

Probability Inequalities and Generating Functions

i) Cauchy - Schwartz's Inequality:

For any two Random variables X and Y then

$$[E(XY)]^2 \leq E(X^2) \cdot E(Y^2)$$

Proof:- Let us consider a real valued function of 't'

$$Z(t) = E(X + tY)^2 \geq 0 \quad \forall X, Y \text{ \& } t$$

$$\text{i.e. } \{Z(t) \geq 0\}$$

$$\Rightarrow E(X + tY)^2 \geq 0$$

$$\Rightarrow E(X^2) + t^2 E(Y^2) + 2tE(XY) \geq 0$$

This is of the form $At^2 + Bt + C \geq 0$

Where $A = E(Y^2)$, $B = 2E(XY)$, $C = E(X^2)$

$$B^2 - 4AC \leq 0$$

$$[2E(XY)]^2 - 4 \cdot E(Y^2) E(X^2) \leq 0$$

$$4[E(XY)]^2 \leq 4 E(Y^2) E(X^2)$$

$$[E(XY)]^2 \leq E(Y^2) E(X^2)$$

Hence the proof.

** Inequalities are useful for bounding the quantities.

Ex: The j.p.d.f of (X, Y) is given by

$$f(x, y) = \frac{x+y}{21} \text{ for } x=1, 2, 3 \text{ \& } y=1, 2$$

Verify whether Cauchy-Schwartz inequality.

Sol:-

$y \backslash x$	1	2	3	Total
1	$2/21$	$3/21$	$4/21$	$9/21$
2	$3/21$	$4/21$	$5/21$	$12/21$
Total	$5/21$	$7/21$	$9/21$	1

$$E(X) = \sum_{x=1}^3 x f_1(x) = 1 \times \frac{5}{21} + 2 \times \frac{7}{21} + 3 \times \frac{9}{21} = \frac{46}{21}$$

$$E(X^2) = \sum_{x=1}^3 x^2 f_x(x) = 1^2 \times \frac{5}{21} + 2^2 \times \frac{7}{21} + 3^2 \times \frac{9}{21} = \frac{114}{21}$$

$$E(Y) = \sum_{y=1}^2 y f_2(y) = 1 \times \frac{9}{21} + 2 \times \frac{12}{21} = \frac{33}{21}$$

$$E(Y^2) = \sum_{y=1}^2 y^2 f_y(y) = 1^2 \times \frac{9}{21} + 2^2 \times \frac{12}{21} = \frac{57}{21}$$

$$E(XY) = \sum_{x=1}^3 \sum_{y=1}^2 xy f(x, y)$$

$$= 1 \times 1 \times \frac{2}{21} + 1 \times 2 \times \frac{3}{21} + 1 \times 3 \times \frac{4}{21} + 2 \times 1 \times \frac{3}{21}$$

$$+ 2 \times 2 \times \frac{4}{21} + 2 \times 3 \times \frac{5}{21}$$

$$E(XY) = \frac{71}{21}$$

$$\text{WKT } [E(xy)]^2 \leq E(y^2) \cdot E(x^2)$$

$$\left(\frac{71}{21}\right)^2 \leq \frac{114}{21} \times \frac{57}{21}$$

$$\underline{11.755} \leq 14.735$$

Ex:- Let x & y be c.r.v.s with j.p.d.f

$$f(x, y) = \begin{cases} x+y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Verify Cauchy-Schwartz inequality.

Sol:- The m.p.d.f of x is given by

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 (x+y) dy = \left(xy + \frac{y^2}{2} \right) \Big|_0^1 = x + \frac{1}{2}$$

$$f_x(x) = \begin{cases} x + 1/2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

being symmetry $f_y(y) = \begin{cases} y + 1/2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

$$E(x^2) = \int_0^1 x^2 (x + 1/2) dx \Rightarrow \int_0^1 \left(x^3 + \frac{x^2}{2} \right) dx = \left. \frac{x^4}{4} \right|_0^1 + \left. \frac{1}{2} \cdot \frac{x^3}{3} \right|_0^1$$

$$= 1/4 + 1/6 = 10/24$$

Similarly $E(y^2) = 10/24$

$$E(xy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(xy) dx dy$$

$$= \int_0^1 \int_0^1 xy(x+y) dx dy$$

$$= \int_0^1 \int_0^1 (x^2y + xy^2) dx dy$$

$$= \int_0^1 \left[\frac{x^3}{3} y \Big|_0^1 + \frac{x^2}{2} y^2 \Big|_0^1 \right] dy$$

$$= \int_0^1 (y/3 + y^2/2) dy$$

$$= \left(\frac{y^2}{2} \times \frac{1}{3} + \frac{y^3}{3} \times \frac{1}{2} \right) \Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 1/3$$

$$(E(xy))^2 \leq E(x^2) E(y^2)$$

$$\left(\frac{1}{3}\right)^2 \leq \frac{10}{24} \times \frac{10}{24}$$

$$0.111 \leq 0.1736$$

Hence the inequality.

