

Unit – 3

Probability Inequalities and Generating Functions

3.1

Probability Inequalities

Inequalities are useful for bounding quantities that might otherwise be hard to compute. They will also be used in the **theory of convergence** and **limit theorems**.

Chebychev's Inequality

When we want to find the probability of an event described by a random variable, its c.d.f or p.d.f. or p.m.f. is required. If it is not known but its *mean* and *variance* are known, we can use **Chebychev's inequality** to find the **upper bound** or **lower bound** for the probability of the event.

Theorem 1: If X is a random variable with mean μ and variance σ^2 , then

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \dots \dots \dots (1)$$

where $\epsilon > 0$

Proof: The proof is given for a continuous random variable. Let X be a continuous r.v. with p.d.f. $f(x)$. Then

$$\begin{aligned} \sigma^2 &= E(X - E(X))^2 = E(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int_{-\infty}^{\mu - \epsilon} (x - \mu)^2 f(x) dx + \int_{\mu - \epsilon}^{\mu + \epsilon} (x - \mu)^2 f(x) dx + \int_{\mu + \epsilon}^{\infty} (x - \mu)^2 f(x) dx \\ &\geq \int_{-\infty}^{\mu - \epsilon} (x - \mu)^2 f(x) dx + \int_{\mu + \epsilon}^{\infty} (x - \mu)^2 f(x) dx \end{aligned}$$

In the first integral,

$$x \leq \mu - \epsilon \Rightarrow -x \geq -\mu + \epsilon \Rightarrow -(x - \mu) \geq \epsilon \Rightarrow (x - \mu)^2 \geq \epsilon^2$$

In the third integral, $x \geq \mu + \epsilon \Rightarrow (x - \mu) \geq \epsilon \Rightarrow (x - \mu)^2 \geq \epsilon^2$

$$\begin{aligned} \therefore \sigma^2 &\geq \epsilon^2 \left[\int_{-\infty}^{\mu-\epsilon} f(x)dx + \int_{\mu+\epsilon}^{\infty} f(x)dx \right] \\ &= \epsilon^2 [P(X \leq \mu - \epsilon) + P(X \geq \mu + \epsilon)] \\ &= \epsilon^2 P[\mu - \epsilon \geq X \geq \mu + \epsilon] = \epsilon^2 P[-\epsilon \geq X - \mu \geq \epsilon] \\ &= \epsilon^2 P[|X - \mu| \geq \epsilon] \end{aligned}$$

$$\text{Thus, } \sigma^2 \geq \epsilon^2 P[|X - \mu| \geq \epsilon] \Rightarrow P[|X - \mu| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$$

Note: The proof is similar as in the case of d.r.v. X except that integration is replaced by summation.

Alternative forms:

$$\begin{aligned} P[|X - \mu| < \epsilon] &= 1 - P[|X - \mu| \geq \epsilon] \geq 1 - \frac{\sigma^2}{\epsilon^2} \\ &\Rightarrow P[|X - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{\epsilon^2} \dots \dots \dots (2) \end{aligned}$$

Let $\epsilon = k\sigma$ for $k > 0$. Then from (1), we have

$$P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2} \dots \dots \dots (3)$$

and from (2), we have

$$P[|X - \mu| < k\sigma] \geq 1 - \frac{1}{k^2} \dots \dots \dots (4)$$

Example 1: If a r.v. X has mean 12 and variance 9 and the probability distribution is unknown, then find $P(6 < X < 18)$.

Solution: Since the probability distribution of X is not known, we can't find the value of the required probability. We can find only a lower bound for probability using Chebychev's inequality. We have, for $\epsilon > 0$.

$$P[|X - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{\epsilon^2}$$

Given $E(X) = \mu = 12$ and $V(X) = \sigma^2 = 9$.

$$\text{Then } P[|X - 12| < \epsilon] \geq 1 - \frac{9}{\epsilon^2} \Rightarrow P[-\epsilon < (X - 12) < \epsilon] \geq 1 - \frac{9}{\epsilon^2}$$

$$\Rightarrow P[12 - \epsilon < X < 12 + \epsilon] \geq 1 - \frac{9}{\epsilon^2}$$

$$\text{Let } \epsilon = 6. \text{ Then } P[6 < X < 18] \geq 1 - \frac{9}{36} = 1 - \frac{1}{4} = 0.75$$

$$\Rightarrow P[6 < X < 18] \geq 0.75$$

Thus, the probability of X lying between 6 and 18 is atleast 75%.

Example 2: A d.r.v. X takes the values $-1, 0$ and 1 with probabilities $\frac{1}{8}, \frac{3}{4}$ and $\frac{1}{8}$ respectively. Evaluate $P[|X - \mu| \geq 2\sigma]$ and compare it with the upper bound given by Chebychev's inequality.

Solution: We have,

X	-1	0	1
$p(x)$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{8}$

$$\text{Then } E(X) = \mu = \sum xp(x) = -1 \times \frac{1}{8} + 0 \times \frac{3}{4} + 1 \times \frac{1}{8} = 0$$

$$\text{and } E(X^2) = \sum x^2p(x) = 1 \times \frac{1}{8} + 0 \times \frac{3}{4} + 1 \times \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

Hence $\sigma^2 = V(X) = E(X^2) - (E(X))^2 = \frac{1}{4} - 0 = \frac{1}{4}$

Consider $P[|X - \mu| \geq 2\sigma] = P\left[|X| \geq 2 \cdot \frac{1}{2}\right] = P[|X| \geq 1]$

$$= P(X = -1, 1)$$

$$= P(X = -1) + P(X = 1)$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$$

$\Rightarrow P[|X - \mu| \geq 2\sigma] \leq 0.25$

On the other hand, by Chebychev's inequality,

$$P[|X - \mu| \geq 2\sigma] \leq \frac{1}{2^2} = \frac{1}{4}$$

Note that the two values are same.

Example 3: Use Chebychev's inequality to find how many times must a fair coin be tossed in order that the probability that the ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55 will be at least 0.95.

Solution: Let X denote the number of heads obtained when a fair coin is tossed n times. Then $X \sim B(n, p)$. That is $E(X) = np$ and $V(X) = npq$.

Let $Y = \frac{X}{n}$. Then $E(Y) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \frac{np}{n} = p$

and $V(Y) = V\left(\frac{X}{n}\right) = E\left(\left(\frac{X}{n}\right)^2\right) - \left(E\left(\frac{X}{n}\right)\right)^2 = \frac{1}{n^2}(E(X^2) - (E(X))^2)$

$$= \frac{1}{n^2}V(X) = \frac{npq}{n^2} = \frac{pq}{n}.$$

Since $p = \frac{1}{2}$ for a fair coin, $E(Y) = \frac{1}{2}$ and $V(Y) = \frac{1}{4n}$

By Chebychev's inequality for Y

$$P\left[\left|Y - \frac{1}{2}\right| < \epsilon\right] \geq 1 - \frac{\frac{1}{4n}}{\epsilon^2} = 1 - \frac{1}{4n\epsilon^2}$$

$$\Rightarrow P\left[\frac{1}{2} - \epsilon < Y < \frac{1}{2} + \epsilon\right] \geq 1 - \frac{1}{4n\epsilon^2}$$

Notice that, if $\epsilon = 0.05$ then $P(0.45 < Y < 0.55) \geq 1 - \frac{1}{4n\epsilon^2}$

Now, find n when $\epsilon = 0.05$ and $1 - \frac{1}{4n\epsilon^2} = 0.95 \Rightarrow 1 - \frac{1}{n \times 4 \times (0.05)^2} = 0.95$

$$\Rightarrow 1 - \frac{1}{0.01 \times n} = 0.95 \Rightarrow \frac{1}{0.01 \times n} = 0.05 \Rightarrow n = \frac{1}{0.01 \times 0.05} = \frac{1}{0.0005} = \frac{10000}{5} = 2000$$

Thus, $n = 2000$

Bienayme – Chebychev's inequality

Theorem 3: Let $g(X)$ be a non-negative function of a r.v. X . Then for every $k > 0$, we have

$$P[g(X) \geq k] \leq \frac{E[g(X)]}{k} \dots \dots \dots (1)$$

Proof: Here we shall prove the theorem for continuous random variable. The proof can be adapted to the case of discrete random variable on replacing integration by summation over the given range of the variable.

Let S be the set of all X for which $g(X) \geq k$. That is, $S = \{x \mid g(x) \geq k\}$.

Now, $E[g(X)] = \int_S g(x) f(x) dx$, where $f(x)$ is the p.d.f. of X

$$\geq k \int_S f(x) dx \quad (\text{on } S, g(x) \geq k)$$

$$= kP[g(X) \geq k]$$

$$\Rightarrow P[g(X) \geq k] \leq \frac{E[g(X)]}{k}$$

Note:

1. If $g(X) = (X - E(X))^2 = (X - \mu)^2$, then $E(g(X)) = V(X) = \sigma^2$ and replacing k by $\epsilon^2 \sigma^2$ in equation (1), we get

$$P[(X - \mu)^2 \geq \epsilon^2 \sigma^2] \leq \frac{\sigma^2}{\epsilon^2 \sigma^2} = \frac{1}{\epsilon^2}$$
$$\Rightarrow P[|X - \mu| \geq \epsilon \sigma] \leq \frac{1}{\epsilon^2}$$

which is **Chebychev's inequality**.

2. If $g(X) = |X|$ in (1), then we get for any $k > 0$,

$$P[|X| \geq k] \leq \frac{E(|X|)}{k}$$

which is known as **Markov's inequality**.

3. If $g(X) = |X|^r$ in (1), then we get

$$P[|X|^r \geq k^r] \leq \frac{E(|X|^r)}{k^r}$$

which is known as **generalized Markov's inequality**.

Cauchy – Schwartz Inequality

When the j.p.d.f. of X and Y is known, upper bound for expected value of the product of X and Y can be found by using Cauchy – Schwartz inequality when second moments about origin of X and Y are given (*i. e.*, $E(X^2)$ and $E(Y^2)$ are given).

Theorem 2: For any two random variables X and Y

$$(E(XY))^2 \leq E(X^2)E(Y^2)$$

Proof: Consider $E(X - tY)^2 \geq 0$ for any real number t . That is,

$$E(X^2 - 2tXY + t^2Y^2) = E(X^2) - 2tE(XY) + t^2E(Y^2) \geq 0$$

which is a quadratic expression in t . This expression is always positive only when t

has complex roots. This is possible only when discriminant of the expression is negative. Thus,

$$4(E(XY))^2 - 4E(X^2)E(Y^2) \leq 0$$

$$\Rightarrow (E(XY))^2 \leq E(X^2)E(Y^2)$$

Hence the result.

Example 4: The j.p.d.f. of (X, Y) is given by

$$f(x, y) = \frac{x+y}{21} \text{ for } x = 1, 2, 3 \text{ and } y = 1, 2.$$

Verify whether Cauchy-Schwartz inequality.

Solution: The joint and marginal p.m.fs $f_1(x)$ and $f_2(y)$ of X and Y respectively are given in the following table.

$\begin{matrix} X \\ Y \end{matrix}$	1	2	3	$f_2(y)$
1	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{9}{21}$
2	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{12}{21}$
$f_1(x)$	$\frac{5}{21}$	$\frac{7}{21}$	$\frac{9}{21}$	1

$$E(X) = \sum_{x=1}^3 x f_1(x) = 1 \times \frac{5}{21} + 2 \times \frac{7}{21} + 3 \times \frac{9}{21} = \frac{46}{21}$$

$$E(X^2) = \sum_{x=1}^3 x^2 f(x) = 1^2 \times \frac{5}{21} + 2^2 \times \frac{7}{21} + 3^2 \times \frac{9}{21} = \frac{114}{21}$$

$$E(Y) = \sum_{y=1}^2 y f_2(y) = 1 \times \frac{9}{21} + 2 \times \frac{12}{21} = \frac{33}{21}$$

$$E(Y^2) = \sum_{y=1}^2 y^2 f_2(y) = 1^2 \times \frac{9}{21} + 2^2 \times \frac{12}{21} = \frac{57}{21}$$

$$\begin{aligned} E(XY) &= \sum_{x=1}^3 \sum_{y=1}^2 xyf(x, y) \\ &= 1 \times 1 \times \frac{2}{21} + 1 \times 2 \times \frac{3}{21} + 1 \times 3 \times \frac{4}{21} + 2 \times 1 \times \frac{3}{21} + 2 \times 2 \times \frac{4}{21} + 2 \times 3 \times \frac{5}{21} \\ &= \frac{1}{21} (2 + 6 + 12 + 6 + 16 + 30) = \frac{71}{21} \end{aligned}$$

Verification of Cauchy-Schwartz inequality:

$$\text{Here } (E(XY))^2 = \left(\frac{71}{21}\right)^2 = 11.755 \text{ and } E(X^2)E(Y^2) = \frac{114}{21} \times \frac{57}{21} = 14.735$$

Note that $(E(XY))^2 \leq E(X^2)E(Y^2)$.

Example 5: Let X and Y be c.r.vs with j.p.d.f.

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Verify Cauchy-Schwartz inequality.

Solution: The m.p.d.f. of X is given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 (x + y) dy = \left[xy + \frac{y^2}{2} \right]_0^1 = x + \frac{1}{2}$$

$$f_1(x) = \begin{cases} x + \frac{1}{2}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Since $f(x, y)$ is symmetric in x and y , the m.p.d.f. of Y is given by

$$f_2(y) = \begin{cases} y + \frac{1}{2}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$E(X^2) = \int_0^1 x^2 \left(x + \frac{1}{2}\right) dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{10}{24}$$

Similarly, $E(Y^2) = \frac{10}{24}$. Now,

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy(x + y) dx dy \\ &= \int_0^1 \left\{ \int_0^1 (x^2y + xy^2) dx \right\} dy = \int_0^1 \left(\frac{x^3y}{3} + \frac{x^2y^2}{2} \right)_0^1 dy \\ &= \frac{1}{6} \int_0^1 (2y + 3y^2) dy = \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^1 = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Thus $(E(XY))^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} = 0.111$, and $E(X^2)E(Y^2) = \frac{10}{24} \times \frac{10}{24} = 0.1736$

Hence $(E(XY))^2 \leq E(X^2)E(Y^2)$.

P1:

A symmetric die is thrown 600 times. Find the lower bound for the probability of getting 80 to 120 sixes.

Solution:

Let X be the total number of sixes.

Then $X \sim B\left(600, \frac{1}{6}\right)$, $E(X) = np = 600 \times \frac{1}{6} = 100$

and $V(X) = np(1 - p) = 600 \times \frac{1}{6} \times \frac{5}{6} = \frac{500}{6}$.

Using Chebychev's inequality, we get

$$P\{|X - E(X)| < k\sigma\} \geq 1 - \frac{1}{k^2} \Rightarrow P\left\{|X - 100| < k\sqrt{\frac{500}{6}}\right\} \geq 1 - \frac{1}{k^2}$$

Therefore, $P\left\{100 - k\sqrt{\frac{500}{6}} < X < 100 + k\sqrt{\frac{500}{6}}\right\} \geq 1 - \frac{1}{k^2}$

Taking $k = \frac{20}{\sqrt{\frac{500}{6}}}$, we get $P(80 < X < 120) \geq 1 - \frac{1}{400 \times \left(\frac{6}{500}\right)} = \frac{19}{24}$

The lower bound for the probability of getting 80 to 120 sixes.

P2:

For geometric distribution $p(x) = 2^{-x}$; $x = 1, 2, 3, \dots$, prove that Chebychev's inequality gives $P\{|X - 2| \leq 2\} > \frac{1}{2}$, while the actual probability is $\frac{15}{16}$.

Solution:

$$\begin{aligned} E(X) &= \sum x p(x) = \sum_{x=1}^{\infty} \frac{x}{2^x} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots \\ &= \frac{1}{2}(1 + 2A + 3A^2 + 4A^3 + \dots) = \frac{1}{2}(1 - A)^{-2} = 2, \left(A = \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum x^2 p(x) = \sum_{x=1}^{\infty} \frac{x^2}{2^x} = \frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \dots \\ &= \frac{1}{2}\{1 + 4A + 9A^2 + \dots\}, \text{ where } A = \frac{1}{2} \\ &= \frac{1}{2}(1 + A)(1 - A)^{-3} = 6 \end{aligned}$$

$$\therefore \text{Var}(X) = \sigma^2 = E(X^2) - \{E(X)\}^2 = 6 - 4 = 2 \Rightarrow \sigma = \sqrt{2}$$

Using Chebychev's inequality, we get $P\{|X - E(X)| > k\sigma\} \leq \frac{1}{k^2}$

With $k = \sqrt{2}$, we get $P\{|X - 2| > \sqrt{2} \cdot \sqrt{2}\} \leq \frac{1}{2} \Rightarrow P[|X - 2| \leq 2] > 1 - \frac{1}{2} = \frac{1}{2}$

The actual probability is given by

$$\begin{aligned} P\{|X - 2| \leq 2\} &= P\{0 \leq X \leq 4\} = P\{X = 1, 2, 3 \text{ or } 4\} \\ &= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{15}{16} \end{aligned}$$

P3:

Does there exist a variate X for which

$$P[\mu - 2\sigma \leq X \leq \mu + 2\sigma] = 0.6 \quad \text{..... (1)}$$

Solution:

We have

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = P(|X - \mu| \leq 2\sigma)$$

By Chebychev's inequality $P(|X - \mu| \leq 2\sigma) \geq 1 - \frac{1}{2^2} = 0.75$

Since lower bound for the probability is 0.75, there does not exist a r.v. X for which the equation (1) holds.

P4:

(a) For a p.m.f

$$p(x) = \begin{cases} \frac{1}{8} & , \quad x = -1 \\ \frac{6}{8} & , \quad x = 0 \\ \frac{1}{8} & , \quad x = 1 \end{cases}$$

Find $P(|X - \mu| \geq 2\sigma)$.

(b) Compare this result with that obtained by using Chebychev's inequality.

