Chapter 1 Random Processes

"Probabilities" is considered an important background to communication study.

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5/06/2021 Chapter-1

Random Process

Review of Random variables :-

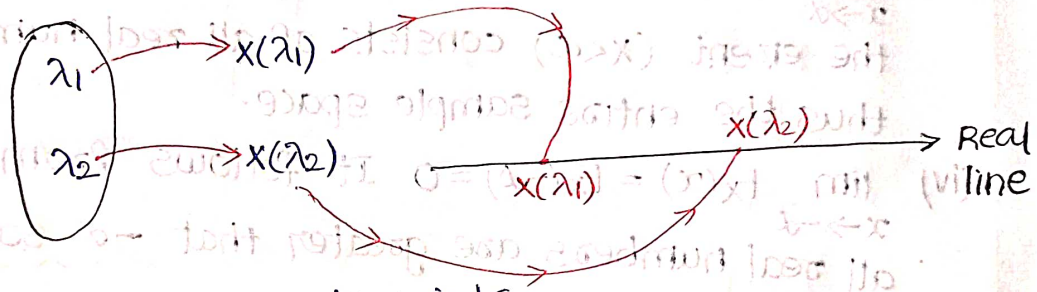
Random variable :

Random variable is a function which performs mapping between 2 sets based upon some prior definitions.

set 1 (sample space elements) → mapping → set 2 (Real line)

Real Random variable :

Real Random variable is a real valued function which assigns a real value to each sample point of a sample space obtained by performing a random experiment.



λ_1 and λ_2 are sample points.

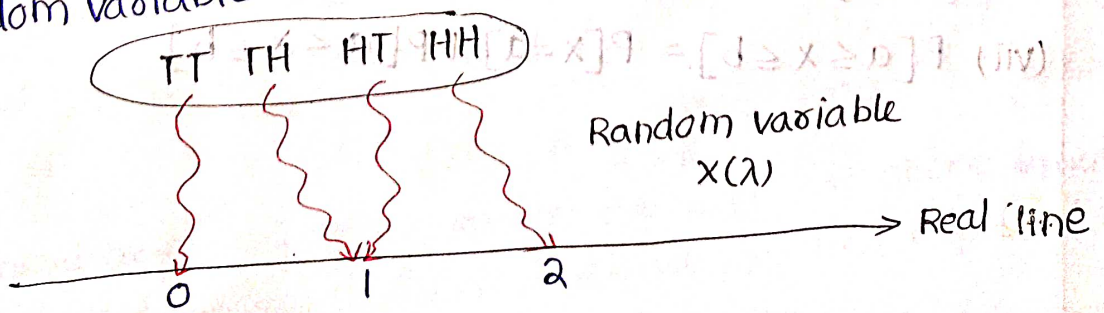
$X(\lambda_1)$ and $X(\lambda_2)$ are values of random variables.

e.g :-

Random Experiment : 2 coins are tossed simultaneously

sample space $S = \{TT, TH, HT, HH\}$

Let number of heads in random experiment represents a random variable



Random variable is denoted by X or $X(\lambda)$

Cumulative distribution function (C.D.F):-

The CDF associated with a random variable is defined as probability that outcome of an experiment will be one of the outcomes for which $X \leq x$, where x is any given number.

The distribution function (or cumulative distribution function (CDF)) of X is the function defined by

$$F_X(x) = P(X \leq x) \quad -\infty < x < \infty$$

Properties of cumulative distribution function:-

(i) $0 \leq F_X(x) \leq 1$ It follows because $F_X(x)$ is a probability

(ii) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$ It shows that $F_X(x)$ is a non decreasing function.

(iii) $\lim_{x \to \infty} F_X(x) = F_X(\infty) = 1$ It follows that the fact that the event $(X < \infty)$ consists of all real numbers and is thus the entire sample space.

(iv) $\lim_{x \to -\infty} F_X(x) = F_X(-\infty) = 0$ It follows from the fact that all real numbers are greater that $-\infty$ so event $\{x \leq -\infty\}$ is an empty set.

(v) $P[a < X \leq b] = F_X(b) - F_X(a)$

(vi) $P[X = b] = F_X(b) - F_X(b^-)$

Thus the probability that an arbitary random variable X takes on specific value, say b is given by the magnitude of the jump of the C.D.F at a point b. It follows that if the CDF is continuous at point b, then the event $\{X = b\}$ has probability zero.

(vii) $P[a \leq X \leq b] = P[X = a] + P[a < X \leq b]$

Probability Density Functions :-

The probability density function of the random variable x is simply the derivative of the cumulative distribution function.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Properties of probability density function $f_X(x)$:-

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ [Total area under PDF is always equal to 1].

(iii) $P(a<x<b) = \int_{a}^{b} f_X(x)\,dx$

The probability of an interval is therefore the area under $f_X(x)$ in that interval.

(iv) The CDF $F_X(x)$ of a continuous random variable x can be obtained by

$$F_X(x) = P(x \leq x) = \int_{-\infty}^{x} f_X(\varepsilon)\,d\varepsilon$$

characteristics parameter of Random variable :-

1) The mean :-

The mean (or expected value) of random variable X, denoted by μ_X or $E[X]$ is denoted by

$$\mu_X = E[X] = \overline{X} = \begin{cases} \sum_i x_i P_X(x_i) & X : \text{discrete} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & X : \text{continuous.} \end{cases}$$

2) Expected value of function of random variable :-

$$\mu_X = E[g(x)] = \overline{X} = \begin{cases} \sum_i g(x_i) P_X[g(x_i)] & X : \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x)\,dx & X : \text{continuous.} \end{cases}$$

There are random variables for which the above equations do not converge. In such cases, we say that expected value does not exist.

e.g:- Cauchy Random variable.

properties of mean/expectation :-

(i) $E[cX] = c\bar{X}$

(ii) $E[c] = c$ where c is constant

(iii) $E[X+Y] = \bar{X} + \bar{Y}$ Expectation is linear operator

3) moment about origin :-

The nth moment of a random variable x is defined as

$$m_n = E(X^n) = \begin{cases} \sum_i x_i^n P_X(x_i) & X: \text{discrete} \\ \int_{-\infty}^{\infty} x^n f_X(x)\,dx & X: \text{continuous} \end{cases}$$

when 'n' represents order of the moment.

(i) For $n=0$

$$E[X^n] = E[X^0] = E[1] = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

(ii) For $n=1$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \longrightarrow \text{mean}$$

(iii) For $n=2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \longrightarrow \text{mean square value}$$

Note :-

The first order moment of x is called mean and the second order moment is called mean square value.

4) central moments or moment about mean :-

moments about mean value of x are called central moments and are given by μ_n

$$\mu_n = E[(X-\bar{X})^n] = \begin{cases} \sum_i (x_i-\bar{X})^n P_X(x_i) & X: \text{discrete} \\ \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)\,dx & X: \text{continuous} \end{cases}$$

5) variance :-

The variance is a measure of the randomness of the random variable X, thereby constraining the width of its P.D.F.

The variance of random variable X, denoted by σ_x^2 or var(x) is defined by

$$\sigma_x^2 = var(x) = E[(x-\bar{x})^2]$$

Note from definition that $var(x) \geq 0$

$$\sigma_x^2 = E[(x-\bar{x})^2]$$

$$= E[x^2 - 2x\bar{x} + \bar{x}^2]$$

$$= E[x^2 - (\bar{x})^2]$$

$$\sigma_x^2 = E[x^2] - (\bar{x})^2$$

properties of variance :-

(i) $var[cx] = c^2\sigma_x^2$

(ii) $var[c] = 0$

(iii) $var[c+x] = \sigma_x^2$

(iv) $var[x+y] = \sigma_x^2 + \sigma_y^2$

if x and y are independent random variables.

6) standard deviation :

The standard deviation of random variable X, denoted by σ_x, is the positive square root of var(x)

$$\sigma_x^2 = var(x)$$

$$\sigma_x = \pm\sqrt{var(x)}$$

Take +ve root only

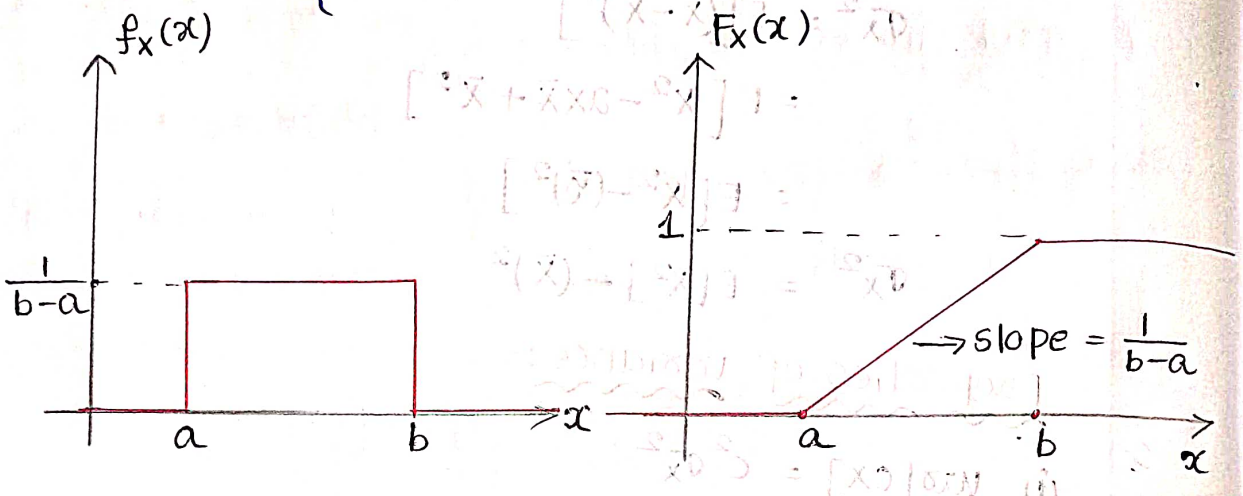
$$\sigma_x = \sqrt{var(x)}$$

some special random variables distribution:-

1) uniformly distribution function:

A random variable x is said to be uniformly distributed if the density function is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



The mean and variance of the uniformly distributed random variable x are

$$\mu_X = E[X] = \frac{b+a}{2}$$

$$\sigma_X^2 = \text{var}(x) = \frac{(b-a)^2}{12}$$

Remember:-

The uniform density finds a number of practical uses. A particularly important application is in the quantization of signal samples prior to encoding in digital communication system. Quantisation amounts to rounding off the actual sample to the nearest of a larger number of discrete quantum levels. The errors introduced in round off process are uniformly distributed.

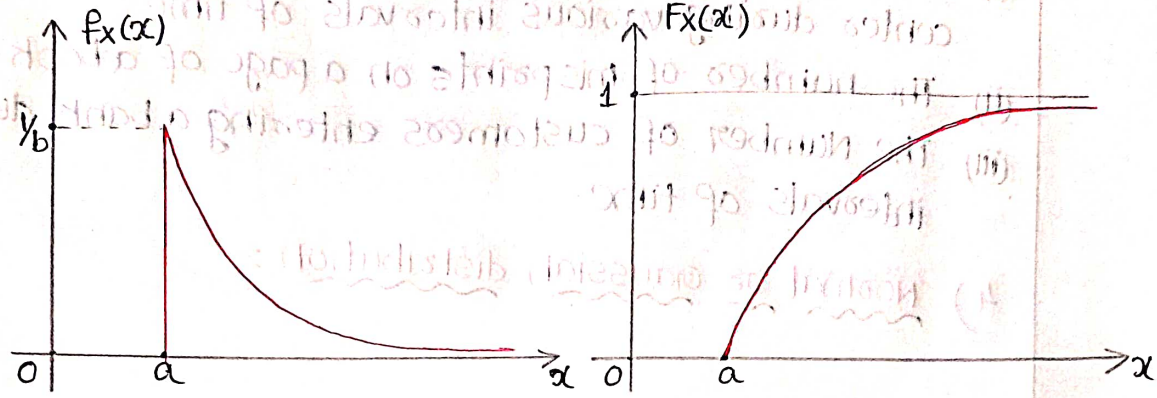
#2) Exponential distribution function :-

A random variable x is said to be exponentially distributed if its density function is given by

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x \geq a \\ 0 & x < a \end{cases}$$

where $-\infty < a < \infty$

$b > 0$



Exponential Distribution

The mean and variance of the exponential random variable x are

i) $\mu_X = E[X] = a+b$

ii) $\sigma_X^2 = Var(x) = b^2$

Remember :-

The exponential density is useful in describing raindrop sizes when a large number of rainstorm measurements are made. It is also used to approximately describe the fluctuation in signal strength received by radar from certain types of aircraft.

#3) Poisson Distribution :-

A random variable x is called poisson random variable if its probability mass function is given by

$$P(X=K) = e^{-\lambda} \frac{\lambda^K}{K!} \quad K = 0, 1, 2, 3, \cdots$$

The probability density function is given by

$$f_X(x) = \sum_{K=0}^{\infty} e^{-\lambda} \frac{\lambda^K}{K!} \delta(x-K)$$

The cumulative distribution function is given by

$$F_X(x) = \sum_{K=0}^{\infty} e^{-\lambda} \frac{\lambda^K}{K!} u(x-K)$$

The mean and variance of poisson random variable x are

$$\mu_x = E[X] = \lambda$$

$$\sigma_x^2 = Var(x) = \lambda$$

Remember :-

Some examples of poisson random variable includes

(i) The number of telephone calls arriving at a switching center during various intervals of time

(ii) The number of misprints on a page of a book.

(iii) The Number of customers entering a bank during various intervals of time.

