5.2: Moment generating functions

Moment generating function (M.G.F):

The moment generating function (mgf) of a random variable X is denoted by $M_X(t)$ & it is defined as

$$M_X(t) = E(e^{tX})$$

$$\therefore M_X(t) = E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots + \infty$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots \text{to} \quad \text{---} (1)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \quad \text{which gives m.g.f in terms of moments.}$$

Here coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is $\mu_r' = E(X^r)$ where $r = 1, 2, \cdots$ which is moment about origin.

→ The m.g.f of X about mean $\mu = \mu_1' = E(X)$ is defined as

$$M_{X-\mu}(t) = E[e^{t(X-\mu)}] = E\left[1 + \frac{t(X-\mu)}{1!} + \frac{t^2}{2!}(X-\mu)^2 + \cdots\right]$$

$$= 1 + \frac{t}{1!}E(X-\mu) + \frac{t^2}{2!}E(X-\mu)^2 + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \cdots$$

where $E(X-\mu)^r = \mu_r$ is known as r^{th} central moment for $r = 1, 2, \cdots$

Note: $\mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$

∵ $M_X(t)$ generates moments, it is called m.g.f

→ If X is a drv with p.m.f $P(x)$ then

$$M_X(t) = E(e^{tx}) = \sum e^{tx} P(x)$$

→ If X is a c.r.v with pdf $f(x)$ then

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Moments Using Moment Generating Function

Differentiation eqn (1) w.r.t t & then putting $t=0$,

$$\left[\frac{d}{dt}[M_X(t)]\right] = 0 + \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots$$

$$\Rightarrow \boxed{\mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0}}$$

$$\frac{d^2}{dt^2}M_X(t) = \mu_2' + \frac{6t}{3!}\mu_3' + \cdots$$

$$\boxed{\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \mu_2'}$$

$$\left(\frac{d^r}{dt^r}M_X(t)\right)_{t=0} = \mu_r', \quad r = 1, 2, \cdots$$

→ Mgf $M_X(t)$ is used to calculate the higher moments.

Theorems on MGF:

Theorem: $M_{aX}(t) = M_X(at)$, where a is a constant

Proof: By def, $M_{aX}(t) = E(e^{taX}) = E(e^{atX}) = M_X(at)$

Theorem: The MGF of the sum of n independent r.v's is equal to the product of their respective mgf, i.e.,

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t)$$

Proof:

$$M_{X_1+X_2+\cdots+X_n}(t) = E\left(e^{t(X_1+X_2+\cdots+X_n)}\right)$$

$$= E[e^{tX_1}\,e^{tX_2}\cdots e^{tX_n}]$$

$$= E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_n})$$

$[\because X_1, X_2 \cdots X_n$ are Independent$]$

$$= M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t)$$

Uniqueness theorem of MGF: The m.g.f of a distribution, if exists, uniquely determines the distribution. This implies that corresponding to a given prob. distribution, there is only one m.g.f (provided it exists) & corresponding to a given m.g.f, there is only one prob. distribution Hence $M_X(t) = M_Y(t) \Rightarrow X$ & Y are identically distributed.

Effect of change of origin & Scale on MGF:

Let a r.v X be transformed to a new variable U by changing both the origin & scale in X as $U = \frac{X-a}{h}$, where a & h are constants.

The m.g.f of U (about origin) is given by

$$M_U(t) = E(e^{tU}) = E\left[e^{t\left(\frac{X-a}{h}\right)}\right] = E\left[e^{\left(\frac{tX}{h} - \frac{ta}{h}\right)}\right]$$

$$= e^{-at/h}\, E\left(e^{\frac{tX}{h}}\right)$$

$$= e^{-at/h}\, M_X\left(\frac{t}{h}\right)$$

Note: If $Y = aX+b$ then $M_Y(t) = E[e^{tY}]$

$$= E\left[e^{t(ax+b)}\right] = E\left[e^{tax}.e^{bt}\right]$$

$$= e^{bt}\, E\left[e^{tax}\right]$$

$$= e^{bt}\, M_X(at)$$

Example: i) If X represents the outcome when a fair die is tossed, find the m.g.f of X & hence, find $E(X)$ & $V(X)$.

Sol: X = outcomes on a fair die tossing

$X = 1, 2, 3, 4, 5, 6$

$$P(x) = P(X=x) = \frac{1}{6},\ x = 1, 2 \cdots 6$$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{6} e^{tx} P(x)$$

$$= \frac{1}{6}\left[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right]$$

$$E(X) = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1+2+3+4+5+6] = \frac{21}{6} = \frac{7}{2}$$

Mean $= E(X) = 7/2$

$$E(X^2) = \mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1+4+9+16+25+36]$$

$= 91/6$

$V(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$

2) find the m.g.f of the r.v X whose probability function $P(X=x) = \frac{1}{2^x}$, $x = 1, 2 \ldots$, & hence find its mean.

Sol: $M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2^x}\right)$

$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots$

$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$

$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$ $\quad (1-x)^{-1} = 1 + x + x^2 + \ldots$

$= \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1} = \frac{e^t}{2} \cdot \frac{2}{2-e^t} = \frac{e^t}{2-e^t}$

$E(X) = \mu_1' = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$

$= \frac{(2-1)(1) - 1(-1)}{(2-1)^2} = 2$

3) let X be a r.v with pdf $f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x > 0 \\ 0, & \text{o.w} \end{cases}$

find (i) $P(X>3)$ (ii) Mgf of X (iii) $E(X)$ & $V(X)$

Sol: (i) $P(X>3) = \int_3^{\infty} f(x)dx = \int_3^{\infty} \frac{1}{3}e^{-x/3} dx = \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty}$

$= -[0 - e^{-1}] = \frac{1}{e}$

(ii) $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^{\infty} e^{tx} \cdot \frac{1}{3}e^{-x/3} dx$

$= \frac{1}{3}\int_0^{\infty} e^{-(\frac{1}{3}-t)x} dx = \frac{1}{3}\left[\frac{e^{-(\frac{1}{3}-t)x}}{-(\frac{1}{3}-t)}\right]_0^{\infty}$

$= \frac{1}{3}\left[\frac{0-1}{-(\frac{1}{3}-t)}\right] = \frac{1}{1-3t} = (1-3t)^{-1}$

(iii) $E(X) = \mu_1' = \left(\frac{d}{dt}[M_X(t)]\right)_{t=0} = \left[-(1-3t)^{-2}(-3)\right]_{t=0}$

$= 3$

$E(X^2) = \mu_2' = \left(\frac{d^2}{dt^2}(M_X(t))\right)_{t=0} = \left[3(-2)(1-3t)^{-3}(-3)\right]_{t=0}$

$= 18$

$V(X) = E(X^2) - (E(X))^2 = 18 - 9 = 9$

4) If $X \sim B(n,p)$, find the m.g.f of X & hence find its mean & variance.

Sol: Since $X \sim B(n,p)$, its P.m.f is given by

$P(x) = P(X=x) = {}^nC_x p^x q^{n-x}$, $x = 0,1,\ldots n$, $q = 1-p$

Then mgf of X is given by

$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} P(x) = \sum_{x=0}^{n} e^{tx} {}^nC_x p^x q^{n-x}$

$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$

$= (q + pe^t)^n$

$(q+p)^n = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$

Mean, $E(X) = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[n(pe^t+q)^{n-1} pe^t\right]_{t=0}$

$= n(p+q)^{n-1} p$

$= np$

$E(X^2) = \mu_2' = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left(np\left[(pe^t+q)^{n-1} e^t + (n-1)(pe^t+q)^{n-2} pe^t\right]\right)_{t=0}$