Solution:

(a)

$x:$	-1	0	1
$p(x):$	$\frac{1}{8}$	$\frac{6}{8}$	$\frac{1}{8}$

$$\therefore \mu = E(X) = \sum x p(x) = -1 \times \frac{1}{8} + 1 \times \frac{1}{8} = 0 \text{ and}$$

$$E(X^2) = \sum x^2 p(x) = 1 \times \frac{1}{8} + 1 \times \frac{1}{8} = \frac{1}{4}$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{4} \Rightarrow \sigma = \frac{1}{2}$$

$$P\{|X - \mu| \geq 2\sigma\} = P\{|X| \geq 1\} = 1 - P(|X| < 1)$$

$$= 1 - P(-1 < X < 1) = 1 - P(X = 0) = \frac{1}{4}$$

$$(b) P\{|X - \mu| \geq 2\sigma\} \leq \frac{1}{2^2} \text{ (By Chebychev's inequality: } P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2} \text{)}$$

In this case, both results are same.

Note: This example shows that, in general, Chebychev's inequality cannot be improved.

3.1 .Probability inequalities

Exercise:

1. The Chebychev's inequality for random variable X is $(-2 < X < \infty) \geq \frac{21}{25}$, find $E(X)$ and $V(X)$.
2. Two unbiased dice are thrown. If X is the sum of the numbers showing up, prove that $P(|X - 7| \geq 3) \leq \frac{35}{54}$. Compare this with the actual probability.
3. If X is the number scored in a throw of a fair die, find the upper bound for $P(|X - \mu| \geq 2.5)$ where $\mu = E(X)$. Also find the actual probability.
4. If X is a r.v. such that $E(X) = 3$ and $E(X^2) = 13$, find the lower bound of $P(-2 \leq X \leq 8)$.
5. A discrete random variable X is specified by $p(-a) = p(a) = \frac{1}{8}$ and $p(0) = \frac{3}{4}$.
Compute
 - (i) $P(|X| \geq 2\sigma)$ and
 - (ii) Chebychev's inequality bound.

Answers:

1. $E(X) = 3$ and $V(X) = 4$
2. Actual probability = $\frac{1}{3}$
3. Upper bound = 0.47 and actual probability = 0
4. $\frac{21}{25}$
5. (i) $\frac{1}{4}$ (ii) $\frac{1}{4}$

1. The average IQ of the students in one calculus class is 110, with a standard deviation of 5; the average IQ of students in another class is 106, with a standard deviation of 4. A student has an IQ of 112, in which class is he ranked higher?

$$Z_1 = \frac{112 - 110}{5}$$

$$Z_1 = 0.4$$

$$Z_2 = \frac{112 - 106}{4}$$

$$Z_2 = 1.5$$



Ranked higher in
2nd Class.

2. The average price of the wagon at Car Dealer A is \$25,000, with a standard deviation of \$4000. The average price at Car Dealer B is \$20,000, with a standard deviation of \$2000. If I spent \$23,000 at dealer A and my sister paid \$18,000 at dealer B, which was a better deal?

$$Z_A = \frac{23,000 - 25,000}{4,000}$$

$$Z_A = -0.5$$

$$Z_B = \frac{18,000 - 20,000}{2,000}$$

$$Z_B = -1$$



Dealer B
was the
better deal.

3. The average score on an English final examination was 85, with a standard deviation of 5; the average score on a history final exam was 110, with a standard deviation of 8. I made an 80 on the English test and a 100 on the history test. Which test was better?

$$Z_E = \frac{80 - 85}{5}$$

$$Z_E = -1$$

$$Z_H = \frac{100 - 110}{8}$$

$$Z_H = -1.25$$



English

4. The average age of the accountants at Three Rivers Corp. is 26, with a standard deviation of 6; the average salary of the accountants is \$31,000 with a standard deviation of \$4000. I'm applying at both places and I'm 31 and want to make \$35,000, in which category will I be higher?

$$Z_{Age} = \frac{31 - 26}{6}$$

$$Z_A = 0.83$$

$$Z_{Salary} = \frac{35,000 - 31,000}{4,000}$$

$$Z_S = 1$$

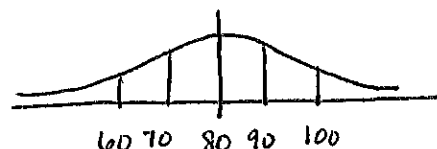


Salary

5. Using Chebyshev's theorem, solve the following problems for a distribution with a mean of 80 and a standard deviation of 10

- a. At least what percentage of values will fall between 60 and 100?

$$K = 2 \rightarrow \left(1 - \frac{1}{2^2}\right)(100) = 75\%$$



- b. At least what percentage of values will fall between 65 and 95?

$$K = 1.5 \rightarrow \left(1 - \frac{1}{1.5^2}\right)(100) = 56\%$$

$$Z = \frac{95 - 80}{10}$$

$$Z = 1.5$$

$$Z = \frac{65 - 80}{10}$$

$$Z = -1.5$$

Thus, $K = 1.5$

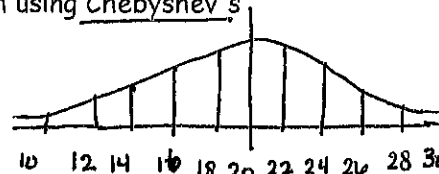
6. The mean of a distribution is 20 and the standard deviation is 2. Answer each using Chebyshev's theorem.

- a. At least what percentage of the values will fall between 10 and 30?

$$\begin{array}{l} K = 5 \\ \text{from Graph} \end{array} \quad \text{or} \quad \begin{array}{l} z = \frac{30-20}{2} \\ z = 5 \end{array} \quad \left\} \quad \left[1 - \frac{1}{(5)^2} \right] (100) = \boxed{96\%}$$

- b. At least what percentage of the values will fall between 12 and 28?

$$\begin{array}{l} K = 4 \\ z = 4 \end{array} \quad \text{or} \quad \begin{array}{l} z = \frac{28-20}{2} \\ z = 4 \end{array} \quad \left\} \quad \left[1 - \frac{1}{(4)^2} \right] (100) = \boxed{93.75\%}$$



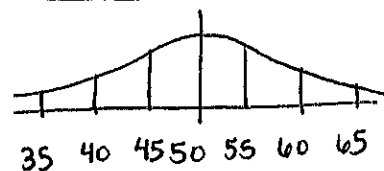
7. The mean of a distribution is 50 and the standard deviation is 5. Answer each using The Empirical Rule.

- a. At least what percentage of the values will fall between 45 and 55?

$$1 \text{ st dev} \rightarrow \boxed{68\%} \quad \text{OR} \quad z = \frac{55-50}{5} = 1$$

- b. At least what percentage of the values will fall between 35 and 65?

$$3 \text{ st dev} \rightarrow \boxed{99.7\%}$$



8. A sample of the hourly wages of employees who work in restaurants in a large city has a mean of \$8.58 and a standard deviation of \$1.29.

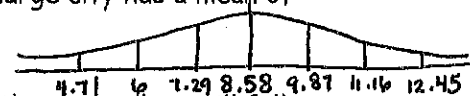
- a. Using Chebyshev's theorem, find the range in which at least 75% of the data values will fall.

$$\begin{array}{l} .75 = 1 - \frac{1}{K^2} \\ -\frac{1}{K^2} = -0.25 \end{array} \rightarrow \begin{array}{l} -0.25K^2 = -1 \\ K^2 = 4 \\ K = 2 \end{array} \quad \therefore \quad 8.58 \pm 2(1.29)$$

$$\boxed{\$6 \text{ to } \$11.16}$$

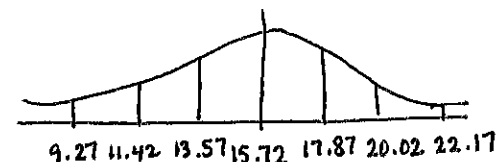
- b. Using the Empirical Rule, find the range in which at least 95% of the data values will fall.

$$95\% \rightarrow 2 \text{ st dev} \quad \text{Thus, range is } \boxed{\$6 \text{ to } \$11.16}$$



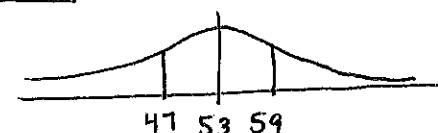
9. A sample of the labor costs per hour to assemble a certain product has a mean of \$15.72 and a standard deviation of \$2.15. Using Chebyshev's theorem, find the values in which at least 88.89% of the data will lie.

$$\begin{array}{l} .8889 = 1 - \frac{1}{K^2} \\ -0.1111 = -\frac{1}{K^2} \\ 0.1111 K^2 = 1 \\ K^2 = 9 \end{array} \rightarrow \begin{array}{l} K = 3 \\ \therefore 15.72 \pm 3(2.15) \\ \boxed{\$9.27 \text{ to } \$22.17} \end{array}$$



10. The average score on a special test of knowledge of wood refinishing has a mean of 53 and a standard deviation of 6. Using the Empirical Rule, find the values in which at least 68% of the data will lie.

$$68\% \rightarrow 1 \text{ st dev} \quad \therefore \quad 53 \pm 1(6) \\ \boxed{47 \text{ to } 59}$$



3.2

Moment Generating Function

Certain derivations presented in modules 2.4, 2.5 and 2.6 have been somewhat heavy on algebra. For example, determining the mean and variance of the **Binomial distribution** turned out to be fairly tiresome. Another example of hard work was determining the set of probabilities associated with a sum , $P(X + Y = t)$. Many of these tasks are greatly simplified by using **probability generating functions**.

Moment Generating Function: The moment generating function (m.g.f) of a random variable X is denoted by $M_X(t)$ and it is defined as

$$M_X(t) = E(e^{tX})$$

$$\begin{aligned}\therefore M_X(t) &= E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots\right] \\ &= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots + \frac{t^r}{r!}E(X^r) + \dots + \infty \\ \therefore M_X(t) &= 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \dots + \frac{t^r}{r!}\mu'_r + \dots + \infty \quad \dots \dots (1)\end{aligned}$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

which gives the m.g.f in terms of moments.

Therefore the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ'_r , where $r = 1, 2, 3, \dots$ and $\mu'_r = E(X^r)$, moment about origin.

The m.g.f of X about mean $\mu = \mu'_1 = E(X)$ is defined as

$$\begin{aligned}M_{X-\mu}(t) &= E[e^{t(X-\mu)}] = E\left[1 + \frac{t}{1!}(X-\mu) + \frac{t^2}{2!}(X-\mu)^2 + \frac{t^3}{3!}(X-\mu)^3 + \dots\right] \\ &= 1 + \frac{t}{1!}E(X-\mu) + \frac{t^2}{2!}E(X-\mu)^2 + \frac{t^3}{3!}E(X-\mu)^3 + \dots\end{aligned}$$

$$= 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \dots$$

where $E(x - \mu)^r = \mu_r$ is known as the r^{th} central moment for $r = 1, 2, \dots$

Note that $\mu_1 = E(X - \mu) = E(X) - \mu = \mu - \mu = 0$

Since $M_X(t)$ generates moments, it is called **moment generating function**.

If X is a discrete random variable with p.m.f. $p(x)$ then

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable with p.d.f. $f(x)$, then

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Moments Using Moment Generating Function:

Differentiating equation (1) with respect to t and then putting $t = 0$, gives

$$\mu'_1 = \left[\frac{d}{dt} M_X(t) \right]_{t=0}$$

$$\mu'_2 = \left[\frac{d^2}{dt^2} M_X(t) \right]_{t=0}$$

In general,

$$\mu'_r = \left[\frac{d^r}{dt^r} M_X(t) \right]_{t=0}, \quad r = 1, 2, 3, \dots$$

Note: Moment generating function $M_X(t)$ is used to calculate the higher moments.

Theorems on Moment Generating Function:

Theorem 1: $M_{aX}(t) = M_X(at)$, where a is a constant.

Proof: By definition $M_{aX}(t) = E(e^{taX}) = E(e^{atX}) = M_X(at)$

Therefore, $M_{aX}(t) = M_X(at)$

Theorem 2: The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, i. e., $M_{X_1+X_2+X_3+\dots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t) \dots M_{X_n}(t)$

Proof: By definition,

$$\begin{aligned} M_{X_1+X_2+X_3+\dots+X_n}(t) &= [E^{t(X_1+X_2+X_3+\dots+X_n)}] \\ &= E(e^{tX_1})E(e^{tX_2})E(e^{tX_3}) \dots E(e^{tX_n}) \\ &\quad \text{(Since } X_1, X_2, \dots, X_n \text{ are independent).} \end{aligned}$$

Therefore, $M_{X_1+X_2+X_3+\dots+X_n}(t) = M_{X_1}(t)M_{X_2}(r) \dots M_{X_n}(t)$
Hence the proof.

Uniqueness Theorem of Moment Generating Function:

The m.g.f. of a distribution, if exists, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one m.g.f (provided it exists) and corresponding to a given m.g.f, there is only one probability distribution. Hence $M_X(t) = M_Y(t) \Rightarrow X$ and Y are identically distributed.

Effect of Change of Origin and Scale on Moment Generating Function:

Let a random variable X be transformed to a new variable U by changing both the origin and scale in X as $= \frac{X-a}{h}$, where a and h are constants.

The m.g.f of U (about origin) is given by

$$\begin{aligned} M_U(t) &= E(e^{tU}) = E\left[e^{t\left(\frac{X-a}{h}\right)}\right] = E\left(e^{\left(\frac{tX}{h}-\frac{ta}{h}\right)}\right) = e^{-\frac{at}{h}}E\left(e^{\left(\frac{tX}{h}\right)}\right) \\ \therefore M_{\frac{X-a}{h}}(t) &= e^{-\frac{at}{h}}M_X\left(\frac{t}{h}\right) \end{aligned}$$

Note: If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$

Example 1: If X represents the outcome when a fair die is tossed, find the m.g.f. of X and hence, find $E(X)$ and $Var(X)$.

Solution: When a fair die is tossed

$$P(X = x) = \frac{1}{6}, \quad x = 1, 2, 3, 4, 5, 6$$

$$\begin{aligned} \therefore M_X(t) &= \sum_{x=1}^6 e^{tx} P(X = x) = \frac{1}{6} \sum_{x=1}^6 e^{tx} \\ &= \frac{1}{6} (e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}) \end{aligned}$$

$$\begin{aligned} E(X) &= \left[\frac{d}{dt} M_X(t) \right]_{t=0} = \frac{1}{6} [e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0} \\ &= \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2} \end{aligned}$$

$$\therefore \text{Mean} = E(X) = \frac{7}{2}$$

$$\begin{aligned} \text{Now, } E[X^2]_{t=0} &= \left\{ \frac{d^2}{dt^2} [M_X(t)] \right\}_{t=0} \\ &= \frac{1}{6} [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0} \\ &= \frac{1}{6} (1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} \end{aligned}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Example 2: Find the m.g.f. of the random variable X whose probability function $P(X = x) = \frac{1}{2^x}, x = 1, 2, 3, \dots$ and hence find its mean.

Solution: By definition,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(X = x) = \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2^x} \right) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2} \right)^x$$

$$\begin{aligned}
&= \left[\frac{e^t}{2} + \left(\frac{e^t}{2} \right)^2 + \left(\frac{e^t}{2} \right)^3 + \dots \right] \\
&= \frac{e^t}{2} \left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2} \right)^2 + \dots \right] = \frac{e^t}{2} \left(1 - \frac{e^t}{2} \right)^{-1} \\
&= \frac{e^t}{2} \left(\frac{2 - e^t}{2} \right)^{-1} = \frac{e^t}{2} \left(\frac{2}{2 - e^t} \right) = \frac{e^t}{2 - e^t}
\end{aligned}$$

Therefore, $M_X(t) = \frac{e^t}{2 - e^t}$

$$\mu'_1 = \left[\frac{d}{dt} M_X(t) \right]_{t=0} = \left[\frac{d}{dt} \left(\frac{e^t}{2 - e^t} \right) \right]_{t=0} = \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2} \right]_{t=0} = \frac{(2 - 1)1 + 1}{(2 - 1)^2} = 2$$

Thus, $E(X) = \text{mean} = 2$

Example 3: If the moments of a random variable X are defined by $E(X^r) = 0.6, r = 1, 2, \dots$. Show that $P(X = 0) = 0.4, P(X = 1) = 0.6$, and $P(X \geq 2) = 0$.

Solution: We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

where $\mu'_r = E(X^r) = 0.6$

$$\begin{aligned}
\therefore M_X(t) &= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} \mu'_r = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6) = 1 + (0.6) \left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \\
&= 1 + (0.6)(e^t - 1) = 1 - 0.6 + 0.6e^t = 0.4 + 0.6e^t \quad \dots \dots (1)
\end{aligned}$$

But by definition,

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} e^{tx} P(X = x)$$

$$M_X(t) = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \dots + \dots \dots (2)$$

From equations (1) and (2), we have

$$0.4 + 0.6e^t = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \dots$$

Equating the coefficients of like terms on both sides,

$$P(X = 0) = 0.4, P(X = 1) = 0.6$$

$$P(X = 2) = P(X = 3) = \dots = 0 \implies P(X \geq 2) = 0$$

Example 4: Find the m.g.f. of a random variable whose moments are $\mu_r = (r + 1)! 2^r$.

