3}} \quad (2.129)$$

It has zeros at -2 and $1.2 \pm j1.6$ and poles at -0.5 and $0.3 \pm j0.4$.

We know that the poles of a casual stable system function lies inside the unit circle. Therefore the zeros of the allpass system function lie outside the unit circle. The phase response shows the discontinuity, whereas the unwrapped phase removes the discontinuity. (See MATLAB program 2.11)

Example 2.50 Two causal systems with impulse responses $h_1(n) = a\delta(n) + \delta(n-1)$ and $h_2(n) = b^n u(n) : |b| < 1$ are connected in cascade. Determine the frequency response $H(e^{j\omega})$ of the overall system. Find the values of a and b for $|H(e^{j\omega})| = 1$.



Solution

If two system with impulse responses $h_1(n)$ and $h_2(n)$ are cascaded then the impulse response of overall system is given by

$$h(n) = h_1(n) * h_2(n)$$

Taking Fourier transform on both sides we get

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

Given

$$h_1(n) = a\delta(n) + \delta(n-1)$$

$$\Rightarrow H_1(e^{j\omega}) = a + e^{-j\omega}$$

and

$$h_2(n) = b^n u(n)$$

$$\Rightarrow H_2(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{a + e^{-j\omega}}{1 - be^{-j\omega}} = \frac{a + \cos \omega - j \sin \omega}{1 - b \cos \omega + jb \sin \omega}$$

$$|H(e^{j\omega})|^2 = \frac{(a + \cos \omega)^2 + \sin^2 \omega}{(1 - b \cos \omega)^2 + b^2 \sin^2 \omega}$$

$$= \frac{a^2 + 2a \cos \omega + 1}{b^2 - 2b \cos \omega + 1}$$

For $b = -a$ $|H(e^{j\omega})|^2 = 1$
 The other value for b can be found by equating

The Z-Transform 2.79

$$\frac{d}{d\omega} |H(e^{j\omega})|^2 = 0$$

$$\Rightarrow (1 - 2b \cos \omega + b^2)(-2a \sin \omega) - (a^2 + 2a \cos \omega + 1)(2b \sin \omega) = 0$$

$$(1 - 2b \cos \omega + b^2)(-a) = (a^2 + 2a \cos \omega + 1)b$$

$$-a + 2ab \cos \omega - ab^2 = a^2 b + 2ab \cos \omega + b$$

$$-ab(a + b) = a + b$$

$$\Rightarrow ab = -1$$

$$\text{or } b = -\frac{1}{a}$$

2.21 Minimum Phase, Maximum Phase and Non-minimum Phase Systems

Let us consider two first order systems with transfer functions

$$H_1(z) = \frac{1 + bz^{-1}}{1 + az^{-1}}; \quad |a| < 1; |b| < 1 \quad (2.130)$$

$$H_2(z) = \frac{b + z^{-1}}{1 + az^{-1}}$$

The pole zero plot of above transfer functions is shown in Fig. 2.23. Both the transfer functions has a pole inside the circle at $z = -a$ whereas the transfer function $H_1(z)$ has a zero at $z = -b$ inside the unit circle and the transfer function $H_2(z)$ has a zero at $z = -1/b$ outside the unit circle. The frequency response of these two systems can be expressed as

$$H_1(e^{j\omega}) = \frac{1 + be^{-j\omega}}{1 + ae^{-j\omega}} \quad \text{and} \quad H_2(e^{j\omega}) = \frac{b + e^{-j\omega}}{1 + ae^{-j\omega}} \quad (2.131)$$

The magnitude and phase response of these two systems are

$$|H_1(e^{j\omega})| = \left[\frac{(1 + b \cos \omega)^2 + b^2 \sin^2 \omega}{(1 + a \cos \omega)^2 + a^2 \sin^2 \omega} \right]^{1/2} = \left(\frac{1 + b^2 + 2b \cos \omega}{1 + a^2 + 2a \cos \omega} \right)^{1/2} \quad (2.132)$$

$$|H_2(e^{j\omega})| = \left[\frac{(b + \cos \omega)^2 + \sin^2 \omega}{(1 + a \cos \omega)^2 + a^2 \sin^2 \omega} \right]^{1/2} = \left(\frac{1 + b^2 + 2b \cos \omega}{1 + a^2 + 2a \cos \omega} \right)^{1/2} \quad (2.133)$$