$= np[1 + (n-1)p]$

$= np[np - p + 1]$

$= n^2p^2 + npq$

$V(X) = E(X^2) - (E(X))^2 = n^2p^2 + npq - n^2p^2 = npq$

5. If $X \sim P(\lambda)$, find its m.g.f & hence find its mean & variance

6. If $X \sim NB(r,p)$, " " " "

5. Sol: $X \sim P(\lambda)$ then Pmf, $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,\ldots$ & $\lambda > 0$

mgf of X, $M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t-1)}$$

$$\frac{d}{dt}(M_X(t)) = e^{\lambda(e^t-1)} \lambda e^t$$

Mean, $\mu_0' = \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = e^{\lambda(1-1)}\lambda(1) = \lambda$

$$E(X^2) = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left(\lambda\left[e^{\lambda(e^t-1)}e^t + e^t e^{\lambda(e^t-1)}\lambda e^t\right]\right)_{t=0}$$

$$= \lambda(1+\lambda) = \lambda + \lambda^2$$

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

6. Sol: $X \sim NB(r,p)$, its pmf, $P(x) = {}^{-r}C_x\, p^r (-q)^x$, $x = 0,1,\ldots$

mgf of X, $M_X(t) = \sum_{x=0}^{\infty} e^{tx}\, {}^{-r}C_x\, p^r (-q)^x$

$$= \sum_{x=0}^{\infty} (qe^t)^x p^r\, {}^{-r}C_x = p^r \sum_{x=0}^{\infty} {}^{-r}C_x (-qe^t)^x$$

$$= p^r (1-qe^t)^{-r} \quad \left[(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r x^r\right]$$

$$\frac{d}{dt}(M_X(t)) = p^r(1-qe^t)^{-r-1}(-r)(-qe^t)$$

$$= qrp^r(1-qe^t)^{-(r+1)} e^t$$

Mean, $\mu = \left(\frac{d}{dt}(M_X(t))\right)_{t=0} = qrp^r p^{-r-1} = rq/p$

$$\frac{d^2}{dt}(M_X(t)) = qrp^r\left[(1-qe^t)^{-r-2}(-r)(-qe^t)e^t + e^t(1-qe^t)^{-r-1}\right]$$

$$(M_X(t))''_{t=0} = qrp^r\left[p^{-r-2}(-r-1)(-q) + p^{-r-1}\right]$$

$= r(r+1)q^2p^{-2} + rqp^{-1} = \frac{rq}{p}\left[\frac{(r+1)q}{p} + 1\right] = \frac{rq(rq+q+p)}{p^2}$

$= \frac{rq(rq+1)}{p^2}$

$V(X) = \frac{r^2q^2}{p^2} + \frac{rq}{p^2} - \frac{r^2q^2}{p^2} = \frac{rq}{p^2}$

7) If the moments of a r.v X are defined by $E(X^r) = 0.6$, $r = 1, 2, \ldots$ S.T $P(X=0) = 0.4$, $P(X=1) = 0.6$ & $P(X \geq 2) = 0$

Sol: WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$ where $\mu_r' = E(X^r) = 0.6$

$M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}\mu_r' = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}(0.6)$

$= 1 + 0.6\left(\frac{t}{1!} + \frac{t^2}{2!} + \ldots\right)$

$= 1 + 0.6(e^t - 1) = 0.4 + 0.6e^t$ — (1)

But by def, $M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X=x)$

$= P(X=0) + e^t P(X=1) + e^{2t} P(X=2) + e^{3t} P(X=3) + \ldots$ — (2)

Compare (1) & (2)

$P(X=0) = 0.4$, $P(X=1) = 0.6$, $P(X=2) = P(X=3) = \ldots = 0$

$P(X \geq 2) = 0$

8) Find the m.g.f a r.v whose moments are $\mu_r = (r+1)!\, 2^r$

Sol: $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\, 2^r = \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)r!\, 2^r$

$= \sum_{r=0}^{\infty} (2t)^r (r+1) = 1 + 2(2t) + 3(2t)^2 + \ldots$

$= (1-2t)^{-2} = \frac{1}{(1-2t)^2}$ $\quad (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

3.3 Characteristic function

Characteristic function:

The characteristic function of a r.v. X is defined by

$$\phi_X(t) = E[e^{itX}] = \begin{cases} \int e^{itx} f(x)\,dx, & \text{if } X \text{ is a c.r.v with pdf } f(x) \\ \sum_x e^{itx} P(x), & \text{if } X \text{ is a d.r.v with pmf } P(x) \end{cases}$$

where $i = \sqrt{-1}$, the imaginary number.

Note: 1. $|\phi_X(t)| = |E(e^{itx})| \le E(|e^{itx}|) = E|\cos tx + i \sin tx|$

$$= E(\sqrt{\cos^2 tx + \sin^2 tx}) = E(1) = 1$$

$\therefore |\phi_{(X)}(t)| \le 1$

$\phi_X(t)$ always exists for any probn. distribution.

2. $\phi_X(t) = E[e^{itX}] = E\left[1 + (it)X + \frac{(it)^2 X^2}{2!} + \frac{(it)^3 X^3}{3!} + \cdots\right]$

$$= 1 + it\,E(X) + \frac{(it)^2}{2!}E(X^2) + \frac{(it)^3}{3!}E(X^3) + \cdots$$

$$= 1 + (it)\mu_1' + \frac{(it)^2}{2!}\mu_2' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mu_r'$$ where $\mu_r' = E(X^r) = r^{th}$ moment about origin for $r = 1, 2 \ldots$

3. If $\phi_X(t)$ is given, then r^{th} moment about origin is

μ_r' = coefficient of $\frac{(it)^r}{r!}$ in $\phi_X(t)$

Properties:

1) $\phi_X(0) = 1$

$[\phi_X(t) = E[e^{itX}] \Rightarrow \phi_X(0) = E[1] = 1]$

2) $|\phi_X(t)| \le 1$ for all real t.

3) $\phi_X(t)$ Continuous function of t in $(-\infty, \infty)$

[For $h \ne 0$, $|\phi_X(t+h) - \phi_X(t)| = |E(e^{i(t+h)X} - e^{itX})| = |E(e^{itX}(e^{ihX} - 1))| \le E(|e^{itX}||e^{ihX} - 1|) = E|e^{ihX} - 1| \to 0$ as $h \to 0$

$\lim_{h \to 0} |\phi_X(t+h) - \phi_X(t)| = 0 \Rightarrow \lim_{h \to 0} \phi_X(t+h) = \phi_X(t)$

$\therefore \phi_x(t)$ is continuous function of t in $(-\infty, \infty)$

4) $\phi_x(-t) = \overline{\phi_x(t)}$ i.e., $\phi_x(-t)$ is complex conjugate of $\phi_x(t)$

$$\overline{\phi_x(t)} = \overline{E[e^{itx}]} = \overline{E[\cos tx + i\sin tx]} = E[\cos tx - i\sin tx]$$

$$\phi_x(-t) = E[\cos(-tx) + i\sin(-tx)] = E[\cos tx - i\sin tx]$$

$$= \overline{\phi_x(t)}$$

5) If the pdf is even i.e., $f(-x) = f(x)$, then CF is real valued & even function of t.

$$\phi_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

Let $x = -y \Rightarrow dx = -dy$

$$\phi_x(t) = \int_{\infty}^{-\infty} e^{-ity} f(-y)(-dy) = \int_{-\infty}^{\infty} e^{-ity} f(y)\,dy$$

$$= E[e^{-itx}] = \phi_x(-t)$$

$$\overline{\phi_x(t)} = \phi_x(-t) = \phi_x(t)$$

Thus $\phi_x(t)$ is real

6) If x is a r.v with CF $\phi_x(t)$ & $\mu_r' = E(x^r)$ exists, then

$$\mu_r' = (-i)^r \frac{d^r}{dt^r}(\phi_x(t))\Big|_{t=0}$$

$$\frac{d^r}{dt^r}(\phi_x(t)) = \frac{d^r}{dt^r}[E(e^{itx})] = i^r E[x^r e^{itx}]$$

$$\frac{d^r}{dt^r}(\phi_x(t))\Big|_{t=0} = i^r E(x^r) \Rightarrow E(x^r) = \frac{1}{i^r}\frac{d^r}{dt^r}(\phi_x(t))\Big|_{t=0}$$

$(-i)^r = (i)^{-1}$

$$\mu_r' = (-i)^r \frac{d^r}{dt^r}\phi_x(t)\Big|_{t=0}$$

7) Effect of change of origin & scale

$U = \frac{x-a}{h}$ where a & h are constant

$$\phi_U(t) = E(e^{itU}) = E\left(e^{it\left(\frac{x-a}{h}\right)}\right) = e^{\frac{-ita}{h}} E\left(e^{\frac{itx}{h}}\right)$$

$$= e^{\frac{-ita}{h}} \phi_x\left(\frac{t}{h}\right)$$

8) If $X_1, X_2, \ldots X_n$ are independent, then

$$\phi_{X_1+X_2+\ldots X_n}(t) = \phi_{X_1}(t)\,\phi_{X_2}(t)\ldots\phi_{X_n}(t)$$

$$\phi_{X_1+X_2+\ldots+X_n}(t) = E\left[e^{it(X_1+X_2+\ldots X_n)}\right]$$

$$= E\left[e^{itX_1}\cdot e^{itX_2}\ldots e^{itX_n}\right]$$

$$= E\left[e^{itX_1}\right].E\left[e^{itX_2}\right]\ldots E\left[e^{itX_n}\right]$$

$$= \phi_{X_1}(t)\cdot\phi_{X_2}(t)\ldots\phi_{X_n}(t)$$

Uniqueness theorem for Characteristic functions:

The c.f uniquely determines the distribution.

i.e., A necessary & sufficient condition for two distributions with pdfs $f_1(\cdot)$ & $f_2(\cdot)$ to be identical is that their c.f's $\phi_1(t)$ & $\phi_2(t)$ are identical.