4) Normal or Gaussian distribution :-

A random variable x is called a normal (or Gaussian) random variable if its PDF is given by

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\mu_x)^2/2\pi\sigma_x^2}$$

properties of Gaussian probability density spectrum are following :-

(i) The peak value of Gaussian probability density spectrum Gaussian (PSD) occurs at $x = \mu_x$ and is given by $f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x}$

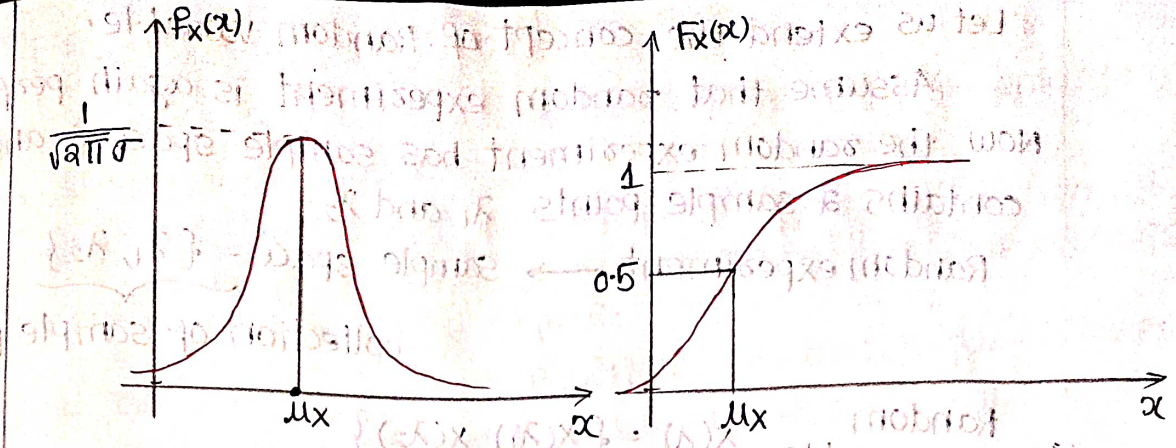
(ii) The plot of Gaussian (PDS) exhibits even symmetry around mean value

i.e, $f_x(\mu_x - \sigma_x) = f_x(\mu_x + \sigma_x)$

(iii) The area under the Gaussian (PDS) curve is one-half for all values x below μ_x and above μ_x respectively

i.e, $P(X \le \mu_x) = P(X > \mu_x) = \frac{1}{2}$

(iv) As σ_x approaches to zero, the gaussian function approaches to impulse function located at $x = \mu_x$.



The mean and variance of normal random variable x are

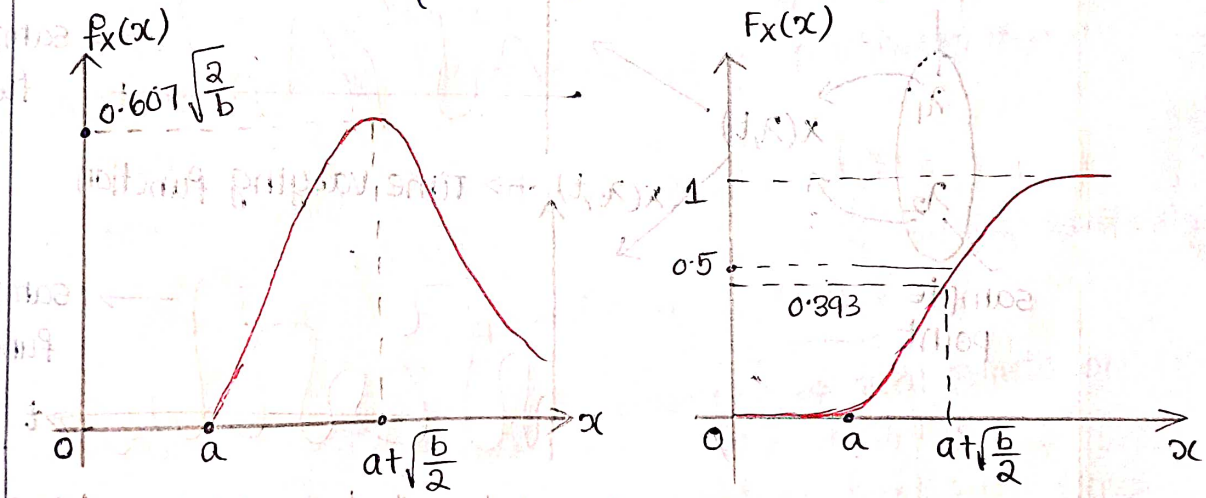
(i) $\mu_X = E(X) = \mu$

(ii) $Var(X) = \sigma_X^2$

5) Rayleigh distribution :-

A random variable x is said to be Rayleigh distribution if its density function is given by

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 0 & x \leq a \end{cases}$$



The mean and variance of the Rayleigh random variable x are

$$\mu_X = E[x] = \sqrt{b}\,\frac{\sqrt{\pi}}{2} + a$$

$$Var(x) = \sigma_X^2 = b\left(1 - \frac{\pi}{4}\right)$$

Remember :-

The rayleigh density describes the envelope of one type of noise when passed through a band pass filter. It is also important in analysis of errors in various measurement systems.

Let us extend the concept of Random variable.

Assume that random experiment is again performed. Now the random experiment has sample space S and contains a sample points λ_1 and λ_2.

Random experiment $\longrightarrow$ sample space = $\underbrace{\{\lambda_1, \lambda_2\}}$ collection of sample points

Random variable $X(\lambda) = \{X(\lambda_1), X(\lambda_2)\}$

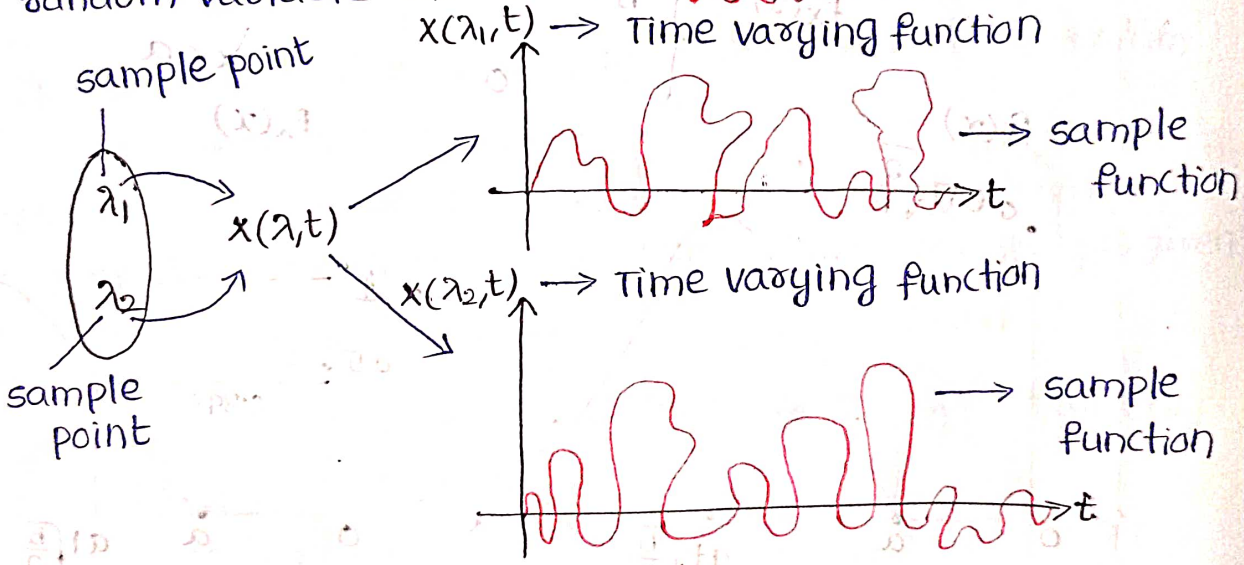
Let us add another dimesion 'time t'

Now the random variable $X(\lambda)$ becomes $X(\lambda, t)$

$$X(\lambda) = \{X(\lambda_1), X(\lambda_2)\}$$

$\downarrow$ Time 't'

$X(\lambda,t) = \{X(\lambda_1,t), X(\lambda_2,t)\}$ This is now no longer random variable. This is called Random process.



λ_1 and λ_2 are sample points which maps to a time varying waveform. The time varying waveform is called sample function.

conclusion :-

Whenever a sample point is passing through a random process, it generates a time varying function. The time varying function particularly called sample function.

$\underbrace{X(\lambda,t)}_{\text{Random process}} = \{\underbrace{X(\lambda_1,t), X(\lambda_2,t)}_{\text{sample function}}\}$

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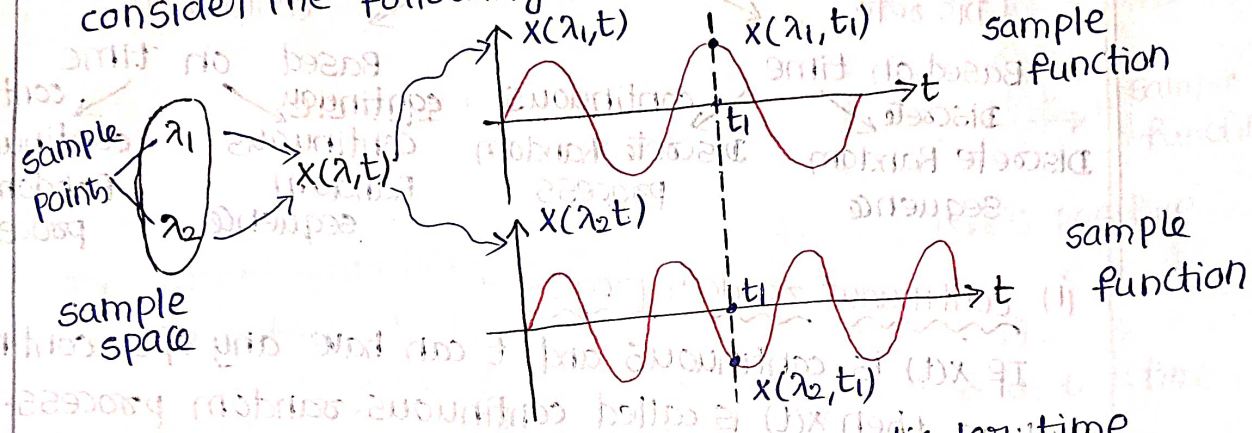
Random process :-

The concept of random variable is based on enlarging the random variable concept to include time.

A random variable x is, by its defintion, a function of possible outcomes λ of an experiment, it now becomes a function of λ and time.

The collection of such functions is called Random process which can be denoted by $x(\lambda,t)$. sometimes random process can also be represented as $x(t)$.

consider the following random process.



suppose random process observed at particular time instant $t=t_1$

∴ Random process $x(\lambda,t) = \{x(\lambda_1,t), x(\lambda_2,t)\}$

(Random process; collection of sample functions)

At $t=t_1$ $x(\lambda,t_1) = \{x(\lambda_1,t_1), x(\lambda_2,t_1)\}$

(Random variable; collection of real points on real line)

When Random process is observed at particular time instants then

i) sample points of sample space maps to real value

ii) Therefore random process now become random variable

$$x(\lambda,t) \xrightarrow{t=t_1} x(\lambda,t_1)$$

Therefore #

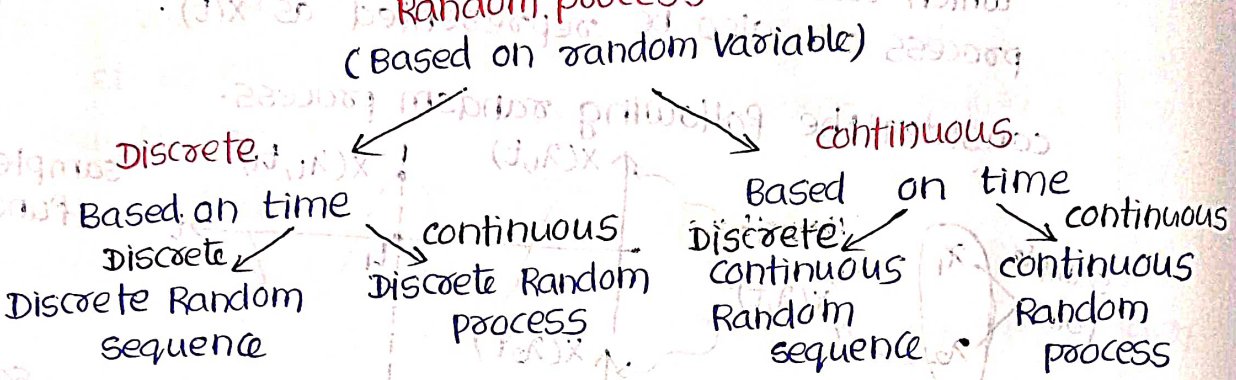
at $t=t_1 \Rightarrow x(\lambda) = \{x(\lambda_1), x(\lambda_2)\}$

Note :-

*** Random process at each instant of time behaves as a random variable.

Classification of Random process :-

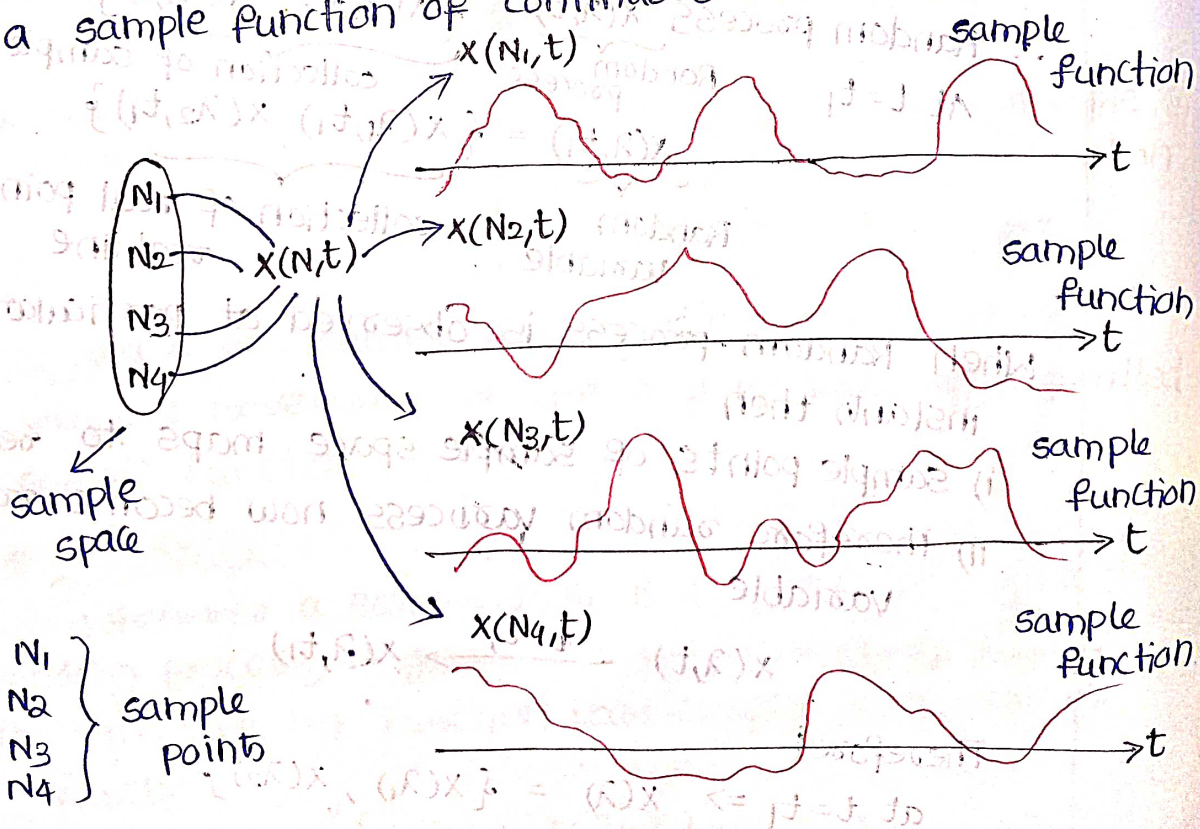
It is convenient to classify random process according to the characteristics of t and the random variable $x = x(t)$ at time t.