Solution: By definition, we have $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)! 2^r = \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= 1 + 2(2t) + 3(2t)^2 + \dots = (1 - 2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$\therefore M_X(t) = \frac{1}{(1-2t)^2}$$

Example 5: If $X \sim B(n, p)$, find the m.g.f of X and hence find its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n \text{ and } q = 1 - p.$$

Then the m.g.f. of X is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^n e^{tx} p(x) = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n \binom{n}{x} (pe^t)^x q^{n-x}$$

$$\implies M_X(t) = (q + pe^t)^n$$

$$\frac{d}{dt} M_X(t) = n(q + pe^t)^{n-1} pe^t \implies \text{Mean} = \mu'_1 = \left[\frac{dM_X(t)}{dt} \right]_{t=0} = np$$

Next, $\frac{d^2}{dt^2}(M_X(t)) = np[(n-1)(q+pe^t)^{n-2}pe^{2t} + (q+pe^t)^{n-1}e^t]$

$$\Rightarrow \mu'_2 = \left[\frac{d^2}{dt^2}(M_X(t)) \right]_{t=0} = np[(n-1)p + 1] = np[np - p + 1] = np(np + q)$$

$$\Rightarrow \mu'_2 = n^2p^2 + npq$$

Now, variance $= \sigma^2 = \mu'_2 - (\mu'_1)^2 = n^2p^2 + npq - n^2p^2 = npq$

Thus, $\mu = np$ and $\sigma^2 = npq$

Note that $\sigma^2 = npq = \mu q$ where $(0 < q < 1)$. Thus, $\mu > \sigma^2$.

Note: For binomial distribution, mean is always greater than variance.

Example 6 : If $X \sim P(\lambda)$, find its m.g.f. and hence find its mean and variance.

Solution: Since $X \sim P(\lambda)$, then its p.m.f. is given by

$$p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, \dots \text{ and } \lambda > 0$$

The m.g.f. of X is given by

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

$$\Rightarrow M_X(t) = e^{\lambda(e^t - 1)}$$

Since $\frac{d}{dt}(M_X(t)) = e^{\lambda(e^t - 1)} \lambda e^t$; Mean $= \mu = \mu' = \left[\frac{d}{dt}(M_X(t)) \right]_{t=0} = \lambda$.

Now, $\frac{d^2}{dt^2}(M_X(t)) = \lambda[e^{\lambda(e^t - 1)} e^t + e^{\lambda(e^t - 1)} \lambda e^{2t}]$

Then $\mu'_2 = \left[\frac{d^2 M_X(t)}{dt^2} \right]_{t=0} = \lambda(1 + \lambda) = \lambda + \lambda^2$

Thus, variance $= \sigma^2 = \mu'_2 - (\mu'_1)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

Therefore, $\mu = \sigma^2 = \lambda$

Note: Mean and variance are same for Poisson distribution.

Example 7: If $X \sim NB(r, p)$, find its m.g.f. and hence find its mean and variance.

Solution: Since $X \sim NB(r, p)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, \quad x = 0, 1, 2, \dots$$

The m.g.f of X is given by

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} p^r (-q)^x \\ &= \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-qe^t)^x = p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^t)^x \end{aligned}$$

$$\Rightarrow M_X(t) = p^r (1 - qe^t)^{-r}.$$

$$\text{Now, } \frac{d}{dt}(M_X(t)) = p^r (-r)(1 - qe^t)^{-(r+1)}(-qe^t) = qrp^r (1 - qe^t)^{-(r+1)} e^t$$

$$\text{Mean} = \mu = \mu'_1 = \left[\frac{d}{dt}(M_X(t)) \right]_{t=0} = qrp^r (1 - q)^{-(r+1)} = qrp^r (p)^{-(r+1)} = \frac{rq}{p}$$

$$\begin{aligned} \text{Further, } \frac{d^2}{dt^2}(M_X(t)) &= rqp^r \frac{d}{dt} \{ (1 - qe^t)^{-(r+1)} e^t \} \\ &= rqp^r \{ -(r+1)(1 - qe^t)^{-(r+2)}(-qe^{2t}) + (1 - qe^t)^{-(r+1)} e^t \} \end{aligned}$$

$$\text{Then } \mu'_2 = \left[\frac{d^2}{dt^2}(M_X(t)) \right]_{t=0} = rqp^r [(r+1)qp^{-(r+2)} + p^{-(r+1)}]$$

$$= r(r+1)q^2p^{-2} + rqp^{-1} = \frac{rq}{p} \left(\frac{(r+1)q}{p} + 1 \right) = \frac{rq}{p^2} (rq + 1)$$

$$\Rightarrow \mu'_2 = \frac{r^2q^2}{p^2} + \frac{rq}{p^2}$$

$$\text{Hence, variance} = \sigma^2 = \mu'_2 - (\mu'_1)^2 = \frac{r^2q^2}{p^2} + \frac{rq}{p^2} - \frac{r^2q^2}{p^2} \Rightarrow \sigma^2 = \frac{rq}{p^2}$$

Example 8: Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find

- (i) $P(X > 3)$
- (ii) **M.g.f. of X**
- (iii) $E(X)$ and $Var(X)$

Solution:

$$(i) \quad P(X > 3) = \int_3^{\infty} f(x) dx = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}} dx = \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_3^{\infty} = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

$$(ii) \quad M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{3} e^{-\frac{x}{3}} dx = \frac{1}{3} \int_0^{\infty} e^{\left(t - \frac{1}{3}\right)x} dx = \frac{1}{3} \int_0^{\infty} e^{-\left(\frac{1}{3} - t\right)x} dx = \frac{1}{3} \left[\frac{e^{-\left(\frac{1}{3} - t\right)x}}{-\left(\frac{1}{3} - t\right)} \right]_0^{\infty}$$

$$= \frac{1}{3} \left[0 - \frac{1}{-\left(\frac{1}{3} - t\right)} \right] = \frac{1}{3} \left[\frac{1}{\left(\frac{1 - 3t}{3}\right)} \right]$$

$$M_X(t) = \frac{1}{1 - 3t} = (1 - 3t)^{-1}$$

$$\frac{d}{dt} [M_X(t)] = -(1 - 3t)^{-2} (-3) = 3(1 - 3t)^{-2}$$

$$(iii) \quad E(X) = \text{Mean} = \left[\frac{d}{dt} M_X(t) \right]_{t=0} = 3$$

$$\frac{d^2}{dt^2} [M_X(t)] = -6(1 - 3t)^{-3} (-3) = 18(1 - 3t)^{-3}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t) \right]_{t=0} = 18$$

$$Var(X) = E(X^2) - [E(X)]^2 = 18 - 9 = 9$$

Example 9: Let X be a discrete random variable with p.d.f.

$$p(x) = \begin{cases} \frac{1}{x(x+1)} & , \quad x = 1, 2, \dots \\ 0 & , \quad \text{otherwise} \end{cases}$$

Show that $E(X)$ does not exist even though m.g.f. exist.

Solution:

$$E(X) = \sum_{x=1}^{\infty} x p(x) = \sum_{x=1}^{\infty} \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_{x=1}^{\infty} \frac{1}{x} - 1$$

But $\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series.

Therefore, $E(x)$ does not exist and hence, no moment exists.

Now, m.g.f. of X is given by

$$M_X(t) = \sum_{x=1}^{\infty} p(x) e^{tx} = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Substituting $z = e^t$,

$$\begin{aligned} M_X(t) &= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)} = \frac{z}{1.2} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \dots \\ &= z \left(1 - \frac{1}{2} \right) + z^2 \left(\frac{1}{2} - \frac{1}{3} \right) + z^3 \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\ &= \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^4}{4} \dots \\ &= -\log(1-z) - \frac{1}{z} \left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \dots \right) \end{aligned}$$

$$= -\log(1-z) + 1 + \frac{1}{z} \log(1-z), |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right) \log(1-z), |z| < 1$$

$$M_X(t) = \begin{cases} 1 + (e^{-t} - 1) \log(1 - e^t) & , \quad t < 0 \\ 1 & , \quad \text{for } t = 0 \end{cases}$$

and $M_X(t)$ does not exist for $t > 0$.

3.2

Moment Generating Function

Certain derivations presented in modules 2.4, 2.5 and 2.6 have been somewhat heavy on algebra. For example, determining the mean and variance of the **Binomial distribution** turned out to be fairly tiresome. Another example of hard work was determining the set of probabilities associated with a sum , $P(X + Y = t)$. Many of these tasks are greatly simplified by using **probability generating functions**.

Moment Generating Function: The moment generating function (m.g.f) of a random variable X is denoted by $M_X(t)$ and it is defined as

$$M_X(t) = E(e^{tX})$$

$$\begin{aligned}\therefore M_X(t) &= E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots\right] \\ &= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots + \frac{t^r}{r!}E(X^r) + \dots + \infty \\ \therefore M_X(t) &= 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \frac{t^3}{3!}\mu'_3 + \dots + \frac{t^r}{r!}\mu'_r + \dots + \infty \quad \dots \dots (1)\end{aligned}$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

which gives the m.g.f in terms of moments.

Therefore the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ'_r , where $r = 1, 2, 3, \dots$ and $\mu'_r = E(X^r)$, moment about origin.

The m.g.f of X about mean $\mu = \mu'_1 = E(X)$ is defined as

$$\begin{aligned}M_{X-\mu}(t) &= E[e^{t(X-\mu)}] = E\left[1 + \frac{t}{1!}(X-\mu) + \frac{t^2}{2!}(X-\mu)^2 + \frac{t^3}{3!}(X-\mu)^3 + \dots\right] \\ &= 1 + \frac{t}{1!}E(X-\mu) + \frac{t^2}{2!}E(X-\mu)^2 + \frac{t^3}{3!}E(X-\mu)^3 + \dots\end{aligned}$$

$$= 1 + \frac{t}{1!}\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \dots$$

where $E(x - \mu)^r = \mu_r$ is known as the r^{th} central moment for $r = 1, 2, \dots$

Note that $\mu_1 = E(X - \mu) = E(X) - \mu = \mu - \mu = 0$

Since $M_X(t)$ generates moments, it is called **moment generating function**.

If X is a discrete random variable with p.m.f. $p(x)$ then

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable with p.d.f. $f(x)$, then

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Moments Using Moment Generating Function:

Differentiating equation (1) with respect to t and then putting $t = 0$, gives

$$\mu'_1 = \left[\frac{d}{dt} M_X(t) \right]_{t=0}$$

$$\mu'_2 = \left[\frac{d^2}{dt^2} M_X(t) \right]_{t=0}$$

In general,

$$\mu'_r = \left[\frac{d^r}{dt^r} M_X(t) \right]_{t=0}, \quad r = 1, 2, 3, \dots$$

Note: Moment generating function $M_X(t)$ is used to calculate the higher moments.

Theorems on Moment Generating Function:

Theorem 1: $M_{aX}(t) = M_X(at)$, where a is a constant.

Proof: By definition $M_{aX}(t) = E(e^{taX}) = E(e^{atX}) = M_X(at)$

Therefore, $M_{aX}(t) = M_X(at)$

Theorem 2: The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, i. e., $M_{X_1+X_2+X_3+\dots+X_n}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t) \dots M_{X_n}(t)$

Proof: By definition,

$$\begin{aligned} M_{X_1+X_2+X_3+\dots+X_n}(t) &= [E^{t(X_1+X_2+X_3+\dots+X_n)}] \\ &= E(e^{tX_1})E(e^{tX_2})E(e^{tX_3}) \dots E(e^{tX_n}) \\ &\quad \text{(Since } X_1, X_2, \dots, X_n \text{ are independent).} \end{aligned}$$

Therefore, $M_{X_1+X_2+X_3+\dots+X_n}(t) = M_{X_1}(t)M_{X_2}(r) \dots M_{X_n}(t)$
Hence the proof.

Uniqueness Theorem of Moment Generating Function:

The m.g.f. of a distribution, if exists, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one m.g.f (provided it exists) and corresponding to a given m.g.f, there is only one probability distribution. Hence $M_X(t) = M_Y(t) \Rightarrow X$ and Y are identically distributed.

Effect of Change of Origin and Scale on Moment Generating Function:

Let a random variable X be transformed to a new variable U by changing both the origin and scale in X as $= \frac{X-a}{h}$, where a and h are constants.

The m.g.f of U (about origin) is given by

$$\begin{aligned} M_U(t) &= E(e^{tU}) = E\left[e^{t\left(\frac{X-a}{h}\right)}\right] = E\left(e^{\left(\frac{tX}{h}-\frac{ta}{h}\right)}\right) = e^{-\frac{at}{h}}E\left(e^{\left(\frac{tX}{h}\right)}\right) \\ \therefore M_{\frac{X-a}{h}}(t) &= e^{-\frac{at}{h}}M_X\left(\frac{t}{h}\right) \end{aligned}$$

Note: If $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$

Example 1: If X represents the outcome when a fair die is tossed, find the m.g.f. of X and hence, find $E(X)$ and $Var(X)$.

Solution: When a fair die is tossed

$$P(X = x) = \frac{1}{6}, \quad x = 1, 2, 3, 4, 5, 6$$

$$\begin{aligned} \therefore M_X(t) &= \sum_{x=1}^6 e^{tx} P(X = x) = \frac{1}{6} \sum_{x=1}^6 e^{tx} \\ &= \frac{1}{6} (e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}) \end{aligned}$$

$$\begin{aligned} E(X) &= \left[\frac{d}{dt} M_X(t) \right]_{t=0} = \frac{1}{6} [e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0} \\ &= \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2} \end{aligned}$$

$$\therefore \text{Mean} = E(X) = \frac{7}{2}$$

$$\begin{aligned} \text{Now, } E[X^2]_{t=0} &= \left\{ \frac{d^2}{dt^2} [M_X(t)] \right\}_{t=0} \\ &= \frac{1}{6} [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0} \\ &= \frac{1}{6} (1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} \end{aligned}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

Example 2: Find the m.g.f. of the random variable X whose probability function $P(X = x) = \frac{1}{2^x}, x = 1, 2, 3, \dots$ and hence find its mean.

Solution: By definition,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(X = x) = \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2^x} \right) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2} \right)^x$$

$$\begin{aligned}
&= \left[\frac{e^t}{2} + \left(\frac{e^t}{2} \right)^2 + \left(\frac{e^t}{2} \right)^3 + \dots \right] \\
&= \frac{e^t}{2} \left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2} \right)^2 + \dots \right] = \frac{e^t}{2} \left(1 - \frac{e^t}{2} \right)^{-1} \\
&= \frac{e^t}{2} \left(\frac{2 - e^t}{2} \right)^{-1} = \frac{e^t}{2} \left(\frac{2}{2 - e^t} \right) = \frac{e^t}{2 - e^t}
\end{aligned}$$

Therefore, $M_X(t) = \frac{e^t}{2 - e^t}$

$$\mu'_1 = \left[\frac{d}{dt} M_X(t) \right]_{t=0} = \left[\frac{d}{dt} \left(\frac{e^t}{2 - e^t} \right) \right]_{t=0} = \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2} \right]_{t=0} = \frac{(2 - 1)1 + 1}{(2 - 1)^2} = 2$$

Thus, $E(X) = \text{mean} = 2$

Example 3: If the moments of a random variable X are defined by $E(X^r) = 0.6, r = 1, 2, \dots$. Show that $P(X = 0) = 0.4, P(X = 1) = 0.6$, and $P(X \geq 2) = 0$.

Solution: We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$$

where $\mu'_r = E(X^r) = 0.6$

$$\begin{aligned}
\therefore M_X(t) &= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} \mu'_r = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6) = 1 + (0.6) \left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) \\
&= 1 + (0.6)(e^t - 1) = 1 - 0.6 + 0.6e^t = 0.4 + 0.6e^t \quad \dots \dots (1)
\end{aligned}$$

But by definition,

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} e^{tx} P(X = x)$$

$$M_X(t) = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \dots + \dots \dots (2)$$

From equations (1) and (2), we have

$$0.4 + 0.6e^t = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \dots$$

Equating the coefficients of like terms on both sides,

$$P(X = 0) = 0.4, P(X = 1) = 0.6$$

$$P(X = 2) = P(X = 3) = \dots = 0 \Rightarrow P(X \geq 2) = 0$$

Example 4: Find the m.g.f. of a random variable whose moments are $\mu_r = (r + 1)! 2^r$.

Solution: By definition, we have $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)! 2^r = \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= 1 + 2(2t) + 3(2t)^2 + \dots = (1 - 2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$\therefore M_X(t) = \frac{1}{(1-2t)^2}$$

Example 5: If $X \sim B(n, p)$, find the m.g.f of X and hence find its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n \text{ and } q = 1 - p.$$

Then the m.g.f. of X is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^n e^{tx} p(x) = \sum_{x=0}^n e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n \binom{n}{x} (pe^t)^x q^{n-x}$$

$$\Rightarrow M_X(t) = (q + pe^t)^n$$

$$\frac{d}{dt} M_X(t) = n(q + pe^t)^{n-1} pe^t \Rightarrow \text{Mean} = \mu'_1 = \left[\frac{dM_X(t)}{dt} \right]_{t=0} = np$$

$$\text{Next, } \frac{d^2}{dt^2} (M_X(t)) = np[(n-1)(q + pe^t)^{n-2}pe^{2t} + (q + pe^t)^{n-1}e^t]$$

$$\Rightarrow \mu'_2 = \left[\frac{d^2}{dt^2} (M_X(t)) \right]_{t=0} = np[(n-1)p + 1] = np[np - p + 1] = np(np + q)$$

$$\Rightarrow \mu'_2 = n^2p^2 + npq$$

$$\text{Now, variance} = \sigma^2 = \mu'_2 - (\mu'_1)^2 = n^2p^2 + npq - n^2p^2 = npq$$

$$\text{Thus, } \mu = np \text{ and } \sigma^2 = npq$$

Note that $\sigma^2 = npq = \mu q$ where $(0 < q < 1)$. Thus, $\mu > \sigma^2$.

Note: For binomial distribution, mean is always greater than variance.

Example 6 : If $X \sim P(\lambda)$, find its m.g.f. and hence find its mean and variance.

Solution: Since $X \sim P(\lambda)$, then its p.m.f. is given by

$$p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, \dots \text{ and } \lambda > 0$$

The m.g.f. of X is given by

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

$$\Rightarrow M_X(t) = e^{\lambda(e^t - 1)}$$

$$\text{Since } \frac{d}{dt} (M_X(t)) = e^{\lambda(e^t - 1)} \lambda e^t; \text{ Mean} = \mu = \mu' = \left[\frac{d}{dt} (M_X(t)) \right]_{t=0} = \lambda.$$

$$\text{Now, } \frac{d^2}{dt^2} (M_X(t)) = \lambda [e^{\lambda(e^t - 1)} e^t + e^{\lambda(e^t - 1)} \lambda e^{2t}]$$

$$\text{Then } \mu'_2 = \left[\frac{d^2 M_X(t)}{dt^2} \right]_{t=0} = \lambda(1 + \lambda) = \lambda + \lambda^2$$

$$\text{Thus, variance} = \sigma^2 = \mu'_2 - (\mu'_1)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Therefore, $\mu = \sigma^2 = \lambda$

Note: Mean and variance are same for Poisson distribution.