Ex: 1) If $X \sim B(n,p)$, find its c.f & hence obtain its μ & σ^2

Sol: $\because X \sim B(n,p)$ then pmf is $P(x) = {}^nC_x\, p^x q^{n-x}$, $x = 0,1,\ldots n$

The CF of X is

$$\phi_X(t) = E\left[e^{itX}\right] = \sum_{x=0}^{n} e^{itx} P(x)$$

$$= \sum_{x=0}^{n} e^{itx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^{it})^x q^{n-x}$$

$$\phi_X(t) = (q+pe^{it})^n$$

$$\frac{d}{dt}\phi_X(t) = n(q+pe^{it})^{n-1} pe^{it}\, i = npi\, e^{it}(q+pe^{it})^{n-1}$$

$$\frac{d^2}{dt^2}\phi_X(t) = npi\left[e^{it}(n-1)(q+pe^{it})^{n-2} pe^{it}\, i + (q+pe^{it})^{n-1} e^{it}\, i\right]$$

$$\text{Mean} = \mu_1' = (-i)^1 \frac{d}{dt}\phi_X(t)\Big|_{t=0} = (-i)\, npi\,(1)(q+p)^{n-1} = np$$

$$\mu_2' = E(X^2) = (-i)^2 \frac{d^2}{dt^2} \phi_X(t)\Big|_{t=0}$$

$$= (-i)^2 npi\left[ip(n-1)(q+p)^{n-2} + i(q+p)^{n-1}\right]$$

$$= np\left[p(n-1)+1\right] = np[np-p+1]$$

$$= n^2p^2 + npq$$

$$\sigma^2 = E(x^2) - (E(x))^2 = n^2p^2 + npq - n^2p^2 = npq$$

2) If $X \sim P(\lambda)$, find the c.f of X & hence obtain its μ & σ^2.

Sol: Since $X \sim P(\lambda)$, its p.m.f is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \cdots$$

The c.f of X is

$$\phi_X(t) = E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} P(x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$$\frac{d}{dt}[\phi_X(t)] = e^{\lambda(e^{it}-1)} \cdot \lambda e^{it} \cdot i$$

$$\frac{d^2}{dt^2}[\phi_X(t)] = \lambda i\left[e^{\lambda(e^{it}-1)} \cdot e^{it} \cdot i + e^{it} e^{\lambda(e^{it}-1)} \cdot \lambda e^{it} \cdot i\right]$$

Mean, $\mu = \mu_1' = E(X) = (-i)\frac{d}{dt}[\phi_X(t)]\Big|_{t=0}$

$$= (-i)(1)\lambda \cdot 1 \cdot i = \lambda$$

$$E(X^2) = (-i)^2 \frac{d^2}{dt^2} \phi_X(t)\Big|_{t=0}$$

$$= (-i)^2 \lambda i^2 [1 \cdot 1 + \lambda] = \lambda[\lambda+1] = \lambda^2 + \lambda$$

$$\sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

3) If $X \sim N(\mu, \sigma^2)$, find c.f of X & its μ & σ^2

Sol: $\because X \sim N(\mu, \sigma^2)$, its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x, \mu < \infty, \ \sigma > 0$$

The C.F of X is given by

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it(\mu+\sigma z)} e^{\frac{-z^2}{2}} \sigma dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}[z^2 - 2it\sigma z]} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}[z^2 - 2it\sigma z + i^2t^2\sigma^2 - t^2i^2\sigma^2]} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} e^{\frac{+1}{2}t^2i^2\sigma^2} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-it\sigma)^2} dz$$

Let $z - it\sigma = u \Rightarrow dz = du$

$$= e^{it\mu - \frac{\sigma^2t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}} du$$

$$= e^{it\mu - \frac{\sigma^2t^2}{2}} \cdot \frac{1}{\sqrt{2}\sqrt{\pi}} 2\int_{0}^{\infty} e^{-u^2/2} du$$

$$\phi_X(t) = e^{it\mu - \frac{\sigma^2t^2}{2}} \frac{1}{\sqrt{2}\sqrt{\pi}} 2' \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$\frac{d}{dt}(\phi_X(t)) = e^{it\mu - \frac{\sigma^2t^2}{2}} (i\mu - \sigma^2 t)$$

$$\frac{d^2}{dt^2}(\phi_X(t)) = e^{it\mu - \frac{\sigma^2t^2}{2}} (-\sigma^2) + (i\mu - \sigma^2 t) e^{it\mu - \frac{\sigma^2t^2}{2}} (i\mu - \sigma^2 t)$$

$$\text{Mean} = E(x) = (-i)\frac{d}{dt}(\phi_X(t))\Big|_{t=0} = (-i)1\,(i\mu) = \mu$$

$$E(x^2) = (-i)^2 \frac{d^2}{dt^2}\phi_X(t)\Big|_{t=0} = (-i)^2[-\sigma^2 + i^2\mu^2] = \sigma^2 + \mu^2$$

$V(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

4) find the CF of a r.v X whose moments are given by $\mu_r' = (r+1)!\, 2^r$.

Sol: $CF, \phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r' = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} (r+1)!\, 2^r$

$= 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \mu_r' = \sum_{r=0}^{\infty} (2it)^r (r+1)$

$= 1 + 2(2it) + 3(2it)^2 + \cdots$

$= (1-2it)^{-2} = \frac{1}{(1-2it)^2}$

5) If $\phi_X(t) = \frac{3}{3-it}$, then find S.D of X.

Sol: $\phi_X(t) = \frac{3}{3-it}$

$\frac{d}{dt}(\phi_X(t)) = 3 \frac{-1}{(3-it)^2} \cdot (-i) = \frac{3i}{(3-it)^2}$

$\frac{d^2}{dt^2}(\phi_X(t)) = 3i \frac{-2}{(3-it)^3} (-i) = \frac{-6}{(3-it)^3}$

$\mu = (-i) \frac{d}{dt} \phi_X(t)\Big|_{t=0} = (-i) \frac{3i}{3^2} = \frac{1}{3}$

$E(X^2) = (-i)^2 \frac{d^2}{dt^2} \phi_X(t)\Big|_{t=0} = -1\left(\frac{-6}{27}\right) = \frac{2}{9}$

$\sigma^2 = \frac{2}{9} - \frac{1}{9} = 1/9$, $\boxed{\sigma = \frac{1}{3}}$

6) find CF of a r.v X whose pdf is

$$f(x) = \begin{cases} x/2 & , 0 \le x \le 2 \\ 0, & 0.w \end{cases}$$

Sol: $CF, \phi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$

$= \int_0^2 e^{itx} \frac{x}{2}\, dx$

$= \frac{1}{2}\left[\left(x \frac{e^{itx}}{it}\right)_0^2 - \int_0^2 \frac{e^{itx}}{it}\, dx\right]$

$$= \frac{1}{2}\left[2\frac{e^{2it}}{it} - \frac{1}{it}\left[\frac{e^{itx}}{it}\right]_0^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{2ie^{2it}}{-t} - \frac{1}{i^2t^2}(e^{2it}-1)\right]$$

$$= \frac{1}{2t^2}\left[-2ite^{2it} + e^{2it} - 1\right]$$

7) Find CF of a r.v x whose pdf is $f(x) = \begin{cases} 2e^{-2x}, x \geq 0 \\ 0, 0.w \end{cases}$ & Find μ & σ^2

Sol:

$$\phi_x(t) = E[e^{itx}] = \int_0^{\infty} e^{itx} 2e^{-2x} dx$$

$$= 2\int_0^{\infty} e^{-(it+2)x} dx$$

$$= 2\left[\frac{e^{-(it+2)x}}{-(it+2)}\right]_0^{\infty}$$

$$= \frac{2[0-1]}{-(2-it)} = \frac{2}{2-it}$$

$$\frac{d}{dt}(\phi_x(t)) = 2 \cdot \frac{-1}{(2-it)^2} \cdot (-i) = \frac{2i}{(2-it)^2}$$

$$\mu = E(x) = (-i)\frac{d}{dt}\phi_x(t)\Big|_{t=0} = (-i)\frac{2i}{4} = \frac{1}{2}$$

$$\frac{d^2}{dt^2}\phi_x(t) = 2i\frac{-2}{(2-it)^3}(-i) = \frac{-4}{(2-it)^3}$$

$$E(x^2) = (-i)^2\frac{d^2}{dt^2}\phi_x(t)\Big|_{t=0} = \frac{4}{8} = \frac{1}{2}$$

$$\sigma^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Finding pmf (or pdf) when CF is known

1) If x is a drv with CF $\phi_x(t)$, then $\phi_x(t) = \sum_x e^{itx} P(x)$

First write the CF in this form & then identify $P(x)$ which is pmf of drv x.

2) If x is a crv with CF $\phi_x(t)$ then its Pdf is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_x(t)\,dt$$

Ex: 1) find the p.m.f of the d.r.v x whose CF is given by $\phi_x(t) = (q+pe^{it})^n$.