2) Chebyshev's Inequality:

To find the probability of an event describe by a r.v if its c.d.f or p.d.f or p.m.f is required.

To find the upper bound or lower bound for the probability of the event we use Chebyshev's inequality.

Statement:- If X is a random variable with mean μ and variance σ^2 , ~~then~~ ^{and} Considering a positive value constant K , then $P(|X - \mu| \geq K\sigma) \leq \frac{1}{K^2}$

$$\textcircled{or} \quad P(|X - \mu| \leq K\sigma) \geq 1 - \frac{1}{K^2}$$

Proof:- Let X be a Continuous Random variable

$$\text{Consider } \text{Var}(X) = \sigma^2 = E(X - E(X))^2$$

$$\sigma^2 = E(X - \mu)^2$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$



$$\sigma^2 = \int_{-\infty}^{\mu - K\sigma} (x - \mu)^2 f(x) dx + \int_{\mu - K\sigma}^{\mu + K\sigma} (x - \mu)^2 f(x) dx + \int_{\mu + K\sigma}^{\infty} (x - \mu)^2 f(x) dx$$

having the limits $\mu - K\sigma$ to $\mu + K\sigma$ is always greater than zero and that can be expressed as

$$\sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} (x - \mu)^2 f(x) dx + \int_{\mu + k\sigma}^{\infty} (x - \mu)^2 f(x) dx \quad \text{--- (1)}$$

limits of ① $x \leq \mu - k\sigma$

$$\begin{aligned} x - \mu &\leq -k\sigma \\ -(x - \mu) &\geq k\sigma \\ (x - \mu)^2 &\geq (k\sigma)^2 \end{aligned}$$

② $x \geq \mu + k\sigma$

$$\begin{aligned} x - \mu &\geq k\sigma \\ (x - \mu)^2 &\geq (k\sigma)^2 \end{aligned}$$

$$\text{①} \Rightarrow \sigma^2 \geq \int_{-\infty}^{\mu - k\sigma} (k\sigma)^2 f(x) dx + \int_{\mu + k\sigma}^{\infty} (k\sigma)^2 f(x) dx$$

$$\sigma^2 \geq k^2 \sigma^2 \left[\int_{-\infty}^{\mu - k\sigma} f(x) dx + \int_{\mu + k\sigma}^{\infty} f(x) dx \right]$$

$$1 \geq k^2 [P(x < \mu - k\sigma) + P(x > \mu + k\sigma)]$$

[$\because$ from the property of density function]

$$\frac{1}{k^2} \geq [P(x - \mu < -k\sigma) + P(x - \mu > k\sigma)]$$

Since $\begin{cases} x < -a \\ x > a \end{cases} \Rightarrow |x| > a$

$$\frac{1}{k^2} \geq P(|x - \mu| \geq k\sigma)$$

$$P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Hence the proof

$$P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Multiplying '-' on both sides

$$-P(|x - \mu| \geq k\sigma) \geq -\frac{1}{k^2}$$

Add One on both sides

$$1 - P(|x - \mu| \geq k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(|x - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

Hence verified.

Ex: A random Variable x has a mean 12, Variance 9 and an unknown probability distribution. Find $P(6 < x < 18)$ and $P(3 < x < 21)$.

Sol:- Given Mean $\mu = 12$, Variance $\sigma^2 = 9$, S.D $\sigma = 3$

$$P(6 < x < 18) = P(6 - 12 < x - 12 < 18 - 12)$$

$$= P(-6 < x - 12 < 6)$$

$$= P(|x - 12| < 6)$$

$$\geq 1 - \frac{1}{k^2}$$

$$\geq 1 - \frac{1}{4}$$

$$\geq 3/4$$

$$\therefore P(|x - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$k\sigma = 6$$

$$k \cdot 3 = 6$$

$$k = 2$$

$$P(3 < x < 21)$$

$$= P(3-12 < x-12 < 21-12)$$

$$= P(-9 < x-12 < 9)$$

$$= P(|x-12| < 9)$$

$$\geq 1 - \frac{1}{k^2}$$

$$K\sigma = 9$$

$$K(3) = 9$$

$$K=3$$

$$\geq 1 - \frac{1}{9}$$

$$\geq 8/9$$

(lower bound)

If x is a random variable such that $E(x) = 3$
 $E(x^2) = 13$, find the lower bound of $P(-2 \leq x \leq 8)$.