Example 7: If $X \sim NB(r, p)$, find its m.g.f. and hence find its mean and variance.

Solution: Since $X \sim NB(r, p)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, \quad x = 0, 1, 2, \dots$$

The m.g.f of X is given by

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \binom{-r}{x} p^r (-q)^x \\ &= \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-qe^t)^x = p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^t)^x \end{aligned}$$

$$\Rightarrow M_X(t) = p^r (1 - qe^t)^{-r}.$$

$$\text{Now, } \frac{d}{dt}(M_X(t)) = p^r (-r)(1 - qe^t)^{-(r+1)}(-qe^t) = qrp^r (1 - qe^t)^{-(r+1)} e^t$$

$$\text{Mean} = \mu = \mu'_1 = \left[\frac{d}{dt}(M_X(t)) \right]_{t=0} = qrp^r (1 - q)^{-(r+1)} = qrp^r (p)^{-(r+1)} = \frac{rq}{p}$$

$$\begin{aligned} \text{Further, } \frac{d^2}{dt^2}(M_X(t)) &= rqp^r \frac{d}{dt} \{ (1 - qe^t)^{-(r+1)} e^t \} \\ &= rqp^r \{ -(r+1)(1 - qe^t)^{-(r+2)}(-qe^{2t}) + (1 - qe^t)^{-(r+1)} e^t \} \end{aligned}$$

$$\text{Then } \mu'_2 = \left[\frac{d^2}{dt^2}(M_X(t)) \right]_{t=0} = rqp^r [(r+1)qp^{-(r+2)} + p^{-(r+1)}]$$

$$= r(r+1)q^2p^{-2} + rqp^{-1} = \frac{rq}{p} \left(\frac{(r+1)q}{p} + 1 \right) = \frac{rq}{p^2} (rq + 1)$$

$$\Rightarrow \mu'_2 = \frac{r^2q^2}{p^2} + \frac{rq}{p^2}$$

$$\text{Hence, variance} = \sigma^2 = \mu'_2 - (\mu'_1)^2 = \frac{r^2q^2}{p^2} + \frac{rq}{p^2} - \frac{r^2q^2}{p^2} \Rightarrow \sigma^2 = \frac{rq}{p^2}$$

Example 8: Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find

- (i) $P(X > 3)$
- (ii) **M.g.f. of X**
- (iii) $E(X)$ and $Var(X)$

Solution:

$$(i) \quad P(X > 3) = \int_3^{\infty} f(x) dx = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}} dx = \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_3^{\infty} = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

$$(ii) \quad M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{3} e^{-\frac{x}{3}} dx = \frac{1}{3} \int_0^{\infty} e^{\left(t - \frac{1}{3}\right)x} dx = \frac{1}{3} \int_0^{\infty} e^{-\left(\frac{1}{3} - t\right)x} dx = \frac{1}{3} \left[\frac{e^{-\left(\frac{1}{3} - t\right)x}}{-\left(\frac{1}{3} - t\right)} \right]_0^{\infty}$$

$$= \frac{1}{3} \left[0 - \frac{1}{-\left(\frac{1}{3} - t\right)} \right] = \frac{1}{3} \left[\frac{1}{\left(\frac{1 - 3t}{3}\right)} \right]$$

$$M_X(t) = \frac{1}{1 - 3t} = (1 - 3t)^{-1}$$

$$\frac{d}{dt} [M_X(t)] = -(1 - 3t)^{-2} (-3) = 3(1 - 3t)^{-2}$$

$$(iii) \quad E(X) = \text{Mean} = \left[\frac{d}{dt} M_X(t) \right]_{t=0} = 3$$

$$\frac{d^2}{dt^2} [M_X(t)] = -6(1 - 3t)^{-3} (-3) = 18(1 - 3t)^{-3}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t) \right]_{t=0} = 18$$

$$Var(X) = E(X^2) - [E(X)]^2 = 18 - 9 = 9$$

Example 9: Let X be a discrete random variable with p.d.f.

$$p(x) = \begin{cases} \frac{1}{x(x+1)} & , \quad x = 1, 2, \dots \\ 0 & , \quad \text{otherwise} \end{cases}$$

Show that $E(X)$ does not exist even though m.g.f. exist.

Solution:

$$E(X) = \sum_{x=1}^{\infty} x p(x) = \sum_{x=1}^{\infty} \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_{x=1}^{\infty} \frac{1}{x} - 1$$

But $\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series.

Therefore, $E(x)$ does not exist and hence, no moment exists.

Now, m.g.f. of X is given by

$$M_X(t) = \sum_{x=1}^{\infty} p(x) e^{tx} = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Substituting $z = e^t$,

$$\begin{aligned} M_X(t) &= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)} = \frac{z}{1.2} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \dots \\ &= z \left(1 - \frac{1}{2} \right) + z^2 \left(\frac{1}{2} - \frac{1}{3} \right) + z^3 \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\ &= \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^4}{4} \dots \\ &= -\log(1-z) - \frac{1}{z} \left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \dots \right) \end{aligned}$$

$$= -\log(1-z) + 1 + \frac{1}{z} \log(1-z), |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right) \log(1-z), |z| < 1$$

$$M_X(t) = \begin{cases} 1 + (e^{-t} - 1) \log(1 - e^t) & , \quad t < 0 \\ 1 & , \quad \text{for } t = 0 \end{cases}$$

and $M_X(t)$ does not exist for $t > 0$.

P1:

Find the m.g.f. of uniform distribution $U[a, b]$ and hence obtain the mean and variance of the distribution.

P2:

Find the m.g.f. of Normal $N(\mu, \sigma^2)$ distribution and hence find its mean and variance.

Solution:

Since $X \sim N(\mu, \sigma^2)$, its p.d.f. is given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2} (x - \mu)^2\right)$$

The m.g.f. of X is given by

$$M_X(t) = E[e^{tx}] = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \exp\left(-\frac{1}{2\sigma^2} (x - \mu)^2\right) dx$$

Let $z = \frac{x - \mu}{\sigma}$. Thus $x = \mu + \sigma z$ and $dx = \sigma \cdot dz$

$$\begin{aligned} \text{Thus, } M_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(t(\mu + \sigma z)) \exp\left(-\frac{z^2}{2}\right) dz \\ &= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2 - 2t\sigma z)\right] dz \\ &= \frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\{(z - \sigma t)^2 - \sigma^2 t^2\}\right] dz \\ &= \frac{1}{\sqrt{2\pi}} e^{t\mu + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z - \sigma t)^2\right] dz \end{aligned}$$

Let $u = z - \sigma t \Rightarrow du = dz$

$$\begin{aligned} &= e^{t\mu + \frac{1}{2}\sigma^2 t^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \\ &= e^{t\mu + \frac{1}{2}\sigma^2 t^2} \end{aligned}$$

$$\Rightarrow M_X(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2} \Rightarrow M'_X(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)$$

$$\Rightarrow \mu = \text{Mean} = \mu'_1 = M'_X(t)|_{t=0} = \mu$$

$$\text{Further, } M''_X(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2} (\mu + \sigma^2 t)^2 + e^{t\mu + \frac{1}{2}\sigma^2 t^2} (\sigma^2)$$

$$\Rightarrow \mu'_2 = M''_X(t)|_{t=0} = \mu^2 + \sigma^2 \Rightarrow \mu'_2 = \mu^2 + \sigma^2$$

The variance is given by

$$\text{Variance} = \mu'_2 - (\mu'_1)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

P3:

Find the m.g.f. of geometric $G(p)$ distribution and hence obtain its mean and variance.

Solution:

If $X \sim G(p)$, its p.m.f is given by $p(x) = q^x p$ for $x = 0, 1, 2, \dots$, $0 < p < 1$.

Then the m.g.f. is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} q^x p = \sum_{x=0}^{\infty} (qe^t)^x = \frac{p}{1 - qe^t}$$

$$\Rightarrow M_X(t) = \frac{p}{1 - qe^t}$$

Then $\mu'_1 = M'_X(t)|_{t=0} = pq(1 - q)^{-2} = \frac{q}{p}$ and $\mu'_2 = M''_X(t)|_{t=0} = \frac{q}{p} + \frac{2q^2}{p^2}$

$$\text{and } \mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{q}{p} + \frac{2q^2}{p^2} - \left(\frac{q}{p}\right)^2 = \frac{q}{p} + \frac{q^2}{p^2} = \frac{qp + q^2}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

Thus, Mean = $\mu = \mu'_1 = \frac{q}{p}$ and variance = $\sigma^2 = \frac{q}{p^2}$

P4:

Find the m.g.f. of exponential $E(\lambda)$ distribution and hence find its mean and variance.

Solution:

Since $X \sim E(\lambda)$, its p.d.f. is given by $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $\lambda > 0$

The m.g.f. of X is given by

$$M_X(t) = E[e^{tX}] = \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r \text{ for } \lambda > t$$

$$\Rightarrow M_X(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r \text{ for } \lambda > t$$

But μ'_r = coefficient of $\frac{t^r}{r!}$ in $M_X(t) = \frac{r!}{\lambda^r}$ for $r = 1, 2, \dots$

Thus, $\mu'_1 = \frac{1}{\lambda}$ and $\mu'_2 = \frac{2}{\lambda^2}$ and hence $\mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Thus, $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$

3.2. Moment Generating Function

Exercise:

1. Find the m.g.f of a.r.v. whose moments are given by $\mu'_r = (r + 1)! 2^r$

2. If $M_X(t) = \frac{3}{3-t}$, find standard deviation of X

3. Find the m.g.f. of a.r.v X whose p.d.f is given by

$$f(x) = \begin{cases} \frac{x}{2} , & 0 \leq x \leq 2 \\ 0 , & \text{otherwise} \end{cases}$$

4. Find the m.g.f and hence find the mean and variance of a.r.v. X whose p.d.f. is given by

i.
$$f(x) = \begin{cases} 2e^{-2x} , & x \geq 0 \\ 0 , & \text{otherwise} \end{cases}$$

ii.
$$f(x) = \begin{cases} \frac{1}{3} , & -1 < x < 2 \\ 0 , & \text{otherwise} \end{cases}$$

iii.
$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}} , & x \geq 0 \\ 0 , & \text{otherwise} \end{cases}$$

iv.
$$f(x) = \begin{cases} \frac{1}{k} , & 0 < x < k \\ 0 , & \text{otherwise} \end{cases}$$

v.
$$f(x) = \lambda e^{-\lambda(x-a)} , x \geq a$$

Answers:

1. $M_X(t) = \frac{1}{(1-2t)^r}$

2. $\frac{1}{3}$

3. $\frac{1}{2t^2}(1 + 2t e^{2t} - e^{2t})$

4.

i. $M_X(t) = \frac{2}{2-t}, \mu = \frac{1}{2}, \sigma^2 = \frac{1}{4}$

ii. $M_X(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & , \quad t \neq 0 \\ 1 & , \quad t = 0 \end{cases}$

iii. $M_X(t) = (1 - 3t)^{-1}, \mu = 3, \sigma^2 = 9$

iv. $M_X(t) = \frac{e^{tk}-1}{kt}, \mu = \frac{k}{2}, \sigma^2 = \frac{k^2}{\sqrt{2}}$

v. $M_X(t) = \frac{\lambda e^{at}}{\lambda-t}, \mu = \frac{9\lambda+1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$

3.3

Characteristic Function

In some cases m.g.f. does not exist. For example, consider the p.m.f. given by

$$p(x) = \begin{cases} \frac{6}{\pi^2 x^2}, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

Its m.g.f. is given by

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{e^{tx}}{x^2},$$

which is divergent. Thus, $M_X(t)$ does not exist. A more serviceable function than the m.g.f. is the **characteristic function**.

Characteristic function: The characteristic function of a.r.v. X is defined by

$$\phi_X(t) = E[e^{itX}] = \begin{cases} \int e^{itx} f(x) dx & \text{if } X \text{ is a c.r.v. with p.d.f } f(x) \\ \sum_x e^{itx} p(x) & \text{if } X \text{ is a d.r.v. with p.m.f } p(x) \end{cases}$$

where $i = \sqrt{-1}$, the imaginary number.

Note:

$$1. \quad |\phi_X(t)| = |E(e^{itX})| \leq E(|e^{itX}|) = E(\sqrt{\cos^2 tX + \sin^2 tX}) = E(1) = 1$$

Since $|\phi_X(t)| \leq 1$, $\phi_X(t)$ always exists for any **probability distribution**.

$$\begin{aligned} 2. \quad \phi_X(t) &= E[e^{itX}] = E\left[1 + (it)X + \frac{(it)^2}{2!}X^2 + \frac{(it)^3}{3!}X^3 + \dots\right] \\ &= 1 + (it)E(X) + \frac{(it)^2}{2!}E(X^2) + \frac{(it)^3}{3!}E(X^3) + \dots \\ &= 1 + (it)\mu'_1 + \frac{(it)^2}{2!}\mu'_2 + \frac{(it)^3}{3!}\mu'_3 + \dots \end{aligned}$$

where $\mu'_r = E(X^r) = r^{th}$ moment about origin for $r = 1, 2, \dots$

3. If $\phi_X(t)$ is given, then the r^{th} moment about origin is given by

$$\mu'_r = \text{coefficient of } \frac{(it)^r}{r!} \text{ in } \phi_X(t).$$

Properties:

1. $\phi_X(0) = 1$

Proof: $\phi_X(t) = E[e^{itX}] = E(1)$ when $t = 0$

$$= 1$$

Thus, $\phi_X(0) = 1$

2. $|\phi_X(t)| \leq 1$ for all real t .

Proof: $|\phi_X(t)| = |E(e^{itX})| \leq E(|e^{itX}|) = E(\sqrt{\cos^2 tX + \sin^2 tX}) = E(1) = 1$

$\Rightarrow |\phi_X(t)| \leq 1$ for all real t

3. $\phi_X(t)$ continuous function of t in $(-\infty, \infty)$.

Proof: For $h \neq 0$,

$$\begin{aligned} |\phi_X(t+h) - \phi_X(t)| &= |E(e^{i(t+h)X}) - E(e^{itX})| = |E(e^{i(t+h)X} - e^{itX})| \\ &= |E\{e^{itX}(e^{ihX} - 1)\}| \leq E(|e^{itX}| |e^{ihX} - 1|) = E(|e^{ihX} - 1|) \rightarrow 0 \text{ as } h \rightarrow 0 \end{aligned}$$

Thus $\lim_{h \rightarrow 0} |\phi_X(t+h) - \phi_X(t)| = 0$

$\Rightarrow \lim_{h \rightarrow 0} \phi_X(t+h) = \phi_X(t)$

$\Rightarrow \phi_X(t)$ is a continuous function of t in $(-\infty, \infty)$.

4. $\phi_X(-t) = \overline{\phi_X(t)}$, i. e., $\phi_X(-t)$ is the complex conjugate of $\phi_X(t)$.

Proof: Here $\overline{\phi_X(t)} = \overline{E[e^{itX}]} = E[\cos tX - i \sin tX]$

$$\Rightarrow \phi_X(-t) = E[\cos(-tX) + i \sin(-tX)] = E[\cos tX - i \sin tX] = \overline{\phi_X(t)}$$

$$\text{Thus, } \phi_X(-t) = \overline{\phi_X(t)}$$

5. If the p.d.f. is even i. e., $f(-x) = f(x)$, then the characteristic function is real valued and even function of t .

Proof: We know that, $\phi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itX} f(x) dx$

Let $x = -y \Rightarrow dx = -dy$. Then

$$\begin{aligned} \phi_X(t) &= \int_{\infty}^{-\infty} e^{-ity} f(-y)(-dy) = \int_{-\infty}^{\infty} e^{-ity} f(y) dy \quad (\because f \text{ is an even function}) \\ &= E[e^{-itX}] = \phi_X(-t) \end{aligned}$$

$$\text{Thus, } \phi_X(t) = \phi_X(-t)$$

$\Rightarrow \phi_X(t)$ is an even function of t .

$$\text{Further, } \overline{\phi_X(t)} = \phi_X(-t) \quad (\text{by property 4})$$

$$= \phi_X(t) \quad (\text{Since } \phi_X(t) \text{ is even function})$$

Thus, $\phi_X(t)$ is real.

6. If X is a r.v. with characteristic function $\phi_X(t)$ and $\mu'_r = E(X^r)$ exists, then

$$\mu'_r = (-i)^r \frac{d^r}{dt^r} (\phi_X(t)) \Big|_{t=0}$$

Proof:

$$\frac{d^r}{dt^r} (\phi_X(t)) = \frac{d^r}{dt^r} (E(e^{itX})) = i^r E[X^r e^{itX}] = i^r E(X^r)$$

$$\text{Now, } \frac{d^r}{dt^r} (\phi_X(t)) \Big|_{t=0} = i^r E(X^r) \text{ and } \mu'_r = E(X^r) = \frac{1}{i^r} \frac{d^r}{dt^r} (\phi_X(t)) \Big|_{t=0}.$$

$$\text{Thus, } \mu'_r = (-i)^r \frac{d^r}{dt^r} (\phi_X(t)) \Big|_{t=0}$$

7. Effect of change of origin and scale .

Let $U = \frac{X-a}{h}$ where a and h are constants.

$$\text{Then } \phi_U(t) = E[e^{itU}] = E\left[e^{it\left(\frac{X-a}{h}\right)}\right] = e^{-\frac{ita}{h}} E\left[e^{i\left(\frac{t}{h}\right)X}\right]$$

$$\Rightarrow \phi_U(t) = e^{-\frac{ita}{h}} \phi_X\left(\frac{t}{h}\right)$$

8. If $X_1, X_2, \dots, X_n$ are independent, then

$$\phi_{X_1+\dots+X_n}(t) = \phi_{X_1}(t) \phi_{X_2}(t) \dots \phi_{X_n}(t)$$

Proof:

$$\phi_{X_1+\dots+X_n}(t) = E[e^{it(X_1+X_2+\dots+X_n)}] = E[e^{itX_1} \cdot e^{itX_2} \dots e^{itX_n}]$$

$$(\because X_1, X_2, \dots, X_n \text{ are independent})$$

$$= E[e^{itX_1}] \cdot E[e^{itX_2}] \dots E[e^{itX_n}]$$

$$= \phi_{X_1}(t) \cdot \phi_{X_2}(t) \dots \phi_{X_n}(t)$$

$$\Rightarrow \phi_{X_1+\dots+X_n}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \dots \phi_{X_n}(t)$$

Note: Converse need not be true.

