

Generating Functions

By using generating functions we can find Mean and Variance of probability binomial distributions.

Generating functions classified into four types

- (a) Moment Generating Functions (M.G.F)
- (b) Characteristic Functions (Ch.F)
- (c) Cumulant Generating Functions (G.G.F)
- (d) Probability Generating Function (P.G.F)

a) Moment Generating Functions (MGF) & its Properties:

MGF Notation: $M_X(t)$
defined as $E(e^{tx}) = \begin{cases} \sum e^{tx} p(x) & \text{for d.r.v} \\ \int e^{tx} f(x) dx & \text{for c.r.v} \end{cases}$

$$M_X(t) = E(e^{tx})$$

$$= E \left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \right]$$

$$= E(1) + E\left(\frac{tx}{1!}\right) + E\left(\frac{(tx)^2}{2!}\right) + \dots$$

$$= 1 + \frac{t}{1!} E(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) + \dots + \frac{t^r}{r!} E(x^r)$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r)$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

which gives the m.g.f in terms of moments

The coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ_r' where $r=1, 2, 3, \dots$

$\mu_r' = E(X^r)$ moment about origin.

Mean defined as $\mu = \mu_1' = E(X)$

$$M_{X-\mu}(t) = E[e^{t(X-\mu)}] = E\left[1 + \frac{t}{1!}(X-\mu) + \frac{t^2}{2!}(X-\mu)^2 + \frac{t^3}{3!}(X-\mu)^3 + \dots\right]$$

$$= 1 + \frac{t}{1!}E(X-\mu) + \frac{t^2}{2!}E(X-\mu)^2 + \frac{t^3}{3!}E(X-\mu)^3 + \dots$$

$$= 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \dots$$

Where $E(X-\mu)^r = \mu_r$ is known as the

r th central moment for $r=1, 2, \dots$

$$\mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$$

Since $M_X(t)$ generates moments, it is called moment generating function.

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$\text{in general } \mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0} \quad r=1, 2, 3, \dots$$

Properties of M.G.F:

(1) $M_{cx}(t) = M_x(t)$

LHS $M_{cx}(t) = E(e^{tcx})$

RHS $M_x(t) = E(e^{tx(t)})$

$\therefore M_{cx}(t) = M_x(t)$

(2) Additive property of MGF

Consider $x_1, x_2, \dots, x_n$ are independent variables

then $M_{x_1 + x_2 + \dots + x_n}(t) = M_{x_1}(t) * M_{x_2}(t) * M_{x_3}(t) \dots M_{x_n}(t)$

$[M_{x+y}(t) = M_x(t) * M_y(t)]$

Proof: $M_{x_1 + x_2 + \dots + x_n}(t) = E(e^{t(x_1 + x_2 + \dots + x_n)})$
 $= E(e^{tx_1} \cdot e^{tx_2} \cdot \dots \cdot e^{tx_n})$
 $= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdot \dots \cdot E(e^{tx_n})$
 $= M_{x_1}(t) \cdot M_{x_2}(t) \cdot \dots \cdot M_{x_n}(t)$

(3) Effect of change of origin and scale of MGF:

Let us consider $U = \frac{x-a}{h}$

a = Shifting the origin
 h = Shifting the scale

$$\begin{aligned}
 M_U(t) &= E(e^{tu}) \\
 &= E(e^{t(\frac{x-a}{h})}) \\
 &= e^{-\frac{ta}{h}} \cdot E(e^{\frac{tx}{h}})
 \end{aligned}$$

$$M_U(t) = e^{-\frac{ta}{h}} M_X(t/h)$$

[It is not independent of shifting origin & scale]

② uniqueness theorem of MGF:

$$M_X(t) = M_Y(t) \Rightarrow X=Y$$

If m.g.f of X & Y are equal then obviously their distributions are also equal, identically distributed.

1] Show that the MGF of the r.v X having the probability density function

$$f(x) = \begin{cases} \frac{1}{3}; & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

sol:- $M_X(t) = E(e^{tx})$

$$M_X(t) = \int_{-1}^2 e^{tx} f(x) dx$$

$$= \int_{-1}^2 e^{tx} \frac{1}{3} dx \quad \text{--- ①}$$

$$M_X(t) = \frac{1}{3} \times \left. \frac{e^{tx}}{t} \right|_{-1}^2$$

$$M_X(t) = \frac{1}{3t} [e^{2t} - e^{-t}]$$

$$M_X(t) = \frac{e^{2t} - e^{-t}}{3t} \quad t \neq 0$$

Put $t=0$ in (1)