Given $E(x) = 3 = M$, $E(x^2) = 13$

$$V(x) = E(x^2) - (E(x))^2 = 13 - 9 = 4$$

$$\sigma^2 = 4 \Rightarrow \sigma = 2$$

$$P(-2 \leq x \leq 8) \Rightarrow P(-2-3 \leq x-3 \leq 8-3)$$

$$\Rightarrow P(-5 \leq x-3 \leq 5)$$

$$= P(|x-3| \leq 5)$$

$$\geq 1 - \frac{1}{k^2}$$

$$K\sigma = 5$$

$$K \cdot 2 = 5$$

$$K = 5/2$$

$$\geq 1 - \frac{1}{(5/2)^2}$$

$$\geq 1 - \frac{4}{25}$$

$$\geq \frac{21}{25}$$

Two unbiased dice are thrown. If X is the sum of the numbers showing up, prove that $P(|X-7| \geq 3) \leq \frac{35}{54}$. Compare this with the actual probability.

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ / & / & / & / & / & / \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

$$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

$$P(X) = \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$$

$$E(X) = \sum_{x=2}^{12} x P(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$E(X) = \frac{252}{36} = 7$$

$$E(X^2) = \sum_{x=2}^{12} x^2 P(X) = 4 \times \frac{1}{36} + 9 \times \frac{2}{36} + 16 \times \frac{3}{36} + 25 \times \frac{4}{36} + 36 \times \frac{5}{36} + 49 \times \frac{6}{36} + 64 \times \frac{5}{36} + 81 \times \frac{4}{36} + 100 \times \frac{3}{36} + 121 \times \frac{2}{36} + 144 \times \frac{1}{36}$$

$$E(X^2) = \frac{1974}{36} = \frac{329}{6}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{329}{6} - 7^2 = \frac{35}{6}$$

$$\mu = 7, \quad V(x) = \sigma^2 = \frac{35}{6} \Rightarrow \sigma = \sqrt{\frac{35}{6}} \quad \sigma \text{ is always +ve}$$

$$K\sigma = 3 \Rightarrow K \times \sqrt{\frac{35}{6}} = 3 \Rightarrow K = 3\sqrt{\frac{6}{35}}$$

$$K^2 = 9 \times \frac{6}{35} = \frac{54}{35}$$

$$\frac{1}{K^2} = \frac{35}{54}$$

$$\therefore P(|x - \mu| \geq K\sigma) \leq \frac{1}{K^2}$$

$$\text{is equal to } P(|x - 7| \geq 3) \leq \frac{35}{54}$$

Actual Probability:

$$P(|x - 7| \geq 3) = 1 - (P(|x - 7| < 3))$$

$$= 1 - (-3 + 7 < x < 3 + 7)$$

$$= 1 - (4 < x < 10)$$

$$= 1 - \left[\frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} \right]$$

$$= 1 - \frac{24}{36}$$

$$= \frac{36 - 24}{36}$$

$$= \frac{12}{36}$$

$$\text{Actual Probability} \equiv \frac{1}{3}$$

If X is the number scored in a throw of a fair die find the upper bound for $P(|X - \mu| \geq 2.5)$ where $\mu = E(X)$. Also find the actual probability.

Sim:	x	1	2	3	4	5	6
	$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\mu = \sum xp(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$

$$= \frac{7}{2}$$

$$E(x^2) = \sum x^2 p(x) = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} + 25 \times \frac{1}{6} + 36 \times \frac{1}{6}$$

$$= \frac{91}{6}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 \Rightarrow \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$$P(|X - \mu| \geq 2.5) \leq \frac{1}{k^2}$$

$$k = 2.5$$

$$k \times \sqrt{\frac{35}{12}} = 2.5$$

$$P(|X - \mu| \geq 2.5) \leq \frac{1}{\left(2.5 \sqrt{\frac{12}{35}}\right)^2}$$

$$k = 2.5 \sqrt{\frac{12}{35}}$$

$$\leq \frac{35}{75}$$

$$\leq 0.4666$$

$$\leq 0.47$$

The actual probability p is given by

$$p = P(|X - 3.5| > 2.5) = P\{x \text{ lies outside the limits } (3.5 - 2.5, 3.5 + 2.5) \text{ i.e. } (1, 6)\}$$