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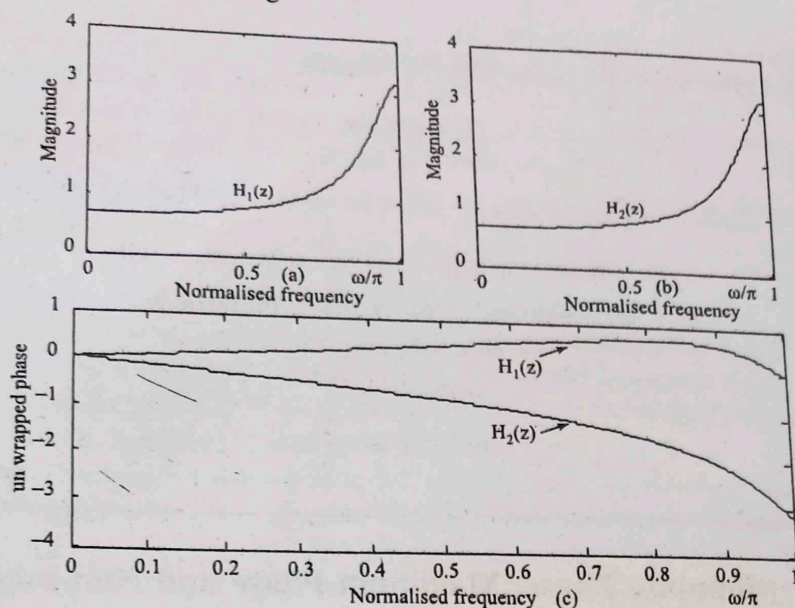


Fig. 2.22 Magnitude and unwrapped phase response of two transfer functions

$$\angle H_1(e^{j\omega}) = \tan^{-1} \left(\frac{-b \sin \omega}{1 + b \cos \omega} \right) - \tan^{-1} \left(\frac{-a \sin \omega}{1 + a \cos \omega} \right) \quad (2.134)$$

$$\text{and } \angle H_2(e^{j\omega}) = \tan^{-1} \left(\frac{-\sin \omega}{b + \cos \omega} \right) - \tan^{-1} \left(\frac{-a \sin \omega}{1 + a \cos \omega} \right) \quad (2.135)$$

From Eqs. (2.132 & 2.133) we can find that the magnitude characteristics of the two systems are identical. The graphs of $\angle H_1(e^{j\omega})$ and $\angle H_2(e^{j\omega})$ are illustrated in Fig. 2.22. (See MATLAB program 2.12) We observe that the phase response $\angle H_1(e^{j\omega})$ first begins at zerophase at the frequency $\omega = 0$ and terminates at zerophase at $\omega = \pi$. On the other hand the phase response $\angle H_2(e^{j\omega})$ starts at zerophase at $\omega = 0$ and terminates at 180° phase at $\omega = \pi$. That is, the system $H_2(z)$ has an excess phase with respect to $H_1(z)$.

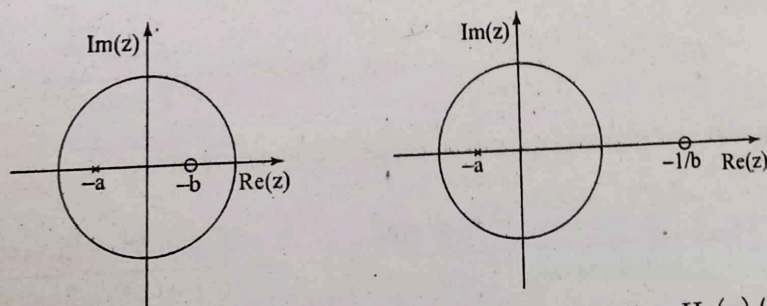


Fig. 2.23 Pole-zero plot of (a) Minimum phase transfer function $H_1(z)$ (b) Maximum phase transfer function $H_2(z)$

That is, a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal stable transfer function with identical magnitudes but having all zeros inside the unit circle. As a consequence, the causal stable

transfer function with all zeros inside the unit circle is known as minimum phase system and the causal stable transfer function with all zeros outside the unit circle is known as maximum phase system. A system that has some zeros inside the unit circle and the remaining zeros outside the unit circle, is called as non-minimum phase system.

