

Random Processes and Spectral Analysis

**By
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chapter 2

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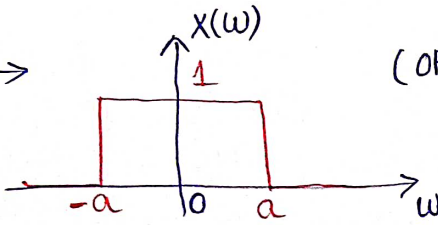
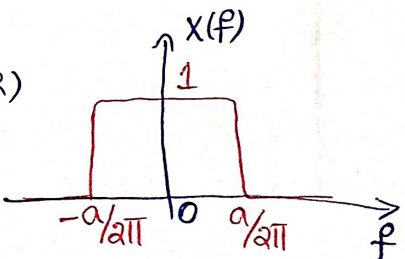
Spectral Properties of Random Process# Density functions :-Basics of Fourier transform and Inverse Fourier transform :-

continuous time Fourier transform of a continuous time signal $x(t)$ is given by $x(\omega)$ or $x(f)$

$$\begin{array}{l|l}
 x(t) \longleftrightarrow x(\omega) & x(t) \longleftrightarrow x(f) \\
 x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt & x(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\
 x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega & x(t) = \int_{-\infty}^{\infty} x(f) e^{j2\pi f t} df
 \end{array}$$

Important Fourier transform pairs :-

- 1) $\delta(t) \longleftrightarrow 1$
- 2) $\cos \omega_0 t \longleftrightarrow \pi \{ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \}$ (OR)
 $\frac{1}{2} \{ \delta(f - f_0) + \delta(f + f_0) \}$
- 3) $\sin \omega_0 t \longleftrightarrow \frac{\pi}{j} \{ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \}$ (OR)
 $\frac{1}{2j} \{ \delta(f - f_0) + \delta(f + f_0) \}$
- 4) e^{-at^2} (Gaussian pulse) $\longleftrightarrow \sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$ (OR)
 $\sqrt{\frac{\pi}{a}} e^{-(2\pi f)^2/4a}$
- 5) $e^{-\pi t^2}$ (Gaussian pulse) $\longleftrightarrow e^{-\omega^2/4\pi}$ (OR) $e^{-\pi f^2}$
- 6) $e^{-a|t|} \longleftrightarrow \frac{2a}{a^2 + \omega^2}$ (OR) $\frac{2a}{a^2 + (2\pi f)^2}$
- 7) $\frac{\sin at}{\pi t} \longleftrightarrow$


(OR)


Density functions

- energy spectral density
units: Joules/Hz
- power spectral density
units: Watt/Hz

Energy spectral density:-

$x(t)$ has to be wide sense stationary random process (WSSRP) and energy signal.

The energy spectral can be calculated as follows:

Since $x(t)$ is an energy signal, Fourier transform is exist.

$$x(t) \xleftrightarrow{F.T} X(\omega) \quad (\text{OR}) \quad x(t) \xleftrightarrow{F.T} X(f)$$

Then energy spectral density is denoted as $G_{xx}(\omega)/G_{xx}(f)$.

$$G_{xx}(\omega) = |X(\omega)|^2 \quad (\text{or})$$

$$G_{xx}(f) = |X(f)|^2$$

Since $x(t)$ is WSSRP, autocorrelation function is strictly depends on time difference.

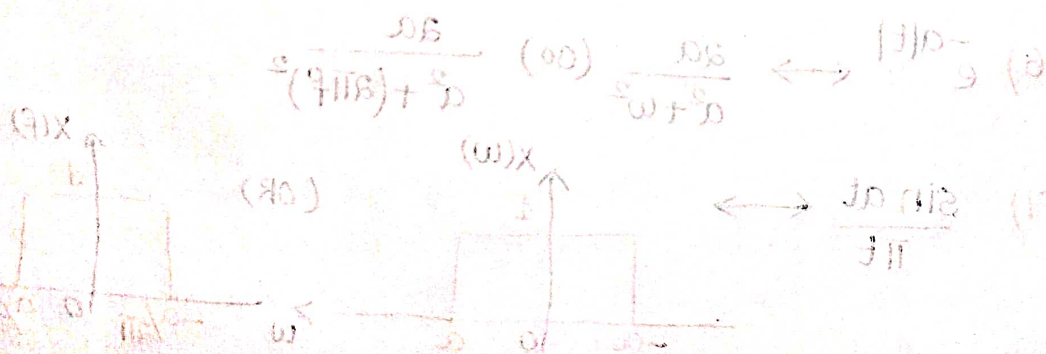
$$E[x(t)x(t+\tau)] = R_{xx}(\tau)$$

when $\tau=0$ $E[x^2(t)] = R_{xx}(0) = \text{Energy}$.

The autocorrelation function $R_{xx}(\tau)$ and the energy density function makes Fourier transform pairs.

$$\left. \begin{aligned} R_{xx}(\tau) &\xleftrightarrow{F.T} G_{xx}(\omega) \\ R_{xx}(\tau) &\xleftrightarrow{F.T} G_{xx}(f) \end{aligned} \right\} \text{Energy spectral density}$$

$$\text{ACF}\{x(t)\} \xleftrightarrow{F.T} \text{ESD}\{x(t)\}$$



methods to calculate Energy spectral density:-

method 1:

$$1) x(t) \xrightarrow{F.T} X(\omega) \Rightarrow |X(\omega)|^2 = G_{xx}(\omega)$$

method 2:

$$a) x(t) \xrightarrow{A.C.F} R_{xx}(\tau) \xrightarrow{F.T} G_{xx}(\omega)$$

We knew that, if $x(t)$ is energy signal and WSSRP then $R_{xx}(\tau)$ and energy spectral density function ($G_{xx}(\omega)/G_{xx}(f)$) forms Fourier transform pairs.