(i) continuous random process :-

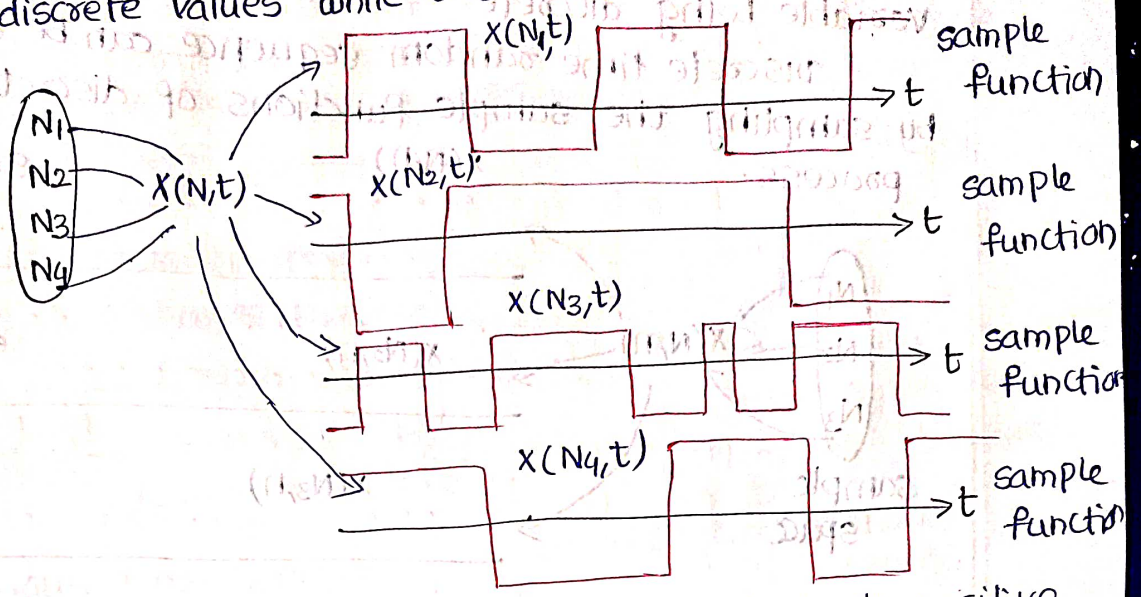
If x(t) is continuous and t can have any of a continuous values, then x(t) is called continuous random process.

e.g :- Thermal noise generated by any realizable network is a practical example of waveform that is modelled as a sample function of continuous random process.



(ii) Discrete random process :-

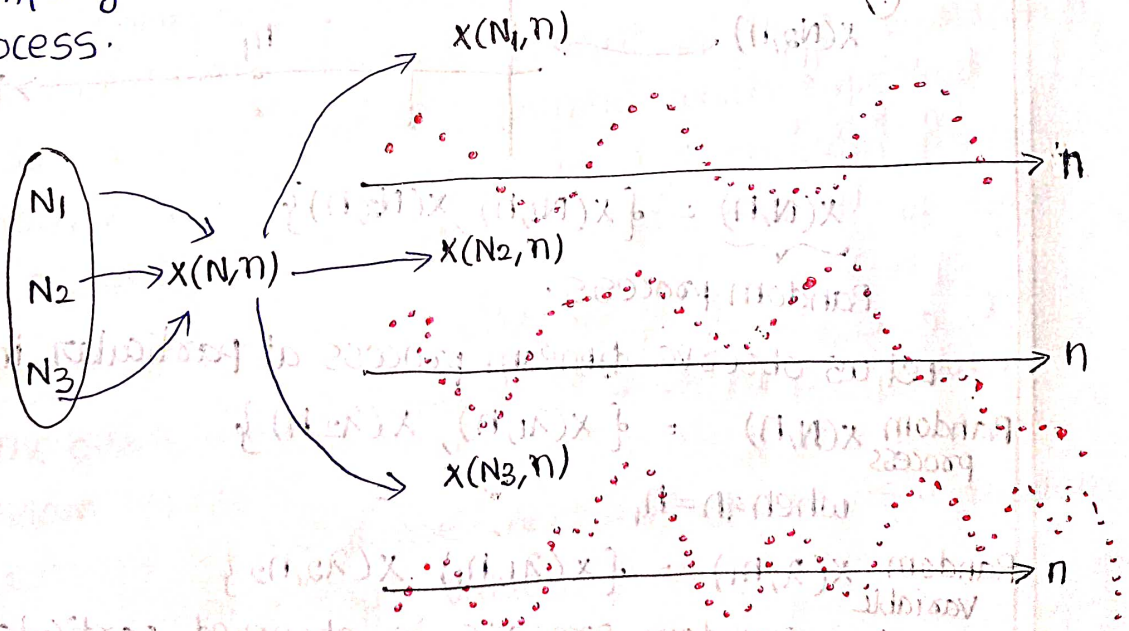
A second class of random process called a discrete random process, corresponds to the random variable x having only discrete values while t is continuous.



The sample functions have only 2 discrete levels positive and negative.

(iii) continuous random sequence :-

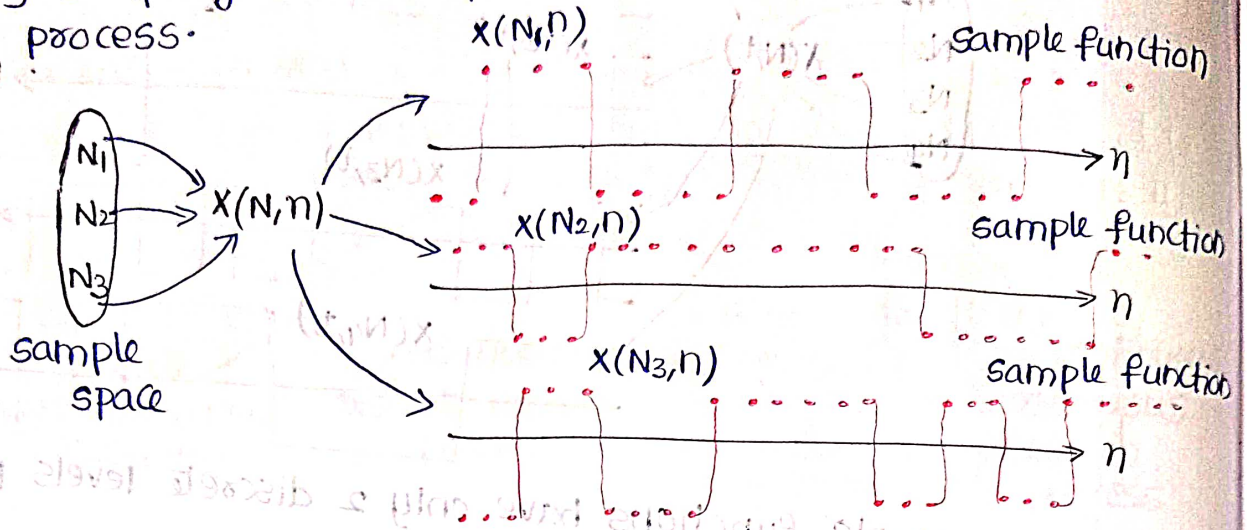
A random process for which x is continuous but time has only discrete values is called continuous random sequence. This sequence can be obtained by periodically sampling the ensemble members of continuous random process.



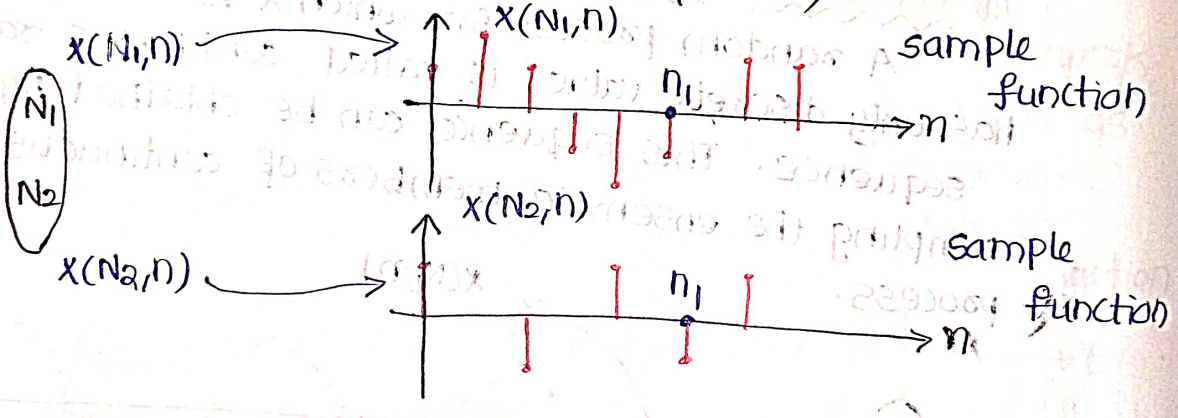
(iv) Discrete Random sequence :-

A fourth class of random process, called a discrete random sequence, corresponds to both time and random variable being discrete.

Discrete time random sequence can be developed by sampling the sample functions of discrete random process.



consider the following discrete time continuous random sequence. (Discrete time Random process)



$$\underbrace{X(N,n)}_{\text{Random process}} = \{X(N_1,n), X(N_2,n)\}$$

Let us observe random process at particular instant of time.

Random process $X(\lambda,n) = \{X(\lambda_1,n), X(\lambda_2,n)\}$

when $n=n_1$

Random variable $X(\lambda,n_1) = \{X(\lambda_1,n_1)\ X(\lambda_2,n_2)\}$

When Random process is observed particular instant of time then

(i) sample points of sample space map to real value.

(ii) Therefore random process now become random variable

Notations :-

Continuous time Random process $x(\lambda,t)$:- continuous time random process (CTRP) maps a sample point on a continuous time waveform

Discrete time Random process $x(\lambda,n)$:- Discrete time random process (DTRP) maps a sample point on a discrete time waveform.

$x(\lambda,t) \rightarrow$ Random process
$x(\lambda_1,t) \rightarrow$ sample function
$x(\lambda,t_1) \rightarrow$ Random variable
$x(\lambda_1,t_1) \rightarrow$ Value taken by Random variable

⇑ continuous time Random process

$x(\lambda,n) \rightarrow$ Random process
$x(\lambda_1,n) \rightarrow$ sample function
$x(\lambda_1,n_1) \rightarrow$ Value taken by Random variable
$x(\lambda,n_1) \rightarrow$ Random variable

⇑ Discrete time Random process

Note :-

If x is a continuous random variable then function of random variable $g(x)$ is also a continuous random variable.

If x is a discrete random variable then function of random variable $f(x)$ is also a discrete random variable.

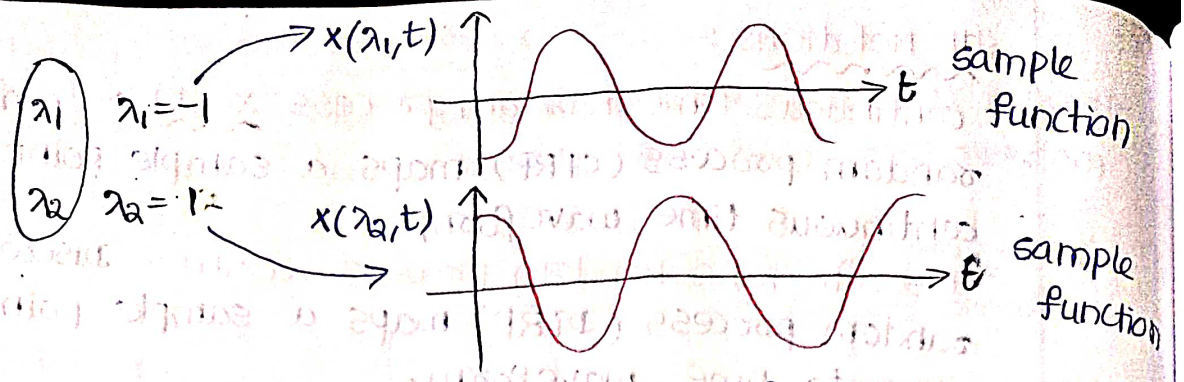
We knew that if we observe a random process (CTRP/DTRP) at particular instant of time, we obtain a random variable. Let us examine which type of random variable is obtained when a random process $x(t)$ or $x(n)$ observed at particular instant of time.

case (i) $x(\lambda,t) = x(t) = A\cos(\omega_0 t + \phi)$

The given random process is continuous time random process.

suppose ω_0 and ϕ are constants and A is a random variable it takes only 2 values -1 and 1

$A = \{-1, 1\}$



$$x(t) = \{-\cos(\omega_0 t + \phi),\ \cos(\omega_0 t + \phi)\}$$

In the random process (CTRP) $x(t) = A\cos(\omega_0 t + \phi)$ the random variable present is A which is discrete in nature.

when the random process observed at particular instant of time then.

at $t = t_1$

$$x(t_1) = A\underbrace{\cos(\omega_0 t_1 + \phi)}$$

constant since ω_0, t_1 and ϕ are constant

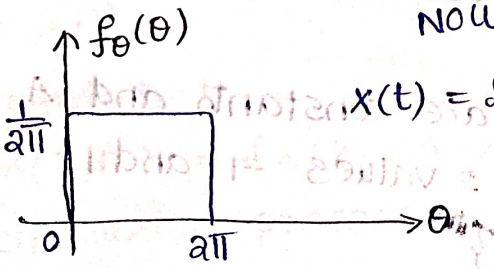
$x(t_1) = \text{func}^n(A)$ → Discrete Random Variable

⇒ continuous random process need not to be result in a continuous random variable, it may result in discrete random variable also.

case (ii) $x(t) = A\cos(\omega_0 t + \phi)$ where A, ω_0 are constants and ϕ is uniformly distributed b/w 0, and 2π.