Sol:
$$\begin{aligned}\phi_x(t) &= (q+pe^{it})^n \\ &= \sum_{x=0}^{n} {}^nC_x (pe^{it})^x q^{n-x} \\ &= \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} e^{itx} \\ &= \sum_{x=0}^{n} P(x=x)\, e^{itx} \\ &= E[e^{itx}] \quad \text{where } P(x=x) = {}^nC_x p^x q^{n-x}\end{aligned}$$

$\therefore P(x) = {}^nC_x p^x q^{n-x}$ for $x = 0,1,\ldots n$

2) Find the pmf of a drv x whose CF is given by $\phi_x(t) = e^{\lambda(e^{it}-1)}$

Sol:
$$\begin{aligned}\phi_x(t) = e^{\lambda(e^{it}-1)} &= e^{-\lambda}e^{\lambda e^{it}} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!} \\ &= \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}e^{itx} \\ &= \sum_{x=0}^{\infty} P(x=x)\,e^{itx} = E[e^{itx}]\end{aligned}$$

where $P(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0,1,\ldots$ which P.D with parameter λ;

3) Find the pdf corresponding to the CF $\phi_x(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}$

Sol:
$$\phi_x(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_x(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, e^{it\mu - \frac{t^2\sigma^2}{2}}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[t^2\sigma^2 + 2it(x-\mu)\right]}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[t^2\sigma^2 + 2it\sigma\left(\frac{x-\mu}{\sigma}\right) + \left(\frac{x-\mu}{\sigma}\right)^2 - \left(\frac{x-\mu}{\sigma}\right)^2\right]} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left\{\left[t\sigma + i\left(\frac{x-\mu}{\sigma}\right)\right]^2 + \left(\frac{x-\mu}{\sigma}\right)^2\right\}} dt$$

Let $t\sigma + i\left(\frac{x-\mu}{\sigma}\right) = u \Rightarrow \sigma dt = du$

$\Rightarrow dt = \frac{du}{\sigma}$

$$= e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-1}{2}[u^2]} \frac{du}{\sigma}$$

$$= \frac{1}{2\pi\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} 2\int_0^{\infty} e^{-u^2/2} du$$

$$= \frac{1}{\pi\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$\therefore X \sim N(\mu, \sigma^2)$

b) The pdf corresponding to the CF defined by

$$\phi(t) = \begin{cases} 1-|t|, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

Sol: $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_x(t)dt$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}(1-|t|)\,dt \qquad [\because -1<t<0, |t| = -t \\ 0<t<1, |t| = t]$$

$$= \frac{1}{2\pi}\int_{-1}^{0} e^{-itx}(1+t)\,dt + \frac{1}{2\pi}\int_0^1 e^{-itx}(1-t)dt$$

$$= \frac{1}{2\pi}\left[\left((1+t)\frac{e^{-itx}}{-ix}\right)_{-1}^{0} - \int_{-1}^{0}\frac{e^{-itx}}{-ix}dt + \left((1-t)\frac{e^{-itx}}{-ix}\right)_0^1 - \int_0^1 (-1)\frac{e^{-itx}}{-ix}dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{ix} + \frac{1}{ix}\left(\frac{e^{-itx}}{-ix}\right)_{-1}^{0} + \frac{1}{ix} - \frac{1}{ix}\left(\frac{e^{-itx}}{-ix}\right)_0^1\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{(ix)^2}(1-e^{ix}) + \frac{1}{(ix)^2}(e^{ix}-1)\right]$$

$$= \frac{1}{2\pi(ix)^2}\left[-1+e^{ix}+e^{-ix}-1\right]$$

$$= \frac{-1}{\pi x^2}\left[\frac{-2+e^{ix}+e^{-ix}}{2}\right]$$

$$= \frac{1}{\pi x^2}\left[1-\frac{e^{ix}+e^{-ix}}{2}\right]$$

$$= \frac{1}{\pi x^2}(1-\cos x), \quad -\infty < x < \infty$$

3) Find the pmf whose CF is given by $\phi_x(t) = \frac{e^{it}}{2-e^{it}}$

Sol:

$$\phi_x(t) = \frac{e^{it}}{2\left(1-\frac{1}{2}e^{it}\right)}$$

$$= \frac{\frac{1}{2}e^{it}}{\left(1-\frac{1}{2}e^{it}\right)}$$

$$= \sum_{x=1}^{\infty}\left(\frac{1}{2}e^{it}\right)^x$$

$$= \sum_{x=1}^{\infty}\frac{1}{2^x}e^{itx}$$

$$\therefore P(x=x) = \frac{1}{2^x}, \quad x = 1, 2 \ldots$$

$(1-x)^{-1} = 1+x+x^2+\ldots = \sum_{x=0}^{\infty} x^x$

$\frac{x}{1-x} = x+x^2+x^3+\ldots = \sum_{n=1}^{\infty} x^n$

5.4 Cumulant Generating function

The logarithm of $M_x(t)$ & $\phi_x(t)$ generates a sequence of numbers called the cumulants of x

Cumulant Generating function (CGF)

Let x be a r.v with Ch.F $\phi_x(t) = E[e^{itx}]$. The cumulant greating function (C.g.F) of x is defined by

$$K_x(t) = \ln(\phi_x(t))$$

$$= K_1(it) + K_2\frac{(it)^2}{2!} + \cdots + K_r\frac{(it)^r}{r!} + \cdots$$

K_j = coeff of $\frac{(it)^j}{j!}$ in $K_x(t)$ & it is called j^{th} cumulant of x.

$$\phi_x(t) = 1 + \mu_1'(it) + \mu_2'\frac{(it)^2}{2!} + \cdots + \mu_r'\frac{(it)^r}{r!} + \cdots$$

$$[\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Now, $K_x(t) = \ln(\phi_x(t))$

$$K_1(it) + K_2\frac{(it)^2}{2!} + K_3\frac{(it)^3}{3!} + \cdots = \ln\left[1 + \left(\mu_1'(it) + \mu_2'\frac{(it)^2}{2} + \cdots\right)\right]$$

$$= \left(\mu_1'(it) + \mu_2'\frac{(it)^2}{2} + \cdots\right) - \frac{1}{2}\left(\mu_1'(it) + \mu_2'\frac{(it)^2}{2} + \cdots\right)^2 + \cdots$$

Comparing the coefficients of like powers of t, we get relationship b/w moments & cumulants

$K_1 = \mu_1' = \text{Mean} = \mu$ & $K_2 = \mu_2' - (\mu_1')^2 = \text{variance} = \sigma^2$

$\mu = K_1$ & $\sigma^2 = K_2$

$$\rightarrow K_x(t) = \sum_{j=1}^{\infty} K_j\frac{(it)^j}{j!}$$

$$\rightarrow K_j = (-i)^j\left.\frac{d^j K_x(t)}{dt^j}\right|_{t=0}$$

Properties of Cumulants:

1) $K_n(cx) = c^n K_n(x)$ for some real constant c.

$$\phi_{cx}(t) = E[e^{itcx}] = E[e^{i(tc)x}] = \phi_x(tc)$$

$$\ln\phi_{cx}(t) = \ln\phi_x(tc) \Rightarrow K_{cx}(t) = K_x(ct)$$

$$\frac{d^n}{dt^n} K_{cx}(t)\Big|_{t=0} = \frac{d^n}{dt^n} K_x(ct)\Big|_{t=0} = c^n \frac{d^n}{ds^n}(K_x(s))\Big|_{s=0}$$, where $ct=s$

$$(-i)^n \frac{d^n}{dt^n} K_{cx}(t)\Big|_{t=0} = (-i)^n c^n \frac{d^n}{dt^n} K_x(t)\Big|_{t=0}$$

$$k_n(cx) = c^n k_n(x)$$

Theorem: $k_n(x+b) = \begin{cases} k_n(x)+b, & \text{if } n=1 \\ k_n(x), & \text{if } n>1 \end{cases}$