Then

$$i) G_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad (OR)$$

$$G_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

$$iii) G_{xx}(0) = \begin{cases} \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau & \xrightarrow{\text{even function}} \frac{1}{2} \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau \\ \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau & \xrightarrow{\text{even function}} \frac{1}{2} \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau \end{cases}$$

iii) We knew that auto correlation function is even function of τ .

$$\text{Then } R_{xx}(\tau) = R_{xx}(-\tau)$$

Now apply Fourier transform on both sides

$$\mathcal{F}\{R_{xx}(\tau)\} = \mathcal{F}\{R_{xx}(-\tau)\}$$

$$\Rightarrow G_{xx}(f) = G_{xx}(-f)$$

$\Rightarrow$ Energy spectral density function is also even function of f .

$$iv) R_{xx}(\tau) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{xx}(\omega) e^{j\omega\tau} d\omega \\ \int_{-\infty}^{\infty} G_{xx}(f) e^{j2\pi f\tau} df \end{cases}$$

$$\text{when } \tau = 0 \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{xx}(\omega) d\omega = \frac{\text{Area under } G_{xx}(\omega)}{2\pi}$$

$$\begin{aligned} R_{xx}(0) &= \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{xx}(\omega) d\omega \\ \int_{-\infty}^{\infty} G_{xx}(f) df = \text{Area under } G_{xx}(f) \end{cases} \\ \swarrow \\ \text{Energy} \end{aligned}$$

power spectral density (P.S.D)

consider the random process $x(t)$ which is wide sense stationary random process and power signal.

* since it is a wide sense stationary random process then autocorrelation function is only function of τ .

$$\Rightarrow E[x(t)x(t+\tau)] = R_{xx}(\tau)$$

* Now $R_{xx}(\tau)$ autocorrelation function and power spectral density function $S_{xx}(\omega)$ / $S_{xx}(f)$ form fourier transform pairs.

$$\Rightarrow R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(\omega)$$

$$R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(f)$$

Auto correlation function $\{x(t)\}$ $\xleftrightarrow{F.T}$ Power spectral density $\{x(t)\}$

$$* x(t) \xleftrightarrow{ACF} R_{xx}(\tau) \xleftrightarrow{FT} S_{xx}(\omega) / S_{xx}(f)$$

$$* 1) S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad \text{or}$$

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau = \int_0^{\infty} R_{xx}(\tau) d\tau$$

zero frequency value of P.S.D

$$* 2) R_{xx}(\tau) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega \\ \int_{-\infty}^{\infty} S_{xx}(f) e^{j2\pi f\tau} df \end{cases}$$

when $\tau=0$

$$R_{xx}(0) = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = \frac{\text{Area under } S_{xx}(\omega)}{2\pi} \\ \int_{-\infty}^{\infty} S_{xx}(f) df = \text{Area under } S_{xx}(f) \end{cases}$$

total power

*3) Since autocorrelation function is even function, then

$$R_{xx}(T) = R_{xx}(-T)$$

Apply Fourier transform on both sides :

$$\mathcal{F}\{R_{xx}(T)\} = \mathcal{F}\{R_{xx}(-T)\}$$

$$\Rightarrow S_{xx}(\omega) = S_{xx}(-\omega)$$

$\Rightarrow$ power spectral density function is also an even function.

*4) methods to calculate power of $x(t)$:-

i) $E[x^2(t)] \rightarrow$ power of $x(t)$

ii) $x(t) \xrightarrow{\text{ACF}} E[x(t)x(t+T)] = R_{xx}(T)$

Then $E[x^2(t)] = R_{xx}(0)$

iii) $x(t) \xrightarrow{\text{ACF}} R_{xx}(T) \xleftrightarrow{\text{FT}} S_{xx}(\omega) \text{ or } S_{xx}(f)$

$$\begin{cases} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega \\ \frac{1}{\pi} \int_0^{\infty} S_{xx}(\omega) d\omega \end{cases} \begin{cases} \int_{-\infty}^{\infty} S_{xx}(f) df \\ 2 \int_0^{\infty} S_{xx}(f) df \end{cases}$$

*5) $E[x^2(t)] = \sigma_{x(t)}^2 + \mu_{x(t)}^2$

$$E[x^2(t)] = \sigma_x^2 + \mu_x^2$$

$\swarrow$ Total power $\downarrow$ AC power $\searrow$ DC power

$$E[x^2(t)] = \sigma_x^2 + \mu_x^2 = \begin{cases} \frac{1}{2\pi} \int_{-\infty}^{0^-} S_{xx}(\omega) d\omega + \underbrace{\frac{1}{2\pi} \int_{0^-}^{0^+} S_{xx}(\omega) d\omega}_{\text{DC power}} + \frac{1}{2\pi} \int_{0^+}^{\infty} S_{xx}(\omega) d\omega \\ \int_{-\infty}^{0^-} S_{xx}(f) df + \underbrace{\int_{0^-}^{0^+} S_{xx}(f) df}_{\text{DC power}} + \int_{0^+}^{\infty} S_{xx}(f) df \end{cases}$$

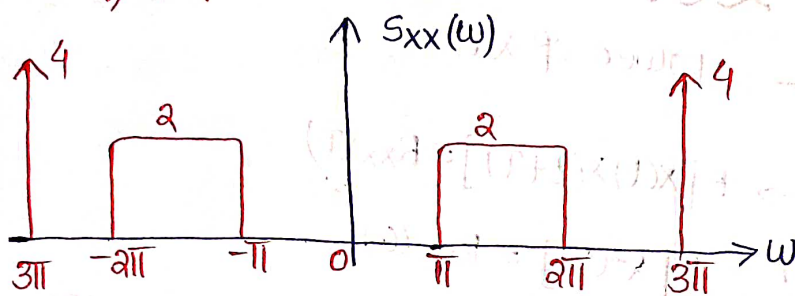
$$\text{DC power } \mu_x^2 = \begin{cases} \frac{1}{2\pi} \int_{0^-}^{0^+} S_{xx}(\omega) d\omega \\ \int_{0^-}^{0^+} S_{xx}(f) df \end{cases} \Rightarrow \text{Non zero only when impulse is present at origin in P.S.D}$$

$$\text{AC power } \sigma_x^2 = \begin{cases} 2\pi \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega \right\} \\ 2\pi \left\{ \int_{-\infty}^{\infty} S_{xx}(f) df \right\} \end{cases}$$

example:

consider the power p.s.d and calculate

i) dc power ii) ac power iii) total power



soln

DC power :-

$$\mu_x^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = 0$$

ac power :-

$$\sigma_x^2 = \frac{2}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$= \frac{2}{2\pi} \left[\int_{-\pi}^{\pi} a d\omega + \int_{-3\pi}^{-3\pi} 4 \delta(\omega) d\omega + \int_{3\pi}^{3\pi} 4 \delta(\omega) d\omega \right]$$

$$= \frac{1}{\pi} [a(a\pi - \pi) + 4]$$

$$= \frac{1}{\pi} [a\pi + 4]$$

$$\sigma_x^2 = a + \frac{4}{\pi}$$

Total power :-

$$E[x^2(t)] = \sigma_x^2 + \mu_x^2 = a + \frac{4}{\pi}$$

Properties of power spectral density :-

- $S_{xx}(f)$ is even function of f $S_{xx}(f) = S_{xx}(-f)$
- If $x(t)$ is real then $S_{xx}(f)$ is also real.
- $S_{xx}(f)$ is non negative $\Rightarrow S_{xx}(f) \geq 0$

#91: consider the following random process $x(t) = u + vt$ where u is zero mean gaussian random variable and v is random variable which is uniformly distributed b/w 0 and 2. u and v are statistically independent. Find the mean value of random process at $t=2$?

soln

$$E[u + vt] = E[u] + E[vt]$$

$$= 0 + t E[v]$$

$$= t \int_{-\infty}^{\infty} v f_v(v) dv$$

$$= t \int_{-\infty}^{\infty} v \left(\frac{1}{2}\right) dv$$

$$= \frac{t}{2} \left. \frac{v^2}{2} \right|_0^2$$

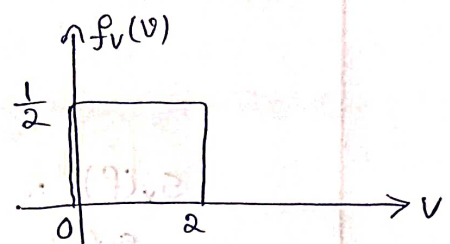
$$= \frac{t}{2} \left\{ \frac{4-0}{2} \right\}$$

$$E[u + vt] = t \Rightarrow \text{mean is depends upon time 't'}$$

$\Rightarrow$ The random process $x(t)$ is not WSSRP.

$$\Rightarrow E[x(t=2)] = t \Big|_{t=2} = 2$$

P.D.F of v :-



#92: consider the following process $x(t) = 3v(t) - 8$ where $v(t)$ is zero mean, ^{stationary} random process and autocorrelation function of $v(t)$ is given by $R_v(\tau) = 4e^{-5|\tau|}$. calculate power in $x(t)$.

soln

$$x^2(t) = 9v^2(t) + 64 - 48v(t)$$

$$E[x^2(t)] = E[9v^2 + 64 - 48v]$$

$$= E[9v^2] + E[64] - E[48v]$$

$$= 9E[v^2] + 64 - 48 \underbrace{E[v]}_{=0}$$

$$= 9(4e^{-5|\tau|}) \Big|_{\tau=0} + 64$$

$$E[x^2(t)] = 36 + 64 = 100 \text{ W}$$

Power = 100 W

#93: Power spectral density of real stationary random process $x(t)$ is given by $S_x(f) = \begin{cases} \frac{1}{W} & |f| \leq W \\ 0 & |f| > W \end{cases}$

calculate: $E[\pi x(t) x(t - \frac{1}{4W})] = ?$

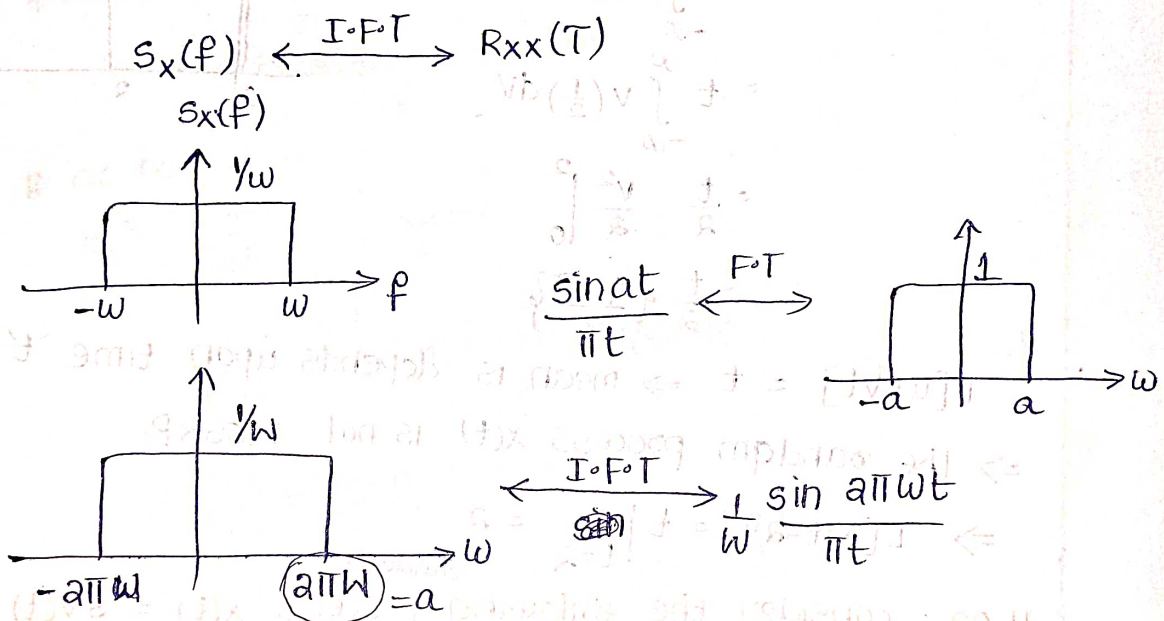
Soln

Given P.S.D $S_x(f) = \begin{cases} \frac{1}{W} & |f| \leq W \\ 0 & |f| > W \end{cases}$