Uniqueness Theorem for Characteristic Functions:

The characteristic function uniquely determines the distribution. That is,

A necessary and sufficient condition for two distributions with p.d.fs $f_1(\cdot)$ and $f_2(\cdot)$ to be identical is that their characteristic function $\phi_1(t)$ and $\phi_2(t)$ are identical.

Example 1: If $X \sim B(n, p)$, find its characteristic function and hence obtain its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x} \text{ for } x = 0, 1, 2, \dots, n$$

The characteristic function of X is given by

$$\phi_X(t) = E[e^{itX}] = \sum_{x=0}^n e^{itx} p(x) = \sum_{x=0}^n e^{itx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n \binom{n}{x} (pe^{it})^x q^{n-x}$$

$$\Rightarrow \phi_X(t) = (q + pe^{it})^n \text{ and } \frac{d}{dt}(\phi_X(t)) = npi(q + pe^{it})^{n-1} e^{it}$$

The mean of X is given by

$$\begin{aligned} \mu = E(X) = \mu' &= (-i) \frac{d}{dt}(\phi_X(t)) \Big|_{t=0} = (-i) [npi(q + pe^{it})^{n-1} e^{it}]_{t=0} \\ &= (-i) npi = np \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{d^2}{dt^2}(\phi_X(t)) &= (npi) \frac{d}{dt} [(q + pe^{it})^{n-1} e^{it}] \\ &= (npi) [(n-1)(q + pe^{it})^{n-2} pie^{2it} + (q + pe^{it})^{n-1} ie^{it}] \end{aligned}$$

$$\text{Thus, } \mu'_2 = (-i)^2 \frac{d^2}{dt^2}(\phi_X(t)) \Big|_{t=0} = (-i)^2 (npi) [(n-1)pi + i]$$

$$= (np)[np - p + 1] = np(np + q) = n^2p^2 + npq$$

$$\Rightarrow \mu'_2 = n^2p^2 + npq$$

Therefore, the variance of X is given by

$$\sigma^2 = V(X) = \mu'_2 - (\mu'_1)^2 = n^2 p^2 + npq - n^2 p^2 \\ \Rightarrow \sigma^2 = npq.$$

Example 2: If $X \sim P(\lambda)$, find the characteristic function X and hence obtain its mean and variance.

Solution: Since $X \sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, \dots$$

The characteristic function of X is given by

$$\phi_X(t) = E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} = e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

$$\Rightarrow j_X(t) = e^{\lambda(e^{it}-1)}$$

$$\text{Now, } \frac{d}{dt}(\phi_X(t)) = e^{\lambda(e^{it}-1)} \lambda i e^{it}$$

Thus, the mean is given by

$$\mu = \mu'_1 = E(X) = (-i) \frac{d}{dt}(\phi_X(t)) \Big|_{t=0} = (-i)(\lambda i) = \lambda$$

$$\Rightarrow \mu = \lambda$$

$$\text{Now, } \frac{d^2}{dt^2}(\phi_X(t)) = (\lambda i) \frac{d}{dt} [e^{\lambda(e^{it}-1)} e^{it}]$$

$$= (\lambda i) [e^{\lambda(e^{it}-1)} \lambda i e^{2it} + e^{\lambda(e^{it}-1)} i e^{it}]$$

Thus, μ'_2 is given by

$$\begin{aligned}\mu'_2 &= (-i)^2 \frac{d^2}{dt^2} (\phi_X(t)) \Big|_{t=0} = (-i)^2 (\lambda i) (\lambda i + i) = (-i)^2 (i^2) \lambda (\lambda + 1) \\ &= \lambda (\lambda + 1) = \lambda^2 + \lambda\end{aligned}$$

Hence, the variance is given by $\sigma^2 = \mu'_2 - (\mu'_1)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \Rightarrow \sigma^2 = \lambda$

Example 3: If $X \sim N(\mu, \sigma^2)$, find the characteristic function of X and hence obtain its mean and variance.

Solution: Since $X \sim N(\mu, \sigma^2)$, its p.d.f. is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right], -\infty < x, \mu < \infty, \sigma > 0$$

The characteristic function of X is given by

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right] dx$$

$$\text{Let } \frac{x - \mu}{\sigma} = z \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$$

$$\begin{aligned}&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right] dx = \int_{-\infty}^{\infty} e^{it(\mu + \sigma z)} e^{-\frac{1}{2}z^2} dz \\ &= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2 - 2i\sigma zt)\right] dz \\ &= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z^2 - 2i\sigma zt + i^2\sigma^2 t^2 - i^2\sigma^2 t^2)\right] dz \\ &= \frac{e^{it\mu - \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(z - i\sigma t)^2\right] dz\end{aligned}$$

$$\text{Let } z - i\sigma t = u \Rightarrow dz = du$$

$$= e^{it\mu - \frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= e^{it\mu - \frac{\sigma^2 t^2}{2}} \left(\because \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = 1 \right)$$

$$\Rightarrow \phi_X(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}$$

$$\text{Now, } \frac{d}{dt}(\phi_X(t)) = e^{it\mu - \frac{\sigma^2 t^2}{2}} (i\mu - \sigma^2 t)$$

$$\text{Then } \mu'_1 = (-i) \frac{d}{dt}(\phi_X(t)) \Big|_{t=0} = (-i) (i\mu) = \mu$$

$$\text{Thus, Mean} = E(X) = \mu.$$

$$\text{Now, } \frac{d^2}{dt^2}(\phi_X(t)) = e^{it\mu - \frac{1}{2}\sigma^2 t^2} (i\mu - \sigma^2 t)^2 + e^{it\mu - \frac{1}{2}\sigma^2 t^2} (-\sigma^2)$$

$$\text{Thus, } \mu'_2 = (-i)^2 \frac{d^2}{dt^2}(\phi_X(t)) \Big|_{t=0} = (-i)^2 [i^2 \mu^2 - \sigma^2] = (-1)(-\mu^2 - \sigma^2)$$

$$\Rightarrow \mu'_2 = \mu^2 + \sigma^2$$

Hence, the variance is given by

$$V(X) = \mu'_2 - (\mu'_1)^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$$\Rightarrow \text{Variance} = V(X) = \sigma^2$$

Finding p.m.f. (or p.d.f.) when characteristic function is known.

If X is a d.r.v. with characteristic function $\phi_X(t)$, then $\phi_X(t) = \sum P(X = j) e^{itj}$.

First write the characteristic function in this form and then identify the $P(X = j)$ which is the p.m.f. of the d.r.v. X .

Example 4: Find the p.m.f. of the d.r.v. X whose characteristic function is given by $\phi_X(t) = (q + pe^{it})^n$.

Solution: We have, $\phi_X(t) = (q + pe^{it})^n$ and

$$\begin{aligned}\phi_X(t) &= (q + pe^{it})^n = \sum_{j=0}^n \binom{n}{j} (pe^{it})^j q^{n-j} \\ &= \sum_{j=0}^n \binom{n}{j} p^j q^{n-j} e^{itj} = \sum_{j=0}^n P(X = j) e^{itj} \\ &= E[e^{itX}] \text{ where } P(X = j) = \binom{n}{j} p^j q^{n-j}\end{aligned}$$

Thus p.m.f. is $p(j) = \binom{n}{j} p^j q^{n-j}$ for $j = 0, 1, 2, \dots, n$.

Example 5: Find the p.m.f. of a d.r.v. X whose characteristic function is given by

$$\phi_X(t) = e^{\lambda(e^{it}-1)}.$$

Solution: We have, $\phi_X(t) = e^{\lambda(e^{it}-1)}$

$$\begin{aligned}\phi_X(t) &= e^{\lambda(e^{it}-1)} = e^{-\lambda} e^{\lambda e^{it}} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} e^{itx} \\ &= \sum_{x=0}^{\infty} P(X = x) e^{itx} = E[e^{itX}]\end{aligned}$$

where $P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$ for $x = 0, 1, 2, \dots$, which is Poisson distribution with parameter λ .

Theorem 1: If X is a continuous random variable with characteristic function $\phi_X(t)$, then its p.d.f. is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt$$

Example 6: Find the p.d.f corresponding to the characteristic function

$$\phi_X(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}.$$

Solution:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{it\mu - \frac{1}{2}t^2\sigma^2} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[t^2\sigma^2 - 2it(x-\mu)]} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2} \left\{ t\sigma - i \left(\frac{x-\mu}{\sigma} \right) \right\}^2 + \left(\frac{x-\mu}{\sigma} \right)^2 \right] dt \\ &= \frac{1}{2\pi} \exp \left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right] \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2} \left\{ t\sigma - i \left(\frac{x-\mu}{\sigma} \right) \right\}^2 \right] dt \end{aligned}$$

$$\text{Let } t\sigma - i \left(\frac{x-\mu}{\sigma} \right) = u \Rightarrow dt = \frac{du}{\sigma}.$$

$$\begin{aligned} &= \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right] \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{u^2}{2} \right) du \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right] \end{aligned}$$

$$\text{Therefore, } f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right], -\infty < x < \infty$$

$$\Rightarrow X \sim N(\mu, \sigma^2)$$

Example 7: Find the p.d.f. corresponding to the characteristic function defined by

$$\phi(t) = \begin{cases} 1 - |t| & , \quad |t| \leq 1 \\ 0 & , \quad |t| > 1 \end{cases}$$

Solution: The p.d.f. of $f(x)$ is given by

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt = \frac{1}{2\pi} \int_{-1}^1 e^{-itx} \phi(t) dt \\
 &= \frac{1}{2\pi} \int_{-1}^0 e^{-itx} (1+t) dt + \frac{1}{2\pi} \int_0^1 e^{-itx} (1-t) dt
 \end{aligned}$$

($\because$ for $-1 < t < 0$, $|t| = -t$ and for $0 < t < 1$, $|t| = t$)

Now,

$$\begin{aligned}
 \int_{-1}^0 e^{-itx} (1+t) dt &= \left[\frac{e^{-itx}}{-ix} (1+t) \right]_{-1}^0 + \frac{1}{ix} \int_{-1}^0 e^{-itx} dt \\
 &= -\frac{1}{ix} + \frac{1}{ix} \left[\frac{e^{-itx}}{-ix} \right]_{-1}^0 \\
 &= -\frac{1}{ix} + \frac{1}{(ix)^2} (e^{ix} - 1)
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \int_0^1 e^{-itx} (1-t) dt &= \frac{1}{ix} + \frac{1}{(ix)^2} (e^{-ix} - 1) \\
 \therefore f(x) &= \frac{1}{2\pi} \left[\frac{1}{(ix)^2} \{e^{ix} - 1 + e^{-ix} - 1\} \right] = \frac{1}{\pi x^2} \left(1 - \frac{e^{ix} + e^{-ix}}{2} \right) \\
 \Rightarrow f(x) &= \frac{1}{\pi x^2} (1 - \cos x), -\infty < x < \infty
 \end{aligned}$$

P1:

If $X \sim NB(r, p)$, find its characteristic function and hence obtain its mean and variance.

Solution:

Since $X \sim NB(r, p)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, x = 0, 1, 2, \dots$$

The characteristic function of X is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} \binom{-r}{x} p^r (-q)^x \\ &= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^{it})^x = p^r (1 - qe^{it})^{-r}\end{aligned}$$

$$\text{Thus, } \phi_X(t) = p^r (1 - qe^{it})^{-r}$$

Now,

$$\frac{d}{dt}(\phi_X(t)) = p^r (-r) (1 - qe^{it})^{-(r+1)} (-q) ie^{it} = ip^r q r e^{it} (1 - qe^{it})^{-(r+1)}$$

$$\text{Thus, mean} = \mu = \mu_1' = (-i) \frac{d}{dt}(\phi_X(t)) \Big|_{t=0}$$

$$= (-i) ip^r q r (1 - q)^{-(r+1)} = p^r q r (p)^{-(r+1)} = \frac{qr}{p}$$

$$\Rightarrow \mu = \frac{rq}{p}$$

$$\text{Now, } \frac{d^2}{dt^2}(\phi_X(t)) = p^r r q i \frac{d}{dt} \left[(1 - qe^{it})^{-(r+1)} e^{it} \right]$$

$$= p^r r q i \left[-(r+1) (1 - qe^{it})^{-(r+2)} (-qi) e^{2it} + (1 - qe^{it})^{-(r+1)} ie^{it} \right]$$

$$\begin{aligned}
\text{Thus, } \mu_2' &= (-i)^2 \frac{d^2}{dt^2} (\phi_X(t)) \Big|_{t=0} \\
&= (-i)^2 p^r r q i [(r+1) q i p^{-(r+2)} + p^{-(r+1)} i] \\
&= (-i)^2 p^r r q i^2 p^{-(r+2)} [(r+1) q + p] \\
&= \frac{r q}{p^2} (r q + q + p) = \frac{r q}{p^2} (r q + 1) \\
\Rightarrow \mu_2' &= \frac{r^2 q^2}{p^2} + \frac{r q}{p^2}
\end{aligned}$$

Thus, the variance is given by

$$\begin{aligned}
\sigma^2 &= \mu_2' - (\mu_1')^2 = \frac{r^2 q^2}{p^2} + \frac{r q}{p^2} - \frac{r^2 q^2}{p^2} \\
\Rightarrow \sigma^2 &= \frac{r q}{p^2}
\end{aligned}$$

P2:

Find the characteristic function of uniform $U[a, b]$ distribution and hence obtain the mean and variance of the distribution.

Solution:

Since $X \sim U(a, b)$, its p.d.f, is given by

$$f(x) = \frac{1}{b-a}, a < x < b$$

The characteristic function is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itX}] = \int_a^b e^{itx} f(x) dx = \int_a^b e^{itx} \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^{itx} dx \\ &= \frac{1}{b-a} \left[\frac{e^{itx}}{it} \right]_a^b = \frac{1}{(b-a)it} (e^{itb} - e^{ita})\end{aligned}$$

$$\Rightarrow \phi_X(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

$$\Rightarrow t\phi_X(t) = \frac{1}{i(b-a)} (e^{itb} - e^{ita}) \quad \dots \dots \dots (1)$$

Differentiating both sides of (1) w.r.t. t, we get

$$t\phi'_X(t) + \phi_X(t) = \frac{1}{i(b-a)} [(ib)e^{itb} - (ia)e^{ita}] \quad \dots \dots (2)$$

Again differentiating both sides of (2) w.r.t. t, we get

$$t\phi''_X(t) + \phi'_X(t) + \phi'_X(t) = \frac{1}{i(b-a)} [(ib)^2 e^{itb} - (ia)^2 e^{ita}]$$

$$\Rightarrow t\phi''_X(t) + 2\phi'_X(t) = \frac{1}{i(b-a)} [(ib)^2 e^{itb} - (ia)^2 e^{ita}]$$

In general,

$$t \frac{d^{k+1}}{dt^{k+1}}(\phi_X(t)) + (k+1) \frac{d^k}{dt^k}(\phi_X(t)) = \frac{1}{i(b-a)} [(ib)^{k+1} e^{itb} - (ia)^{k+1} e^{ita}]$$

If $t = 0$, then

$$(k+1) \frac{d^k}{dt^k}(\phi_X(t)) \Big|_{t=0} = \frac{1}{i(b-a)} [(ib)^{k+1} - (ia)^{k+1}]$$

$$\Rightarrow \frac{d^k}{dt^k}(\phi_X(t)) \Big|_{t=0} = \frac{1}{k+1} \frac{1}{i(b-a)} [(ib)^{k+1} - (ia)^{k+1}] \dots \dots \dots (3)$$

From (3), the mean is given by

$$\begin{aligned} \mu = \text{mean} = \mu_1' &= (-i) \frac{d}{dt}(\phi_X(t)) \Big|_{t=0} \\ &= (-i) \frac{1}{2i(b-a)} [(ib)^2 - (ia)^2] = \frac{1}{2(b-a)} (b^2 - a^2) \\ &= \frac{1}{2(b-a)} (b-a)(b+a) = \frac{b+a}{2} \\ \Rightarrow \mu &= \frac{b+a}{2} \end{aligned}$$

Again from (3),

$$\begin{aligned} \mu_2' &= (-i)^2 \frac{d^2}{dt^2}(\phi_X(t)) \Big|_{t=0} = (-i)^2 \frac{1}{3i(b-a)} ((ib)^3 - (ia)^3) = \\ &= \frac{1}{3(b-a)} (b-a)(b^2 + ab + a^2) \\ \Rightarrow \mu_2' &= \frac{b^2 + ab + a^2}{3} \end{aligned}$$

Thus, the variance is given by

$$\begin{aligned} \sigma^2 &= \mu_2' - (\mu_1')^2 = \frac{b^2 + ab + a^2}{3} - \frac{(b+a)^2}{4} = \frac{4(b^2 + ab + a^2) - 3(b^2 + 2ab + a^2)}{12} = \frac{(b-a)^2}{12} \\ \Rightarrow \sigma^2 &= \frac{(b-a)^2}{12} \end{aligned}$$

P3:

Find the characteristic function of exponential $E(\lambda)$ distribution and hence find its mean and variance.