Pf: $\phi_{x+b}(t) = E[e^{it(x+b)}] = e^{itb} E[e^{itx}] = e^{itb}\phi_x(t)$

$$\ln[\phi_{x+b}(t)] = itb + \ln \phi_x(t)$$

$$K_{x+b}(t) = itb + K_x(t)$$

$$(-i)^n \frac{d^n}{dt^n}(K_{x+b}(t))\Big|_{t=0} = (-i)^n \frac{d^n}{dt^n}(itb)\Big|_{t=0} + (-i)^n \frac{d^n}{dt^n}(K_x(t))\Big|_{t=0}$$

$$k_n(x+b) = (-i)^n \frac{d^n}{dt^n}(itb)\Big|_{t=0} + k_n(x)$$

If $n=1$, then $k_1(x+b) = b + k_1(x)$

If $n>1$, then $k_n(x+b) = k_n(x)$

Theorem: If X & Y are I.r.v's & $S = X+Y$ then

$$k_n(S) = k_n(X) + k_n(Y)$$

Pf: $\because$ X & Y are I rv's

$$\phi_S(t) = \phi_X(t)\,\phi_Y(t)$$

$$\ln(\phi_S(t)) = \ln \phi_X(t) + \ln \phi_Y(t)$$

$$K_S(t) = K_X(t) + K_Y(t)$$

$$(-i)^n \frac{d^n}{dt^n} K_S(t)\Big|_{t=0} = (-i)^n \frac{d^n}{dt^n}(K_X(t))\Big|_{t=0} + (-i)^n \frac{d^n}{dt^n}(K_Y(t))\Big|_{t=0}$$

$$k_n(S) = k_n(X) + k_n(Y)$$

→ If $S = X_1 + X_2 + \cdots + X_m$ where $X_1, X_2, \ldots X_m$ are IRV's then

$$k_n(S) = k_n(X_1) + k_n(X_2) + \cdots + k_n(X_m)$$

Theorem: Let $\mu_j' = E(X^j)$ be the j^{th} moment of X about zero for $j = 1, 2, \ldots n$ where $\mu_0' = 1$. Let $k_1, k_2 \ldots k_n$ be the cumulants of X. Then

$$\mu'_{r+1} = \sum_{j=0}^{r} {}^{r}C_j \, \mu'_j \, k_{(r+1-j)} \quad \text{for } r = 0, 1, \ldots n-1$$

pf: For $j = 0, 1, \ldots n$, we have

$$\mu'_j = (-i)^j \frac{d^j}{dt^j} \phi_x(t)\Big|_{t=0} \quad \& \quad k_j = (-i)^j \frac{d^j}{dt^j} K_x(t)\Big|_{t=0}$$

where $\phi_x(t) = E[e^{itx}]$ & $K_x(t) = \ln[\phi_x(t)]$

$$\phi_x(t) = e^{K_x(t)}$$

$$\phi_x'(t) = e^{K_x(t)} K_x'(t) \quad -(1)$$

At $t=0$ $\quad \phi_x'(0) = e^{K_x(0)} K_x'(0)$

$\Rightarrow i\mu'_1 = ik_1 \Rightarrow \mu'_1 = k_1$ holds for $r = 0$

Diff (1) for r times, gives

$$\phi_x^{(r+1)}(t) = \sum_{j=0}^{r} {}^{r}C_j \, \phi_x^{(j)}(t) K_x^{(r+1-j)}(t)$$

At $t = 0$

$$\mu'_{r+1} = \sum_{j=0}^{r} {}^{r}C_j \, \mu'_j \, k_{r+1-j} \quad \text{for } r = 0, 1, \ldots n-1$$

Note: Taking $r = 0, 1, 2, 3$ in above

$$\mu'_1 = k_1$$

$$\mu'_2 = k_2 + \mu'_1 k_1$$

$$\mu'_3 = k_3 + 2\mu'_1 k_2 + \mu'_2 k_1$$

$$\mu'_4 = k_4 + 3\mu'_1 k_3 + 3\mu'_2 k_2 + \mu'_3 k_1$$

Let $\mu_j = E[x - E(x)]^j = E(x - \mu'_1)^j$ for $j = 1, 2 \ldots$ are known as central moments

$$\mu_2 = E(x - \mu'_1)^2 = k_2$$

$$\mu_3 = E(x - \mu'_1)^3 = E[x^3 - 3x^2\mu'_1 + 3x\mu'^2_1 - \mu'^3_1]$$

$$= E(x^3) - 3\mu'_1 E(x^2) + 3\mu'^2_1 E(x) - \mu'^3_1$$

$$= \mu'_3 - 3\mu'_1\mu'_2 + 3\mu'^2_1\mu'_1 - \mu'^3_1$$

$$= \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1 = k_3$$

$$\mu_4 = k_4 + 3k_2^2$$

$\therefore \mu = k_1, \; \sigma^2 = k_2 = \mu_2$

Examples:

1) If $X \sim B(n,p)$, then obtain c.g.f of x & hence obtain its mean & variance.

Sol: $\because X \sim B(n,p)$ then its pmf is $P(x) = {}^nC_x p^x q^{n-x}$, $x=0,1,\ldots,n$

Then the ch.f of x is given by

$$\phi_x(t) = (q+pe^{it})^n$$

Thus the c.g.f of x is given by

$$K_x(t) = \ln(\phi_x(t)) = \ln(q+pe^{it})^n$$

$$= n\ln(q+pe^{it})$$

$$= n\ln\left[q+p\left(1+\frac{it}{1!}+\frac{(it)^2}{2!}+\cdots\right)\right]$$

$$= n\ln\left[q+p+p\frac{(it)}{1!}+p\frac{(it)^2}{2!}+\cdots\right]$$

$$= n\ln\left[1+\left(\frac{it}{1!}p+\frac{(it)^2}{2!}p+\cdots\right)\right]$$

$$= n\left[\left((it)p+\frac{(it)^2}{2!}p+\cdots\right)-\frac{1}{2}\left((it)p+\frac{(it)^2}{2!}p+\cdots\right)^2+\cdots\right] \quad \left[\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}\right]$$

$$K_x(t) = (it)np+\frac{(it)^2}{2!}(np-np^2)+\cdots$$

$\therefore k_1 =$ coeff of $(it) = np$, $k_2 =$ coeff of $\frac{(it)^2}{2!} = np-np^2$

$$= np(1-p)$$

$$= npq$$

$\therefore \mu = k_1 = np$ & $\sigma^2 = k_2 = npq$

2) If $X \sim P(\lambda)$, then find c.g.f of x & hence obtain its μ & σ^2.

Sol: $\because X \sim P(\lambda)$ then its pmf is $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,\ldots,\lambda>0$

Then ch.F of x is given by

$$\phi_x(t) = E[e^{itx}] = \sum_{x=0}^{\infty} e^{itx}P(x) = \sum_{x=0}^{\infty} e^{itx}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$\phi_x(t) = e^{\lambda(e^{it}-1)}$

The c.g.f of x is given by

$K_x(t) = \ln(\phi_x(t)) = \ln\left[e^{\lambda(e^{it}-1)}\right]$

$= \lambda(e^{it}-1)$

$= \lambda\left[1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots - 1\right]$

$K_x(t) = (it)\lambda + \frac{(it)^2}{2!}\lambda + \cdots$

K_1 = Coeff of (it) in $K_x(t)$

$= \lambda$

K_2 = Coeff of $\frac{(it)^2}{2!}$ in $K_x(t)$

$= \lambda$

$\mu = \sigma^2 = \lambda$

$\left[K_1 = (-i)\frac{d}{dt}(K_x(t))\Big|_{t=0}\right.$

$= (-i)\, i\lambda = \lambda$

$K_2 = (-i)^2\frac{d^2}{dt^2}(K_x(t))\Big|_{t=0}$

$= (-i)^2\left[0 + \frac{2i^2}{2}\lambda\right]$

$\left.= \lambda\right]$

3) If $X \sim NB(r,p)$, then find the c.g.f of X & hence obtain μ & σ^2

Sol: $\because X \sim NB(r,p)$, its p.m.f is $P(x) = {}^{-r}C_x\, p^r(-q)^x, x = 0,1,\cdots$

The ch. F of X is given by

$\phi_x(t) = E[e^{itx}] = \sum_{x=0}^{\infty} e^{itx} P(x) = \sum_{x=0}^{\infty} e^{itx}\, {}^{-r}C_x\, p^r(-q)^x$

$= p^r \sum_{x=0}^{\infty} {}^{-r}C_x (-qe^{it})^x$

$\phi_x(t) = p^r(1-qe^{it})^{-r}$

The c.g.f of X is $K_x(t) = \ln(\phi_x(t)) = \ln\left[p^r(1-qe^{it})^{-r}\right]$

$= r\ln p - r\ln(1-qe^{it})$

$\frac{d}{dt}(K_x(t)) = 0 - r\,\frac{1}{1-qe^{it}}(-qe^{it}\, i) = \frac{irqe^{it}}{1-qe^{it}}$

$\frac{d^2}{dt^2}(K_x(t)) = irq\left[\frac{(1-qe^{it})e^{it}\, i - e^{it}(-qe^{it})\, i}{(1-qe^{it})^2}\right]$

$$K_1 = (-i)\frac{d}{dt}(K_x(t))\Big|_{t=0} = (-i)\frac{irq}{1-q} = \frac{rq}{p}$$

$$K_2 = (-i)^2\frac{d^2}{dt^2}(K_x(t))\Big|_{t=0} = (-i)^2 irq\left[\frac{1-q+q}{(1-q)^2}\right] = rq\left(\frac{p+q}{p^2}\right)$$

$$= \frac{rq}{p^2}$$

$$\mu = \frac{rq}{p}, \quad \sigma^2 = \frac{rq}{p^2}$$

4) If $X \sim N(\mu, \sigma^2)$, then obtain C.g.f of x & hence find its μ & σ^2

Sol: $\because X \sim N(\mu, \sigma^2)$, then its Ch. F is $\phi_x(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

Hence C.g.f of X is $K_x(t) = \ln(\phi_x(t))$

$$= \ln\left[e^{it\mu - \frac{t^2\sigma^2}{2}}\right]$$

$$K_x(t) = it\mu - \frac{t^2\sigma^2}{2} = it\mu + \frac{(it)^2}{2!}\sigma^2$$

$$\frac{d}{dt}(K_x(t)) = i\mu - \frac{2t\sigma^2}{2} \qquad K_1 = \mu$$

$$K_2 = \sigma^2$$

$$\frac{d^2}{dt^2}(K_x(t)) = -\sigma^2$$

Mean $= K_1 = (-i)\frac{d}{dt}(K_x(t))\Big|_{t=0} = (-i)[i\mu - 0] = \mu$

Variance $= K_2 = (-i)^2\frac{d^2}{dt^2}(K_x(t))\Big|_{t=0} = (-i)^2(-\sigma^2) = \sigma^2$

5) If $X \sim E(\lambda)$, then obtain the c.g.f of x & hence obtain its μ & σ^2.