$x(t)$ is real and stationary random process:

$$\Rightarrow E[\pi x(t) x(t - \frac{1}{4W})] = \pi E[x(t) x(t - \frac{1}{4W})]$$

$$= \pi R_{xx}(\frac{1}{4W}) \quad (\because \text{stationary R.P.})$$



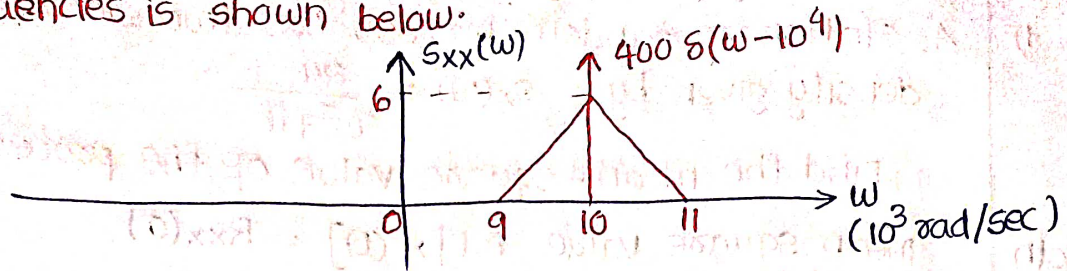
$$\Rightarrow R_{xx}(\tau) = \frac{1}{W} \left\{ \frac{\sin a\pi W \tau}{\pi \tau} \right\}$$

$$\tau = \frac{1}{4W} \quad R_{xx}(\frac{1}{4W}) = \frac{1}{W} \left\{ \frac{\sin a\pi W (\frac{1}{4W})}{\pi \frac{1}{4W}} \right\}$$

$$R_{xx}(\frac{1}{4W}) = \frac{4}{\pi} \sin \frac{\pi}{2} = \frac{4}{\pi}$$

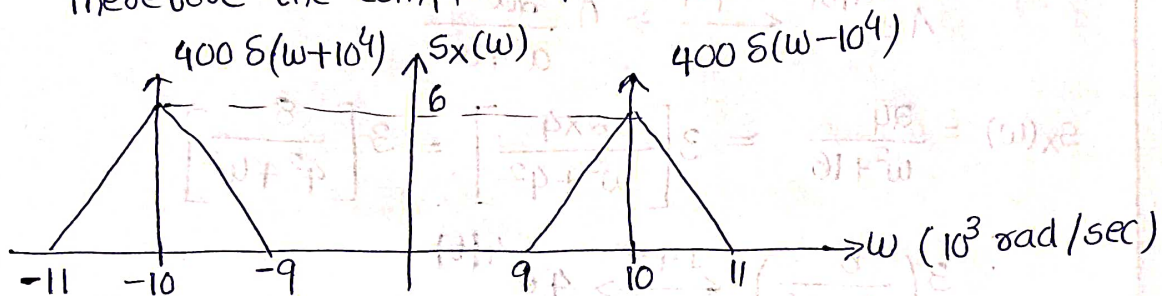
$$E[\pi x(t) x(t - \frac{1}{4W})] = \pi R_{xx}(\frac{1}{4W}) = \pi \times \frac{4}{\pi} = 4$$

#94: The power spectral density of real process for positive frequencies is shown below.



calculate $E[x^2(t)]$ and $|E[x(t)]|$

we knew that $S_x(\omega) = S_x(-\omega)$ even symmetry
Therefore the complete PSD can be drawn as shown below.



$$\mu_x^2 = \frac{1}{2\pi} \int_{0^-}^{0^+} S_x(\omega) d\omega = 0$$

$$\Rightarrow |E[x(t)]| = 0$$

$$E[x^2(t)] = R_{xx}(0) = \sigma_x^2 + \mu_x^2$$

$$= \frac{2\pi}{2\pi} \int_0^\infty S_x(\omega) d\omega$$

$$= \frac{1}{\pi} \int_0^\infty S_x(\omega) d\omega$$

$$= \frac{1}{\pi} \left\{ \text{Area of triangle} + \text{Area of impulse} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{2} \times 2 \times 10^3 \times 6 + 400 \right\}$$

$$= \frac{6000 + 400}{\pi}$$

$$E[x^2(t)] = \frac{6400}{\pi}$$

Assignment problems :-

- 1) A stationary random process has two sided power spectral density given by $S_x(\omega) = \frac{24}{\omega^2 + 16}$

Find the mean-square value of the process.

Soln

mean square value $E[x^2(t)] = R_{xx}(0)$

method 1

We knew that $S_x(\omega)$ and $R_x(\tau)$ forms fourier transform pairs.

$$Ae^{-a|\tau|} \xleftrightarrow{F.T} A \frac{2a}{a^2 + \omega^2}$$

$$S_x(\omega) = \frac{24}{\omega^2 + 16} = 3 \left[\frac{2 \times 4}{\omega^2 + 4^2} \right] = 3 \left[\frac{8}{4^2 + \omega^2} \right]$$

$$3 \left(\frac{8}{4^2 + \omega^2} \right) \xleftrightarrow{I.F.T} 3e^{-4|\tau|}$$

$$\Rightarrow R_{xx}(\tau) = 3e^{-4|\tau|}$$

when $\tau = 0$

$$R_{xx}(0) = 3e^{-4|0|} = 3 \text{ watts}$$

$$E[x^2(t)] = R_{xx}(0) = 3 \text{ watts}$$

method 2:

$$E[x^2(t)] = \sigma_x^2 + \mu_x^2$$

since there is no impulse signal present at $\omega = 0$

$$\mu_x^2 = 0$$

$$\Rightarrow E[x^2(t)] = \sigma_x^2 = \frac{1}{2\pi} \int_0^\infty S_x(\omega) d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \frac{24}{\omega^2 + 16} d\omega = \frac{24}{\pi} \int_0^\infty \frac{d\omega}{\omega^2 + 4^2}$$

$$= \frac{24}{\pi} \left[\frac{1}{4} \tan^{-1} \frac{\omega}{4} \right]_0^\infty$$