$$M_X(t) = \int_{-1}^2 e^{0x} \cdot \frac{1}{3} dx$$

$$= \frac{1}{3} \int_{-1}^2 dx$$

$$= \frac{1}{3} (x) \Big|_{-1}^2$$

$$= \frac{1}{3} [2 - (-1)]$$

$$= \frac{3}{3}$$

$$M_X(t) = 1 \quad ; \quad t=0$$

$$M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

② If $\mu_r' = r! 2^r$ then find MGF

$$\text{Soln:- } M_X(t) = E(e^{tx}) = E\left[1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \dots\right]$$

$$= 1 + t E(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) + \dots$$

$$M_X(t) = 1 + (1! 2^1)t + \frac{(2! 2^2)t^2}{2!} + \frac{(3! 2^3)t^3}{3!} + \dots$$

$$M_X(t) = 1 + 2t + (2t)^2 + (2t)^3 + \dots$$

$$M_X(t) = (1 - 2t)^{-1} \quad \text{or} \quad \frac{1}{1 - 2t}$$

③ If $M_X' = (x+1)! 2^x$ then find MGF

$$\text{Soln: } M_X(t) = E(e^{tx}) = E\left[1 + tx + \frac{t^2 x^2}{2!} + \frac{t^3 x^3}{3!} + \dots\right]$$

$$= 1 + t E(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) + \dots$$

$$= 1 + M_1' (t) + M_2' \frac{(t^2)}{2!} + \frac{t^3}{3!} M_3' + \dots$$

$$M_1' = (2+1)! 2^1$$

$$M_2' = (2+1)! 2^2$$

$$M_3' = (3+1)! 2^3$$

$$= 1 + (2! 2^1)t + \frac{(3! 2^2)t^2}{2!} + \frac{(4! 2^3)t^3}{3!} + \dots$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \dots$$

$$= (1 - 2t)^{-2}$$

$$M_X(t) = \frac{1}{(1 - 2t)^2}$$

4) Obtain the moment generating function of the random variable x having probability distribution

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Also determine mean, variance, standard deviation.

Sol: $M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

$$M_x(t) = \int_{-\infty}^0 e^{tx} \cdot 0 dx + \int_0^1 e^{tx} x dx + \int_1^2 e^{tx} (2-x) dx + \int_2^{\infty} e^{tx} \cdot 0 dx$$

$$M_x(t) = \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx$$

$$= \left[x \cdot \frac{e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^1 + \left[(2-x) \frac{e^{tx}}{t} - (-1) \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \left[\left(\frac{e^t}{t} - \frac{e^t}{t^2} \right) - \left(0 - \frac{1}{t^2} \right) \right] + \left[\left(0 + \frac{e^{2t}}{t^2} \right) + \left(\frac{e^t}{t} + \frac{e^t}{t^2} \right) \right]$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t} + 1 - 2e^t}{t^2}$$

$$= \frac{(e^t - 1)^2}{t^2}$$

(using e^x expansion)

$$= \frac{\left[\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right] - 1 \right]^2}{t^2}$$

$$\begin{aligned}
 M_x(t) &= \frac{\left[t + \frac{t^2}{2} + \frac{t^3}{6} + \dots \right]^2}{t^2} \\
 &= \frac{t^2 \left[t \left(1 + \frac{t}{2} + \frac{t^2}{6} + \dots \right) \right]^2}{t^2} \\
 &= \cancel{t^2} \frac{\left[1 + \frac{t}{2} + \frac{t^2}{6} + \dots \right]^2}{\cancel{t^2}} \\
 &= \left(1 + \frac{t}{2} + \frac{t^2}{6} + \dots \right) \left(1 + \frac{t}{2} + \frac{t^2}{6} + \dots \right)
 \end{aligned}$$

$$= \left(1 + \frac{t}{2} + \frac{t^2}{6} \right) + \left(\frac{t}{2} + \frac{t}{2} \cdot \frac{t}{2} + \frac{t}{2} \cdot \frac{t^2}{6} \right) + \left(\frac{t^2}{6} + \frac{t^2 \cdot t}{2 \cdot 6} + \frac{t^2 \cdot t^2}{6 \cdot 6} + \dots \right) + \dots$$

$$= \left(1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{12} + \frac{t^2}{6} + \frac{t^3}{12} + \frac{t^4}{36} + \dots \right)$$

$$= \left(1 + \left(\frac{t}{2} + \frac{t}{2} \right) + \left(\frac{t^2}{4} + \frac{t^2}{6} + \frac{t^2}{6} \right) + \dots \right)$$