Sol: $X \sim E(\lambda)$ then its Pdf, $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & o.w \end{cases}$

ch.f of x is $\phi_x(t) = E[e^{itx}] = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

$$= \int_0^{\infty} e^{itx}\,\lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{-x(\lambda - it)}\,dx$$

$$= \lambda\left(\frac{e^{-(\lambda - it)x}}{-(\lambda - it)}\right)_0^{\infty} = \lambda\left[\frac{0-1}{-(\lambda - it)}\right]$$

$$= \frac{\lambda}{\lambda - it} = \frac{\lambda}{\lambda(1 - \frac{it}{\lambda})} = \left(1 - \frac{it}{\lambda}\right)^{-1}$$

The c.g.f of X is $K_X(t) = \ln(\phi_X(t))$

$$= \ln\left[\left(1-\frac{it}{\lambda}\right)^{-1}\right]$$

$$= -\ln\left(1-\frac{it}{\lambda}\right) \quad \left[\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right]$$

$$= -\left[-\frac{it}{\lambda} - \frac{1}{2}\left(\frac{it}{\lambda}\right)^2 - \cdots\right]$$

$$K_X(t) = \frac{it}{\lambda} + \frac{1}{2!}\left(\frac{it}{\lambda}\right)^2 + \cdots$$

$K_1 = \frac{1}{\lambda}$, $K_2 = \frac{1}{\lambda^2}$

$\Rightarrow$ Mean $= \mu = K_1 = \frac{1}{\lambda}$ & Variance, $\sigma^2 = K_2 = \frac{1}{\lambda^2}$

6) Let $U = \frac{X-a}{h}$ where a & h are real constants. Obtain the cumulants of U in terms of cumulants of X.

Sol: $\phi_U(t) = e^{\frac{-ita}{h}} \phi_X\left(\frac{t}{h}\right)$

$$K_U(t) = \ln(\phi_U(t)) = \ln\left[e^{\frac{-ita}{h}} \phi_X\left(\frac{t}{h}\right)\right]$$

$$= -\frac{ita}{h} + \ln \phi_X\left(\frac{t}{h}\right)$$

$$= -\frac{ita}{h} + K_X\left(\frac{t}{h}\right)$$

$$(-i)^n \frac{d^n}{dt^n} K_U(t)\Big|_{t=0} = (-i)^n \frac{d^n}{dt^n}\left(\frac{-ita}{h}\right)^n + (-i)^n \frac{d^n}{dt^n} K_X\left(\frac{t}{h}\right)\Big|_{t=0} \rightarrow$$

$\Rightarrow k_n(U) = (-i)^n \frac{d^n}{dt^n}\left(\frac{-ita}{h}\right) + \frac{1}{h^n} K_n(X)$

If $n=1$, $k_1(U) = \frac{-a}{h} + \frac{k_1(X)}{h} = \frac{k_1(X)-a}{h}$

If $n>1$, $k_n(U) = \frac{k_n(X)}{h^n}$.

7) For a distribution, the cumulants are given by $k_r = n(r-1)!$, $n>0$ find ch.f

Sol:

$$K_X(t) = \sum_{r=1}^{\infty} k_r \frac{(it)^r}{r!} = \sum_{r=1}^{\infty} n(r-1)! \frac{(it)^r}{r!}$$

$$= \sum_{r=1}^{\infty} \frac{n(r-1)!}{r(r-1)!} (it)^r$$

$$= -n \sum_{r=1}^{\infty} -\frac{(it)^r}{r}$$

$$= -n\left[-it - \frac{(it)^2}{2} - \frac{(it)^3}{3} - \cdots\right]$$

$$= -n \ln(1-it)$$

$$K_X(t) = \ln(1-it)^{-n} = \ln(\phi_X(t)) \Rightarrow \phi_X(t) = (1-it)^{-n}$$

8) The pmf of 'x' is $P(x) = q^{x-1}p, x=1,2,\cdots$ find cgf & μ & σ^2

Sol:

$$P(x) = q^{x-1}p \quad , x = 1,2\cdots$$

$$\phi_x(t) = \sum_{x=1}^{\infty} e^{itx}\, pq^{x-1}$$

$$= \frac{p}{q}\sum_{x=1}^{\infty}(qe^{it})^x = \frac{p}{q}\left[(1-qe^{it})^{-1} = 1\right]$$

$$K_x(t) = \ln(\phi_x(t)) = \ln\left(\frac{p}{q}\right) + \ln\left[(1-qe^{it})^{-1} - 1\right]$$

$$= \ln p - \ln q + \ln \frac{qe^{it}}{1-qe^{it}}$$

$$\mu = (-i)\frac{d}{dt}(K_x(t))\Big|_{t=0} = 0 + \frac{1-qe^{it}}{qe^{it}}\left[\frac{(1-qe^{it})qe^{it}i - qe^{it}(-qe^{it}i)}{(1-qe^{it})^2}\right]$$

$$= (-i)\frac{1-q}{q}\left[\frac{(1-q)qi - q(-qi)}{(1-q)^2}\right]$$

$$\mu = (-i)\frac{p}{q}\left[\frac{qi}{p^2}\right] = \frac{1}{p}$$

$$K_x'(t) = \frac{qe^{it}i}{qe^{it}(1-qe^{it})} = \frac{i}{1-qe^{it}}$$

$$= \frac{1}{q}\left[\frac{-1}{(e^{it}(1-qe^{it}))^2}\left[e^{it}2(1-qe^{it})(-qe^{it}i) + (1-qe^{it})^2 ie^{it}\right]\right]$$

$$= \frac{1}{q}\left[\frac{-1}{p^4}(2p(-qi) + p^2 i)\right]$$

$$= \frac{i(p^2 - 2pq)}{p^4 q} = \frac{(2pq - p^2)i}{p^4 q}(-i^2)$$

$$(-i)^2\frac{d^2}{dt^2}(K_x(t))_{t=1} = (-i)^2 i\,\frac{-1}{(1-qe^{it})^2}\left[-qe^{it}i\right]\Big|_{t=0}$$

$$= \frac{q}{(1-q)^2} = \frac{q}{p^2}$$

5.5 probability Generating function

Let X be a non-negative integer valued r.v with P.m.f $P(x) = P(X=x)$. Then the P.G.F of X is defined by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x P(x)$$ where $-1 \le t \le 1$ is a dummy variable.

P.g.f only applies to discrete random variables.

Theorem: let X be a drv with pgf $G_X(t)$. Let $Y = a+bX$ where a b are real constants. Then $G_Y(t) = t^a G_X(t^b)$

Pf: $G_Y(t) = E(t^Y) = E(t^{a+bX}) = E[t^a \cdot t^{bX}]$

$= t^a E(t^{bX}) = t^a G_X(t^b)$

Theorem: (Additive property): If X & Y are IRV's then for constants a,b we have $G_{aX+bY}(t) = G_X(t^a) + G_Y(t^b)$

Pf: $G_{aX+bY}(t) = E[t^{aX+bY}] = E[t^{aX} \cdot t^{bY}]$

$= E[t^{aX}] \cdot E[t^{bY}]$

$= G_X(t^a) G_Y(t^b)$

If $a = b = 1$ then $G_{X+Y}(t) = G_X(t) \cdot G_Y(t)$

→ If $X_1, X_2 \ldots X_n$ are IRV then $G_{X_1+X_2+\ldots X_n}(t) = G_{X_1}(t) G_{X_2}(t) \ldots G_{X_n}(t)$

Relationship b/w pgf & mgf:

The pgf & mgf a r.v X are defined by

$G_X(t) = E[t^X]$ & $M_X(t) = E[e^{tX}]$ respectively.