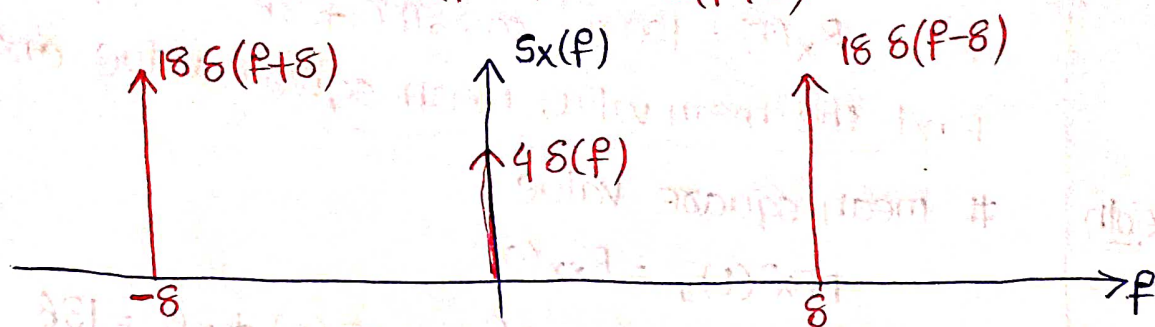
$$= \frac{6}{\pi} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{6}{\pi} \times \frac{\pi}{2} = 3 \text{ watts}$$

- 2) A stationary random process has spectral density given below. Find the autocorrelation function of the process

$$S_x(f) = 4\delta(f) + 18\delta(f-8) + 18\delta(f+8)$$

Soln



$S_x(f)$ and $R_x(\tau)$ form Fourier transforms pairs.

$$\begin{aligned} R_x(\tau) &= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau} df \\ &= \int_{-8^-}^{-8^+} 18\delta(f+8) e^{j2\pi f\tau} df + \int_{0^-}^{0^+} 4\delta(f) e^{j2\pi f\tau} df + \\ &\quad \int_{8^-}^{8^+} 18\delta(f-8) e^{j2\pi f\tau} df \end{aligned}$$

$$\left[\int_{-\infty}^{\infty} x(t) \delta(t-t_0) dt = x(t_0) \right]$$

$$= 18e^{j2\pi(-8)\tau} + 4e^{j2\pi(0)\tau} + 18e^{j2\pi(8)\tau}$$

$$= 4 + 18 \left\{ e^{j2\pi(8)\tau} + e^{-j2\pi(8)\tau} \right\}$$

$$= 4 + 36 \left\{ \frac{e^{j2\pi(8)\tau} + e^{-j2\pi(8)\tau}}{2} \right\}$$

$$= 4 + 36 \left\{ \cos 16\pi\tau \right\}$$

- 7) A stationary random process has an autocorrelation function of the form

$$R_x(\tau) = 100 e^{-\tau^2} \cos 2\pi\tau + 36$$

Find the mean value, mean square value and variance.

Soln

mean square value

$$E[x^2(t)] = R_{xx}(0)$$

$$= 100 e^{-0} \cos 2\pi(0) + 36 = 136$$

mean:

$$\mu_x = \left\{ \lim_{T \rightarrow \infty} R_{xx}(T) \right\}^{1/2}$$

$$= \left\{ \lim_{T \rightarrow \infty} \left\{ 100 e^{-T^2} \cos 2\pi T + 36 \right\} \right\}^{1/2}$$

$$\mu_x = \pm 6$$

variance:

$$\sigma_x^2 = E[x^2(t)] - \mu_x^2$$

$$= 136 - (6)^2$$

$$\sigma_x^2 = 100$$

- 8) A stationary random process has a spectral density of
- $$S_x(\omega) = \frac{1}{\omega^4 + 5\omega^2 + 4}$$

Find the autocorrelation function and mean square value of the process.

Soln

$S_x(\omega)$ and $R_{xx}(\tau)$ forms fourier transform pairs.

$$\Rightarrow S_x(\omega) = \frac{1}{\omega^4 + 5\omega^2 + 4}$$

$$= \frac{1}{\omega^4 + 4\omega^2 + \omega^2 + 4}$$

$$= \frac{1}{\omega^2(\omega^2 + 4) + 1(\omega^2 + 4)}$$

$$S_x(\omega) = \frac{1}{(\omega^2 + 4)(\omega^2 + 1)}$$

Let $\omega^2 = \lambda$

$$S_X(\omega) = \frac{1}{(\lambda+1)(\lambda+4)} = \frac{A}{\lambda+1} + \frac{B}{\lambda+4} \quad \left\{ \begin{array}{l} \text{simple and} \\ \text{unrepeated} \\ \text{poles} \end{array} \right\}$$

$$A = (\lambda+4) \left(\frac{1}{(\lambda+1)(\lambda+4)} \right)_{\lambda=-1} = \frac{1}{3}$$

$$B = (\lambda+1) \left(\frac{1}{(\lambda+1)(\lambda+4)} \right)_{\lambda=-3} = -\frac{1}{3}$$

Back substituting $\lambda = \omega^2$

$$S_{X*}(\omega) = \frac{1}{3(\omega^2+1)} - \frac{1}{3(\omega^2+4)}$$

$$S_X(\omega) = \frac{1}{3 \times 2} \left[\frac{2}{\omega^2+1^2} \right] - \frac{1}{3 \times 4} \left[\frac{4}{\omega^2+2^2} \right] = \frac{1}{6} \left(\frac{2}{\omega^2+1} \right) - \frac{1}{12} \left(\frac{4}{\omega^2+2^2} \right)$$

Now by taking Fourier transform

$$R_{XX}(\tau) = \frac{1}{6} e^{-|\tau|} - \frac{1}{12} e^{-2|\tau|}$$

mean square value $E[X^2(t)] = R_{XX}(0)$

$$E[X^2(t)] = \frac{1}{6} e^{-0} - \frac{1}{12} e^{-0} = \frac{1}{6} - \frac{1}{12} = \frac{2-1}{12} = \frac{1}{12} \text{ Watts}$$