$$M_x(t) = \left(1 + \frac{2t}{2} + \frac{7t^2}{12} + \dots \right)$$

$$\text{mean} = \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$= \left[0 + 1 + \frac{7}{12} \cdot 2t + \dots \right]_{t=0}$$

$$\text{Mean} = \mu = 1$$

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0}$$

$$E(X^2) = \left[0 + \frac{14}{12} + \dots \right]_{t=0}$$

$$E(X^2) = \frac{14}{12} = \frac{7}{6}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{7}{6} - 1$$

$$V(X) = \frac{7-6}{6} = \frac{1}{6}$$

$$S.D = \sqrt{V(X)}$$

$$S.D = \sqrt{1/6}$$

5) Find the m.g.f & hence find the mean & variance of a r.v x whose p.d.f is given by.

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Sol: } M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{1}{3} e^{-x/3} dx$$

$$= \frac{1}{3} \int_0^{\infty} e^{(t-1/3)x} dx$$

$$= \frac{1}{3} \times \frac{e^{(t-1/3)x}}{t-1/3} \Big|_0^{\infty}$$

$$= \frac{1}{3} \times \frac{1}{t-1/3} \times [e^{\infty} - e^0]$$

$$= -\frac{1}{3} \times \frac{3}{3t-1} = \frac{-1}{3t-1} = \frac{1}{1-3t}$$

$$\text{Mean } \mu = \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$= \left[\frac{d}{dt} \left[\frac{1}{1-3t} \right] \right]_{t=0}$$

$$= \left[\frac{d}{dt} (1-3t)^{-1} \right]_{t=0}$$

$$= \left[(-1) (1-3t)^{-2} (-3) \right]_{t=0}$$

$$\mu = \underline{\underline{3}}$$

$$E(x^2) = \left[\frac{d^2}{dt^2} \left[3(1-3t)^{-2} \right] \right]_{t=0}$$

$$= \left[3(-2)(1-3t)^{-3} \times (-3) \right]_{t=0}$$

$$= 18$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= 18 - 9$$

$$V(x) = \underline{\underline{9}}$$

$$SD = \sqrt{V(x)}$$

$$\underline{\underline{SD = 3}}$$

Find the m.g.f of a r.v x whose pdf is given by

$$f(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{Sol: } M_x(t) &= E(e^{tx}) \\ &= \int_0^2 e^{tx} \cdot \frac{x}{2} dx \\ &= \frac{1}{2} \int_0^2 x \cdot e^{tx} dx \end{aligned}$$

$$\text{let } u=x, \quad \begin{aligned} dv &= e^{tx} \\ du &= 1 \\ v &= \frac{e^{tx}}{t} \end{aligned}$$

$$\int u dv = uv - \int v du$$

$$M_x(t) = \frac{1}{2} \left[x \cdot \frac{e^{tx}}{t} - \int \frac{e^{tx}}{t} \cdot 1 \right]_0^2$$

$$= \frac{1}{2} \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^2$$

$$= \frac{1}{2} \left[\frac{x t e^{tx} - e^{tx}}{t^2} \right]_0^2$$

$$= \frac{1}{2t^2} \left[(2te^{2t} - e^{2t}) - (0 - 1) \right]$$

$$M_x(t) = \frac{1}{2t^2} [2te^{2t} - e^{2t} + 1]$$

Find the m.g.f, mean & variance of the p.d.f

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{o.w} \end{cases}$$

$$\text{Sol}^n:- M_x(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx} \cdot 2e^{-2x} dx$$

$$= 2 \int_0^{\infty} e^{x(t-2)} dx$$

$$= 2x \frac{e^{x(t-2)}}{t-2} \Big|_0^{\infty}$$

$$= \frac{2}{t-2} [e^{\infty} - e^0]$$

$$= \frac{2}{t-2} (0 - 1)$$

$$= \frac{-2}{t-2}$$

$$M_x(t) = \frac{2}{2-t}$$

$$\text{Mean } \mu = \frac{d}{dt} \left[\frac{2}{2-t} \right]_{t=0} \Rightarrow \frac{d}{dt} [2(2-t)^{-1}]_{t=0}$$

$$= [2(-1)(2-t)^{-2}(-1)]_{t=0}$$

$$= \left(\frac{2}{(2-t)^2} \right)_{t=0}$$

$$= \frac{2}{4} = \underline{\underline{\frac{1}{2}}}$$

$$E(X^2) = \frac{d^2}{dt^2} \left[\frac{2}{(2-t)^2} \right]_{t=0}$$

$$= \frac{d^2}{dt^2} [2(2-t)^{-2}]_{t=0}$$

$$= [2(-2)(2-t)^{-3}(-1)]_{t=0}$$

$$= \left[\frac{4}{(2-t)^3} \right]_{t=0}$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{2}{8} - \frac{1}{8} = \frac{1}{8}$$

If $M_X(t) = \frac{3}{3-t}$, find standard deviation of X .