The given random process is continuous time random process. suppose given that A, ω_0 are constants and ϕ is uniformly distributed b/w 0 and 2π.

The P.D.F of ϕ is given by



NOW

$$x(t) = \{A\cos(\omega_0 t + 0),\ A\cos(\omega_0 t + 0.001)$$
$$\ldots\ A\cos(\omega_0 t + 2\pi)\}$$

When at time $t = t_1$

$X(t_1) = A\cos(\omega_0 t_1 + \phi)$

since A, ω_0, t_1 are constants $X(t_1) = \text{func}^n(\phi) \rightarrow$ Continuous Random variable

⇒ Here continuous random process result in a continuous random variable.

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case (iii) $X(n) = A\cos(\omega_0 n + \phi)$ where ω, ϕ are constants and A is a random variable which takes only 2 values −1 and 1.

The given random process is discrete time random process. And also stated that ω, ϕ are constants and A is a random variable which takes only 2 values −1 and 1.

$A = \{-1, 1\}$ is a discrete random variable.

when at time $n = n_1$

$X(n_1) = A\cos(\omega_0 n_1 + \phi)$

since ϕ, ω_0, n_1 are constants $X(n_1) = \text{func}^n(A) \rightarrow$ Discrete Random variable.

⇒ Here discrete time random process results in a discrete random variable

case (iv) $X(n) = A\cos(\omega_0 n + \phi)$ where ω_0 and A are constants and ϕ is a random variable uniformly distributed b/w 0 and 2π.

The given random process is discrete time random process. And also stated that ω_0, A are constants and ϕ is a random variable with uniform distribution from 0 to 2π.

$\phi = [0, 2\pi]$ is a continuous random variable

when at time $n = n_1$

$X(n_1) = A\cos(\omega_0 n_1 + \phi)$

since A, ω_0, n_1 are constants $X(n_1) = \text{func}^n(\phi) \rightarrow$ continuous Random variable

⇒ Here discrete time random process results in a continuous random variable.

conclusion :-

i) Random variable type present in a random process does not depends upon the type of random process. It depends only on randomness present in random process.

Summary :-

1) Random process (Based on time)
   - continuous time Random process CTRP : $X(\lambda,t)$ OR $X(t)$
   - Discrete time Random process DTRP : $X(\lambda,n)$ OR $X(n)$

2) Random process maps sample point on to real functions (sample functions)

3) Random process can represented as collection of sample functions.

$$X(\lambda,t) = \{ x(\lambda_1,t)\ x(\lambda_2,t)\ x(\lambda_3,t)\ x(\lambda_4,t) \text{ ---} \}$$

$$X(\lambda,n) = \{ x(\lambda_1,n)\ x(\lambda_2,n)\ x(\lambda_3,n)\ x(\lambda_4,n) \text{ ---} \}$$

4) Random process can also be represented as collection of random variables.

$$X(\lambda,t) = [x(\lambda,t_1), x(\lambda,t_2), x(\lambda,t_3) \text{ ---} x(\lambda,t_n) \text{ ----}]$$

$$X(\lambda,t) = \{ x(\lambda,n_1), x(\lambda,n_2), x(\lambda,n_3) \text{ ----} \}$$

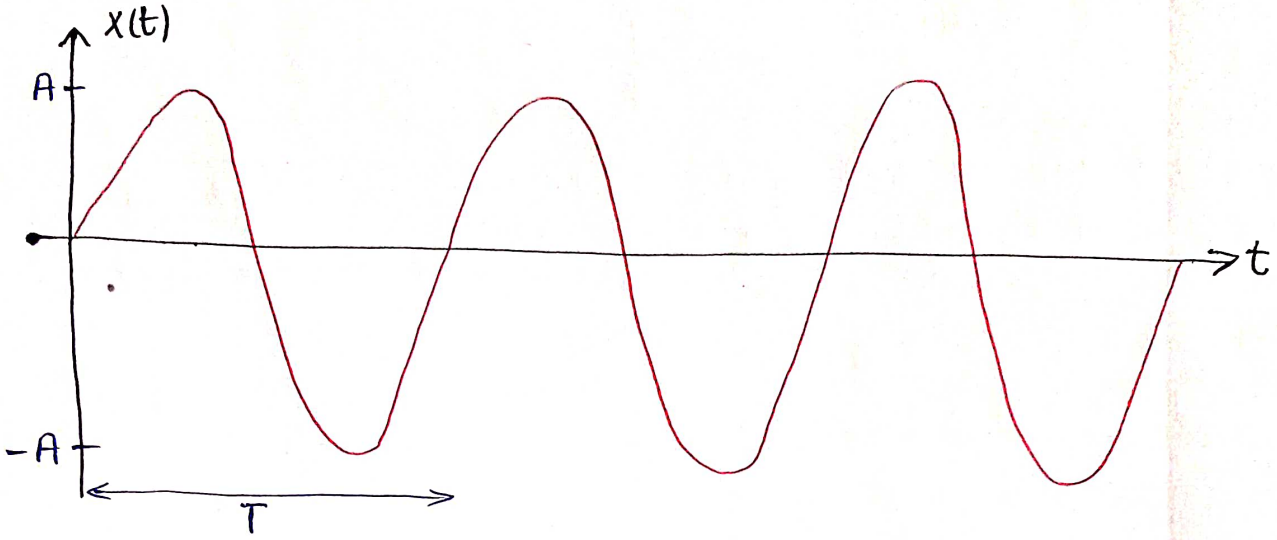
where $\Delta t = t_{i+1} - t_i \to 0$

5) $X(t) = f(\lambda,t)$ — continuous time Random Process
   - $t=t_1 \to f(\lambda,t_1)$ → continuous random variable if λ is continuous random variable
   - $t=t_1 \to f(\lambda,t_1)$ → Discrete random variable if λ is discrete random variable

6) $X(n) = f(\lambda,n)$ — Discrete time Random process
   - $n=n_1 \to f(\lambda,n_1)$ → Discrete random variable if λ is discrete random variable
   - $n=n_1 \to f(\lambda,n_2)$ → Continuous random variable if λ is discrete random variable

Deterministic Random process :-

A process is called deterministic random process if future values of sample function can be predicted from its past values.



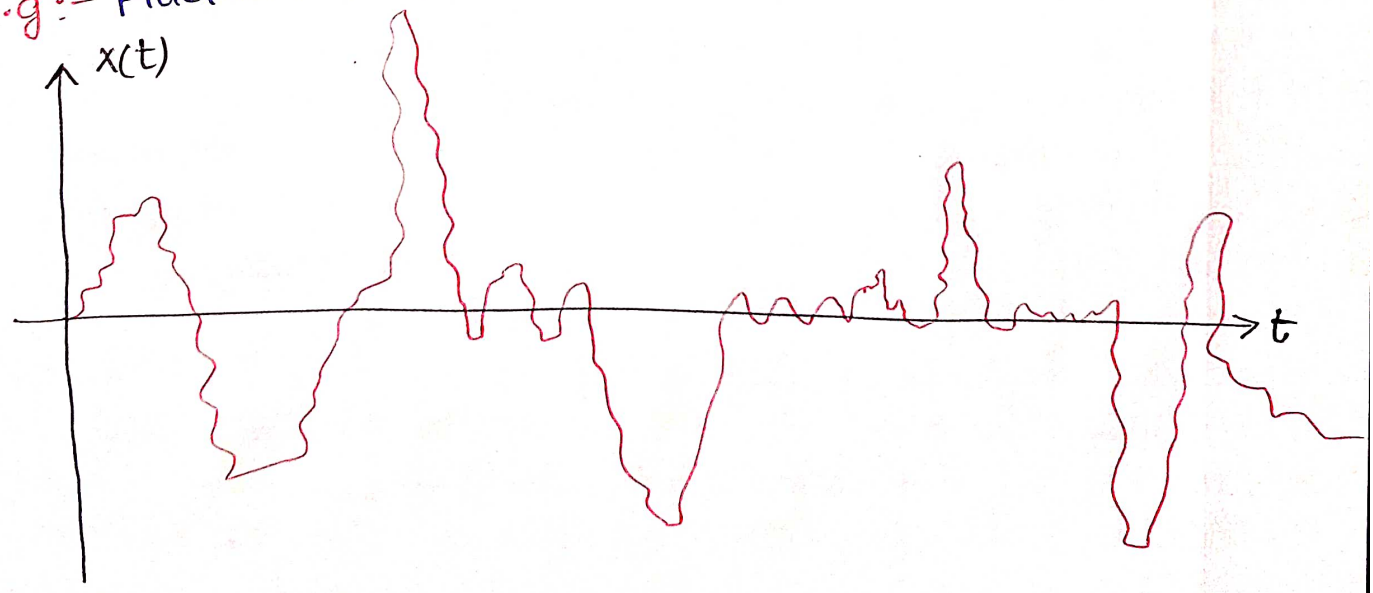
$x(t) = A \sin(\omega_0 t + \theta)$

where the parameters A, ω_0 and θ may be different for different sample functions.

Non deterministic random process :-

A process is called a non deterministic random process if the future values of a sample function cannot be predicted from the observed past values.

e.g :- Fluctuation of noise voltage in any network.

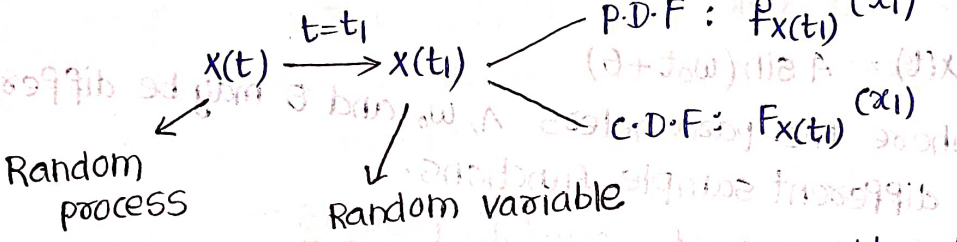


Statistical parameters of Random Process :-

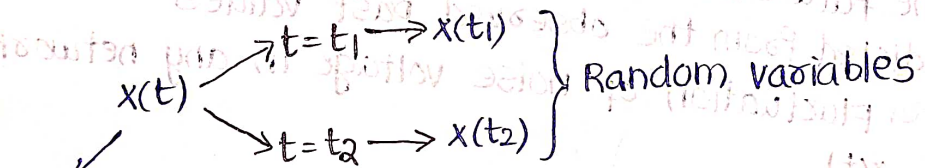
1) mean
2) mean square value
3) variance
4) correlation
5) covariance

The statistical parameters of random process can be studied with help of random variables.

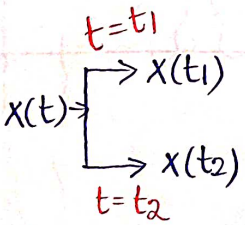
continuous time random process $x(t)$ is observed at particular instant of time results in a random variable.



Let observe $x(t)$ at 2 instants of time then we can obtain 2 random variables.



$x(t_1)$ and $x(t_2)$ are random variables



2nd order Joint P.D.F : $f_{x(t_1)x(t_1)}(x_1, x_2)$

2nd order Joint C.D.F : $F_{x(t_1)x(t_2)}(x_1, x_2)$

For discrete Random there is another parameter called PMF

$x(\lambda)$ → PMF : $P_{x(t_1)}(x)$ or $P\{x(t_1) = x_1\}$

$P_{x(n_1)}(x)$ or $P\{x(n_1) = x_1\}$

↓ Discrete Random variable

1) Mean of Random process :-

case (i) consider the continuous time random process $x(t)$

$x(t)$ is function of time 't' and random variable 'ϕ'.

$$x(t) = f(t, \phi)$$

If $x(t)$ is observed at $t = t_0$ then

$$x(t) = f(t, \phi) \xrightarrow{t=t_0} x(t_0) = f(t_0, \phi)$$

Let ϕ is continuous random variable then $x(t_0)$ is also a continuous random variable.

Then mean is defined as

$$E[x(t_0)] = \int_{-\infty}^{\infty} x_0 \, f_{x(t_0)}(x_0) \, dx_0$$

$$E[f(t, \phi)] = \int_{-\infty}^{\infty} f(t, \phi) \, f_{\phi}(\phi) \, d\phi$$

case (ii) consider the continuous time random process $x(t)$

$x(t)$ is function of time 't' and random variable ϕ.

$$x(t) = f(t, \phi)$$

If $x(t)$ is observed at $t = t_0$ then

$$x(t) = f(t, \phi) \xrightarrow{t=t_0} x(t_0) = f(t_0, \phi)$$

Let ϕ is discrete random variable then $x(t_0)$ is also a discrete random variable.

Then mean is defined as

$$E[x(t_0)] = \sum_i x_{0i} \, P\{x(t_0) = x_{0i}\}$$

case (iii) consider the discrete time random process $x(n)$

$x(n)$ is function of time 'n' and random variable ϕ.

$$x(n) = x(n, \phi)$$

If $x(n)$ is observed at $n = n_0$ then

$$x(n) = f(n, \phi) \xrightarrow{n=n_0} x(n_0) = f(n_0, \phi)$$

Let ϕ is continuous random variable then $x(n_0)$ is also a continuous random variable.