- 9) Find the power in the frequency band of 100 kHz centered at 0.5 MHz for the random process having two sided power spectral density given below.
(consider both positive & negative frequencies for power calculation)

soln

$$\text{power} = E[X^2(t)]$$

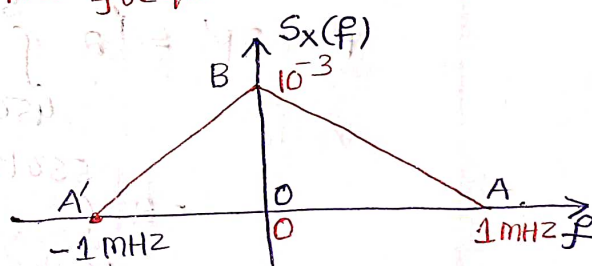
$$= \int_{-\infty}^{\infty} S_X(f) df$$

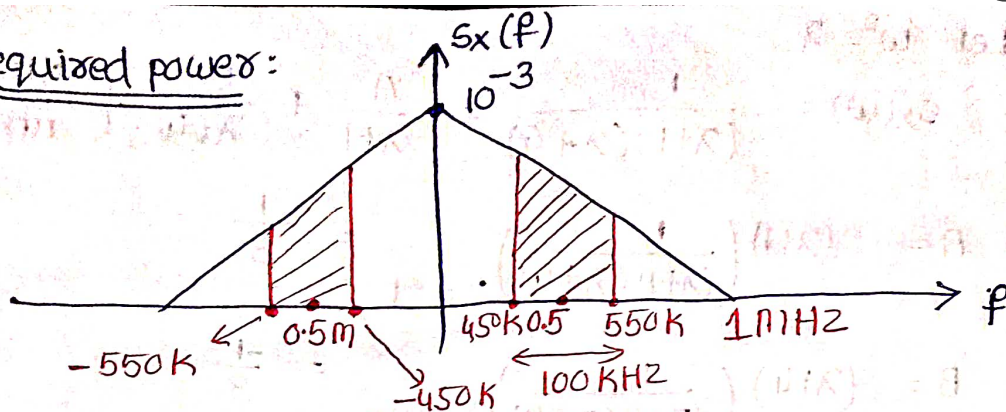
$$= \int_{-\infty}^{0^-} S_X(f) df + \underbrace{\int_{0^-}^{0^+} S_X(f) df}_{=0} + \int_{0^+}^{\infty} S_X(f) df$$

$$\text{Power} = \int_{-\infty}^{0^-} S_X(f) df + \int_{0^+}^{\infty} S_X(f) df \quad \left(\begin{array}{l} \text{No impulse type} \\ \text{function present at} \\ 0 \end{array} \right)$$

$$= \frac{1}{2} \times 10^6 \times 10^{-3} + \frac{1}{2} \times 10^6 \times 10^{-3} = \frac{10^3 + 10^3}{2} = 10^3 \text{ W}$$

$$\text{power} = 1000 \text{ W}$$



Required power:

slope of straight line on right side:

$$(0, 10^{-3}) \quad (10^6, 0)$$

$$\Delta = \frac{0 - 10^{-3}}{10^6 - 0} = -10^{-9}$$

straight line eqⁿ:

$$S_x(f) = 10^{-3} = -10^{-9}(f - 0)$$

$$S_x(f) = -10^{-9}f + 10^{-3}$$

Now power under 100 kHz band width:-
on one side:

$$P' = \int_{450\text{ k}}^{550\text{ k}} S_x(f) df$$

Now total power on both sides:-

$$P = 2P' = 2 \int_{450\text{ k}}^{550\text{ k}} S_x(f) df$$

$$= 2 \int_{450\text{ k}}^{550\text{ k}} (-10^{-9}f + 10^{-3}) df$$

$$= 2 \left\{ \int_{450\text{ k}}^{550\text{ k}} -10^{-9}f df + \int_{450\text{ k}}^{550\text{ k}} 10^{-3} df \right\}$$

$$= 2 \left\{ -10^{-9} \frac{f^2}{2} + 10^{-3}f \right\}_{450\text{ k}}^{550\text{ k}}$$

$$= 2 \left\{ \frac{-10^{-9}}{2} \{ 550^2 \times 10^6 - 450^2 \times 10^6 \} + 10^{-3} (550\text{ k} - 450\text{ k}) \right\}$$

$$= 2 \left\{ -\frac{10^{-9}}{2} \times 10^6 (302500 - 202500) + 10^{-3} 10^3 (550 - 450) \right\}$$