Sol:- Mean = $\mu = \frac{d}{dt} \left[\frac{3}{3-t} \right]_{t=0} \Rightarrow \frac{d}{dt} [3(3-t)^{-1}]_{t=0}$

$$[3(-1)(3-t)^{-2}(-1)]_{t=0} \Rightarrow \left[\frac{3}{(3-t)^2} \right]_{t=0} = \frac{3}{9} = \frac{1}{3}$$

$$E(X^2) = \frac{d^2}{dt^2} \left[\frac{3}{(3-t)} \right]_{t=0} \Rightarrow \frac{d^2}{dt^2} [3(3-t)^{-1}]_{t=0}$$

$$= [3(-2)(3-t)^{-3}(-1)]_{t=0} \Rightarrow \left[\frac{6}{(3-t)^3} \right]_{t=0} = \frac{6}{27} = \frac{2}{9}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$S.D = \sqrt{V(X)} = \sqrt{1/9} = \underline{\underline{1/3}}$$

If $X \sim B(n, p)$ find the m.g.f of X and hence find its mean and variance.

Sol:- Since $X \sim B(n, p)$ its p.m.f is given by

$$p(x) = {}^n C_x p^x q^{n-x} \quad x = 0, 1, 2, \dots, n, \quad q = 1-p$$

then the m.g.f of X is given by

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^n e^{tx} p(x)$$

$$= \sum_{x=0}^n e^{tx} {}^n C_x p^x q^{n-x}$$

$$= \sum_{x=0}^n {}^n C_x (e^t)^x \cdot p^x q^{n-x}$$

$$= \sum_{x=0}^n {}^n C_x (pet)^x q^{n-x}$$

$$\left[\text{W.K.T } \sum_{x=0}^n {}^n C_x p^x q^{n-x} = (q+p)^n \right]$$

$$M_X(t) = \sum_{x=0}^n (q + pet)^n$$

$$\text{mean } \mu = \frac{d}{dt} [(q + pet)^n]_{t=0}$$

$$= [n (q + pet)^{n-1} \times pe^t]_{t=0}$$

$$= [n (q+p)^{n-1} \cdot p]$$

Since $q+p=1$

$$\text{mean} = \underline{\underline{n \cdot p}}$$

$$\begin{aligned}
 E(x^2) &= \left[\frac{d^2}{dt^2} \left[n(q+pet)^{n-1} \cdot pet \right] \right]_{t=0} \\
 &= n \cdot p \left[\frac{d}{dt} (q+pet)^{n-1} \cdot e^t + e^t \cdot (n-1)(q+pet)^{n-2} \cdot pe^t \right]_{t=0} \\
 &= np \left[(q+p)^{n-1} \cdot 1 + 1 \cdot (n-1)(q+p)^{n-2} \cdot p \right] \quad p+q=1 \\
 &= np [1 + (n-1)p]
 \end{aligned}$$

$$E(x^2) = np + n(n-1)p^2$$

$$\begin{aligned}
 V(x) &= E(x^2) - (E(x))^2 \\
 &= np + n^2p^2 - np^2 - n^2p^2 \\
 &= np[1-p]
 \end{aligned}$$

$$V(x) = \underline{\underline{npq}}$$

In binomial distribution, mean is always greater than variance.

If $x \sim P(\lambda)$, find its m.g.f & hence find its mean and variance.