Notch Filter

Eliminating 50 Hz power supply frequency noise is one of the most requirements in instrumentation and recording system. A notch filter is useful for such type of applications. It is a second order filter whose magnitude response is zero or null at $\omega = \pm\omega_0$ and roughly equals one for all the remaining frequencies; that is $-\pi$ to π . The null can be obtained at a desired $\pm\omega_0$ if the transfer function has zeros at $e^{\pm j\omega_0}$. To achieve a constant magnitude response for all remaining frequencies other than $\pm\omega_0$, we have to place the poles very close to the two zeros. The resultant transfer function is

$$H(z) = \frac{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}{(1 - ae^{j\omega_0} z^{-1})(1 - ae^{-j\omega_0} z^{-1})} \quad (2.136)$$

If 'a' is small the notch will be wide. In order to have a narrow bandwidth notch, the value of 'a' can be selected near to one. The magnitude response of a notch filter with a null at $\omega(0) = \frac{\pi}{2}$ for different values of a is shown in Fig. 2.24.

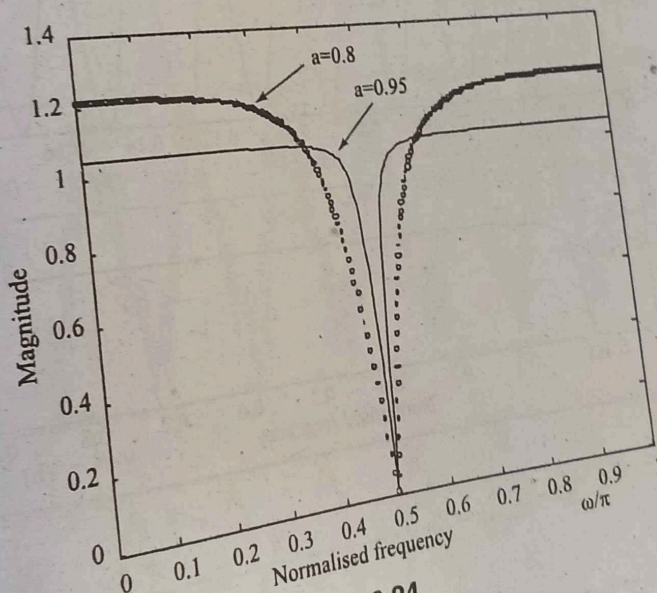


Fig. 2.24

Example 2.51 Design a notch filter to eliminate the 50 Hz sinusoidal noise.

Solution

Given $f = 50$ Hz; The time period $= \frac{1}{50} = 0.02$ sec

2.82 Digital Signal Processing

The Sampling rate

$$F \geq 2f_m$$

$$\geq 100 \text{ Hz}$$

The Sampling period must be less than 0.01, hence let us select $T = 0.001$. Then the frequency range is $\left(-\frac{\pi}{T} \text{ to } \frac{\pi}{T}\right) = (-1000\pi \text{ to } 1000\pi)$ and the normalized frequency is $= 50 \times 2\pi \times 0.001 = 0.314$.

The notch filter transfer function is

$$H(z) = \frac{(1 - e^{j0.314} z^{-1})(1 - e^{-j0.314} z^{-1})}{(1 - ae^{j0.314} z^{-1})(1 - ae^{-j0.314} z^{-1})}$$

Let $a = 0.95$, then

$$H(z) = \frac{1 - 1.9022z^{-1} + z^{-2}}{1 - 1.807z^{-1} + 0.9025z^{-2}}$$

Fig. 2.25a shows the sinusoidal signal of frequency 50 Hz and Fig. 2.25b shows the magnitude response of notch filter which has null at 0.1π . The output of the notch filter is shown in Fig. 2.25c when the sinusoidal signal of frequency 50 Hz is applied. (See MATLAB program 2.13)