$$= 2 \left\{ -\frac{10^{-3}}{2} (10^5) + (100) \right\}$$

$$= 2 \left\{ -\frac{10^2}{2} + 100 \right\}$$

$$= 2 \{ 50 \}$$

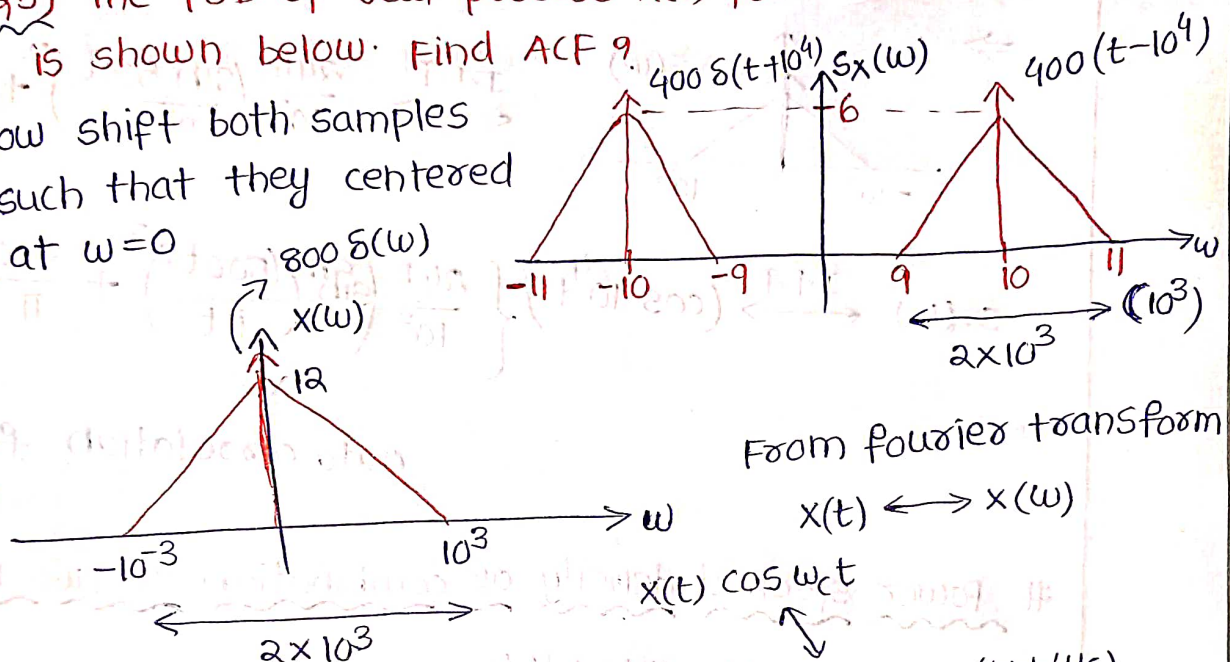
$$P = 100 \text{ W}$$

→ Assignment problems will continue after next topic

#95) The PSD of real process $x(t)$ for frequencies is shown below. Find ACF?

soln

Now shift both samples such that they centered at $\omega = 0$



From Fourier transform

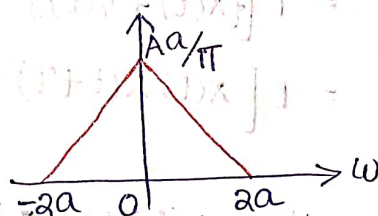
$$x(t) \longleftrightarrow x(\omega)$$

$$x(t) \cos \omega_c t$$

$$\frac{x(\omega - \omega_c) + x(\omega + \omega_c)}{2}$$

$$\text{Now } \omega_c = 10 \times 10^3 \text{ rad/sec}$$

$$A \left\{ \frac{\sin at}{\pi t} \right\}^2 \longleftrightarrow$$



$$2a = 10^3$$

$$a = \frac{10^3}{2}$$

$$\frac{Aa}{\pi} = 12$$

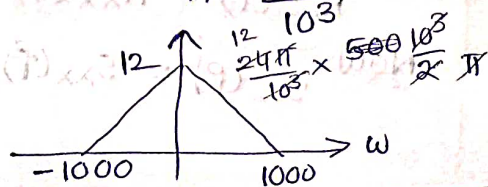
$$\frac{Aa}{\pi} = 12$$

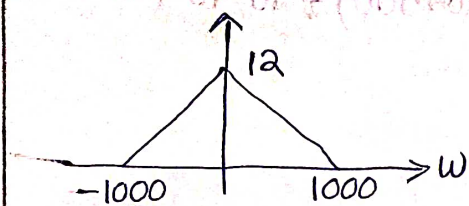
$$\frac{Aa}{\pi} = \frac{A 10^3}{2\pi} = 12$$

⇒

$$A = \frac{24\pi}{10^3}$$

$$\frac{24\pi}{10^3} \left(\frac{\sin 500t}{\pi t} \right)^2 \longleftrightarrow$$





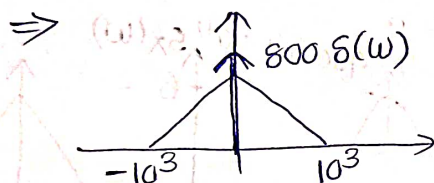
$$\xleftrightarrow{\text{IFT}} \frac{2\pi}{10^3} \times 12 \left(\frac{\sin\left(\frac{10^3}{2}t\right)}{\pi t} \right)^2$$

we knew that

$$1 \longleftrightarrow 2\pi \delta(\omega)$$

$$400 \longleftrightarrow 800\pi \delta(\omega)$$

$$\frac{400}{\pi} \longleftrightarrow 800 \delta(\omega)$$



$$\xleftrightarrow{\text{I.F.T}} \frac{24\pi}{10^3} \left(\frac{\sin 500t}{\pi t} \right)^2 + \frac{400}{\pi}$$

$$s_x(\omega) \xleftrightarrow{\text{I.F.T}} (\cos 10^4 t) \left\{ \frac{24\pi}{10^3} \left(\frac{\sin(500t)}{\pi t} \right)^2 + \frac{400}{\pi} \right\}$$

Auto correlation function
 $R_{xx}(T)$

Power spectral density of combination of two WSSRP:-

case ii) $z(t) = x(t) + y(t)$

Auto correlation function $R_{zz}(T) = E[z(t)z(t+T)]$

$$= E[(x(t) + y(t))\{x(t+T) + y(t+T)\}]$$

$$= E[x(t)x(t+T) + x(t)y(t+T) + y(t)x(t+T) + y(t)y(t+T)]$$

$$R_{zz}(T) = E[x(t)x(t+T)] + E[x(t)y(t+T)] + E[y(t)x(t+T)] + E[y(t)y(t+T)]$$

$$R_{zz}(T) = R_{xx}(T) + R_{xy}(T) + R_{yx}(T) + R_{yy}(T)$$

$$R_{zz}(T) = R_{xx}(T) + R_{yy}(T) + R_{xy}(T) + R_{yx}(T)$$

NOW $S_{zz}(f) = S_{xx}(f) + S_{yy}(f) + S_{xy}(f) + S_{yx}(f)$

$\Rightarrow x(t)$ and $y(t)$ are orthogonal.

orthogonal implies $R_{xy} = E[xy] = 0$

Therefore $E[x(t)y(t+T)] = R_{xy}(T) = 0$ and

$$E[y(t)x(t+T)] = R_{yx}(T) = 0$$

$$\text{So } R_{zz}(T) = R_{xx}(T) + R_{yy}(T)$$

$$\Rightarrow S_{zz}(f) = S_{xx}(f) + S_{yy}(f)$$

$\Rightarrow x(t)$ and $y(t)$ are uncorrelated and at least one of them has zero mean.