$$\begin{aligned}
 \text{Soln: } M_x(t) &= E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot p(x) \\
 &= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} \\
 &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t)^x \cdot \lambda^x}{x!}
 \end{aligned}$$

$$\begin{aligned}
 M_x(t) &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
 &= e^{-\lambda} \cdot e^{\lambda e^t} \\
 &= e^{-\lambda + \lambda e^t} \\
 &= e^{\lambda(e^t - 1)}
 \end{aligned}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$\begin{aligned}
 \sum_{x=0}^{\infty} \frac{y^x}{x!} &\Rightarrow e^{y^x} \\
 e^{\lambda} &= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}
 \end{aligned}$$

$$\begin{aligned}
 \text{Mean: } \mu &= \left[\frac{d}{dt} (M_x(t)) \right]_{t=0} \\
 &= \left[\frac{d}{dt} (e^{\lambda(e^t - 1)}) \right]_{t=0} \\
 &= \left[e^{\lambda(e^t - 1)} \cdot \lambda e^t \right]_{t=0} \\
 &= e^{\lambda(e^0 - 1)} \cdot \lambda e^0 \\
 &= \underline{\lambda}
 \end{aligned}$$

$$\begin{aligned}
 E(x^2) &= \left[\frac{d^2}{dt^2} (M_x(t)) \right]_{t=0} \\
 &= \left[\frac{d^2}{dt^2} (e^{\lambda(e^t - 1)} \cdot \lambda e^t) \right]_{t=0} \\
 &= \lambda \left[e^{\lambda(e^t - 1)} \cdot e^t + e^t \cdot e^{\lambda(e^t - 1)} \cdot \lambda e^t \right]_{t=0} \\
 &= \lambda [1 + \lambda] \\
 &= \lambda + \lambda^2
 \end{aligned}$$

$$e^{y^2}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \lambda + \lambda^2 - \lambda^2$$

$$V(X) = \lambda$$

Mean and variance are same for Poisson distribution.

If $X \sim NB(r, p)$ find its m.g.f and hence find its mean and variance.

$$S_1:- P(X=x) = \binom{x+r-1}{r-1} p^r q^x$$

$$\text{another form is } P(X=x) = \binom{-r}{x} p^r (-q)^x \quad x=0, 1, 2, \dots$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} p^r (-q)^x$$

$$= p^r \sum_{x=0}^{\infty} (e^t)^x (-q)^x \binom{-r}{x}$$

$$= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-e^t q)^x$$

$$\left. \begin{array}{l} \text{We know that} \\ \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^x = (1-q)^{-r} \end{array} \right\} \begin{array}{l} = p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-e^t q)^x \\ = p^r (1 - qe^t)^{-r} \end{array}$$

$$M_X(t) = \frac{p^r}{(1 - qe^t)^r} = \left(\frac{p}{1 - qe^t} \right)^r$$

$$\text{Mean } \mu = \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$= \left[\frac{d}{dt} (p^r (1-qe^t)^{-r}) \right]_{t=0}$$

$$= \left[p^r (-r) (1-qe^t)^{-r-1} (-qe^t) \right]_{t=0}$$

$$= r q p^r (1-q)^{-(r+1)}$$

$$= r q p^r p^{-(r+1)}$$

$$= \frac{r q p^r}{p^{r+1}}$$

$$= \frac{r q p^r}{p^r \cdot p} = \underline{\underline{\frac{r q}{p}}}$$

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2} (r q p^r (1-qe^t)^{-(r+1)} e^t) \right]_{t=0}$$

$$p+q=1 \\ 1-q=p$$

$$= \left[r q p^r \left[-(r+1) (1-qe^t)^{-(r+2)} (-qe^t) e^t + (1-qe^t)^{-(r+1)} e^t \right] \right]_{t=0}$$

$$= \left[r q p^r \left((r+1) p^{-(r+2)} \cdot q + p^{-(r+1)} \right) \right]$$

$$= \frac{r q^2 p^r (r+1)}{p^{(r+2)}} + \frac{r q p^r}{p^{(r+1)}}$$

$$= \frac{\gamma q^2 p^{\gamma} (\gamma+1)}{p^{\gamma} \cdot p^{2\gamma}} + \frac{\gamma q p^{\gamma}}{p^{\gamma} \cdot p}$$

$$= \frac{\gamma q}{p} \left[\frac{q(\gamma+1)}{p} + 1 \right]$$

$$= \frac{\gamma q}{p} \left[\frac{\gamma q + q + p}{p} \right]$$

$$= \frac{\gamma q}{p^2} [\gamma q + 1]$$

$$E(x^2) = \frac{\gamma^2 q^2}{p^2} + \frac{\gamma q}{p^2}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{\gamma^2 q^2}{p^2} + \frac{\gamma q}{p^2} - \frac{\gamma^2 q^2}{p^2}$$

$$V(x) = \frac{\gamma q}{p^2}$$

$$p+q=1$$

$$1-q=p$$

Characteristic Function:

If m.g.f doesnot exists Then c.f is used to Calculate Moments.