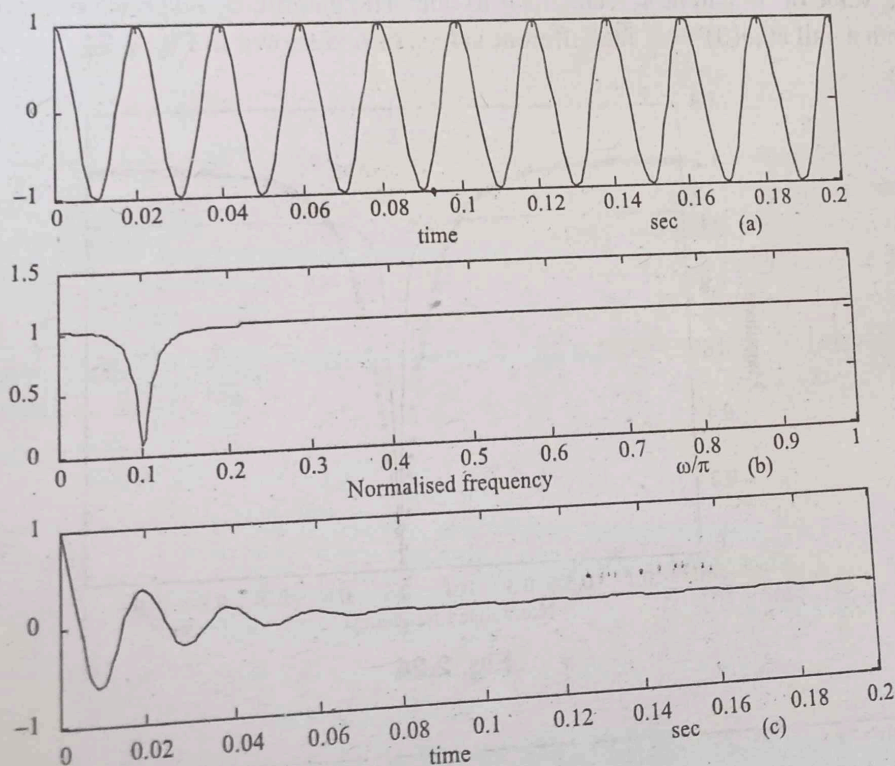


Fig. 2.25

After the transient, the notch filter eliminates the sinusoidal.

Practice Problem

Example 2.52 De
the continuous-tim

Solution To d
frequency alias
Given

$$T \leq \frac{1}{2 \times \frac{30}{2}}$$

Let us sele

The frequ

That is, -

Practice Problem 2.34 Design a notch filter to eliminate 60 Hz sinusoidal noise.

The Z-Transform 2.83

Example 2.52 Design a digital filter to eliminate the high frequency component in the continuous-time signal.

$$x(t) = 2 \cos 5t + \cos 300t$$

Solution To design a lowpass filter, first we select a sampling period that avoids frequency aliasing.

Given

$$\omega_m = 300$$

$$f_m = \frac{300}{2\pi} \text{ we know Sampling period } T \leq \frac{1}{2f_m}, \text{ hence}$$

$$T \leq \frac{1}{2 \times \frac{300}{2\pi}}, \text{ that is } T \leq 0.01047.$$

Let us select $T = 0.008$.

The frequency range of $x(n)$ is $\frac{-\pi}{T}$ to $\frac{\pi}{T}$.

That is, -392.7 to 392.7 .