Then mean is defined as

$$E[x(n_0)] = \int_{-\infty}^{\infty} x_0 \, f_{x(n_0)}(x_0) \, dx_0$$

$$E[f(n_0, \phi)] = \int_{-\infty}^{\infty} f(n_0, \phi) \, f_{\phi}(\phi) \, d\phi$$

case (iv) consider the discrete time random process $x(n)$ $x(n)$ is function of time 'n' and random variable 'ø'.

$$x(n) = f(n, \phi)$$

If $x(n)$ is observed at $n = n_0$ then

$$x(n) = f(n, \phi) \xrightarrow{n=n_0} x(n_0) = f(n_0, \phi)$$

Let ø is discrete random variable

Then mean is defined as

$$E[x(n_0)] = \sum_i x_{oi} P\{x(n_0) = x_{oi}\}$$

2) Autocorrellation and Auto covariance :-

consider a random process $x(t)$ which is continuous time random process and observe $x(t)$ at a different time instants t_1 and t_2.

$x(t)$ → Random process

- $t = t_1$ — $x(t_1)$: Random variable
- $t = t_2$ — $x(t_2)$: Random variable

Autocorrellation function can be defined as

$$E[x(t_1)\, x(t_2)] = R_{x(t_1)\, x(t_2)} = R_{xx}(t_1, t_2) \Rightarrow \text{Auto correlation.}$$

Autocovariance function can be defined as

$$Cov[x(t_1)\, x(t_2)] = \sigma_{x(t_1)\, x(t_2)} = R_{xx}(t_1, t_2) - \mu_{x(t_1)} \mu_{x(t_2)}$$

$$\sigma_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - \mu_{x(t_1)} \mu_{x(t_2)} \rightarrow \text{Auto covariance}$$

consider a random process $x(n)$ which is discrete time random process and observe $x(n)$ at 2 different time instants n_1 and n_2

$x(n)$ → Random process

- $x(n_1)$: Random variable
- $x(n_2)$: Random variable

Auto correlation function can be defined as :

$$E[x(n_1) x(n_2)] = R_{xx}(n_1, n_2) \longrightarrow \text{Auto correlation.}$$

Auto covariance can be defined as

$$\text{cov}[x(n_1)\,x(n_2)] = \sigma_{xx}(n_1, n_2) = R_{xx}(n_1, n_2) - \mu_x(n_1)\,\mu_x(n_2)$$

→ Autocovariance

3) mean square value :-

For continuous random process, auto correlation can be defind as

$$E[x(t_1)x(t_2)] = R_{xx}(t_1, t_2)$$

when $t_1 = t_2$

$$E[x(t_1)\,x(t_1)] = E[x^2(t_1)] = \text{mean square value } (x(t_1))$$

For discrete time random process, autocorrelation can be defined as

$$E[x(n_1)x(n_2)] = R_{xx}(n_1, n_2)$$

when $n_1 = n_2$

$$E[x(n_1)x(n_2)] = E[x^2(n_1)] = \text{mean square value } (x(n_1))$$

3) cross correlation and cross covariance :-

consider a continuous time random processes $x(t)$ and $y(t)$. Each random process is observed at particular instant of time.

$x(t) \longrightarrow t = t_1 \longrightarrow x(t_1)$: Random variable

$y(t) \longrightarrow t = t_2 \longrightarrow y(t_2)$: Random variable

cross correlation function can be defined as

$$E[x(t_1)\,y(t_2)] = R_{xy}(t_1, t_2) \longrightarrow \text{cross correlation}$$

$$\text{cov}[x(t_1)\,y(t_2)] = R_{xy}(t_1, t_2) - \mu_x(t_1)\,\mu_x(t_2) \longrightarrow \text{cross covariance}$$

consider a discrete time random processes $x(n)$ and $y(n)$. Each random process is observed at particular instant of time

$x(n) \longrightarrow n = n_1 \longrightarrow x(n_1)$: Random variable

$y(n) \longrightarrow n = n_2 \longrightarrow y(n_2)$: Random variable

cross correlation function can be defined as

$$E[x(n_1)\,y(n_2)] = R_{xy}(n_1, n_2) \longrightarrow \text{cross correlation.}$$

cross covariance function can be defined as

$$\text{cov}[x(n_1)\,y(n_2)] = R_{xy}(n_1, n_2) - \mu_x(n_1)\,\mu_x(n_2) \longrightarrow \text{cross covariance}$$

Types of Random process :-

1) Strict sense stationary random process (SSSRP)
2) Wide sense stationary random process (WSSRP)
3) Ergodic Random process
4) cyclostationary Random process

Strict sense stationary Random process (SSSRP)

Consider the continuous time random process $X(t)$. $X(t)$ is observed at several time instances so we can obtain several random variables.

$$\underbrace{X(t)}_{\text{Random process}} = \underbrace{\{X(t_1), X(t_2), X(t_3), \ldots X(t_K)\}}_{\text{collection of random variables}}$$

since there are 'K' random variables :

K^{th} order joint P.D.F :- $f_{X(t_1)\,X(t_2)\ldots X(t_K)}(x_1, x_2, \ldots x_K)$ ↘(i)

Now observe each sample functions at τ duration a part.

$$\underbrace{X(t+\tau)}_{\text{Random process}} = \underbrace{\{X(t_1+\tau), X(t_2+\tau), X_3(t_3+\tau), \ldots X(t_K+\tau)\}}_{\text{collection of random variables.}}$$

NOW,

K^{th} order joint P.D.F : $f_{X(t_1+\tau)\,X(t_2+\tau)\ldots X(t_K+\tau)}(x_1, x_2, \ldots x_K)$ ↘(ii)

If random process $X(t)$ is said to be strict sense stationary random process then

i) K^{th} order joint P.D.F is independent of time shifting
$\Rightarrow$ eqn (i) = eqn (ii)

ii) K^{th} order joint C.D.F is independent of time shifting

iii) K^{th} order joint PMF is independent of time shifting

1st order P.D.F :-

$X(t) \xrightarrow{t=t_1} X(t_1)$ — Random process → Random variable

P.D.F : $f_{X(t_1)}(x_1) \longrightarrow (a)$

By observing sample function τ duration apart then

$X(t) \xrightarrow{t=t_1} X(t_1) \xrightarrow{+\tau} X(t_1+\tau)$ P.D.F : $f_{X(t_1+\tau)}(x_1) \longrightarrow (b)$

for stationary

$eq^n(a) = eq^n(b)$

$f_{X(t_1)}(x_1) = f_{X(t_1+\tau)}(x_1) = \text{constant}$

$\Rightarrow$ 1st order is independent of time

2nd order P.D.F :-

$X(t)$ — Random process

$t=t_1 \longrightarrow X(t_1)$

$t=t_2 \longrightarrow X(t_2)$

} Random variables

2nd order joint P.D.F :

$f_{X(t_1)X(t_2)}(x_1, x_2) \longrightarrow (c)$

By observing sample functions τ duration apart then

$X(t)$ — $t=t_1 \longrightarrow X(t_1) \xrightarrow{\tau} X(t_1+\tau)$

$t=t_2 \longrightarrow X(t_2) \xrightarrow{\tau} X(t_2+\tau)$

2nd order joint P.D.F :

$f_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2) \longrightarrow (d)$

For stationary,

$eq^n(c) = eq^n(d)$

$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2)$ $\Rightarrow$ Independent of time shift

$\tau = -t_1$

$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$ $\Rightarrow$ Neither depends on t_1 nor depends on t_2 but depends on difference b/w them.

2nd order P.D.F :

$\Rightarrow f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$

$\Downarrow$

Depends on difference b/w t_1 and t_2

3rd order Joint P.D.F :-

$X(t)$ → $t=t_1 \to X(t_1)$, $t=t_2 \to X(t_2)$, $t=t_3 \to X(t_3)$ } Random variables

Random process

3rd order joint P.D.F = $f_{X(t_1)X(t_2)X(t_3)}(x_1, x_2, x_3)$ → (e)

By observing sample function τ duration apart then

$X(t)$ → $t=t_1 \to X(t_1) \xrightarrow{+\tau} X(t_1+\tau)$, $t=t_2 \to X(t_2) \xrightarrow{+\tau} X(t_2+\tau)$, $t=t_3 \to X(t_3) \xrightarrow{+\tau} X(t_3+\tau)$

Random process

Random variables

3rd order joint P.D.F = $f_{X(t_1+\tau)X(t_2+\tau)X(t_3+\tau)}(x_1, x_2 x_3)$ → (f)

For stationary

eq^n (e) = eq^n (f)

$$f_{X(t_1)X(t_2)X(t_3)}(x_1, x_2, x_3) = f_{X(t_1+\tau)X(t_2+\tau)X(t_3+\tau)}(x_1, x_2, x_3)$$

When $t = -t_1$

$$f_{X(t_1)X(t_2)X(t_3)}(x_1, x_2, x_3) = f_{X(0)X(t_2-t_1)X(t_3-t_1)}(x_1, x_2, x_3)$$

⇓

Depends on difference of time instants but not on time instants.

Summary :-

1) 1st order pdf → Independent of time shift → Independent of time

2) 2nd order joint P.D.F → Independent of time shift → Independent of t_1, t_2 ; depends on $t_1 \sim t_2$

3) 3rd order joint P.D.F → Independent of time shift → Independent of t_1, t_2 and t_3 ; depends on $t_3 - t_1$, $t_2 - t_1$

Wide sense stationary Random process :-

The random process which is strict sense stationary upto 2nd order is called wide sense stationary random process.

strict sense stationary random process : I, II, III, IV, - - -

wide sense stationary random process : I, II.

For WSSRP,

i) I order pdf → Independent of time shift → Independent of time

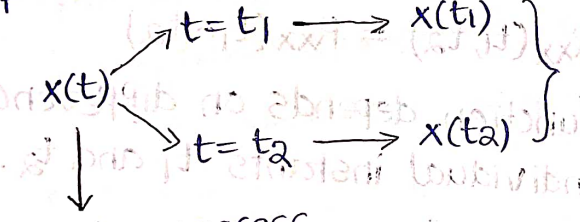
ii) II order joint pdf → Independent of time shift → Independent of t_1, t_2

→ Depends on $t_1 \sim t_2$

Every SSSRP must be WSSRP.

Let x(t) be wide sense stationary random process.

If we observe x(t) at a different time instants then



1st order P.D.F : $f_{x(t_1)}(x_1) \rightarrow$ constant

2nd order Joint P.D.F : $f_{x(t_1)x(t_2)}(x_1, x_2) = f_{x(0)x(t_2-t_1)}(x_1, x_2)$

mean :-

$E[x(t_1)] = \int_{-\infty}^{\infty} x_1 f_{x(t_1)}(x_1)\, dx_1$ = mean is constant / independent of t. $= \mu_x$

$E[x(t_2)]$ = constant $= \mu_x$.

$E[x(t)] = \mu_x$ = constant.

⇒ mean of random process = mean of random variable $= \mu_x$.

Mean square value :-

$$E[x^2(t_1)] = \int_{-\infty}^{\infty} x_1^2 f_{x(t_1)}(x_1)\, dx_1 \longrightarrow \text{constant}$$

$\Rightarrow E[x^2(t)] = \text{constant}$.

If $x(t)$ is WSSRP then :

i) mean $= \mu_x = \text{constant}$

ii) mean square value $E[x^2(t)] = \text{constant}$

iii) variance $\sigma_x^2 = E[x^2(t)] - \mu_x^2 = \text{constant}$

Auto correlation function :-

$$E[x(t_1)\,x(t_2)] = \iint x_1 x_2 f_{x(t_1)x(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

$$E[x(t_1)\,x(t_2)] = \iint x_1 x_2 f_{x(0)x(t_2-t_1)}(x_1, x_2)\, dx_1 dx_2 = f(t_2 - t_1)$$

$$E[x(t_1)\,x(t_2)] = R_{xx}(t_1, t_2) = R_{xx}(t_1 \sim t_2)$$

$\Rightarrow$ Autocorrelation function depends on difference of t_1 and t_2 but not on individual instants t_1 and t_2.

Auto covariance function :-

$$\text{cov}(x(t_1)\,x(t_2)) = R_{xx}(t_1, t_2) - \mu_{x(t_1)}\mu_{x(t_2)}$$

$$\text{cov}(x(t_1)\,x(t_2)) = R_{xx}(t_1 \sim t_2) - \mu_x^2 = f(t_2 - t_1)$$

Wide sense stationary Random process

- mean, mean square value, variance are constant
- Auto correlation, Auto covariance are function of difference b/w t_1 and t_2

Wide sense stationary Random Process :-

1) $X(t)$ be continuous time random process

$X(t)$ → $t = t_1 \longrightarrow X(t_1)$ Random variable

$X(t)$ → $t = t_2 \longrightarrow X(t_1)$ Random variable

2) $X(t+\tau)$ → $t = t_1 \longrightarrow X(t_1+\tau) \longrightarrow$ Random variable

$X(t+\tau)$ → $t = t_2 \longrightarrow X(t_2+\tau) \longrightarrow$ Random variable

3) $E[X(t_1)] = E[X(t_2)] = \mu_x$

$E[X(t_1+\tau)] = E[X(t_2+\tau)] = \mu_x$

4) $E[X(t)] = \mu_x = E[X(t_1)] = E[X(t_2)] = \cdots$

For WSSRP

$E[\text{Random process}] = E[\text{Random variable}]$

mean of random process = mean of random variable

5) $E[X(t+\tau)] = E[X(t)] = \mu_x$

6) $E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2) = R_{XX}(t_1 \sim t_2)$ $(\because$ WSSRP$)$

Let Random process $x(t)$ is observed τ duration apart : $x(t)$, $x(t+\tau)$

(i) $E[X(t)X(t+\tau)] = R_{XX}(\tau) = R_{XX}(-\tau)$

Auto correlation function is even function of time

(ii) $E[X(t)X(t-\tau)] = R_{XX}(-\tau) = R_{XX}(\tau)$

(iii) $E[X(t-\tau)X(t)] = R_{XX}(\tau) = R_{XX}(-\tau)$

(iv) $E[X(t+\tau)X(t)] = R_{XX}(-\tau) = R_{XX}(\tau)$

(v) $E[X(5)X(1)] = R_{XX}(4) = R_{XX}(-4)$

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7) $E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$

when $\tau = 0$

$E[X(t)\,X(t)] = R_{XX}(0)$

$E[X^2(t)] = R_{XX}(0) \longrightarrow$ constant / Independent of time.