Solution:

Since $X \sim E(\lambda)$, its p.d.f. is given by

$$f(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$$

The characteristic function of X is given by

$$\phi_X(t) = E[e^{itX}] = \int_0^{\infty} e^{itx} f(x) dx = \lambda \int_0^{\infty} e^{itx} e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - it)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - it)x}}{-(\lambda - it)} \right]_0^{\infty} = \frac{\lambda}{\lambda - it} = \lambda(\lambda - it)^{-1} = \left(1 - \frac{it}{\lambda}\right)^{-1}$$

$$= \sum_{r=0}^{\infty} \left(\frac{it}{\lambda}\right)^r = \sum_{r=0}^{\infty} \frac{(it)^r r!}{r! \lambda^r}$$

$$\Rightarrow \phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r r!}{r! \lambda^r}$$

Thus, $\mu_r' = \text{coef. of } \frac{(it)^r}{r!} \text{ in } \phi_X(t) = \frac{r!}{\lambda^r}$.

Hence, the mean is given by $\mu = \mu_1' = \frac{1}{\lambda}$

Further, $\mu_2' = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$ and variance is given by

$$\sigma^2 = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Thus, $\sigma^2 = \frac{1}{\lambda^2}$

P4:

Find the p.m.f. of r.v. whose characteristic function is given by

$$\phi_X(t) = \frac{p}{1 - qe^{it}}$$

Solution:

$$\text{Given } \phi_X(t) = \frac{p}{1 - qe^{it}} = p(1 - qe^{it})^{-1}$$

$$= p \sum_{x=0}^{\infty} (qe^{it})^x = \sum_{x=0}^{\infty} pq^x e^{itx}$$

$$= \sum_{x=0}^{\infty} P(X = x) e^{itx} = E[e^{itX}]$$

where $P(X = x) = pq^x, x = 0, 1, 2, \dots$ which is geometric distribution.

3.3. Characteristic Function

Exercise

1. Find the characteristic function of a r.v. X whose moments are given by $\mu_r' = (r + 1)! 2^r$.

2. If $\phi_X(t) = \frac{3}{3-it}$, then find standard deviation of X .

3. Find the characteristic function of a r.v. X , whose p.d.f. is given by

$$f(x) = \begin{cases} \frac{x}{2} & , \quad 0 \leq x \leq 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

4. Find the characteristic function and hence obtain the mean and variance of a r.v. X , whose p.d.f. is given by

$$f(x) = \begin{cases} 2e^{-2x} & , \quad x \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

5. Find the characteristic function of a r.v. X whose p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{3} & , \quad -1 < x < 2 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

6. Find the characteristic function and hence find the mean and variance of a r.v. X whose p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}} & , \quad x \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

7. A r.v. X has the p.d.f. $f(x) = \begin{cases} \frac{1}{k} & , \quad 0 < x < k \\ 0 & , \quad \text{otherwise} \end{cases}$.

Find the characteristic function, mean and variance of X .

8. Find the characteristic function and hence find the mean and variance of a r.v. X whose p.d.f. is given by $f(x) = \lambda e^{-\lambda(x-a)}$, $x \geq a$.

9. Find the p.m.f. whose characteristic function is given by $\phi_X(t) = \frac{e^{it}}{2-e^{it}}$.

Answers:

1. $\phi_X(t) = \frac{1}{(1-2it)^2}$

2. $\frac{1}{3}$

3. $\frac{1}{2t^2}(e^{2it} - 2ite^{2it} - 1)$

4. $\phi_X(t) = \frac{2}{2-it}, \mu = \frac{1}{2}, \sigma^2 = \frac{1}{4}$

5. $\phi_X(t) = \begin{cases} \frac{e^{2it}-e^{it}}{3it} & , \quad t \neq 0 \\ 1 & , \quad t = 0 \end{cases}$

6. $(1-3it)^{-1}, \mu = 3, \sigma^2 = 9$

7. $\phi_X(t) = \frac{e^{itk}-1}{kit}, \mu = \frac{k}{2}, \sigma^2 = \frac{k^2}{12}$

8. $\phi_X(t) = \frac{\lambda e^{ait}}{\lambda-it}, \mu = \frac{a\lambda+1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$

9. $p(x) = \left(\frac{1}{2}\right)^k, k = 1, 2, 3, \dots$

3.4

Cumulant Generating Function

Just as the moment generating function (m.g.f.) $M_X(t)$ or characteristic function (ch.f.) $\phi_X(t)$ of a r.v. X generates its moments, the logarithm of $M_X(t)$ or $\phi_X(t)$ generates a sequence of numbers called the **Cumulants of X** . Cumulants are of interest for the following two reasons.

1. Moments in terms of cumulants can be obtained easily when compared to obtaining them from m.g.f. or ch.f.
2. j^{th} cumulant of a sum of independent r.v.s is simply the sum of the j^{th} cumulants of the summand.

Since the ch.f. exists for every r.v. (the m.g.f. need not exist for some r.v.s), the cumulant generating function (c.g.f.) is defined as the logarithm of the ch.f.

Cumulant generating function: Let X be a r.v. with characteristic function $\phi_X(t) = E[e^{itX}]$. The cumulant generating function (c.g.f.) of X is defined by

$$K_X(t) = \ln(\phi_X(t)) \quad \dots (1)$$

for all t in some open interval about 0 in $\mathbf{R}$, provided the RHS can be expanded as a convergent series in powers of t .

Thus,

$$K_X(t) = k_1(it) + k_2 \frac{(it)^2}{2!} + \dots + k_r \frac{(it)^r}{r!} \quad \dots (2)$$

Note that, $k_j = \text{coef of } \frac{(it)^j}{j!} \text{ in } K_X(t)$ and it is called the j^{th} **Cumulant** of X

We have,

$$\phi_X(t) = 1 + \mu_1'(it) + \mu_2' \frac{(it)^2}{2!} + \dots + \mu_r' \frac{(it)^r}{r!} \quad \dots (3)$$

From (1), (2) and (3), we have

$$\begin{aligned}
k_1(it) + k_2 \frac{(it)^2}{2!} + \dots &= \ln[1 + \mu_1'(it) + \dots + \dots] \\
&= \left(\mu_1'(it) + \mu_2' \frac{(it)^2}{2!} + \dots \right) - \frac{1}{2} \left(\mu_1'(it) + \mu_2' \frac{(it)^2}{2!} + \dots \right)^2 + \frac{1}{3} \left(\mu_1'(it) + \mu_2' \frac{(it)^2}{2!} + \dots \right)^3 - \dots \\
&\quad (\because \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots)
\end{aligned}$$

Comparing the coefficients of like powers of t , we get the relationship between moments and cumulants. Hence, we have

$k_1 = \mu_1' = \text{Mean} = \mu$ and $k_2 = \mu_2' - (\mu_1')^2 = \text{variance} = \sigma^2$.
Thus, $\mu = k_1$ and $\sigma^2 = k_2$.

Note:

1. From (2), $K_X(t)$ can be written as

$$K_X(t) = \sum_{j=1}^{\infty} k_j \frac{(it)^j}{j!}$$

Thus j^{th} cumulant $= k_j = \text{coef. of } \frac{(it)^j}{j!} \text{ in } K_X(t)$.

2. From (2), j^{th} cumulant is obtained as

$$k_j = (-i)^j \left. \frac{d^j K_X(t)}{dt^j} \right|_{t=0}$$

Example 1: If $X \sim B(n, p)$, then obtain the c.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \dots, n$$

Then the characteristic function of X is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{\infty} \binom{n}{x} (pe^{it})^x q^{n-x} \\ &= (q + pe^{it})^n\end{aligned}$$

$$\Rightarrow \phi_X(t) = (q + pe^{it})^n$$

Thus, the c.g.f. of X is given by

$$K_X(t) = \ln(\phi_X(t)) = \ln[q + pe^{it}]^n$$

$$\Rightarrow K_X(t) = n \ln(q + pe^{it})$$

$$= n \ln \left[q + p \left(1 + (it) + \frac{(it)^2}{2!} + \dots \right) \right]$$

$$= n \ln \left[1 + (it)p + \frac{(it)^2}{2!}p + \dots \right]$$

$$\Rightarrow K_X(t) = n \left[\left\{ (it)p + \frac{(it)^2}{2!}p + \dots \right\} - \frac{1}{2} \left\{ (it)p + \frac{(it)^2}{2!}p + \dots \right\}^2 + \dots \right]$$

$$\Rightarrow K_X(t) = (it)(np) + \frac{(it)^2}{2!}(np - np^2) + \dots$$

$$\therefore k_1 = \text{coef. of } (it) = np \text{ and } k_2 = \text{coef. of } \frac{(it)^2}{2!} = np - np^2 = np(1 - p) = npq$$

Thus mean and variance are given by $\mu = k_1 = np$ and $\sigma^2 = k_2 = npq$

Example 2: If $X \sim \text{Poisson } P(\lambda)$, then find the c.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, \dots$$

The characteristic function of X is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}\end{aligned}$$

Thus, $\phi_X(t) = e^{\lambda(e^{it}-1)}$

The c.g.f. of X is given by

$$\begin{aligned}K_X(t) &= \ln(\phi_X(t)) = \ln[e^{\lambda(e^{it}-1)}] = \lambda(e^{it} - 1) \\ &= \lambda \left[1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \dots - 1 \right]\end{aligned}$$

$$\Rightarrow K_X(t) = (it)\lambda + \frac{(it)^2}{2!}\lambda + \dots$$

Thus, $k_1 = \text{coef. of } (it) \text{ in } K_X(t) = \lambda$ and $k_2 = \text{coef. of } \frac{(it)^2}{2!} \text{ in } K_X(t) = \lambda$

Hence, mean = variance = λ

Example 3: If $X \sim NB(r, p)$, then find the c.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim NB(r, p)$, its p.m.f is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, x = 0, 1, 2, \dots$$

The characteristic function of X is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} \binom{-r}{x} p^r (-q)^x \\ &= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-qe^{it})^x = p^r (1 - qe^{it})^{-r}\end{aligned}$$

$$\Rightarrow \phi_X(t) = p^r (1 - qe^{it})^{-r}$$

The c.g.f. is given by

$$\begin{aligned} K_X(t) &= \ln(\phi_X(t)) = \ln(p^r(1 - qe^{it})^{-r}) \\ &= r \ln p - r \ln(1 - qe^{it}) \end{aligned}$$

$$\text{Now, } \frac{d}{dt}(K_X(t)) = (-r) \frac{-iqe^{it}}{1 - qe^{it}} = \frac{irqe^{it}}{1 - qe^{it}}$$

$$\therefore k_1 = (-i) \frac{d}{dt}(K_X(t)) \Big|_{t=0} = (-i) \frac{(irq)}{1 - q} = \frac{rq}{p}$$

$$\text{And } \frac{d^2}{dt^2}(K_X(t)) = irq \frac{d}{dt} \left[\frac{e^{it}}{1 - qe^{it}} \right] = irq \left[\frac{(1 - qe^{it})ie^{it} + e^{it}qie^{it}}{(1 - qe^{it})^2} \right]$$

$$\therefore k_2 = (-i)^2 \frac{d^2}{dt^2}(K_X(t)) \Big|_{t=0} = (-i)^2(irq)(i) \left[\frac{p + q}{p^2} \right]$$

$$\Rightarrow k_2 = \frac{rq}{p^2}$$

Thus mean and variance are given by

$$\mu = k_1 = \frac{rq}{p} \text{ and } \sigma^2 = k_2 = \frac{rq}{p^2} \text{ respectively.}$$

Example 4: If $X \sim N(\mu, \sigma^2)$, then obtain the c.g.f. of X and hence find its mean and variance.

Solution: If $X \sim N(\mu, \sigma^2)$, then its characteristic function can be shown that

$$\phi_X(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2} \quad (\text{recall!})$$

Hence, the c.g.f. is given by

$$K_X(t) = \ln \left[e^{it\mu - \frac{1}{2}\sigma^2 t^2} \right] = it\mu - \frac{1}{2}\sigma^2 t^2$$

$$\Rightarrow K_X(t) = (it)\mu + \frac{(it)^2}{2!} \sigma^2$$

$\therefore k_1 = \text{coef. of } (it) \text{ in } K_X(t) = \mu \text{ and } k_2 = \text{coef. of } \frac{(it)^2}{2!} \text{ in } K_X(t) = \sigma^2$

Thus, mean = μ and variance = σ^2 .

Example 5: If $X \sim E(\lambda)$, then obtain the c.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim E(\lambda)$, its ch.f. can be shown that

$$\phi_X(t) = \frac{\lambda}{\lambda - it} = \left(1 - \frac{it}{\lambda}\right)^{-1} \quad (\text{recall!})$$

The c.g.f. of X is given by

$$\begin{aligned} K_X(t) &= \ln(\phi_X(t)) = (-1) \ln\left(1 - \frac{it}{\lambda}\right) = (-1) \left[-\frac{it}{\lambda} - \frac{1}{2} \left(\frac{it}{\lambda}\right)^2 - \dots\right] \\ \Rightarrow K_X(t) &= \left[(it) \frac{1}{\lambda} + \frac{(it)^2}{2!} \frac{1}{\lambda^2} + \dots\right] \end{aligned}$$

Thus, $k_1 = \text{coef. of } (it) \text{ in } K_X(t) = \frac{1}{\lambda} \text{ and } k_2 = \text{coef. of } \frac{(it)^2}{2!} \text{ in } K_X(t) = \frac{1}{\lambda^2}$.

Thus, the mean and variance are given by $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$ respectively.

Properties of Cumulants: Here we develop some useful properties of cumulants.

Let $k_n(X)$ be the n^{th} cumulant of a r.v. X .

Theorem 1: $k_n(cX) = c^n k_n(X)$ for some real constant c .

Proof: Consider $\phi_{cX}(t) = E[e^{itcX}] = E[e^{i(tc)X}] = \phi_X(tc)$

$$\Rightarrow \phi_{cX}(t) = \phi_X(ct)$$

$$\Rightarrow \ln \phi_{cX}(t) = \ln \phi_X(ct)$$

$$\Rightarrow K_{cX}(t) = K_X(ct)$$

Then, $\frac{d^n}{dt^n} (K_{cX}(t)) \Big|_{t=0} = \frac{d^n}{dt^n} (K_X(ct)) \Big|_{t=0} = \frac{d^n}{ds^n} (K_X(s)) \Big|_{s=0} c^n$, where $ct = s$

Therefore, $(-i)^n \frac{d^n}{dt^n} (K_{cX}(t)) \Big|_{t=0} = (-i)^n c^n \frac{d^n}{dt^n} (K_X(t)) \Big|_{t=0}$

$$\Rightarrow k_n(cX) = c^n k_n(X)$$

$$\textbf{Theorem 2: } k_n(X + b) = \begin{cases} k_n(X) + b & , \text{ if } n = 1 \\ k_n(X) & , \text{ if } n > 1 \end{cases}$$

$$\textbf{Proof: } \phi_{X+b}(t) = E[e^{it(X+b)}] = e^{itb} E[e^{itX}] = e^{itb} \phi_X(t)$$

$$\Rightarrow \phi_{X+b}(t) = e^{itb} \phi_X(t)$$

$$\Rightarrow \ln[\phi_{X+b}(t)] = itb + \ln[\phi_X(t)]$$

$$\Rightarrow K_{X+b}(t) = itb + K_X(t)$$

$$\Rightarrow (-i)^n \frac{d^n}{dt^n} (K_{X+b}(t)) \Big|_{t=0} = (-i)^n \frac{d^n}{dt^n} (itb) \Big|_{t=0} + (-i)^n \frac{d^n}{dt^n} (K_X(t)) \Big|_{t=0}$$

$$\text{If } n = 1, \text{ then } K_n(X + b) = b + K_n(X).$$

$$\text{If } n > 1, K_n(X + b) = K_n(X).$$

Theorem3: If X and Y are independent random variables and $S = X + Y$, then

$$k_n(S) = k_n(X) + k_n(Y).$$

Proof: Since X and Y are independent,

$$\phi_S(t) = \phi_X(t) \phi_Y(t)$$

$$\Rightarrow \ln[\phi_S(t)] = \ln[\phi_X(t)] + \ln[\phi_Y(t)] \Rightarrow K_S(t) = K_X(t) + K_Y(t)$$

$$\Rightarrow (-i)^n \frac{d^n}{dt^n} (K_S(t)) \Big|_{t=0} = (-i)^n \frac{d^n}{dt^n} (K_X(t)) \Big|_{t=0} + (-i)^n \frac{d^n}{dt^n} (K_Y(t)) \Big|_{t=0}$$

$$\Rightarrow k_n(S) = k_n(X) + k_n(Y)$$

Generalization: If $S = X_1 + \dots + X_m$ where $X_1, X_2, \dots, X_m$ are independent random variables, then

$$k_n(S) = k_n(X_1) + k_n(X_2) + \dots + k_n(X_m)$$

Theorem 4: Let $\mu_j' = E(X^j)$ be the j^{th} moment of X about zero for $j = 1, 2, 3, \dots, n$ where $\mu_0' = 1$. Let $k_1, k_2, \dots, k_n$ be the n cumulants of X . Then

$$\mu_{r+1}' = \sum_{j=0}^r \binom{r}{j} \mu_j' k_{(r+1-j)} \quad \dots (1)$$

for $r = 0, 1, \dots, n-1$.

Proof: For $j = 0, 1, 2, \dots, n$, we have

$$\mu_j' = \left. \frac{d^j}{dt^j} (\phi_X(t)) \right|_{t=0} \text{ and } k_j = (-i)^j \left. \frac{d^j}{dt^j} (K_X(t)) \right|_{t=0}$$

where $\phi_X(t) = E[e^{itX}]$ and $K_X(t) = \ln[\phi(t)]$ or equivalently, $\phi_X(t) = e^{K_X(t)}$.

Differentiating this last identify w.r.t. t gives

$$\phi_X'(t) = e^{K_X(t)} K_X'(t) \quad \dots (2)$$

and evaluating this at $t = 0$ gives $i\mu_1' = ik_1 \Rightarrow \mu_1' = k_1$ holds for $r = 0$.