Characteristic Function is denoted by $\phi_X(t)$

& defined as

$$\phi_X(t) = E(e^{itx}) \quad \begin{matrix} i = \sqrt{-1} \\ i = \text{imaginary number} \end{matrix}$$

$$E(e^{itx}) = \begin{cases} \sum e^{itx} p(x) & \text{for d.r.v} \\ \int_{-\infty}^{\infty} e^{itx} f(x) dx & \text{for c.r.v} \end{cases}$$

If $\phi_X(t)$ is given, then the r^{th} moment about origin is given

$$\mu'_r = \text{Coefficient of } \frac{(it)^r}{r!} \text{ in } \phi_X(t).$$

Properties:

① $\phi_X(0) = 1$

Proof:- $\phi_X(0) = E(e^{i(0)x}) = E(1) = 1$

② $|\phi_X(t)| \leq 1$

Proof:- $|\phi_X(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f(x) dx \right| \leq \int_{-\infty}^{\infty} |e^{itx} f(x)| dx$

$$\leq \int_{-\infty}^{\infty} |e^{itx}| |f(x)| dx$$

Since $f(x) \geq 0$

$$\leq \int_{-\infty}^{\infty} |e^{itx}| f(x) dx$$

NKT $|e^{itx}| = \sqrt{\cos^2 x + \sin^2 x} = 1$

$$\leq \int_{-\infty}^{\infty} |f(x)| dx$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\therefore |\phi_x(t)| \leq 1$$

Necessary Condition

$\phi_x(0) = 1$ & $|\phi_x(t)| \leq 1$
 $\phi_x(t)$ is uniform cont in t
 $\phi_x(t)$ & $\phi_x(-t)$ are complex conjugate

$$\phi: \mathbb{R} \rightarrow \mathbb{C}$$

③ $\phi_{cx}(t) = \phi_x(ct)$

Proof: $\phi_{cx}(t) = E(e^{itcx})$

$$\phi_x(ct) = E(e^{ictx})$$

$$\therefore \phi_{cx}(t) = \phi_x(ct)$$

④ Additive Property of C.F:

If n independent r.v $x_1, x_2, \dots, x_n$ then

$$\phi_{x_1 + x_2 + \dots + x_n}(t) = \phi_{x_1}(t) * \phi_{x_2}(t) \times \dots \times \phi_{x_n}(t)$$

Proof:- $\phi_{x_1 + x_2 + \dots + x_n}(t) = E(e^{it(x_1 + x_2 + \dots + x_n)})$

$$= E(e^{itx_1}) \cdot E(e^{itx_2}) \cdot \dots \cdot E(e^{itx_n})$$

$$= \phi_{x_1}(t) \cdot \phi_{x_2}(t) \cdot \dots \cdot \phi_{x_n}(t)$$

⑤ Effect of Change of origin & scale on CF:

Consider $U = \frac{x-a}{h}$ $\phi_U(t) = E(e^{itu})$

$$= E(e^{it(\frac{x-a}{h})})$$

$$= e^{-\frac{ita}{h}} \cdot E(e^{\frac{itx}{h}})$$

$$= e^{-\frac{iat}{h}} \phi_x(t/h)$$

CF is not independent of shifting of origin



⑥ Uniqueness theorem of CF:

Two r.v. X & Y have two CF $\phi_X(t)$ & $\phi_Y(t)$ are equal then their distributions are also equal

$$\phi_X(t) = \phi_Y(t)$$

$$X \Leftrightarrow Y$$

⑦ $\phi_X(-t) = \overline{\phi_X(t)}$

i.e. $\phi_X(-t)$ is the Complex Conjugate of $\phi_X(t)$.

Proof:- RHS $\overline{\phi_X(t)} = \overline{E[e^{itx}]} = E[\cos tx - i \sin tx]$

LHS $\phi_X(-t) = E[\cos(-tx) + i \sin(-tx)]$

$$= E[\cos tx - i \sin tx]$$

$$= \overline{\phi_X(t)}$$

Thus $\phi_X(-t) = \overline{\phi_X(t)}$

⑧ If the p.d.f is even i.e. $f(-x) = f(x)$, then the Characteristic function is real valued & even function of t .

Proof:- WKT $\phi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$

Let $x = -y \Rightarrow dx = -dy$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{it(-y)} \cdot f(-y) -dy$$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{-ity} f(y) dy \quad \because f \text{ is an even fun}$$

$$= E(e^{-itx})$$

$$\phi_X(t) = \phi_X(-t)$$

$\Rightarrow \phi_X(t)$ is an even function of t .

$$\text{WRT} \quad \overline{\phi_X(t)} = \phi_X(-t)$$

$$= \phi_X(t)$$

Since $\phi_X(t)$ is even fun

Thus $\phi_X(t)$ is real.