↙ mean square value of random process $X(t)$

$\therefore$ mean square value of $X(t)$ is $E[X^2(t)] = R_{XX}(0)$

$\Rightarrow E[X^2(t)] = E[X^2(t+T)] = E[X^2(t_1)] = E[X^2(t_2)] = R_{XX}(0)$

$\Rightarrow$ mean square value of random process = mean square value of random variable $= R_{XX}(\tau)$

8) $\sigma^2_{X(t)} = E[X^2(t)] - (E[X(t)])^2$

$\sigma^2_{X(t)} = R_{XX}(0) - \mu_X^2 \longrightarrow$ constant

$\sigma^2_{X(t)} = \sigma^2_{X(t+T)} = \sigma^2_{X(t_1)} = \sigma^2_{X(t_2)}$

↙ RP, ↙ RP, ↙ RV, ↙ RV

$\Rightarrow$ variance (Random process) = variance (Random variable)

$\sigma^2_{X(t)} = \sigma^2_{X(t_1)}$

$\sigma^2_{X(t)} = \sigma^2_{X(t+T)}$

9) zero mean WSSRP :—

$E[X(t)] = \mu_X = 0$

$\Rightarrow \sigma^2_{X(t)} = E[X^2(t)] = R_{XX}(0)$

$\Rightarrow$ mean square value $(X(t))$ = variance $(X(t))$

$\Rightarrow$ mean square value $(X(t=t_1))$ = variance $(X(t=t_1))$

10) consider the following wide sense stationary random process $x(t)$.

If $x(t)$ is power signal then

$$E[x^2(t)] = \sigma^2_{x(t)} + \mu_x^2$$

$$R_{xx}(0) = \sigma^2_{x(t)} + \mu_x^2 \longrightarrow \text{dc power of } x(t)$$

$R_{xx}(0)$ ↙ Total power of $x(t)$

$\sigma^2_{x(t)}$ ↓ AC power of $x(t)$

11) covariance :-

$x(t)$ — $t=t_1$ — $x(t_1)$ → Random variable

$x(t)$ — $t=t_2$ — $x(t_2)$ → Random variable

$$\text{cov}[x(t_1)\,x(t_2)] = \sigma_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - \mu_{x(t_1)}\,\mu_{x(t_2)}$$

$$\Rightarrow \text{cov}[x(t_1)\,x(t_2)] = \sigma_{xx}(t_1 \sim t_2) = R_{xx}(t_1 \sim t_2) - \mu_x^2$$

$$\Rightarrow \text{cov}[x(t)\,x(t+\tau)] = \sigma_{xx}(\tau) = R_{xx}(\tau) - \mu_x^2$$

case (ii)

1) $x(n)$ be discrete time Random process.

$x(n)$ — $n=n_1 \longrightarrow x(n_1)$ — $n=n_2 \longrightarrow x(n_2)$ } Random variables

$x(n)$ ↙ Random process

2) $x(n+K)$ — $n=n_1 \longrightarrow x(n_1+K)$ — $n=n_2 \longrightarrow x(n_2+K)$ } Random variables

$x(n+K)$ ↙ Random process

3) $E[x(n)] = E[x(n+K)] = E[x(n_1)] = E[x(n_2)] = \mu_x$

$E[x(n)]$ ↙ RP; $E[x(n+K)]$ ↙ RP; $E[x(n_1)]$ ↓ RV; $E[x(n_2)]$ ↘ RV

4) $E[x(n)] = E[x(n_1)]$

mean of random process = mean of random variables

5) $E[x(n)] = E[x(n+K)]$

6) Let random process $x(n)$ is observed K duration apart $\begin{cases} x(n) \\ x(n+K) \end{cases}$

$$E[x(n_1)\,x(n_2)] = R_{xx}(n_1, n_2) = R_{xx}(n_1 \sim n_2)$$

(i) $E[x(n)\,x(n+K)] = R_{xx}(K) = R_{xx}(-K)$

(ii) $E[x(n+K)\,x(n)] = R_{xx}(-K) = R_{xx}(K)$

(iii) $E[x(n)\,x(n-K)] = R_{xx}(-K) = R_{xx}(K)$

(iv) $E[x(n-K)\,x(n)] = R_{xx}(K) = R_{xx}(-K)$

(v) $E[x(7)\,x(18)] = R_{xx}(11) = R_{xx}(-11)$

7) $E[x(n)\,x(n+K)] = R_{xx}(K)$

when $K=0$

$E[x(n)\,x(n)] = E[x^2(n)] = R_{xx}(0)$

$\Rightarrow E[x^2(n)]$ (mean square value) $= R_{xx}(0)$

$\Rightarrow E[x^2(n)] = E[x^2(n+K)] = E[x^2(n_1)] = E[x^2(n_2)] = R_{xx}(0)$

$\Rightarrow E[x^2(n)] = E[x^2(n+K)] = R_{xx}(0)$

$\Rightarrow$ mean square value (random variable) = mean square value (random process)

8) Variance :-

$$\sigma^2_{x(n)} = E[x^2(n)] - (E[x(n)])^2$$

$$\sigma^2_{x(n)} = R_{xx}(0) - \mu_x^2$$

$$\sigma^2_{x(n)} = \sigma^2_{x(n+K)} = \sigma^2_{x(n_1)}$$

RP RP RV

$\Rightarrow$ variance of Random process = variance of random variable

$\Rightarrow E[x^2(n)] = E[x^2(n+K)]$

9) zero mean WSSRP :-

$$E[x(n)] = 0$$

$$\sigma^2_{x(n)} = E[x^2(n)]$$

$$\sigma^2_{x(n)} = R_{xx}(0)$$

variance of $x(n)$ = mean square value of $x(n)$

10) If $x(n)$ is power signal then

$$E[x^2(n)] = \sigma^2_{x(n)} + \mu^2_{x(n)}$$

↙ total power ↓ ac power ↓ dc power

11) covariance :-

$$\text{cov}(x(n)\, x(n+k)) = R_{xx}(k) - \mu_x^2 = \sigma^2_{xx}(k)$$

9/6/2021

Important points :-

$$\int_0^{2\pi} \cos(\theta + \omega_0 t)\, d\theta = 0$$

$$\int_0^{2\pi} \cos(2\theta + \omega_0 t)\, d\theta = 0$$

$$\int_0^{2\pi} \cos(3\theta + \omega_0 t + \phi)\, d\theta = 0$$

Let x and y are random variables and c be constant (deterministic value)

Expectation operator operates only on random variables / random process.

$E[x] \neq x$

$E[y] \neq y$

$E[c] = c$

$E[g(x)] \neq g(x)$

$E[t \cdot x] = t\,E[x]$ where t is time

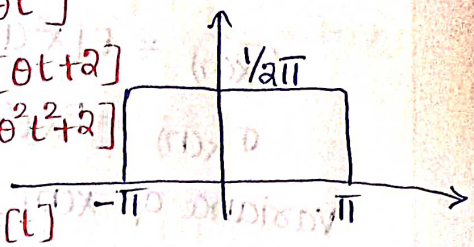
$E[t \cdot \theta] = t\,E[\theta]$ θ is random process.

$E[x(t,\theta)] \neq x(t,\theta)$

#Q1) If θ is uniformly distributed random variable in b/w $-\pi$ and π. Find mean. And calculate

i) $E[2\theta]$ v) $E[\theta t^2]$
ii) $E[\theta^2]$ vi) $E[\theta t+2]$
iii) $E[\theta+2]$ vii) $E[\theta^2 t^2+2]$
iv) $E[\theta t]$ viii) $E[t]$

P.D.F:-



soln

$$E[\theta] = \int_{-\infty}^{\infty} \theta f_\theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \theta \left(\frac{1}{2\pi}\right) d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\theta^2}{2}\right]_{-\pi}^{\pi}$$

$$E[\theta] = \frac{1}{4\pi}\left[\pi^2 - (-\pi)^2\right] = \frac{1}{4\pi}\left[\pi^2 - \pi^2\right] = 0$$

i) $E[2\theta] = 2E[\theta] = 0$

ii) $E[\theta^2] = \int_{-\infty}^{\infty} \theta^2 f_\theta(\theta) d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{1}{6\pi}\theta^3\Big|_{-\pi}^{\pi}$

$$= \frac{1}{6\pi}\left\{\pi^3 - (-\pi^3)\right\} = \frac{2\pi^3}{6\pi} = \frac{\pi^2}{3}$$

iii) $E[\theta+2] = E[\theta] + E[2] = 0+2 = 2$

iv) $E[\theta t] = t\,E[\theta] = 0$

v) $E[\theta t^2] = t^2 E[\theta] = 0$

vi) $E[\theta t+2] = E[\theta t] + E[2] = 0+2 = 2$

vii) $E[\theta^2 t^2+2] = E[\theta^2 t^2] + 2 = t^2 E[\theta^2] + 2$

$$= \frac{t^2\pi^2}{3} + 2$$

viii) $E[t] = t$

(ix) $E[\cos\theta] = \int_{-\infty}^{\infty} \cos\theta\, f_\theta(\theta)\, d\theta$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\sin\theta\Big|_{-\pi}^{\pi} = 0$$

(x) $E[t^2\cos\theta] = t^2 E[\cos\theta] = 0$

(xi) $E[\cos(\theta t)] = \int_{-\infty}^{\infty} \cos(\theta t) f_\theta(\theta)\, d\theta$

$= \int_{-\pi}^{\pi} \cos(\theta t) \frac{1}{2\pi} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta t)\, d\theta$

$= \frac{1}{2\pi} \frac{\sin\theta t}{t} \Big|_{-\pi}^{\pi}$

$= \frac{1}{2\pi t} \{ \sin \pi t - \sin(-\pi t) \}$

$= \frac{1}{2\pi t} \{ \sin \pi t + \sin \pi t \}$

$= \frac{2 \sin \pi t}{2\pi t}$

$= \text{sinc}(t)$

(xii) $E\left[\cos\left(\omega_0 t + \frac{\theta}{2}\right)\right] = \int_{-\infty}^{\infty} \cos\left(\frac{\theta}{2} + \omega_0 t\right) f_\theta(\theta)\, d\theta$

$= \int_{-\infty}^{\infty} \cos\left(\frac{\theta}{2} + \omega_0 t\right) \frac{1}{2\pi} d\theta$

$= \frac{2}{2\pi} \sin\left(\frac{\theta}{2} + \omega_0 t\right) \Big|_{-\pi}^{\pi}$

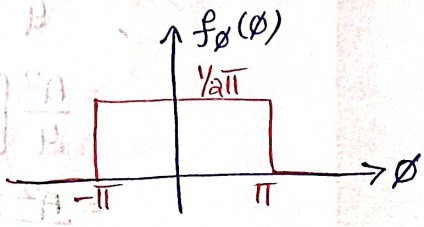
$= \frac{1}{\pi} \left[\sin\left(\pi/2 + \omega_0 t\right) - \sin\left(-\frac{\pi}{2} + \omega_0 t\right) \right]$

$= \frac{1}{\pi} \left[\sin\left(\frac{\pi}{2} + \omega_0 t\right) + \sin\left(\frac{\pi}{2} - \omega_0 t\right) \right]$

$= \frac{1}{\pi} \left[\cos(\omega_0 t) + \cos(\omega_0 t) \right]$

$= \frac{2}{\pi} \cos(\omega_0 t)$

#Q2) consider the random process $X(t) = A\cos(\omega_0 t + \phi)$ where A, ω_0 are constant and ϕ is random variable uniformly distributed from $-\pi$ to π.



i) $E[X(t)]$

Soln $E[X(t)] = E[A\cos(\omega_0 t + \phi)]$

$= \int_{-\infty}^{\infty} A\cos(\omega_0 t + \phi) f_\phi(\phi)\, d\phi$

$$= \int_{-\pi}^{\pi} A\cos(\omega_0 t+\phi)\frac{1}{2\pi}\,d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos(\omega_0 t+\phi)\,d\phi$$

$$= \frac{A}{2\pi}\sin(\omega_0 t+\phi)\Big|_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}\{\sin(\pi+\omega_0 t)-\sin(-\pi+\omega_0 t)\}$$

$$= \frac{A}{2\pi}\{\sin(\pi+\omega_0 t)+\sin(\pi-\omega_0 t)\}$$

$$= \frac{A}{2\pi}\{-\sin(\omega_0 t)+\sin(\omega_0 t)\}$$

$$E[X(t)] = 0$$

(ii) $E[X^2(t)] = E[A^2\cos^2(\omega_0 t+\phi)]$

$$= E\left[A^2\left(\frac{1+\cos(2\omega_0 t+2\phi)}{2}\right)\right]$$

$$= \frac{A^2}{2}\left[E\left(1+\cos(2\omega_0 t+2\phi)\right)\right]$$

$$= \frac{A^2}{2}\left[E(1)+E\left(\cos(2\omega_0 t+2\phi)\right)\right]$$

$$= \frac{A^2}{2}\left[1+\int_{-\infty}^{\infty}\cos(2\omega_0 t+2\phi)f_\phi(\phi)\,d\phi\right]$$

$$= \frac{A^2}{2}\left[1+\int_{-\pi}^{\pi}\cos(2\omega_0 t+2\phi)\left(\frac{1}{2\pi}\right)d\phi\right]$$

$$= \frac{A^2}{2}\left[1+\frac{1}{2\pi}\,\frac{\sin(2\omega_0 t+2\phi)}{2}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{A^2}{2}\left[1+\frac{1}{4\pi}\{\sin(2\pi+2\omega_0 t)-\sin(-2\pi+2\omega_0 t)\}\right]$$

$$= \frac{A^2}{2}\left[1+\frac{1}{4\pi}\{\sin(2\pi+2\omega_0 t)+\sin(2\pi-2\omega_0 t)\}\right]$$

$$= \frac{A^2}{2}\left[1+\frac{1}{4\pi}\{\sin(2\omega_0 t)-\sin(2\omega_0 t)\}\right]$$

$$= \frac{A^2}{2}$$

(iii) $Var(X(t)) = \sigma^2_{X(t)}$

$\sigma^2_{X(t)} = E[X^2(t)] - (E[X(t)])^2 = \frac{A^2}{2} - 0 = \frac{A^2}{2}$

(iv) $E[X(t)X(t+T)] = E[A\cos(\omega_0 t+\phi) A \cos(\omega_0 t+\omega_0 T+\phi)]$