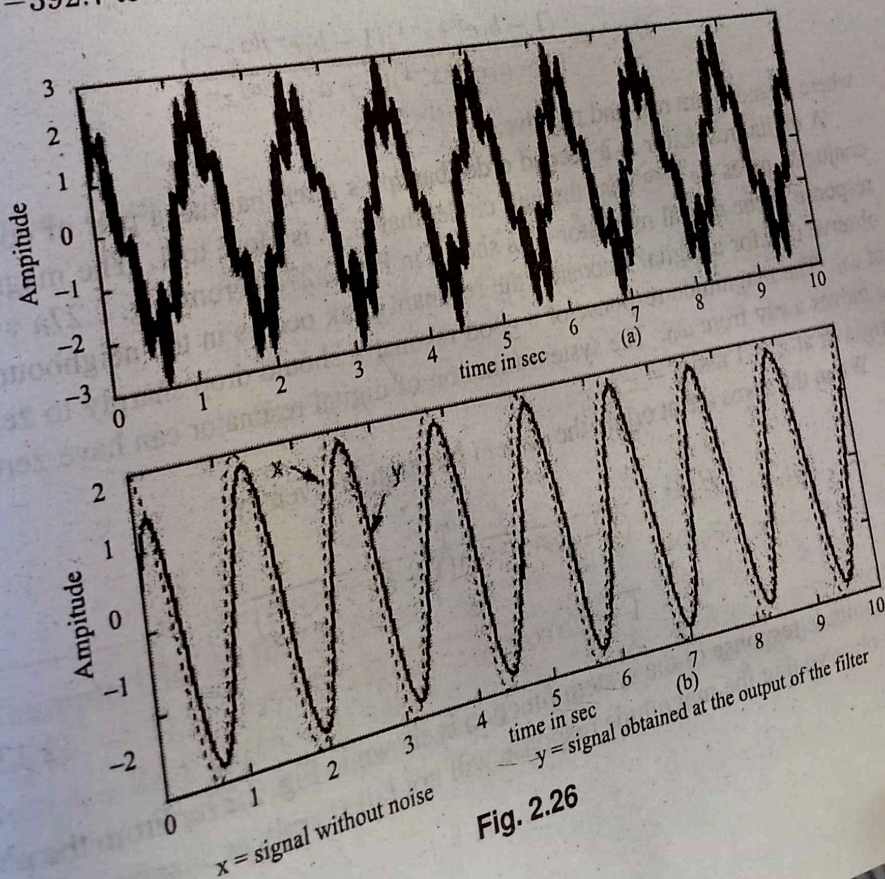


Fig. 2.26

2.84 Digital Signal Processing

Normalize the frequency range of $x(n)$ to $(-\pi$ to $\pi)$ which can be achieved by multiplying 5 and 300 by $T = 0.008$ which is 0.040 and 2.4.

Now we have to design a digital lowpass filter which allows 0.04 and rejects 2.4. Select bandwidth of the filter as 0.1, then from Eq. (2.85) we have $a = 0.9$ and the filter transfer function

$$\begin{aligned} H(z) &= \frac{1 - 0.9}{2} \frac{1 + z^{-1}}{1 - 0.9z^{-1}} \\ &= \frac{0.05(1 + z^{-1})}{1 - 0.9z^{-1}} \end{aligned}$$

The input signal that is applied to a lowpass filter is shown in Fig. 2.26a. The filtered output is shown in Fig. 2.26b, from which we can see that the high frequency signal is completely blocked by the filter. (See MATLAB program 2.14).

Practice Problem 2.35 Design a digital filter to eliminate the low frequency component in the continuous-time signal

$$x(t) = \sin 10\pi t + 2 \sin 1000\pi t$$

Second order digital filter

The second order digital filter has two poles and the maximum of two zeros. If it has two poles and two zeros the transfer function can be written as

$$H(z) = \frac{(1 - b_1 e^{jb_2} z^{-1})(1 - b_1 e^{-jb_2} z^{-1})}{(1 - a_1 e^{ja_2} z^{-1})(1 - a_1 e^{-ja_2} z^{-1})} \quad (2.137)$$

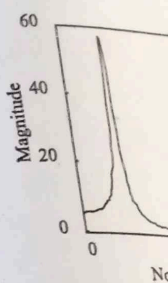
where a_i and b_i are real and positive.

A digital resonator is a second order bandpass filter that has a pair of complex conjugate poles $ae^{\pm j\omega_0}$ near the unit circle, that is, a is close to 1. The magnitude response of the digital resonator is as shown in Fig. 2.27a. From Fig. 2.27a we can observe that for a digital resonator, the resonant peak occurs in the neighbourhood of ω_0 . The magnitude response of a good resonator should drop sharply to zero as ω moves away from ω_0 . The system function of digital resonator can have zeros at origin or at $z = 1$ and $z = -1$.