$M_X(t) = E[e^{tX}] = E[(e^t)^X] = G_X(e^t)$

(or) $G_X(t) = E[t^X] = E[e^{\ln(t^X)}] = E[e^{X \ln t}] = M_X(\ln t)$

Theorem: (pmf from pgf) Let $G_X(t)$ be the p.g.f of a drv X that can take values $0,1,2\ldots$ Then the pmf of X is given by

$$P(x) = P(X=x) = \frac{1}{x!} G_X^{(x)}(t)\Big|_{t=0}$$

Pf: $G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x P(x)$

$$= P(X=0)t^0 + P(X=1)t^1 + P(X=2)t^2 + \ldots + P(X=x)t^x + \ldots$$

$$G_X'(t) = P(X=0).0 + P(X=1)1 + P(X=2)2t + \ldots + P(X=x)xt^{x-1} + \ldots$$

$$G_X'(t)\Big|_{t=0} = P(X=1)$$

$$G_X''(t) = 0 + 0 + P(X=2)\cdot 2\cdot 1 + P(X=3)3.2t + \ldots + P(X=x)x(x-1)t^{x-2} + \ldots$$

$$\frac{1}{2!} G_X''(t)\Big|_{t=0} = P(X=2)$$

$$\vdots$$

$$G_X^{(x)}(t)\Big|_{t=0} = P(X=x)\,x! \Rightarrow P(X=x) = \frac{1}{x!} G_X^{(x)} t\Big|_{t=0}$$

Computation of moments using pgf:

In the derivation of moments we use Taylor's expansion

Suppose $f(x)$ has derivatives of all orders at $x=a$. Then the Taylor's expansion of $f(x)$ at the point $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

$$= \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$$

Taylor's expansion of $f(t) = t^x$ about $t=1$ is

$$t^x f(t) = f(1) + f'(1)(t-1) + \frac{f''(1)}{2!}(t-1)^2 + \ldots$$

$$t^x = 1 + x(t-1) + x(x-1)\frac{(t-1)^2}{2!} + x(x-1)(x-2)\frac{(t-1)^3}{3!} + \ldots$$

$$G_X(t) = E[t^X] = 1 + (t-1)E(X) + \frac{(t-1)^2}{2!}E[x(x-1)] + \frac{(t-1)^3}{3!}E[x(x-1)(x-2)] + \ldots$$

Diff (1) wrt t r times & put $t=1$,

$$G_X^{(r)}(t)\big|_{t=1} = E[X(X-1)\cdots(X-r+1)] = G_X^{(r)}(1)$$

which is known as r^{th} factorial moment of X

moments about origin as follows

If $r=1$

$$G_X'(t)\big|_{t=1} = E(X) = \mu_1' = \mu \text{ (mean)}$$

If $r=2$ $\quad G_X''(t)\big|_{t=1} = E(X(X-1)) = E(X^2-X) = E(X^2)-E(X)$

$$\mu_2' = E(X^2) = G_X''(1) + G_X'(1)$$

$$\text{Variance } \sigma^2 = E(X^2) - (E(X))^2 = G_X''(1) + G_X'(1) - [G_X'(1)]^2$$

Convolution formula:

Theorem: If X & Y are independent integer-valued random variables with $P(X=x) = P_1(x)$ & $P(Y=y) = P_2(y)$, $x = 0,1,2\ldots$ & $y = 0,1,\ldots$

then $P(X+Y=z) = P(z) = \sum_{x=0}^{z} P_1(x) P_2(z-x)$

Pf: $G_X(t) = \sum_{x=0}^{\infty} t^x P_1(x)$ & $G_Y(t) = \sum_{y=0}^{\infty} t^y P_2(y)$

$$G_{X+Y}(t) = G_X(t)\, G_Y(t) = \sum_{x=0}^{\infty} t^x P_1(x) \sum_{y=0}^{\infty} t^y P_2(y)$$

$$= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} P_1(x) P_2(y)\, t^{x+y} \quad -(1)$$

Let $Z = X+Y$ then

$$G_Z(t) = E[t^Z] = \sum_{z=0}^{\infty} t^z P(z) \quad -(2)$$

$$\sum_{z=0}^{\infty} t^z P(z) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} P_1(x) P_2(y)\, t^{x+y}$$

$$= \sum_{z=0}^{\infty}\left(\sum_{x=0}^{z} P_1(x) P_2(z-x)\right) t^z$$

$$P(z) = \sum_{x=0}^{z} P_1(x) P_2(z-x), \text{ for } z = 0,1,2\ldots$$

Ex: 1) If $X \sim B(n,p)$ then find the pgf of X & hence obtain its mean & variance.

Sol: $\because X \sim B(n,p)$ its pmf $P(x) = {}^nC_x p^x q^{n-x}, x = 0,1,\ldots n$

The pgf of X, $G_X(t) = E[t^X] = \sum_{x=0}^{n} t^x P(x)$

$$= \sum_{x=0}^{n} t^x {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (tp)^x q^{n-x}$$

$$= (q+tp)^n$$

$$G_X^{(1)}(t) = n(q+tp)^{n-1} \cdot p$$

$$G_X''(t) = pn(n-1)(q+tp)^{n-2} p$$

$$\mu = G_X'(1) = n(q+p)^{n-1} p = np$$

$$E(X^2) = G_X''(t) + G_X'(1) = np^2(n-1) + np = n^2p^2 - np^2 + np$$

$$\sigma^2 = E(X^2) - (E(X))^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

2) If $X \sim P(\lambda)$, then find the pgf of X & hence obtain its μ & σ^2.

Sol: $\because X \sim P(\lambda)$, its pmf $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,\ldots$ & $\lambda > 0$

The pgf of X is

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x P(x) = \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{t\lambda} = e^{\lambda(t-1)}$$

$$G_X'(t) = e^{\lambda(t-1)} \cdot \lambda \Rightarrow G_X'(1) = \lambda$$

$$G_X''(t) = \lambda^2 e^{\lambda(t-1)} \Rightarrow G_X''(1) = \lambda^2$$

$$\mu = G_X'(1) = \lambda \quad \& \quad \sigma^2 = G_X''(1) + G_X'(1) - (G_X'(1))^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

3) If $X \sim NB(r,p)$ then find pgf of X & hence obtain its mean & σ^2

Sol: $\because X \sim NB(r,p)$ then its pmf $P(x) = {}^{-r}C_x p^r (-q)^x, x=0,1,\ldots$

The pgf of X is $G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x P(x)$

$$= \sum_{x=0}^{\infty} t^x \, {}^{-r}C_x \, p^r (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} {}^{-r}C_x (-qt)^x$$

$$= p^r (1-qt)^{-r}$$

$$G_X'(t) = p^r(-r)(1-qt)^{-r-1}(-q) = rq p^r (1-qt)^{-r-1}$$

$$G_X''(t) = rq p^r (-r-1)(1-qt)^{-r-2}(-q)$$

$$= r(r+1)q^2 p^r (1-qt)^{-r-2}$$

Mean, $\mu = G_X'(1) = rq p^r (1-q)^{-r-1} = rq p^r p^{-r-1} = \frac{rq}{p}$

Variance, $\sigma^2 = G_X''(1) + G_X'(1) - (G_X'(1))^2$

$$= r(r+1)\frac{q^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2} = \frac{rq^2}{p^2} + \frac{rq}{p} = \frac{rq}{p}\left[\frac{q}{p} + \right]$$

$$= \frac{rq}{p^2}$$

4) If $X \sim G(p)$, then find the pgf of X & hence obtain its μ & σ^2

Sol: $\because X \sim G(p)$, its pmf $P(x) = pq^x, x = 0,1,\ldots$

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x P(x) = \sum_{x=0}^{\infty} t^x pq^x$$

$$= p \sum_{x=0}^{\infty} (tq)^x$$

$$= p(1-tq)^{-1}$$

$$G_X'(t) = p(-1)(1-tq)^{-2}(-q), \quad G_X''(t) = pq(-2)(1-tq)^{-3}(-q)$$

$$G_X''(1) = pq p^{-2} = \frac{q}{p}, \quad G_X^{(2)}(1) = 2pq^2 p^{-3} = \frac{2q^2}{p^2}$$

$\mu = \text{Mean} = G_X'(1) = \frac{q}{p}$

$$\sigma^2 = G_X''(1) + G_X'(1) - [G_X'(1)]^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p}$$

$= \frac{9}{p}\left[\frac{9}{p}+1\right] = \frac{9}{p^2}$

5) The jpmf of (x,y) is given in the following table. prove & disprove $G_{X+Y}(t) = G_X(t)\,G_Y(t)$ iff x & y are independent.

x \ y	0	1	2	$P_1(x)$
0	1/9	2/9	0	3/9
1	0	1/9	2/9	3/9
2	2/9	0	1/9	3/9
$P_2(y)$	3/9	3/9	3/9	1

Sol: $\because P(X=1,Y=2) = \frac{2}{9} \neq P_1(1)\,P_2(2) = \frac{3}{9}\frac{3}{9} = \frac{1}{9}$,