$= E[A^2 \cos(\omega_0 t+\phi)\cos(\omega_0 t+\omega_0 T+\phi)]$

$= E\left[\frac{A^2}{2}\{\cos(2\omega_0 t+2\phi+\omega_0 T)+\cos(\omega_0 T)\}\right]$

$= E\left\{\frac{A^2}{2}\cos(2\omega_0 t+2\phi+\omega_0 T)\right\} + \underbrace{E\left\{\frac{A^2}{2}\cos(\omega_0 T)\right\}}_{\text{Deterministic}}$

$= \frac{A^2}{2} E\{\cos(2\omega_0 t+2\phi+\omega_0 T)\} + \frac{A^2}{2}\cos\omega_0 T$

$= \left\{\frac{A^2}{2}\int_{-\infty}^{\infty} \cos(2\omega_0 t+2\phi+\omega_0 T) f_\phi(\phi)\, d\phi\right\} + \frac{A^2}{2}\cos\omega_0 T$

$= \frac{A^2}{2}\left\{\frac{\sin(2\omega_0 t+2\phi+\omega_0 T)}{2}\left(\frac{1}{2\pi}\right)\right\}_{-\pi}^{\pi} + \frac{A^2}{2}\cos\omega_0 T$

$= \frac{A^2}{8\pi}\{\sin(2\pi+2\omega_0 t+\omega_0 T) - \sin(-2\pi+2\omega_0 t+\omega_0 T)\} + \frac{A^2}{2}\cos\omega_0 T$

$= \frac{A^2}{8\pi}\{\sin(2\pi+2\omega_0 t+\omega_0 T) + \sin(2\pi-2\omega_0 t-\omega_0 T)\} + \frac{A^2}{2}\cos\omega_0 T$

$= \frac{A^2}{8\pi}\{\sin(2\omega_0 t+\omega_0 T) - \sin(2\omega_0+\omega_0 T)\} + \frac{A^2}{2}\cos\omega_0 T$

$= \frac{A^2}{2}\cos\omega_0 T$

$E[X(t)X(t+T)] = R_{XX}(t, t+T) = \frac{A^2}{2}\cos\omega_0 T = R_{XX}(T)$

$\Rightarrow$ Autocorrelation function is completely depends on T.

$R_{XX}(t, t+T) = \frac{A^2}{2}\cos\omega_0 T = R_{XX}(T) \Rightarrow$ WSSRP

If X(t) is periodic with period T_0 then $R_{XX}(T)$ is also periodic with period T_0

We observed that

$E[x(t)] = \text{mean} = 0$ {constant}

$E[x(t)\,x(t+\tau)] = \text{Auto correlation} = R_{xx}(\tau)$

$\Downarrow$

Depends only on τ but not on t, $t+\tau$

} WSSRP

(iv) $\text{cov}(x(t)\,x(t+\tau))$

$$\text{cov}[x(t)x(t+\tau)] = E[x(t)x(t+\tau)] - (E[x(t)])(E[x(t+\tau)])$$

($E[x(t)] = 0$, $E[x(t+\tau)] = 0$)

$$= R_{xx}(\tau) - 0$$

$$= R_{xx}(\tau)$$

$$\text{cov}[x(t)x(t+\tau)] = R_{xx}(\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

$\text{cov}[x(t)x(t+\tau)]$ — Auto covariance

$R_{xx}(\tau)$ — Auto correlation

summary :-

consider the following random process

$x(t) = A\cos(\omega_0 t + \phi)$ or $x(t) = A\sin(\omega_0 t + \phi)$

(i) mean :- $E[x(t)] = 0$

(ii) mean square value :- $E[x^2(t)] = \frac{A^2}{2}$

(iii) variance :- $\sigma^2_{x(t)} = \frac{A^2}{2}$

(iv) Autocorrelation :- $E[x(t)\,x(t+\tau)] = \frac{A^2}{2}\cos\omega_0\tau$

(v) Auto covariance :- $\text{cov}[x(t)\,x(t+\tau)] = \frac{A^2}{2}\cos\omega_0\tau$

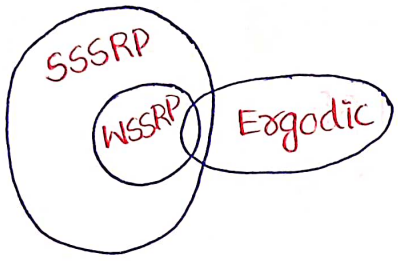
Ergodic Random process :-

Statistical average of Random process = Time average of random process.

$$\Rightarrow E[x(t)] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\,dt$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t) f_{x(t)}(x)\,dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\,dt$$

venn diagram representing strict sense stationary random process, wide sense stationary random process, ergodic random process :



properties of Auto correlation :-

Let $x(t)$ be WSSRP :-

1) $E[x(t)\,x(t+\tau)] = R_{xx}(\tau)$

2) $R_{xx}(\tau) = R_{xx}(-\tau) \Rightarrow$ Even function

3) $R_{xx}(0) = E[x^2(t)]$ = mean square value of $x(t)$

4) $|R_{xx}(\tau)| \le R_{xx}(0)$

$$-R_{xx}(0) \le R_{xx}(\tau) \le R_{xx}(0)$$

5) $\{R_{xx}(\tau)\}_{max} = R_{xx}(0)$

6) $x(t)$: WSSRP
- is periodic then $R_{xx}(\tau)$ is periodic with same fundemental time period
- is aperiodic then $R_{xx}(\tau)$ is also aperiodic

7) $R_{xx}(\tau)$
- $\lim_{\tau\to 0} R_{xx}(\tau)$ = mean square value = $E[x^2(t)]$
- $\lim_{\tau\to\infty} R_{xx}(\tau) = \mu_x^2$ when $x(t)$ is ergodic random process

$\lim_{\tau \to 0} R_{XX}(\tau) = R_{XX}(0) = E[X^2(t)]$ = mean square value → if $x(t)$ is WSSRP.

$\lim_{\tau \to \infty} R_{XX}(\tau) = \mu_X^2 = (E[X(t)])^2$ = if $x(t)$ is WSSRP + Ergodic process.

If $x(t)$ is WSSRP and ergodic then

$$\sigma^2_{X(t)} = E[X^2(t)] - (E[X(t)])^2$$

$$\sigma^2_{X(t)} = \lim_{\tau \to 0} R_{XX}(\tau) - \lim_{\tau \to \infty} R_{XX}(\tau)$$

#93: $X(t)$ is ergodic random process, having auto correlation function

(a) $R_{XX}(\tau) = \frac{50}{2+\tau^2} + 25$

calculate (i) $E[X(t)]$

(ii) $E[X^2(t)]$

(iii) $E[X(2)\,X(4)]$

(iv) $E[X(91)]$

Soln (i) The autocorrelatio function is function of τ only. and it is given that $X(t)$ is ergodic random process.

⇒ $X(t)$ is both ergodic & WSSRP.

Therefore $\mu_X^2 = \lim_{\tau \to \infty} \left(\frac{50}{2+\tau^2} + 25\right)$

$$\mu_X^2 = 25$$

$$\mu_X = \sqrt{25}$$

$$E[X(t)] = \mu_X = 5$$

(ii) $E[X^2(t)] = \lim_{\tau \to 0} \left(\frac{50}{2+\tau^2} + 25\right) = \frac{50}{2+0} + 25 = 50$

(iii) $E[X(2)\,X(4)] = R_{XX}(2,4) = R_{XX}(4-2) = R_{XX}(2)$

$$E[X(2)X(4)] = \frac{50}{2+4} + 25 = \frac{50}{6} + 25 = \frac{200}{6}$$

(iv) mean/ expectation value is independent of time/ constant.

⇒ $E[X(91)] = 5$

#94) $X(t)$ is ergodic random process with auto correlation function.

$$R_{XX}(\tau) = \begin{cases} \frac{-25|\tau|}{2} + 100 & |\tau| \le 4 \\ 50 & \text{elsewhere.} \end{cases}$$

i) Is it WSSRP or not?

ii) $E[X(t)]$ iii) $E[X^2(t)]$ iv) $E[X(2)X(8)]$

v) $E[X^2(1001)]$ vi) $E[X(101)]$

Soln i) mean of a random process

$$\mu_X^2 = \lim_{\tau \to \infty} R_{XX}(\tau) = 50$$

$\mu_X = \sqrt{50}$ = constant and auto correlation function is function of τ and hence it is WSSRP

ii) $E[X(t)] = \mu_X = \sqrt{50}$

iii) $E[X^2(t)] = R_{XX}(0) = 100$

iv) $E[X(2)X(8)] = R_{XX}(6) = 50$

v) $E[X^2(1001)] = 100$

vi) $E[X(101)] = \sqrt{50}$

Important observations :-

When $X(t)$ is WSSRP :-

→ $E[X(t)] = \mu_X$ = constant

→ $E[X^2(t)] = R_{XX}(0)$ = constant

→ $\sigma^2_{X(t)} = R_{XX}(0) - \mu_X^2$ = constant

→ $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

When $X(n)$ is WSSRP :-

→ $E[X(n)] = \mu_X$ = constant

→ $E[X^2(n)] = R_{XX}(0)$ = constant

→ $\sigma^2_{X(n)} = R_{XX}(0) - \mu_X^2$

→ $E[X(n)X(n+k)] = R_{XX}(k)$

Let's investigate some cases : { x(t), x(n) are WSSRP }

case (i) Y(t) = x(at+b)

1) $E[Y(t)] = E[x(at+b)] = \mu_x$ (Independent of time)

2) $E[Y^2(t)] = E[x^2(at+b)] = R_{xx}(0)$

3) $\sigma^2_{Y(t)} = R_{xx}(0) - \mu_x^2$

4) $R_{yy}(\tau) = E[y(t)\, y(t+\tau)] = E[x(at+b)\, x(at+a\tau+b)]$

$= R_{xx}(at+a\tau+b-at-b) = R_{xx}(a\tau)$,

$R_{yy}(\tau) = R_{xx}(a\tau)$

case (ii) y(n) = x(an+b)

1) $E[y(n)] = E[x(an+b)] = \mu_x$ {Independent of time}

2) $E[y^2(n)] = E[x^2(an+b)] = R_{xx}(0)$

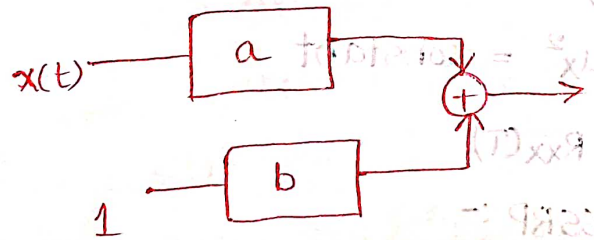
3) $\sigma^2_{y(n)} = R_{xx}(0) - \mu_x^2$

4) $R_{yy}(\tau) = E[y(n)\, y(n+k)] = E[x(an+b)\, x(an+ak+b)]$

$= R_{xx}(an+ak+b-an-b) = R_{xx}(ak)$

$R_{yy}(\tau) = R_{xx}(a\tau)$

case (iii) y(t) = ax(t) + b



1) $E[y(t)] = E[ax(t)+b] = a\,E[x(t)] + E[b]$

$= a\mu_x + b$

2) $E[y^2(t)] = E[a^2x^2(t) + b^2 + 2ax(t)\,b]$

$= E[a^2x^2(t)] + E[b^2] + E[2ab\,x(t)]$

$= a^2E[x^2(t)] + E[b^2] + 2ab\,E[x(t)]$

$$E[y^2(t)] = a^2 R_{xx}(0) + b^2 + 2ab\mu_x$$

3) $\sigma^2_{y(t)} = E[y^2(t)] - \{E[y(t)]\}^2$

$= a^2 R_{xx}(0) + b^2 + 2ab\mu_x - (a\mu_x + b)^2$

$= a^2 R_{xx}(0) + b^2 + 2ab\mu_x - (a^2\mu_x^2 + b^2 + 2ab\mu_x)$

$= a^2 R_{xx}(0) + b^2 + 2ab\mu_x - a^2\mu_x^2 - b^2 - 2ab\mu_x$

$= a^2 R_{xx}(0) - a^2\mu_x^2$

$\sigma^2_{y(t)} = a^2\{R_{xx}(0) - \mu_x^2\}$

4) $R_{yy} = E[y(t)\, y(t+T)]$

$= E[(ax(t)+b)(ax(t+T)+b)]$

$= E[a^2 x(t)x(t+T) + ab\,x(t) + ab\,x(t+T) + b^2]$

$= E[a^2 x(t)x(t+T)] + E[ab\,x(t)] + E[ab\,x(t+T)] + E[b^2]$

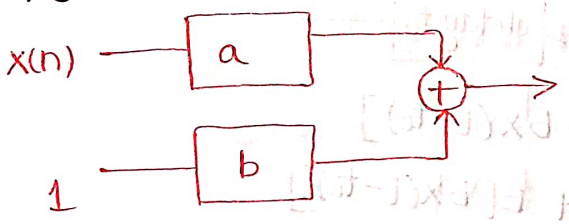
$= a^2 E[x(t)x(t+T)] + abE[x(t)] + ab\,E[x(t+T)] + E[b^2]$