1) If $X \sim B(n, p)$ find its characteristic function and hence obtain its mean and variance.

Sol: Since $X \sim B(n, p)$ the p.m.f is given by

$$p(x) = {}^nC_x p^x q^{n-x} \quad \text{for } x = 0, 1, 2, \dots, n$$

The characteristic function of X is given by

$$\begin{aligned} \phi_X(t) &= E[e^{itx}] \\ &= \sum_{x=0}^n e^{itx} p(x) \\ &= \sum_{x=0}^n e^{itx} {}^nC_x p^x q^{n-x} \\ &= \sum_{x=0}^n {}^nC_x (e^{it})^x p^x q^{n-x} \\ &= \sum_{x=0}^n {}^nC_x (e^{it} p)^x q^{n-x} \end{aligned}$$

$$\phi_x(t) = (q + pe^{it})^n$$

Mean of x is given by

$$E(x) = \mu = (-i) \left[\frac{d}{dt} (\phi_x(t)) \right]_{t=0}$$

$$= -i \left[\frac{d}{dt} (q + pe^{it})^n \right]_{t=0}$$

$$= -i \left[n (q + pe^{it})^{n-1} \cdot pe^{it} \cdot i \right]_{t=0}$$

$$= -i [n \cdot pi]$$

$$= -i^2 n \cdot p$$

$$\mu = \underline{\underline{np}}$$

$$E(x^2) = (-i)^2 \left[\frac{d^2}{dt^2} (\phi_x(t)) \right]_{t=0}$$

$$= (-i)^2 \left[\frac{d}{dt} (npi (q + pe^{it})^{n-1} \cdot e^{it}) \right]_{t=0}$$

$$= (-i)^2 \left[npi \left\{ (n-1)(q + pe^{it})^{n-2} \cdot pe^{it} \cdot e^{it} \cdot i \right\} + (q + pe^{it})^{n-1} e^{it} i \right]_{t=0}$$

$$= (-i)^2 [npi \{ (n-1)pi \} + i]$$

$$= (-i)^2 [npi^2 \{ (n-1)p + 1 \}]$$

$$= (-i)^2 [np^2 i^2 (np - p + 1)]$$

$$E(x^2) = np[np + q]$$

$$E(x^2) = n^2 p^2 + npq$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= n^2 p^2 + npq - n^2 p^2$$

$$\underline{V(x) = npq}$$

2) If $X \sim P(\lambda)$ find the characteristic function ϕ and hence obtain its mean and variance.

sol: $\phi_x(t) = E(e^{itx})$

$$= \sum_{x=0}^{\infty} e^{itx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{it} \lambda)^x}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{it}}$$

$$= e^{\lambda e^{it} - \lambda}$$

$$= e^{\lambda [e^{it} - 1]}$$

$$\phi_x(t) = e^{\lambda [e^{it} - 1]}$$

$$\text{mean } \mu = (-i) \left[\frac{d}{dt} (\phi_x(t)) \right]_{t=0}$$

$$\sum_{x=0}^{\infty} \frac{y^x}{x!} = e^y$$

$$\begin{aligned}
 \text{Mean} &= -i \left[\frac{d}{dt} (e^{-\lambda} \cdot e^{\lambda e^{it}}) \right]_{t=0} \\
 &= -i \cdot e^{-\lambda} \left[\frac{d}{dt} (e^{\lambda} e^{i\lambda t} e^{\lambda e^{it}}) \right]_{t=0} \\
 &= -i e^{-\lambda} \left[\lambda \cdot e^{i\lambda t} \cdot e^{\lambda e^{it}} \cdot i \right]_{t=0} \\
 &= -i e^{-\lambda} \left[\lambda \cdot e^{\lambda} \cdot i \right]
 \end{aligned}$$