Differentiating (2) for r times, it gives

$$\phi_X^{(r+1)}(t) = \sum_{j=0}^r \binom{r}{j} \phi_X^{(j)}(t) K_X^{(r+1-j)}(t)$$

(Use Leibnitz theorem for the n^{th} derivative of the product of two functions)

and evaluating this at $t = 0$ gives

$$\mu_{r+1}' = \sum_{j=0}^r \binom{r}{j} \mu_j' k_{(r+1-j)} \text{ for } r = 0, 1, \dots, n-1$$

Note: Taking $r = 0, 1, 2, 3$ in (1) produces

$$\left. \begin{aligned} \mu_1' &= k_1 \\ \mu_2' &= k_2 + \mu_1' k_1 \\ \mu_3' &= k_3 + 2\mu_1' k_2 + \mu_2' k_1 \\ \mu_4' &= k_4 + 3\mu_1' k_3 + 3\mu_2' k_2 + \mu_3' k_1 \end{aligned} \right\} \quad \dots (3)$$

These recursive formulae can be used to calculate the (μ') s efficiently from k s and vice versa.

Let $\mu_j = E[(X - E(X))^j] = E[(X - \mu_1')^j]$ for $j = 1, 2, \dots$ are unknown as **central moments**.

Then formulae (3) simplify to

$$\mu_2 = k_2, \mu_3 = k_3, \mu_4 = k_4 + 3k_2^2 \text{ and } k_2 = \mu_2, k_3 = \mu_3, k_4 = \mu_4 - 3\mu_2^2$$

Note: Mean = $\mu = k_1$ and variance = $\mu_2 = \sigma^2 = k_2$.

P1:

Let X be a.r.v with p.d.f

$$f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Find the c. g. f. of X and hence find its mean and variance.

Solution:

The characteristic function of X is given by

$$\begin{aligned} \phi_X(t) &= E[e^{itx}] = \int_{-a}^a e^{itx} f(x) dx = \frac{1}{2a} \int_{-a}^a e^{itx} dx \\ &= \frac{1}{2a} \left[\frac{e^{itx}}{it} \right]_{-a}^a = \frac{1}{2ait} [e^{ita} - e^{-ita}] \\ &= \frac{1}{2ait} \left[\left(1 + ita + \frac{(ita)^2}{2!} + \frac{(ita)^3}{3!} + \dots \right) - \left(1 - ita + \frac{(ita)^2}{2!} - \frac{(ita)^3}{3!} + \dots \right) \right] \\ &= \frac{1}{2ait} \left[2ita + 2 \frac{(ita)^3}{3!} + 2 \frac{(ita)^5}{5!} + \dots \right] \\ \Rightarrow \phi_X(t) &= \left[1 + \frac{(it)^2 a^2}{2! \cdot 3} + \frac{(it)^4 a^4}{4! \cdot 5} + \dots \right] \end{aligned}$$

The c.g.f. of X is given by

$$\begin{aligned} K_X(t) &= \ln[\phi_X(t)] = \ln \left[1 + \frac{(it)^2 a^2}{2! \cdot 3} + \frac{(it)^4 a^4}{4! \cdot 5} + \dots \right] \\ \Rightarrow K_X(t) &= \left(\frac{(it)^2 a^2}{2! \cdot 3} + \frac{(it)^4 a^4}{4! \cdot 5} + \dots \right) - \frac{1}{2} \left[\frac{(it)^2 a^2}{2! \cdot 3} + \frac{(it)^4 a^4}{4! \cdot 5} + \dots \right]^2 + \dots \\ \Rightarrow k_1 &= \text{coeff. of } (it) \text{ in } K_X(t) = 0 \text{ and } k_2 = \text{coeff. of } \frac{(it)^2}{2!} \text{ in } K_X(t) = \frac{a^3}{3} \end{aligned}$$

Thus, the mean and variance are given by $\mu = 0$ and $\sigma^2 = \frac{a^3}{3}$

P2:

Let X be r.v. with p.d.f

$$f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty$$

Find the c.g.f. of and hence find its mean and variance.

Solution:

The characteristic function of X is given by

$$\begin{aligned}\phi_X(t) &= E[e^{itx}] = \int_{-\infty}^{\infty} e^{itx} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{itx} e^{-|x|} dx \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} (\cos tx + i \sin tx) e^{-|x|} dx \right] = \frac{1}{2} 2 \int_0^{\infty} (\cos tx) e^{-x} dx\end{aligned}$$

(Since the integrals in the first and second integrals are even and odd functions of x respectively).

$$\begin{aligned}&= \int_0^{\infty} e^{-x} \cos tx dx \\ &= \frac{1}{t^2} - \frac{1}{t^2} \int_0^{\infty} e^{-x} \cos tx dx \quad (\text{On integration by parts}) \\ \Rightarrow \phi_X(t) &= \frac{1}{t^2} - \frac{1}{t^2} \phi_X(t) \Rightarrow \phi_X(t) = \frac{1}{1+t^2}\end{aligned}$$

The c.g.f. is given by $K_X(t) = \ln(\phi_X(t)) = \ln\left[\frac{1}{1+t^2}\right] = -\ln(1+t^2)$

$$= -\left(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \dots\right) = -t^2 + \frac{t^4}{2} - \frac{t^6}{3} + \dots$$

$$\Rightarrow K_X(t) = \frac{(it)^2}{2!} 2 + \dots$$

$$\Rightarrow k_1 = \text{coeff. of } (it) \text{ in } K_X(t) = 0 \text{ and } k_2 = \text{coeff. of } \frac{(it)^2}{2!} \text{ in } K_X(t) = 2$$

Thus, the mean and variance are given by $\mu = 0$ and $\sigma^2 = 2$

P3:

Let X be a.r.v. with p.m.f given by

$$p(x) = \begin{cases} q^x p & , \quad x = 0, 1, 2, \dots, \quad 0 < p < 1, \quad q = 1 - p \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the c.g.f. and hence find its mean and variance.

Solution:

The characteristic function of X is given by

$$\begin{aligned} \phi_X(t) &= E[e^{itX}] = \sum_{x=0}^{\infty} e^{itx} p(x) = \sum_{x=0}^{\infty} e^{itx} q^x p \\ &= p \sum_{x=0}^{\infty} (qe^{it})^x = p(1 - qe^{it})^{-1} \end{aligned}$$

$$\Rightarrow \phi_X(t) = p(1 - qe^{it})^{-1}$$

The c.g.f. is given by $K_X(t) = \ln[\phi_X(t)] = \ln p - \ln(1 - qe^{it})$

$$\text{Now, } \frac{d}{dt}(K_X(t)) = \frac{qie^{it}}{1 - qe^{it}} = (iq) \left[\frac{e^{it}}{1 - qe^{it}} \right]$$

$$\therefore k_1 = (-i) \frac{d}{dt}(K_X(t)) \Big|_{t=0} = (-i)(iq) \frac{1}{p} = \frac{q}{p}$$

$$\text{Further, } \frac{d^2}{dt^2}(K_X(t)) = (iq) \left[\frac{(1 - qe^{it})ie^{it} + e^{2it}iq}{(1 - qe^{it})^2} \right] \text{ and}$$

$$k_2 = (-i)^2 \frac{d^2}{dt^2}(K_X(t)) \Big|_{t=0} = (-i)^2(iq) \frac{(ip + iq)}{p^2} \Rightarrow k_2 = \frac{q}{p^2}$$

Thus, the mean and variance are given by $\mu = \frac{q}{p}$ and $\sigma^2 = \frac{q}{p^2}$ respectively.

P4:

Let X be a r.v. with p.d.f.

$$f(x) = \frac{1}{2\lambda} \exp\left[-\frac{|x-\mu|}{\lambda}\right], -\infty < x, \mu < \infty, \sigma > 0$$

Find the c.g.f. and hence obtain its mean and variance.

Solution:

It can be shown that the characteristic function is given by

$$\phi_X(t) = \frac{e^{it\mu}}{1+\lambda^2 t^2} \quad (\text{do it!})$$

$$\begin{aligned} \text{The c.g.f. is given by } K_X(t) &= \ln[\phi_X(t)] = \ln\left[\frac{e^{it\mu}}{1+\lambda^2 t^2}\right] \\ &= it\mu - \ln(1 + \lambda^2 t^2) \\ &= it\mu - (\lambda^2 t^2 + \dots) \end{aligned}$$

$$\Rightarrow K_X(t) = it\mu + \frac{(it)^2}{2!} 2\lambda^2 + \dots$$

$$\Rightarrow k_1 = \text{coef. of } (it) = \mu \text{ and } k_2 = \text{coef. of } \frac{(it)^2}{2!} = 2\lambda^2$$

Thus, the mean and variance are given by μ and $2\lambda^2$ respectively.

3.4. Cumulant Generating Function

Exercise

1. Let $U = \frac{X-a}{h}$ where a and h are real constants. Obtain the cumulants of U in terms of cumulants of X .

2. For a distribution, the cumulants are given by

$$k_r = n(r-1)!, n > 0$$

Find the ch.f.

3. The p.m.f of X is given by

$$p(x) = q^{r-1}p, r = 1, 2, 3, \dots$$

Find the c.g.f. and hence find its mean and variance.

4. The p.d.f. of X is given by

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty$$

Find the c.g.f. and hence find its mean and variance.

Answers

1. $k_1(U) = \frac{k_1(X)-a}{h}$ and $k_r(U) = \frac{k_r(X)}{h^r}$ for $r = 2, 3 \dots$

2. $\phi_X(t) = (1 - it)^{-n}$

3. $\mu = \frac{1}{p}$ and $\sigma^2 = \frac{1+q}{p^2}$

4. $\mu = 0$ and $\sigma^2 = \frac{\pi^2}{3}$

3.5

Probability Generating Function

Let X be a non-negative integer valued random variable with p.m.f. $p(x) = P(X = x)$. Then the **probability generating function** (p.g.f.) of X is defined by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x)$$

where $-1 \leq t \leq 1$ is a dummy variable.

Advantages:

1. It is easy to compute.
2. Moments and some probabilities can be obtained easily.
3. The p.m.f. can be obtained easily from p.g.f.
4. It is easy to handle with sum of independent r.vs.

Effect of linear transformation of p.g.f:

Theorem 1: Let X be a discrete random variable with p.g.f. $G_X(t)$. Let $Y = a + bX$ where a and b are real constants. Then $G_Y(t) = t^a G_X(t^b)$

Proof: By the definition of probability generating function, we have,
 $G_X(t) = E[t^X]$. Then

$$\begin{aligned} G_Y(t) &= E[t^{(a+bX)}] = E[t^a t^{bX}] = t^a E[(t^b)^X] = t^a G_X(t^b) \\ \Rightarrow G_Y(t) &= t^a G_X(t^b) \end{aligned}$$

Theorem 2: Additive Property: If X and Y are independent random variables, then for constants a, b , we have

$$G_{(aX+bY)}(t) = G_X(t^a) + G_Y(t^b)$$

Proof: $G_{aX+bY}(t) = E[t^{aX+bY}]$ (by P.g.f.)

$$= E[(t^a)^X (t^b)^Y]$$

$$= E[(t^a)^X] E[(t^b)^Y] \quad (\because X \& Y \text{ are independent.})$$

$$= G_X(t^a) G_Y(t^b)$$

Thus, $G_{aX+bY}(t) = G_X(t^a) G_Y(t^b)$

Note: In particular, if $a = b = 1$, then $G_{X+Y}(t) = G_X(t) G_Y(t)$

Generalization: If $X_1, X_2, \dots, X_n$ are independent random variables, then

$$G_{(X_1+\dots+X_n)}(t) = G_{X_1}(t) G_{X_2}(t) \dots G_{X_n}(t)$$

Relationship between p.g.f. and m.g.f :

The p.g.f. and m.g.f. of a random variable X are defined by $G_X(t) = E[t^X]$ and $M_X(t) = E[e^{tX}]$ respectively.

Now, $M_X(t) = E[e^{tX}] = E[(e^t)^X] = G_X(e^t)$

$$\Rightarrow M_X(t) = G_X(e^t)$$

Further, $G_X(t) = E[t^X] = E[e^{\ln(t^X)}] = E[e^{X \ln t}] = M_X(\ln t)$

$$\Rightarrow G_X(t) = M_X(\ln t)$$

Theorem 3: p.m.f. from p.g.f : Let $G_X(t)$ be the p.g.f. of a discrete r.v. X that can take the values $0, 1, 2, \dots$. Then the p.m.f. of X is given by

$$p(x) = P(X = x) = \frac{1}{x!} G_X^{(x)}(t) \Big|_{t=0}$$

Proof: By definition, we have

$$G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x p(x)$$

$$= P(X = 0)t^0 + P(X = 1)t^1 + P(X = 2)t^2 + \dots + P(X = x)t^x + \dots$$

It can be observed that the coefficient of t^x in $G_X(t)$ is $P(X = x)$. To obtain coefficient of t^x , differentiate $G_X(t)$, x times and substitute $t = 0$. Thus,

$$G_X^{(x)}(t) = x(x-1)(x-2) \dots 2 \cdot 1 \cdot P(X = x) + (x+1)(x) + \dots 2 \cdot 1 \cdot t \cdot P(X = x+1) + \dots$$

When $t = 0$, all terms after the first vanish. Thus,

$$P(X = x) = \frac{1}{x!} G_X^{(x)}(t) \Big|_{t=0} = \frac{1}{x!} G_X^{(x)}(0)$$

Computation of moments using p.g.f:

In the derivation of moments, we use *Taylor's expansion*:

Suppose $f(x)$ has derivatives of all orders at $x = a$. The Taylor's expansion of $f(x)$ at the point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n + \dots$$

$$\Rightarrow f(x) = \sum_{i=0}^{\infty} \frac{f^i(a)}{i!}(x-a)^i$$

The Taylor's expansion of $f(t) = t^X$ about $t = 1$ is given by

$$t^X = 1 + X(t-1) + X(X-1)\frac{(t-1)^2}{2!} + X(X-1)(X-2)\frac{(t-1)^3}{3!} + \dots$$

$$\Rightarrow G_X(t) = E[t^X]$$

$$= 1 + (t-1)E(X) + \frac{(t-1)^2}{2!}E[X(X-1)] + \frac{(t-1)^3}{3!}E[X(X-1)(X-2)] + \dots$$

Differentiating (1) w.r.t., t r times and setting $t = 1$, we get

$$G_X^{(r)}(t) \Big|_{t=1} = E[X(X-1) \dots (X-r+1)]$$

$$\Rightarrow E[X(X-1) \dots (X-r+1)] = G_X^{(r)}(1) \quad \dots(2)$$

which is known as **r^{th} factorial moment of X** . Using these, we can find the moments about origin as follows:

If $r = 1$ in (2), we have

$$\mu_1' = E(X) = G_X^{(1)}(1)$$

If $r = 2$ in (2), we have

$$E[X(X-1)] = E[X^2 - X] = E(X^2) - E(X) = G_X^{(2)}(1)$$

$$\Rightarrow E(X^2) = G_X^{(2)}(1) + E(X) = G_X^{(2)}(1) + G_X^{(1)}(1)$$

Thus, the second moment about origin is given by

$$\mu_2' = E(X^2) = G_X^{(2)}(1) + G_X^{(1)}(1)$$

Similarly, we can find any moment about origin.

Computation of mean and variance using p.g.f:

Theorem 4: If the r.v. X has p.g.f. $G_X(t)$, then the mean and variance of X are given by

$$\mu = E(X) = G_X^{(1)}(1) \text{ and}$$

$$\sigma^2 = V(X) = G_X^{(2)}(1) + G_X^{(1)}(1) - [G_X^{(1)}(1)]^2$$

respectively.