When the zeros are at origin the system function is given by

$$\begin{aligned} H(z) &= \frac{b_0}{(1 - ae^{j\omega_0} z^{-1})(1 - ae^{-j\omega_0} z^{-1})} \\ &= \frac{b_0}{1 - (2a \cos \omega_0)z^{-1} + a^2 z^{-2}} \end{aligned} \quad (2.138)$$

The magnitude response of the system function is shown in Fig. 2.27a. From the plot we can observe that the magnitude response will not fall sharply as ω approaches to



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b_o can be c

Example
of an LTI

Solution

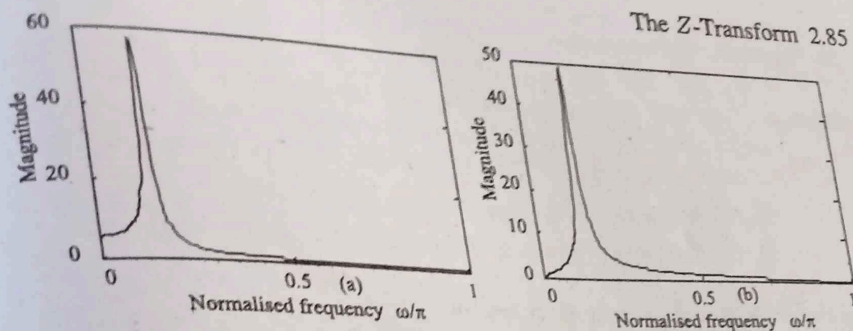


Fig. 2.27

zero. But the magnitude response falls sharply as ω approaches to π . Similarly if the two zeros are complex the magnitude response has sharp fall for $\omega > \omega_0$ but of $\omega < \omega_0$ the magnitude response does not have a sharp fall. If one zero is located at $z = 1$ and other at $z = -1$, the magnitude response is zero at frequencies $\omega = 0$ and π . This type of resonator is a better one than the previous one and is useful for many practical applications including simple bandpass filtering and speech generation.

Thus the transfer function of a good digital resonator is

$$H(z) = b_0 \frac{(1 + z^{-1})(1 - z^{-1})}{(1 - ae^{j\omega_0}z^{-1})(1 - ae^{-j\omega_0}z^{-1})} \quad (2.139)$$

b_0 can be chosen so that the peak magnitude is equal to one.

Example 2.53 Determine a closed-form expression for the frequency response $H(e^{j\omega})$ of an LTI-DT system with impulse response

$$h(n) = \begin{cases} a^n & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases} \quad |a| < 1$$

Solution Given

$$\begin{aligned} h(n) &= \begin{cases} a^n & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases} \\ H(e^{j\omega}) &= \sum_{n=0}^{M-1} a^n e^{-j\omega n} = \sum_{n=0}^{M-1} (ae^{-j\omega})^n \\ &= \frac{1 - a^M e^{-j\omega M}}{1 - ae^{-j\omega}} \end{aligned}$$

Example 2.54 An IIR LTI DT system is described by the difference equation $y(n) + a_1y(n-1) + a_2y(n-2) + a_3y(n-3) = b_0x(n) + b_1x(n-1) + b_2x(n-2) + b_3x(n-3)$ where $x(n)$ and $y(n)$ are input and output respectively. Find the frequency response. For what values of b_i will the magnitude response is constant for all values of ω ?

2.86 Digital Signal Processing

Solution

Given:

$$\begin{aligned} y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 y(n-3) \\ = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) \end{aligned}$$

Taking Fourier transform on both sides we get

$$\begin{aligned} Y(e^{j\omega}) + a_1 e^{-j\omega} Y(e^{j\omega}) + a_2 e^{-j2\omega} Y(e^{j\omega}) + a_3 e^{-j3\omega} Y(e^{j\omega}) \\ = b_0 X(e^{j\omega}) + b_1 e^{-j\omega} X(e^{j\omega}) + b_2 e^{-j2\omega} X(e^{j\omega}) + b_3 e^{-j3\omega} X(e^{j\omega}) \\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + b_3 e^{-j3\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega} + a_3 e^{-j3\omega}} \end{aligned}$$

The magnitude response is constant for all frequencies if

$$\begin{aligned} b_0 &= a_3 \\ b_1 &= a_2 \\ b_2 &= a_1 \text{ and } b_3 = 1 \end{aligned}$$

Example 2.55 Determine the frequency response $H(e^{j\omega})$ of the LTIDT system with impulse response $h(n) = \delta(n) - a\delta(n-k)$ where $|a| < 1$. Find the maximum and minimum value of the magnitude response. How many peaks and dips of the magnitude response occur in the range $0 \leq \omega < 2\pi$? Find the locations of the peak and the dips.