$$\text{Mean } \mu = \underline{\underline{\lambda}}$$

$$\begin{aligned}
 E(x^2) &= (-i)^2 \left[\frac{d^2}{dt^2} (\phi_x(t)) \right]_{t=0} \\
 &= i^2 \left[\frac{d}{dt} \left[\frac{d}{dt} (e^{-\lambda} e^{\lambda e^{it}}) \right] \right]_{t=0} \\
 &= i^2 e^{-\lambda} \left[\frac{d}{dt} (\lambda e^{i\lambda t} \cdot e^{\lambda e^{it}} \cdot i) \right]_{t=0} \\
 &= i^2 \cdot i e^{-\lambda} \cdot \lambda \left[\frac{d}{dt} (e^{i\lambda t} \cdot e^{\lambda e^{it}}) \right]_{t=0} \\
 &= i^3 \lambda e^{-\lambda} \left[e^{i\lambda t} \cdot \lambda e^{i\lambda t} \cdot e^{\lambda e^{it}} \cdot i + e^{\lambda e^{it}} \cdot e^{i\lambda t} \cdot i \right]_{t=0} \\
 &= i^3 \lambda e^{-\lambda} \left[\lambda \cdot e^{\lambda} \cdot i + e^{\lambda} \cdot i \right] \\
 &= i^3 \lambda e^{-\lambda} (i e^{\lambda} (\lambda + 1))
 \end{aligned}$$

$$\begin{aligned}
 i^4 &= i^2 \cdot i^2 \\
 &= -1 \times -1 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda (\lambda + 1) \\
 E(x^2) &= \lambda^2 + \lambda
 \end{aligned}$$

$$V(x) = E(x^2) - E(x)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\underline{\underline{V(x) = \lambda}}$$

3) If $X \sim N(\mu, \sigma^2)$ find the Characteristic function of X and hence obtain its mean and variance.

Sol:- WKT $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2}$$

$$-\infty \leq x \leq \infty$$

$$\sigma > 0$$

Characteristic function of X is given by

$$\begin{aligned} \phi_X(t) &= E[e^{itx}] \\ &= \int_{-\infty}^{\infty} e^{itx} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{itx} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2} dx \end{aligned}$$

$$\text{Let } \frac{x-\mu}{\sigma} = z \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$$

$$\begin{aligned} \phi_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it(\mu + \sigma z)} e^{-\frac{1}{2} z^2} \cdot \sigma dz \\ &= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\sigma z} e^{-\frac{1}{2} z^2} dz \end{aligned}$$

$$\phi_x(t) = \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz - \frac{1}{2}z^2} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2itcz)} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left[z^2 - 2itcz + i^2 c^2 t^2 - i^2 c^2 t^2 \right]} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} [z^2 - 2itcz] - \frac{i^2 c^2 t^2}{2}} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \times e^{\frac{i^2 c^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} [z - ict]^2} dz$$

$$\text{let } z - ict = u$$

$$dz = du$$

$$= e^{it\mu} \cdot e^{\frac{i^2 c^2 t^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$\text{Since } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = 1$$

$$\phi_x(t) = e^{it\mu} \cdot \frac{e^{i^2 c^2 t^2}}{2} \times 1$$

$$\phi_X(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

$$\text{Mean } \mu = (-i) \left[\frac{d}{dt} [\phi_X(t)] \right]_{t=0}$$

$$\mu = (-i) \left[\frac{d}{dt} \left(e^{it\mu - \frac{\sigma^2 t^2}{2}} \right) \right]_{t=0}$$

$$= (-i) \left[e^{it\mu - \frac{\sigma^2 t^2}{2}} \left(i\mu - \frac{\sigma^2 \times 2t}{2} \right) \right]_{t=0}$$

$$= (-i) [i\mu]$$

$$\text{Mean} = \underline{\underline{\mu}}$$

$$E(X^2) = (-i)^2 \left[\frac{d^2}{dt^2} (\phi_X(t)) \right]_{t=0}$$

$$= (-i)^2 \left[\frac{d}{dt} \left[\frac{d}{dt} \phi_X(t) \right] \right]_{t=0}$$

$$= (-i)^2 \left[\frac{d}{dt} \left(e^{it\mu - \frac{\sigma^2 t^2}{2}} (i\mu - \sigma^2 t) \right) \right]_{t=0}$$

$$= (-i)^2 \left[e^{it\mu - \frac{\sigma^2 t^2}{2}} (i\mu - \sigma^2 t)^2 + e^{it\mu - \frac{\sigma^2 t^2}{2}} \left(\frac{-\sigma^2}{2} \right) \right]_{t=0}$$

$$= (-i)^2 [i^2 \mu^2 - \sigma^2]$$

$$= (-1) [-\mu^2 - \sigma^2]$$

$$E(X^2) = \mu^2 + \sigma^2$$

$$V(x) = E(x)^2 - (E(x))^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$\underline{\underline{V(x) = \sigma^2}}$$

If X is a Continuous random variable with characteristic function $\phi_X(t)$, then its p.d.f is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt.$$

Find the p.d.f corresponding to the characteristic