$\therefore$ x & y are not independent.

$$G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x P_1(x) = \sum_{x=0}^{2} t^x P_1(x)$$

$$= t^0 P_1(0) + t^1 P_1(1) + t^2 P_2(1) = \frac{1}{3} + \frac{t}{3} + \frac{t^2}{3}$$

$$= \frac{1}{3}(1+t+t^2)$$

Similarly $G_Y(t) = \frac{1}{3}(1+t+t^2)$

Let $Z = X+Y$. Then $Z = 0,1,2,3,4$ & its pmf

$$P(Z) = \sum_{x=0} P_1(x)\,P_2(Z-x)$$

$P(Z=0) = P(0) = P(X=0,Y=0) = \frac{1}{9}$

$P(Z=1) = P(1) = P(X=0,Y=1) + P(X=1,Y=0) = \frac{2}{9}$

$P(Z=2) = P(2) = P(X=0,Y=2) + P(X=1,Y=1) + P(X=2,Y=0) = 0 + \frac{1}{9} + \frac{2}{9} = 3/9$

$P(Z=3) = P(3) = P(X=1,Y=2) + P(X=2,Y=1) = \frac{2}{9} + 0 = \frac{2}{9}$

$P(Z=4) = P(4) = P(X=2,Y=2) = \frac{1}{9}$

$$G_Z(t) = G_{X+Y}(t) = \sum_{z=0}^{4} t^z P(z) = t^0 P(0) + t^1 P(1) + t^2 P(2) + t^3 P(3) + t^4 P(4)$$

$$= \frac{1}{9} + \frac{2}{9}t + \frac{3}{9}t^2 + \frac{2}{9}t^3 + \frac{1}{9}t^4$$

$$G_{X+Y}(t) = \frac{1}{9}(1+2t+3t^2+2t^3+t^4)$$

$$= \left[\frac{1}{3}(1+t+t^2)\right]^2$$

$$G_{X+Y}(t)\, G_X(t)\, G_Y(t) = \frac{1}{3}(1+t+t^2) \cdot \frac{1}{3}(1+t+t^2)$$

$$= \left[\frac{1}{3}(1+t+t^2)\right]^2$$

$G_{X+Y}(t) = G_X(t) \cdot G_Y(t)$ but X & Y are not indepd

Thus the statement is disproved.

6) Can $G_X(t) = \frac{2}{1+t}$ be a pgf of r.v X? Give reasons.

Sol: $G_X(t) = \frac{2}{1+t} = 2(1+t)^{-1} = 2(1-t+t^2-t^3+\cdots)$

$$= 2\sum_{x=0}^{\infty}(-1)^x t^x$$

$P(x) = 2(-1)^x$, $x = 0,1,2 \cdots$

which takes -ve values also.

Hence $G_X(t)$ is not a pgf.

7) A fair die is thrown n times. Let S be the total no. of points. S.T $P(S=n+5) = {}^{n+4}C_5 \left(\frac{1}{6}\right)^n$

Sol: The pgf of a single throw is

$$G_X(t) = \sum_{x=1}^{6} t^x P(x) = \sum_{x=1}^{6} \frac{t^x}{6}$$

$$= \frac{1}{6}(t+t^2+\cdots+t^6)$$

$$= \frac{t}{6}(1+t+\cdots+t^5)$$

$$= \frac{t}{6}\left[\frac{1(1-t^6)}{1-t}\right]$$

$$= \frac{t}{6}(1-t^6)(1-t)^{-1}$$

n = no. of terms
r = ratio

$S_n = \frac{a(1-r^n)}{1-r}$

$S = \frac{a}{1-r}$ if

Since n throws are identical & independent

$$G_S(t) = [G_X(t)]^n = \frac{t^n(1-t^6)^n(1-t)^{-n}}{6^n}$$

$$= \frac{t^n}{6^n} \sum_{j=0}^{n} {}^nC_j (-t)^j \sum_{k=0}^{\infty} {}^{n+k-1}C_k \, t^k$$

$$= \frac{1}{6^n} \sum_{j=0}^{n} \sum_{k=0}^{\infty} (-1)^j {}^nC_j \, {}^{n+k-1}C_k \, t^{k+6j+n}$$

$k+6j=5$

$$= \sum_{k=0}^{\infty} P(S = k+6j+n) \, t^{k+6j+n}$$

When $P(S = k+6j+n) = \frac{1}{6^n} \sum_{j=0}^{n} (-1)^j {}^nC_j \, {}^{n+k-1}C_k$

$P(S = n+5)$ with $j=0$ & $k=5$

$$= \frac{1}{6^n} (-1)^0 \, {}^nC_0 \, {}^{n+5-1}C_5$$

$$= \frac{1}{6^n} \, {}^{n+4}C_5$$

8) let x be a r.v with the following pmf find pgf.

x	1	2	3	4
P(x)	1/3	1/6	1/8	3/8

Sol: Pgf, $G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x P(x)$

$$= \sum_{x=1}^{4} t^x P(x)$$

$$= t^1 P(1) + t^2 P(2) + t^3 P(3) + t^4 P(4)$$

$$= \frac{t}{3} + \frac{t^2}{6} + \frac{t^3}{8} + \frac{3t^4}{8}$$

9) If $P(x) = P(X = x) = \frac{1}{2^{x-2}}$ for $x = 3, 4, 5 \ldots$ find Pgf of X.

Sol: Pgf, $G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x P(x)$

$$= \sum_{x=3}^{\infty} t^x P(x) = \sum_{x=3}^{\infty} t^x \frac{1}{2^{x-2}}$$

$$= 4 \sum_{x=3}^{\infty} \left(\frac{t}{2}\right)^x$$

$$= 4 \left[\left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \ldots \right]$$

$$= 4 \, \frac{t^3/8}{1 - t/2} = \frac{t^3/2}{2 - t}$$

10) Let X be a rv with pgf $G_X(t)$. Find pgf of

(i) X+1 (ii) 2X

Sol:

$$G_{X+1}(t) = E[e^{t(X+1)}] = E[e^{tX}.e^{t}]$$

$$= E[t^{X+1}] = E[t^X.t] = t\,E(t^X)$$

$$= t\,G_X(t)$$

$$G_{2X}(t) = E[t^{2X}] = E[(t^2)^X] = G_X(t^2)$$

11) If $G_X(t)$ is Pgf of X, then find Pgf of $\frac{X-a}{b}$ where a,b are real constants.

Sol:

Let $U = \frac{X-a}{b}$

$$G_U(t) = E[t^U] = E[t^{\frac{X-a}{b}}]$$

$$= E[t^{\frac{X}{b}}.t^{-a/b}] = t^{-a/b}\,E[(t^{1/b})^X]$$

$$= t^{-a/b}\,G_X(t^{1/b})$$

12) Let $G_X(t) = \frac{\left(\frac{1}{3}t+\frac{2}{3}\right)^4}{t}$. what is range of X? what is pmf.

Sol:

$$G_X(t) = \frac{\left(\frac{1}{3}t+\frac{2}{3}\right)^4}{t} = \frac{1}{t}\left(\frac{t}{3}+\frac{2}{3}\right)^4$$

$$P(x) = \frac{1}{x!}G_X^{(x)}(t)\Big|_{t=0} \Rightarrow x = 0,1,\ldots$$

$$= \frac{1}{t}\sum_{x=0}^{4} {}^4C_x \left(\frac{t}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$$

$$= \sum_{x=0}^{4} {}^4C_x \left(\frac{2}{3}\right)^{4-x} \frac{1}{3^x}\, t^{x-1}$$

$$= \sum_{x-1=-1}^{3} {}^4C_x \left(\frac{2}{3}\right)^{4-x} \frac{1}{3^x}\, t^{x-1}$$

$P(X=x-1) = P(x) = {}^4C_x \left(\frac{2}{3}\right)^{4-x} \frac{1}{3^x}$ for $x \leq 4$

$x-1 = -1, 0, 1, 2, 3$

$\Rightarrow x = 0, 1, 2, 3, 4$

$P(X=-1) = P(0) = {}^4C_0 \left(\frac{2}{3}\right)^4 \cdot \frac{1}{3^0} = \frac{16}{81}$

$P(X=0) = P(1) = {}^4C_1 \left(\frac{2}{3}\right)^3 \frac{1}{3} = 4\,\frac{8}{81} = \frac{32}{81}$

$P(X=1) = P(2) = {}^4C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3^2}\right) = 6\,\frac{4}{81} = \frac{24}{81}$

$P(X=2) = P(3) = {}^4C_3 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3^3}\right) = 4\,\frac{2}{81} = \frac{8}{81}$

$P(X=3) = P(4) = {}^4C_4 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3^4}\right) = \frac{1}{81}$

x	-1	0	1	2	3
$P(x)$	$16/81$	$32/81$	$24/81$	$8/81$	$1/81$