$= a^2 R_{xx}(\tau) + ab\mu_x + ab\mu_x + b^2$

$= a^2 R_{xx}(\tau) + 2ab\mu_x + b^2$

$R_{yy}(T) = a^2 R_{xx}(\tau) + 2ab\mu_x + b^2$

case (iv) $y(t) = ax(n) + b$:-



1) $E[y(n)] = E[ax(n)+b] = a\,E[x(n)] + b$

$= a\mu_x + b$

2) $E[y^2(n)] = E[(ax(n)+b)^2] = E[a^2x^2(n) + b^2 + 2abx(n)]$

$= E[a^2x^2(n)] + E[b^2] + E[2abx(n)]$

$= a^2 E[x^2(n)] + E[b^2] + 2ab\,E[x(n)]$

$E[y^2(n)] = a^2 R_{xx}(0) + b^2 + 2ab\mu_x$

3) $\sigma^2_{y(n)} = E[y^2(n)] - (E[y(n)])^2$

$= E[(ax(n)+b)^2] - (a\mu_x+b)^2$

$= E[a^2x^2(n) + b^2 + 2abx(n)] - (a^2\mu_x^2 + b^2 + 2ab\mu_x)$

$= E[a^2x^2(n)] + E[b^2] + E[2abx(n)] - (a^2\mu_x^2 + b^2 + 2ab\mu_x)$

$= a^2 E[x^2(n)] + E[b^2] + 2ab\,E[x(n)] - (a^2\mu_x^2 + b^2 + 2ab\mu_x)$

$= a^2 R_{xx}(0) + b^2 + 2ab\mu_x - a^2\mu_x^2 - b^2 - 2ab\mu_x$

$\sigma^2_{y(n)} = a^2\{R_{xx}(0) - \mu_x^2\}$

4) $R_{yy}(K) = E[y(n)\,y(n+K)]$

$= E[(ax(n)+b)(ax(n+K)+b)]$

$= E[a^2x(n)x(n+K) + abx(n) + abx(n+K) + b^2]$

$= E[a^2x(n)x(n+K)] + E[abx(n)] + E[abx(n+K)] + E[b^2]$

$= a^2E[x(n)x(n+K)] + abE[x(n)] + abE[x(n+K)] + E[b^2]$

$= a^2R_{xx}(K) + ab\mu_x + ab\mu_x + b^2$

$R_{yy}(K) = a^2R_{xx}(K) + 2ab\mu_x + b^2$

case (V) $y(t) = ax(t) + bx(t-t_0)$



1) $E[y(t)] = E[ax(t) + bx(t-t_0)]$

$= E[ax(t)] + E[bx(t-t_0)]$

$= a\,E[x(t)] + b\,E[x(t-t_0)]$

$= a\mu_x + b\mu_x$

$E[y(t)] = (a+b)\mu_x$

2) $E[y^2(t)] = E[(ax(t) + bx(t-t_0))^2]$

$E[a^2x^2(t) + b^2x^2(t-t_0) + 2abx(t)x(t-t_0)]$

$$= E[a^2x^2(t)] + E[b^2x^2(t-t_0)] + E[2ab\,x(t)\,x(t-t_0)]$$

$$= a^2E[x^2(t)] + b^2E[x^2(t-t_0)] + 2ab\,E[x(t)\,x(t-t_0)]$$

$$= a^2R_{xx}(0) + b^2R_{xx}(0) + 2ab\,R_{xx}(t_0)$$

$$E[y^2(t)] = (a^2+b^2)\,R_{xx}(0) + 2ab\,R_{xx}(t_0)$$

3) $Var(y(t)) = \sigma^2_{y(t)}$

$$= E[y^2(t)] - (E[y(t)])^2$$

$$= (a^2+b^2)\,R_{xx}(0) + 2ab\,R_{xx}(t_0) - (a\mu_x + b\mu_x)^2$$

$$= (a^2+b^2)\,R_{xx}(0) + 2ab\,R_{xx}(t_0) - a^2\mu_x^2 - b^2\mu_x^2 - 2ab\mu_x^2$$

$$= (a^2+b^2)\,R_{xx}(0) - (a^2+b^2)\mu_x^2 + 2ab(R_{xx}(t_0) - \mu_x^2)$$

$$= (a^2+b^2)\{R_{xx}(0) - \mu_x^2\} + 2ab\{R_{xx}(t_0) - \mu_x^2\}$$

$$\sigma^2_{y(t)} = (a^2+b^2)\,\sigma_x^2 + 2ab\{R_{xx}(t_0) - \mu_x^2\}$$

4) $E[y(t)\,y(t+\tau)] = E[\{ax(t)+bx(t-t_0)\}\{ax(t+\tau)+bx(t+\tau-t_0)\}]$

$$= E[a^2x(t)x(t+\tau) + abx(t)x(t+\tau-t_0) + 2ab\,x(t-t_0)x(t+\tau) + b^2x(t-t_0)x(t+\tau-t_0)]$$

$$= E[a^2x(t)x(t+\tau)] + E[abx(t)x(t+\tau-t_0)] + E[abx(t-t_0)x(t+\tau)] + E[b^2x(t-t_0)x(t+\tau-t_0)]$$

$$= a^2E[x(t)x(t+\tau)] + ab\,E[x(t)x(t+\tau-t_0)] + ab\,E[x(t-t_0)x(t+\tau)] + b^2E[x(t-t_0)x(t+\tau-t_0)]$$

$$= a^2R_{xx}(\tau) + ab\,R_{xx}(\tau-t_0) + ab\,R_{xx}(\tau+t_0) + b^2R_{xx}(\tau)$$

$$R_{xx}(\tau) = R_{xx}(\tau)(a^2+b^2) + ab\,R_{xx}(\tau-t_0) + ab\,R_{xx}(\tau+t_0)$$

Cross correlation:

If the random processes are jointly stationary in wide sense and if sample functions from these process are designated as $x(t)$ and $y(t)$ then for two random variables $x_1 = x(t_1)$ and $y_2 = y(t_1+\tau)$

It is possible to define the cross correlation function

$$R_{xy}(\tau) = E[x_1 y_2] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} x_1 y_2 f(x_1, y_2)\, dy_2$$

There is also another cross correlation function that can be defined for the same two time instants. Thus

Let $y_1 = y(t_1)$ and $x_2 = x(t_1+\tau)$

$$R_{yx}(\tau) = E[y_1 x_2] = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} y_1 x_2 f(y_1, x_2)\, dx_2$$

Note that because both random process are assumed to be jointly stationary, these cross correlation function depend only upon the time difference τ.

The time crosscorrelation function may be defined for particular pair of sample function as

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y(t+\tau)\, dt \quad \text{and}$$

$$R_{yx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, x(t+\tau)\, dt$$

If the random process are jointly ergodic, the above two equations yield the same value for every pair of sample functions.

$$R_{xy}(\tau) = R_{xy}(\tau)$$

$$R_{yx}(\tau) = R_{yx}(\tau)$$

properties:-

1) The quantities $R_{xy}(0)$ and $R_{yx}(0)$ have no particular physical significance and do not represent mean square values. It is true however $R_{xy}(0) = R_{yx}(0)$

2) crosscorrelation functions are not generally even function of τ. There is a type of symmetry and

indicated by relation

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

3) The crosscorrelation function does not necessarily have its maximum value at $\tau = 0$

$$|R_{xy}(\tau)| \leq \{R_x(0)\, R_y(0)\}^{1/2}$$

4) If two random process are statistically independent then $R_{xy}(\tau) = E[x_1, y_2] = E[x_1]\, E[y_2] = \bar{x}\,\bar{y} = R_{yx}(\tau)$

5) If $x(t)$ is stationary random process $\dot{x}(t)$ is its derivative with respective to time, the cross correlation function of $x(t)$ and $\dot{x}(t)$ is give by

$$R_{x\dot{x}}(\tau) = \underbrace{\frac{d}{d\tau} R_{xx}(\tau)}_{\text{derivative of auto correlation func}^n \text{ w.r.t } \tau.}$$

11/06/2021

Covariance and correlation matrices :-

In previous all discussions, we consider only a random variables. There are many practical situations in which there may be many random variables and it is necessary to develop a convenient method for representing the many auto correlation and cross correlation that araise.

In those situations vector notation is useful in representing a signal arises in the case of single time function that is sampled at periodic time instants. If only a finite number of such samples are to be considered, say N, then each sample value can become a component of (Nx1) vector. Thus if the sample times are $t_1, t_2, \ldots t_N$ the vector representing the time function $x(t)$ can be expressed as

$$X = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix}$$

indicated by relation

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

3) The crosscorrelation function does not necessarily have its maximum value at $\tau = 0$

$$|R_{XY}(\tau)| \leq \{R_X(0)\, R_Y(0)\}^{1/2}$$

4) If two random process are statistically independent then $R_{XY}(\tau) = E[X_1, Y_2] = E[X_1]\, E[Y_2] = \bar{X}\,\bar{Y} = R_{YX}(\tau)$

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$$X = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix}$$

If $x(t)$ is a sample function from a random process, then each of the components of the vector x is a random variable.

Now it is possible to define a correlation matrix that is $(N \times N)$ and gives the correlation b/w every pair of random variables. Thus

$$R_X = E[XX^T] = E\left\{ \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix} [x(t_1)\, x(t_2) \cdots x(t_N)] \right\}$$

$$R_X = E[XX^T] = E\begin{bmatrix} x(t_1)x(t_1) & x(t_1)x(t_2) & \cdots & x(t_1)x(t_N) \\ x(t_2)x(t_1) & x(t_2)x(t_2) & \cdots & x(t_2)x(t_N) \\ \vdots & & & \\ x(t_N)x(t_1) & x(t_N)x(t_2) & \cdots & x(t_N)x(t_N) \end{bmatrix}$$

X^T is the transpose of X.

When the expected value of each element of the matrix is taken, that element becomes a particular value of the autocorrelation function of the random process from which $x(t)$ came. Thus

$$R_X = \begin{bmatrix} R_X(t_1,t_1) & R_X(t_1,t_2) & \cdots & R_X(t_1,t_N) \\ R_X(t_2,t_1) & R_X(t_2,t_2) & \cdots & R_X(t_2,t_N) \\ \vdots & & & \\ R_X(t_N,t_1) & R_X(t_N,t_2) & \cdots & R_X(t_N,t_N) \end{bmatrix}$$

When the random process from which $x(t)$ come is wide sense stationary then all the components of R_X become function of time difference only.

If the interval between sample values is Δt then

$$t_2 = t_1 + \Delta t$$
$$t_3 = t_1 + 2\Delta t$$
$$\vdots$$
$$t_N = t_1 + (N-1)\Delta t$$

$$R_X = \begin{bmatrix} R_X(0) & R_X(\Delta t) & \cdots & R_X((N-1)\Delta t) \\ R_X(-\Delta t) & R_{XX}(0) & & R_X((N-2)\Delta t) \\ \\ R_X(-(N-1)\Delta t) & R_X(-(N-2)\Delta t) & & R_X(0) \end{bmatrix}$$

since R_X is even function then

$$R_{XX} = \begin{bmatrix} R_X(0) & R_X(\Delta t) & & R_X(N-1) \\ R_X(\Delta t) & R_X(0) & & R_{XX}((N-2)\Delta t) \\ \vdots & & & \\ R_X((N-1)\Delta t) & R_X((N-2)\Delta t) & & R_{XX}(0) \end{bmatrix}$$

*** $\Rightarrow$ R_X matrix symmetric matrix $\Rightarrow$ $a_{ij} = a_{ji}$ ***

covariance matrix :-

The general covariance b/w two random variables is defined as

$$E\left\{\left[\left(X(t_i) - \bar{X}(t_i)\right)\left(X(t_j) - \bar{X}(t_j)\right)\right]\right\} = \sigma_i \sigma_j \rho_{ij}$$

$\bar{X}(t_i)$ = mean value of $x(t_i)$

$\bar{X}(t_j)$ = mean value of $x(t_j)$

σ_i^2 = variance of $x(t_i)$

σ_j^2 = variance of $x(t_j)$

ρ_{ij} = normalized covariance coefficient of $x(t_i)$ and $x(t_j)$

$= 1$ when $i = j$

The covariance matrix is defined as

$$A_X = E\left[(X - \bar{X})(X^T - \bar{X}^T)\right] = R_{XX} - \bar{X}\,\bar{X}^T$$

for zero mean random process $A_X = R_{XX}$

$$A_X = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_N\rho_{1N} \\ \sigma_2\sigma_1\rho_{21} & \sigma_2^2 & & \\ & & & \\ \sigma_N\sigma_1\rho_{N1} & & & \sigma_N^2 \end{bmatrix}$$

This is valid for stationary and non stationary random process.

For WSSRP, all variances are same and correlation coefficients in a diagonal are same. Thus

$$\sigma_i^2 = \sigma_j^2 = \sigma^2 \qquad i,j = 1,2,3,\ldots N$$

$$\rho_{ij} = \rho_{|i-j|} \qquad i,j = 1,2,3,\ldots N$$

$$A_X = \begin{bmatrix} \sigma^2 & \sigma^2\rho_1 & \sigma^2\rho_2 & \cdots & \sigma^2\rho_{N-1} \\ \sigma^2\rho_1 & \sigma^2 & \sigma^2\rho_1 & & \sigma^2\rho_{N-2} \\ \vdots & & & & \\ \rho_{N-1} & & & & \sigma^2 \end{bmatrix}$$

$$A_X = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & & \rho_{N-2} \\ & & & & \\ \rho_{N-1} & & & & 1 \end{bmatrix}$$

$\Rightarrow$ Toe plitz matrix

$\Rightarrow$ covariance matrix is also a symmetric matrix for WSSRP.

Gaussian Random process :-

Gaussian random process is a special class of random process. consider a random process $X(t)$.

$X(t)$ is a gaussian random process if $X(t_1), X(t_2), \ldots$ $X(t_n)$ at time instants $t_1, t_2, \ldots t_n$ are jointly gaussian.

multivariate density :

$f_{X(t_1)\,X(t_2)\,\ldots\,X(t_n)}(x_1, x_2, \ldots x_n)$ has to be gaussian

$\forall\ t_1, t_2, \ldots t_n$

In particular for $n=1$,

The p.d.f of $x(t)$ at any time instant t must be gaussian.

$$\Rightarrow f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

μ = mean $= E\{x(t)\}$

$\sigma^2 = E\{(x(t)-\mu)^2\}$

consider a widesense stationary gaussian random process

$\Rightarrow$ mean $= E\{x(t)\} = \mu_x \longrightarrow$ constant

$\Rightarrow$ Autocorrelation $= E\{x(t)\,x(t+\tau)\} = R_{XX}(\tau)$

$\Rightarrow$ when $\tau = 0$

mean square value $E\{x^2(t)\} = R_{XX}(0)$

$\Rightarrow$ variance $= \sigma^2 = E\{x^2(t)\} - \mu_x^2 = R_{XX}(0) - \mu_x^2$

Now p.d.f function become

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\,(R_{XX}(0) - \mu_x^2)}}\, e^{-\left(\frac{(x-\mu)^2}{2(R_{XX}(0)-\mu_x^2)}\right)}$$