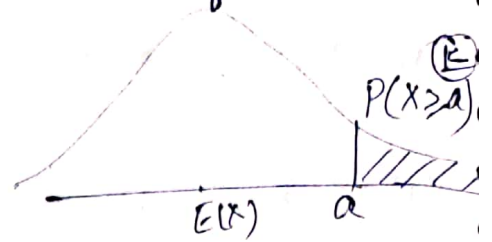
But since X is the number on the dice when thrown it cannot lie outside the limits of 1 & 6.

$$\therefore p = P(\emptyset) = \underline{\underline{0}}$$

3) Markov's Inequality:

Statement:- If X is a random variable then for every $K > 0$, then

$$P(|X| \geq K) \leq \frac{E(|X|)}{K}$$



Proof:- Let X be a r.v and let S be the set of all random variables such that $|X| \geq K$

$$\text{i.e. } S = \{X \mid |X| \geq K\}$$

Then by the definition of mean

$$E(|X|) = \int_S |x| f(x) dx$$

$$E(|X|) \geq K \int_S f(x) dx$$

$$\frac{E(|X|)}{K} \geq \int_S f(x) dx$$

$$\frac{E(|X|)}{K} \geq P(|X| \geq K) \quad (\because \text{By the defn of 'S'})$$

$$\Rightarrow P(|X| \geq K) \leq \frac{E(|X|)}{K}$$

Hence the proof:

* Markov's inequality provides an upper bound to the probability

If we wanna to find least upper bound apply Chebyshev's inequality.

* Markov's ineq is a non-negative r.v.

* Only have to know expected value

* But in Chebyshev's inequality we need both mean & Variance.

Ex:- let $X \sim \text{Exp}(\lambda)$ using Markov's inequality find an upper bound for $P(X \geq a)$ where $a > 0$. Compare the upper bound with the actual value of $P(X \geq a)$.

$$\text{Sim:- } X \sim \text{Exp}(\lambda)$$

$$E(X) = 1/\lambda$$

$$\text{by Markov's inequality } P(X \geq a) \leq \frac{E(X)}{a}$$

$$\leq \frac{1}{\lambda a}$$

The actual value of

$$\begin{aligned} P(X \geq a) &= 1 - P(X < a) \Rightarrow 1 - F(a) \\ &= 1 - [1 - e^{-\lambda a}] = \underline{\underline{e^{-\lambda a}}} \end{aligned}$$

CDF of exponen
is $1 - e^{-\lambda x}$

On Comparing

$$e^{-\lambda a} \leq \frac{1}{\lambda a}$$

or $\frac{1}{\lambda a} \geq e^{-\lambda a}$

Ex:- Suppose that the average grade on the upcoming statistics exam is 70%. Given an upper bound on the proportion of students who score at least 90%?

Sol:- Let x be the proportion of students

Required probability = $P(x \geq 0.9)$

$$P(x \geq 0.9) \leq \frac{E(x)}{0.9}$$

Given $E(x) = 0.70$

$$P(x \geq 0.9) \leq \frac{7}{9}$$

It means that at most 77.8% of students can possible score this high.

Ex: A coin is weighted so that its probability of landing on heads is 20%. Suppose the coin is flipped 20 times. Find a bound for the probability it lands on heads at least 16 times.

Solⁿ:- Let X be the number of heads

Required probability = $P(X \geq 16)$

Given that $n = 20$, $p = 20\% = \frac{20}{100}$
follows binomial distribution

$$E(X) = n \cdot p \Rightarrow 20 \times \frac{20}{100} = 4$$

Thus, by Markov's inequality, we have

$$P(X \geq 16) \leq \frac{E(X)}{16}$$

$$\leq \frac{4}{16}$$

$$\leq \frac{1}{4}$$

$$P(X \geq 16) \leq 25\%$$

Consider a r.v X that takes the value 0 with ~~prob~~ $\frac{24}{25}$
and the value 1 with ~~prob~~ $\frac{1}{25}$. Find a bound on the
probability that X is at least 5.

$$\text{Sol}^n:- \begin{array}{cc} X & 0 & 1 \\ P(X) & \frac{24}{25} & \frac{1}{25} \end{array}$$

$$\begin{aligned} E(X) &= 0 \times \frac{24}{25} + 1 \times \frac{1}{25} \\ &= \frac{1}{25} \end{aligned}$$

Required probability = $P(X \geq 5)$

by Markov's inequality

$$P(X \geq 5) \leq \frac{E(X)}{5}$$

$$\leq \frac{1}{25 \cdot 5}$$

$$P(X \geq 5) \leq \frac{1}{125}$$