Solution

Given: $h(n) = \delta(n) - a\delta(n-k)$

Taking Fourier transform on both sides

$$\begin{aligned} H(e^{j\omega}) &= 1 - ae^{-j\omega k} \\ &= 1 - a(\cos \omega k - j \sin \omega k) \\ &= (1 - a \cos \omega k + ja \sin \omega k) \\ |H(e^{j\omega})|^2 &= (1 - a \cos \omega k)^2 + a^2 \sin^2 \omega k \\ &= 1 + a^2 - 2a \cos \omega k \end{aligned}$$

For $a = 0.5$

$$\begin{aligned} |H(e^{j\omega})|^2 &= 1.25 - \cos \omega k \\ |H(e^{j\omega})| &= \sqrt{1.25 - \cos \omega k} \end{aligned}$$

when $\omega k = (2\pi$

and

There are k

Questions

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when $\omega k = (2n + 1)\pi$

The Z-Transform 2.87

$$\text{or } \omega = \frac{(2n + 1)\pi}{k} \quad n = 0, 1, 2, 3 \dots k - 1$$

$$|H(e^{j\omega})| = 1.5$$

and

$$\text{for } \omega = \frac{2n\pi}{k} \quad n = 0, 1, 2, 3 \dots k - 1$$

$$|H(e^{j\omega})| = 0.5$$

There are k peaks and k dips.

Questions and Answers

1. Define the z-transform.

(EIE AU' 03)

The z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

where z is a complex variable. In polar form z can be expressed as $z = re^{j\omega}$, where r is the radius of the circle.

2. What is meant by Region of convergence?

(EIE AU' 03)

The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

3. ROC of a causal signal is the _____ of a circle of same radius r .

Exterior

4. ROC of an anticausal signal is the interior of a circle of same radius r .
True/False

True

5. Explain the linearity property of the z-transform.

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$ then

$$Z\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z)$$

6. Explain the time shifting property of the z-transform.

If $Z\{x(n)\} = X(z)$, then $Z\{x(n - k)\} = z^{-k}X(z)$

7. Explain the scaling property of the z-transform.

If $Z\{x(n)\} = X(z)$ ROC: $r_1 < |z| < r_2$, then $Z\{a^n x(n)\} = X(a^{-1}z)$
ROC: $|a| r_1 < |z| < |a| r_2$

8. Explain the time reversal property of the z-transform.

If $Z\{x(n)\} = X(z)$ ROC: $r_1 < |z| < r_2$ then

$$Z\{x(-n)\} = X(z^{-1}) \quad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

9. Explain the convolution property of the z-transform.

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$, then

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z)$$

10. Explain the multiplication property of the z-transform.

If $Z\{x_1(n)\} = X_1(z)$ and $Z\{x_2(n)\} = X_2(z)$ then

$$Z\{x_1(n)x_2(n)\} = \frac{1}{2\pi j} \oint_C X_1(v)X_2\left(\frac{z}{v}\right) v^{-1} dv$$

11. State Parseval's relation in z-transform.

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

12. State the initial value theorem and the final value theorem.

Initial value theorem: If $x(n)$ is causal, then

$$x(0) = \lim_{z \rightarrow \infty} X(z)$$

Final value theorem: If $x(n)$ is causal, $Z[x(n)] = X(z)$, where the ROC for $X(z)$ includes, but is not necessarily confined to $|z| > 1$ and $(z-1)X(z)$ has no poles on or outside the unit circle, then

$$x(\infty) = \lim_{z \rightarrow 1} (z-1)X(z)$$

13. An LTI system with the system function $H(z)$ is BIBO stable if and only if the ROC for $H(z)$ contains the unit circle. True/False

True

14. What are the properties of Region of convergence?

(AU EEE'07)

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. The ROC of an LTI stable system contains the unit circle.