Proof: From the above, we have

$$\mu_1' = G_X^{(1)}(1), \mu_2' = G_X^{(2)}(1) + G_X^{(1)}(1)$$

Thus, the mean $\mu = \mu_1' = G_X^{(1)}(1)$ and variance $\sigma^2 = \mu_2' - (\mu_1')^2$

$$\Rightarrow \sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left(G_X^{(1)}(1)\right)^2$$

Convolution formula:

Theorem 5: If X and Y are independent integer-valued random variables with $P(X = x) = p_1(x)$ and $P(Y = y) = p_2(y)$, $x = 0, 1, 2, \dots$ and $y = 0, 1, 2, \dots$, then

$$P(X + Y = z) = p(z) = \sum_{x=0}^z p_1(x)p_2(z-x)$$

Proof: We have ,

$$G_X(t) = \sum_{x=0}^{\infty} t^x p_1(x) \quad \text{and} \quad G_Y(t) = \sum_{y=0}^{\infty} t^y p_2(y)$$

Now, $G_{X+Y}(t) = G_X(t)G_Y(t)$ (Since X and Y are independent)

$$= \left(\sum_{x=0}^{\infty} t^x p_1(x) \right) \left(\sum_{y=0}^{\infty} t^y p_2(y) \right)$$

$$\Rightarrow G_{X+Y}(t) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} p_1(x)p_2(y)t^{x+y} \quad \dots (1)$$

Let $Z = X + Y$. Then

$$G_Z(t) = E[t^Z] = \sum_{z=0}^{\infty} t^z p(z) \quad \dots (2)$$

From (1) and (2), we have

$$\sum_{z=0}^{\infty} t^z p(z) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} p_1(x)p_2(y)t^{x+y}$$

$$\Rightarrow \sum_{z=0}^{\infty} t^z p(z) = \sum_{z=0}^{\infty} \left(\sum_{x=0}^z p_1(x) p_2(z-x) \right) t^z$$

$$\Rightarrow p(z) = \sum_{x=0}^z p_1(x) p_2(z-x), \text{ for } z = 0, 1, 2, \dots$$

Example 1: If $X \sim B(n, p)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim B(n, p)$, its p.m.f. is given by

$$p(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \dots, n, \quad 0 < p < 1, q = 1 - p$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^n t^x p(x) = \sum_{x=0}^n t^x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^n \binom{n}{x} (tp)^x q^{n-x} = (q + tp)^n$$

$$\Rightarrow G_X(t) = (q + tp)^n$$

Differentiating both sides w.r.t., t we get

$$G_X^{(1)}(t) = n(q + tp)^{n-1} p$$

$$\Rightarrow \mu = \text{mean} = \mu_1' = G_X^{(1)}(1) = np \text{ and variance is given by}$$

$$\sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1) \right]^2$$

$$\text{But } G_X^{(2)}(t) = np(n-1)(q + tp)^{n-2} p$$

$$\Rightarrow G_X^{(2)}(1) = n(n-1)p^2 = n^2 p^2 - np^2$$

$$\text{Therefore, } \sigma^2 = n^2 p^2 - np^2 + np - n^2 p^2 = np(1 - p) \Rightarrow \sigma^2 = npq$$

Example 2: If $X \sim P(\lambda)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim P(\lambda)$, its p.m.f. is given by

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \dots \text{ and } \lambda > 0$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(t\lambda)^x}{x!} = e^{-\lambda} e^{t\lambda} = e^{\lambda(t-1)}$$

$$\Rightarrow G_X(t) = e^{\lambda(t-1)}$$

Differentiating both sides w.r.t. t , we get

$$G_X^{(1)}(t) = e^{\lambda(t-1)} \lambda \text{ and } G_X^{(2)}(t) = e^{\lambda(t-1)} \lambda^2$$

$$\text{Thus, } G_X^{(1)}(1) = \lambda \text{ and } G_X^{(2)}(1) = \lambda^2$$

$$\text{Hence, the mean and variance are given by } \mu = G_X^{(1)}(1) = \lambda$$

$$\text{and } \sigma^2 = G_X^{(2)}(1) + G_X^{(1)}(1) - [G_X^{(1)}(1)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda \text{ respectively.}$$

Example 3: If $X \sim NB(r, p)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim NB(r, p)$, its p.m.f. is given by

$$p(x) = \binom{-r}{x} p^r (-q)^x, \quad x = 0, 1, 2, \dots$$

The p.g.f. of X is given by

$$\begin{aligned} G_X(t) &= E[t^X] = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x \binom{-r}{x} p^r (-q)^x \\ &= p^r \sum_{x=0}^{\infty} \binom{-r}{x} (-tq)^x = p^r (1 - tq)^{-r} \end{aligned}$$

$$\Rightarrow G_X(t) = p^r(1 - tq)^{-r}$$

$$\Rightarrow G_X^{(1)}(t) = p^r(-r)(1 - tq)^{-(r+1)}(-q) = rqp^r(1 - tq)^{-(r+1)}$$

$$\Rightarrow G_X^{(2)}(t) = rqp^r(-(r+1))(1 - tq)^{-(r+2)}(-q) = r(r+1)q^2p^r(1 - tq)^{-(r+2)}$$

$$\text{Thus, } G_X^{(1)}(t) = rqp^r p^{-(r+1)} = \frac{rq}{p} \text{ and}$$

$$G_X^{(2)}(t) = r(r+1)q^2p^r p^{-(r+2)} = (r^2 + r) \frac{q^2}{p^2}$$

$$\Rightarrow G_X^{(2)}(t) = \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2}$$

$$\text{Thus, } \mu = \text{mean} = G_X^{(1)}(1) = \frac{rq}{p} \text{ and}$$

$$\begin{aligned} \sigma^2 = \text{variance} &= G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1) \right]^2 \\ &= \frac{r^2q^2}{p^2} + \frac{rq^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2} = \frac{rq}{p^2}(q + p) \Rightarrow \sigma^2 = \frac{rq}{p^2} \end{aligned}$$

Example4: If $X \sim G(p)$, then find the p.g.f. of X and hence obtain its mean and variance.

Solution: Since $X \sim G(p)$, its p.m.f. is given by

$$p(x) = q^x p, \quad x = 0, 1, 2, \dots$$

The p.g.f. of X is given by

$$G_X(t) = E[t^X] = \sum_{x=0}^{\infty} t^x p(x) = \sum_{x=0}^{\infty} t^x q^x p = p \sum_{x=0}^{\infty} (tq)^x = \frac{p}{1 - tq}$$

$$\Rightarrow G_X(t) = p(1 - tq)^{-1}$$

$$G_X^{(1)}(t) = p(-1)(1 - tq)^{-2}(-q) = pq(1 - tq)^{-2}$$

$$\Rightarrow G_X^{(1)}(1) = \frac{pq}{p^2} = \frac{q}{p}$$

$$\text{Now, } G_X^{(2)}(t) = pq(-2)(1-tq)^{-3}(-q) = 2pq^2(1-tq)^{-3}$$

$$\Rightarrow G_X^{(2)}(1) = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

Hence, the mean μ and variance σ^2 of X are given by:

$$\mu = G_X^{(1)}(1) = \frac{q}{p} \text{ and}$$

$$\begin{aligned} \sigma^2 &= G_X^{(2)}(1) + G_X^{(1)}(1) - \left[G_X^{(1)}(1) \right]^2 \\ &= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p^2} (q + p) = \frac{q}{p^2}. \end{aligned}$$

Example 5: The j.p.m.f. of (X, Y) is given in the following table. Prove or disprove

$G_{X+Y}(t) = G_X(t)G_Y(t)$ iff X and Y are independent.

$X \backslash Y$	0	1	2	Total
0	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{1}{3}$
1	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$
2	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{3}$
Total	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Solution: Since $P(X = 1, Y = 2) = \frac{2}{9} \neq P(X = 1)P(Y = 2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$, it follows that X and Y are not independent.

$$\text{Now, } G_X(t) = G_Y(t) = \frac{1}{3}(1 + t + t^2)$$

Let $Z = X + Y$. Then $Z = 0, 1, 2, 3, 4$. Let $p_i = P(Z = i), i = 0, 1, 2, 3, 4$.

$$p_0 = P(Z = 0) = P(X + Y = 0) = P(X = 0, Y = 0) = \frac{1}{9}$$

$$p_1 = P(Z = 1) = P(X + Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 0) = \frac{2}{9} + 0 = \frac{2}{9}$$

$$\begin{aligned} p_2 = P(Z = 2) &= P(X + Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0) \\ &= 0 + \frac{1}{9} + \frac{2}{9} = \frac{3}{9} \end{aligned}$$

$$p_3 = P(Z = 3) = P(X + Y = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{2}{9} + 0 = \frac{2}{9}$$

$$p_4 = P(Z = 4) = P(X + Y = 4) = P(X = 2, Y = 2) = \frac{1}{9}$$

The p.d.f. of $Z = X + Y$ is given by

$$G_{X+Y}(t) = \frac{1}{9} + \frac{2}{9}t + \frac{3}{9}t^2 + \frac{2}{9}t^3 + \frac{1}{9}t^4$$

$$\Rightarrow G_{X+Y}(t) = \frac{1}{9}(1 + 2t + 3t^2 + 2t^3 + t^4) = \left[\frac{1}{3}(1 + t + t^2)\right]^2$$

$\Rightarrow G_{X+Y}(t) = G_X(t)G_Y(t)$ but X and Y are not independent. Thus the statement is disproved.

Example 6: Can $G_X(t) = \frac{2}{1+t}$ be the p.d.f. of .r.v. X ? Give reasons.

Solution: We have $G_X(1) = \frac{2}{1+1} = \frac{2}{2} = 1$

Further, $G_X(t) = \frac{2}{1+t} = 2(1+t)^{-1} = 2(1 - t + t^2 - t^3 + \dots)$

$$\Rightarrow G_X(t) = 2 \sum_{x=0}^{\infty} (-1)^x t^x$$

Thus, $p(x) = P(X = x) = \text{coef. of } t^x \text{ in } G_X(t) = 2(-1)^x$

$$\Rightarrow p(x) = 2(-1)^x, x = 0, 1, 2, \dots$$

Note that it takes negative values also. Hence, $G_X(t)$ is not a p.g.f.

Example 7 : A fair die is thrown n times. Let S be the total number of points.

Show that $P(S = n + 5) = \binom{n+4}{5} \left(\frac{1}{6}\right)^n$.

Solution: The p. g. f. of a single throw is given by:

$$G_X(t) = \sum_{x=1}^6 t^x p(x) = \sum_{x=1}^6 \frac{t^x}{6}$$

$$= \frac{1}{6} (t + t^2 + \dots + t^6) = \frac{t}{6} (1 + t + \dots + t^5) = \frac{t}{6} \frac{(1-t^6)}{1-t}$$

$$\Rightarrow G_X(t) = \frac{t}{6} (1 - t^6) (1 - t)^{-1}$$

Since the n throws are identical and independent,

$$G_S(t) = [G_X(t)]^n = \frac{t^n (1-t^6)^n (1-t)^{-n}}{6^n}$$

$$= \frac{t^n}{6^n} \sum_{j=0}^n \binom{n}{j} (-t^6)^j \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k$$

$$\Rightarrow G_S(t) = \frac{1}{6^n} \sum_{j=0}^n \sum_{k=0}^{\infty} (-1)^j \binom{n}{j} \binom{n+k-1}{k} t^{k+6j+n}$$

$$= \sum_{k=0}^{\infty} P(S = k + 6j + n) t^{k+6j+n}$$

where,

$$P(S = k + 6j + n) = \frac{1}{6^n} \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{n+k-1}{k}$$

Now , $P(S = n + 5) = P(S = k + 6j + n)$ with $j = 0$ and $k = 5$

$$= \frac{1}{6^n} (-1)^0 \binom{n}{0} \binom{n+5-1}{5} = \frac{1}{6^n} \binom{n+4}{5}$$

$$\Rightarrow P(S = n + 5) = \frac{1}{6^n} \binom{n+4}{5}$$

P1:

Urns U_1 and U_2 have the following distribution

U_1				
x	0	1	2	3
$p_1(x)$	0.4	0.2	0.1	0.3

U_2			
y	1	2	3
$p_1(y)$	0.3	0.6	0.1

A ball is drawn from each urn and the numbers X, Y appearing are added. Find the p.m.f of $X + Y$ by using p.g.f.

Solution:

Here X and Y are independent. The p.g.fs of X and Y are given by

$$G_X(t) = \sum_{x=0}^3 t^x p_1(x) = 0.4 + 0.2t + 0.1t^2 + 0.3t^3$$

and

$$G_Y(t) = \sum_{y=1}^3 t^y p_2(y) = 0.3t + 0.6t^2 + 0.1t^3$$

respectively. Since X and Y are independent

$$\begin{aligned} G_{X+Y}(t) &= G_X(t)G_Y(t) = (0.4 + 0.2t + 0.1t^2 + 0.3t^3)(0.3t + 0.6t^2 + 0.1t^3) \\ &= 0.12t + 0.30t^2 + 0.19t^3 + 0.17t^4 + 0.19t^5 + 0.03t^6 \end{aligned}$$

Thus the p.m.f. of $X + Y$ is given by

r	1	2	3	4	5	6
$P(X + Y = r)$	0.12	0.30	0.19	0.17	0.19	0.03

P2:

Lottery tickets bear numbers from 000000 to 999999. Find the probability that a ticket bears a number whose sum of the first three digits equals the sum of the last three digits.

Solution:

Let the ticket bear the number $Y_1Y_2Y_3Y_4Y_5Y_6$. Obviously, all Y s are identically and independently distributed with

$$p(x) = P(Y_j = x) = \frac{1}{10}, \quad x = 0, 1, 2, \dots, 9$$

The p.g.f. of each Y is given by

$$\begin{aligned} G_Y(t) &= \frac{1}{10} (1 + t + \dots + t^9) \\ \Rightarrow G_Y(t) &= \frac{1}{10} \frac{(1-t^{10})}{(1-t)} \end{aligned}$$

Let $X_1 = Y_1 + Y_2 + Y_3$ and $X_2 = Y_4 + Y_5 + Y_6$.

Then $G_{X_1}(t) = G_{Y_1}(t)G_{Y_2}(t)G_{Y_3}(t)$

$$= \left(\frac{1}{10}\right)^3 (1 - t^{10})^3 (1 - t)^{-3}$$

Similarly $G_{X_2}(t) = \left(\frac{1}{10}\right)^3 (1 - t^{10})^3 (1 - t)^{-3}$

Let $G_{X_1}(t) = G_{X_2}(t) = G(t)$

Then $P(X_1 = r) = \text{coef. of } t^r \text{ in } G(t)$

and $P(X_2 = r) = \text{coef. of } t^{-r} \text{ in } G(t^{-1})$

$\therefore p = P(X_1 = X_2) = \text{coef. of } t^0 \text{ in } G(t)G(t^{-1})$

$$= \text{coef. of } t^0 \text{ in } \left(\frac{1}{10}\right)^3 (1 - t^{10})^3 (1 - t)^{-3} \cdot \left(\frac{1}{10}\right)^3 \left(1 - \frac{1}{t^{10}}\right)^3 \left(1 - \frac{1}{t}\right)^{-3}$$

$$\Rightarrow p = \text{coef. of } t^0 \text{ in } \left(\frac{1}{10}\right)^3 (1 - t^{10})^6 \cdot t^{-27} (1 - t)^{-6}$$

$$\text{Now, } (1 - t^{10})^6 = 1 - \binom{6}{1} t^{10} + \binom{6}{2} t^{20} - \dots$$

$$\text{and } (1 - t)^{-6} = 1 + \binom{6}{5} t + \binom{7}{5} t^2 + \dots \binom{5+r}{5} t^r + \dots$$

$$\therefore p = \left(\frac{1}{10}\right)^6 \left[\binom{32}{5} - \binom{6}{1} \binom{22}{5} + \binom{6}{2} \binom{12}{5} \right]$$

P3:

Find the p.g.f. of $P(X > n)$.

Solution:

The p.g.f. of X is defined by

$$G(t) = \sum_{x=0}^{\infty} p(x)t^x, \text{ where } p(x) \text{ is the p.m.f. of } X.$$

$$\text{Let } f_n = P(X > n) \text{ and } p(n) = P(X = n)$$

$$\text{Consider } P(X = n + 1) = P(X > n) - P(X > n + 1)$$

$$\Rightarrow p(n + 1) = f_n - f_{n+1}$$

$$\therefore t^n f_n - \left(\frac{1}{t}\right) t^{n+1} f_{n+1} = \frac{1}{t} p(n + 1)$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n f_n - \frac{1}{t} \sum_{n=0}^{\infty} t^{n+1} f_{n+1} = \frac{1}{t} \sum_{n=0}^{\infty} t^{n+1} p(n + 1) \quad \dots (1)$$

$$\text{Let } H(t) = \sum_{n=0}^{\infty} t^n f_n \quad \dots (2)$$

$$\text{Now, } \sum_{n=0}^{\infty} t^{n+1} p(n + 1) = tp(1) + t^2p(2) + t^3p(3) + \dots$$

$$\Rightarrow \sum_{n=0}^{\infty} t^{n+1} p(n + 1) = G(t) - p(0) \quad \dots (3)$$

$$\text{Note that } \sum_{n=0}^{\infty} t^{n+1} f_{n+1} = H(t) - f_0 \quad \dots (4)$$

$$\text{But } f_0 = P(X > 0) = 1 - P(X = 0) = 1 - p(0) \quad \dots (5)$$

From (4) and (5), we have

$$\sum_{n=0}^{\infty} t^{n+1} f(n+1) = H(t) - 1 + p(0) \quad \dots (6)$$

On substituting (2), (3) and (6) in (1), we get

$$H(t) - \frac{1}{t} [H(t) - 1 + p(0)] = \frac{1}{t} [G(t) - p(0)]$$

$$tH(t) - H(t) + 1 - p(0) = G(t) - p(0)$$

$$\Rightarrow (t - 1)H(t) + 1 = G(t)$$

$$\Rightarrow H(t) = \frac{G(t)-1}{t-1} \Rightarrow H(t) = \frac{1-G(t)}{1-t}$$

P4:

Find the p.g.f. of $P(X < n)$.

Solution:

From the relation $P(X < n + 1) - P(X < n) = P(X = n)$,

we get $\phi_{n+1} - \phi_n = p(n)$ where $\phi_n = P(X < n)$

$$\therefore \frac{1}{t} \left(\sum_{n=0}^{\infty} t^{n+1} \phi_{n+1} \right) - \left(\sum_{n=0}^{\infty} t^n \phi_n \right) = \left(\sum_{n=0}^{\infty} t^n p(n) \right)$$

or $t^{-1}[H(t) - \phi_0] - H(t) = G(t)$

Since, $\phi_0 = P(X < 0) = 0$, the above result is

$$t^{-1}H(t) - H(t) = G(t) \Rightarrow (1 - t)H(t) = tG(t) \Rightarrow H(t) = \frac{t \cdot G(t)}{1 - t}$$

3.5. Probability Generating Function

Exercise

1. Let X be a r. v. with the following p.m.f.

x	1	2	3	4
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{3}{8}$

Find the p.g.f.

2. If $p(x) = P(X = x) = \frac{1}{2^{x-2}}$ for $x = 3, 4, 5, \dots$. Find the p.g.f. of X
3. Let X be a r. v. with p.g.f. $G_X(t)$. Find the p.g.f. of
- (i) $X + 1$
 - (ii) $2X$
4. If $G(t)$ is the p.g.f. of X , then find the p.g.f. of $\frac{X-a}{b}$ where a and b are real constants.
5. Let $G_X(t) = \frac{\left(\frac{1}{3}t + \frac{2}{3}\right)^4}{t}$. What is the range of X ? What is its p.m.f.?

Answers

1. $G_X(t) = \frac{1}{3}t + \frac{1}{6}t^2 + \frac{1}{8}t^3 + \frac{3}{8}t^4$

2. $G_X(t) = \frac{t^3}{2-t}$

3. (i) $tG_X(t)$

(ii) $G_X(t^2)$

4. $t^{-\frac{1}{b}}G\left(t^{\frac{1}{b}}\right)$

5.

x	-1	0	1	2	3
$p(x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$