4. The ROC must be a connected region.

The Z-Transform 2.89

15. Determine z-transform and ROC of the finite-duration signal.

$$x(n) = \{2, 4, 5, 7, 0, 1\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\text{We have } x(0) = 5; x(-1) = 4; x(-2) = 2; x(1) = 7$$

$$x(2) = 0; x(3) = 1$$

substituting the sequence values we get

$$X(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$$

ROC: entire z-plane except $z = 0$ and $z = \infty$

16. Find the z-transform and ROC of the signal

$$x(n) = -b^n u(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} -b^n u(-n-1)z^{-n}$$

$$= - \sum_{n=-\infty}^{\infty} b^n z^{-n} \quad \boxed{\because u(-n-1) = 0 \text{ for } n > 1}$$

$$= - \sum_{n=1}^{\infty} (b^{-1}z)^n$$

The series converges if $|b^{-1}z| < 1$ i.e., $|z| < b$

$$= - \frac{b^{-1}}{1 - b^{-1}z} = \frac{-z}{b - z} = \frac{z}{z - b} \quad \text{ROC: } |z| < b$$

17. Find the z-transform of (a) A digital impulse (b) a digital step.

(a) Since $x(n)$ is zero except for $n = 0$, where $x(n)$ is 1, we find $X(z) = 1$

(b) Since $x(n)$ is zero except for $n \geq 0$, where $x(n)$ is 1, we find

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

18. The z-transform of a sequence $x(n)$ is $X(z)$, what is the z-transform of

$nx(n)$?

$$\text{If } Z\{x(n)\} = X(z)$$

$$Z\{nx(n)\} = -z \frac{d}{dz} X(z)$$

(AU EEE'07)

the origin.

circle.

19. What is the relationship between z -transform and DTFT? (MU Oct' 96)

The z -transform of $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (1)$$

where $z = re^{j\omega}$

Substituting Eq. (2) in Eq. (1) we get

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\omega n} \quad (3)$$

The Fourier transform of $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad (4)$$

Eq. (3) and Eq. (4) are identical, when $r = 1$. In the z -plane this corresponds to the locus of points on the unit circle $|z| = 1$. Hence $X(e^{j\omega})$ is equal to $X(z)$ evaluated along the unit circle, or

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

For $X(e^{j\omega})$ to exist, the ROC of $X(z)$ must include the unit circle.

20. What are the different methods of evaluating inverse z -transform?

The inverse z -transform can be evaluated using several methods

1. Long division Method
2. Partial fraction expansion Method
3. Residue Method
4. Convolution Method

21. Express the z -transform of $y(n) = \sum_{k=-\infty}^n x(k)$ in terms of $X(z)$.

$$y(n) = \sum_{k=-\infty}^n x(k) = \sum_{k=-\infty}^{n-1} x(k) + x(n)$$

$$y(n-1) = \sum_{k=-\infty}^{n-1} x(k)$$

$$y(n) = y(n-1) + x(n)$$

Taking z -transform on both sides

$$Y(z) = z^{-1}Y(z) + X(z)$$

$$Y(z) = \frac{X(z)}{1 - z^{-1}}$$

22. Find the convolution of the following using z-transform
 $x(n) = \{ \underset{\uparrow}{1}, 2, 1 \}$; $h(n) = \{ \underset{\uparrow}{1}, 1, 1 \}$ We know

$$Z\{x(n) * h(n)\} = X(z)H(z)$$

$$X(z) = (1 + 2z^{-1} + z^{-2})$$

$$H(z) = (1 + z^{-1} + z^{-2})$$

$$X(z)H(z) = 1 + 3z^{-1} + 4z^{-2} + 3z^{-3} + z^{-4}$$

$$x(n) * h(n) = \{ \underset{\uparrow}{1}, 3, 4, 3, 1 \}$$

23. Define system function.

Let $x(n)$ and $y(n)$ is the input and output sequences of an LTI system with impulse response $h(n)$. Then the system function of the LTI system is defined as the ratio of $Y(z)$ and $X(z)$, i.e.,

$$H(z) = \frac{Y(z)}{X(z)}$$

where $Y(z)$ is the z-transform of the output signal $y(n)$ and $X(z)$ is the z-transform of the input signal $x(n)$.

24. An LTI system having system function $H(z)$ is stable if and only if all the poles of $H(z)$ are inside the unit circle. True/False.

True.

25. Determine the Z transform of the signal $x(n) = \{1, 2, 5, 7, 0, 1\}$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

26. Draw the direct form I structure for the second order system function

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