$$\text{cov}(xy) = 0 \Rightarrow \text{uncorrelated}$$

$$\text{Therefore } \text{cov}(x(t)y(t+T)) = E[x(t)y(t+T)] - E[x(t)]E[y(t+T)] = 0$$

$$\Rightarrow E[x(t)y(t+T)] - \underbrace{E[x(t)]}_0 \text{ OR } \underbrace{E[y(t+T)]}_0 = 0$$

$$\Rightarrow R_{xy}(T) = 0$$

$$\Rightarrow R_{zz}(T) = R_{xx}(T) + R_{yy}(T)$$

$$\Rightarrow S_{zz}(f) = S_{xx}(f) + S_{yy}(f)$$

$\Rightarrow x(t)$ and $y(t)$ are independent and at least one of them has zero mean :-

$$\text{cov}(x(t)y(t+T)) = R_{xy}(T) - \underbrace{\mu_x \mu_y}_{=0}$$

Independent $\Rightarrow$ uncorrelated.

$$\Rightarrow R_{xy}(T) = 0$$

$$R_{zz}(T) = R_{xx}(T) + R_{yy}(T)$$

$$S_{zz}(f) = S_{xx}(f) + S_{yy}(f)$$

Cross correlation of a WSSRP :-

$$\text{i) } E[x(t)y(t+T)] = R_{xy}(T)$$

$$\text{ii) } E[x(t)y(t-T)] = R_{xy}(-T)$$

$$\text{iii) } E[x(t+T)y(t)] = R_{xy}(-T)$$

$$\text{iv) } E[y(t+T)x(t)] = R_{yx}(-T)$$

$$R_{xy}(T) = R_{yx}(-T)$$

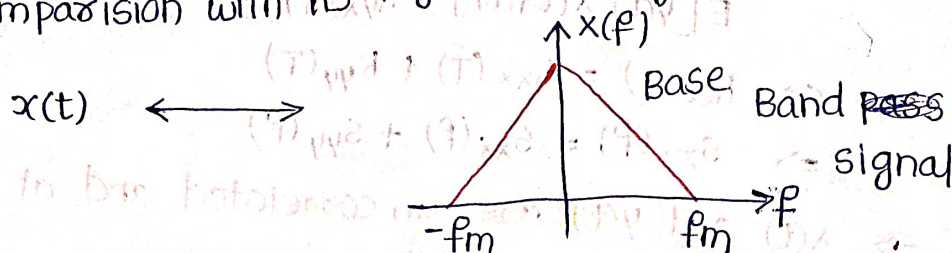
$$R_{xy}(T) \neq R_{yx}(T)$$

$$R_{xy}(T) \leq \sqrt{R_{xx}(0)R_{yy}(0)}$$

$$|R_{xy}(T)| \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$$

Base band signals :-

A base band signal or low pass signal is a signal that can include frequencies that are very near by zero, by comparison with its high frequency.

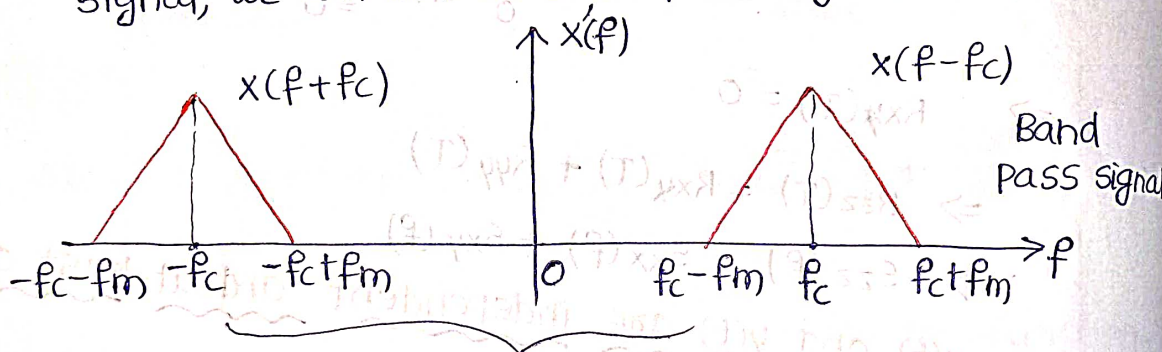


Voice signal : 300 Hz - 3.5 kHz

Audio signal : 20 Hz - 20 kHz

video signal : 0 Hz - 4.5 MHz

When we multiply base band signal by high frequency signal, we can obtain band pass signal.



$$x(t) \cos 2\pi f_c t \longleftrightarrow x'(f)$$

Taking product of $x(f-f_c)$ and $x(f+f_c)$

$$\Rightarrow x(f-f_c) x(f+f_c) = 0$$

Bandpass density functions :-

$\Rightarrow$ Bandpass energy density functions :-

consider $x(t)$ is a base band random process.

$$x(t) \longleftrightarrow x(f) \Rightarrow |x(f)|^2 = G_{xx}(f) = E \cdot S \cdot D$$

Base band

Let us make new random process

$$y(t) = x(t) \cos 2\pi f_c t$$

Band pass random process.

NOW

$$y(t) = x(t) A \cos \omega_c t \xleftrightarrow{F.T} Y(f) = \frac{A}{2} \{ X(f-f_c) + X(f+f_c) \}$$

$$\text{NOW } G_Y(f) = |Y(f)|^2$$

$$G_Y(f) = |Y(f)|^2$$

$$= \left| \frac{A}{2} \{ X(f-f_c) + X(f+f_c) \} \right|^2$$

$$|A+B|^2 = |A|^2 + |B|^2 + A^*B + AB^*$$

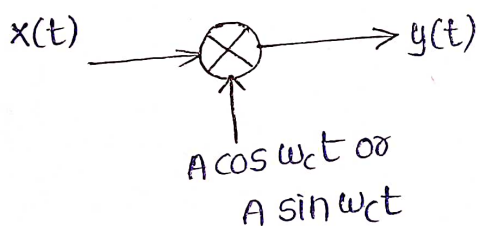
$$= \frac{A^2}{4} |X(f-f_c)|^2 + \frac{A^2}{4} |X(f+f_c)|^2 + \underbrace{\frac{A^2}{4} X(f-f_c) X^*(f+f_c)}_{=0} + \underbrace{\frac{A^2}{4} X^*(f-f_c) X(f+f_c)}_{=0}$$

$$G_Y(f) = |Y(f)|^2 = \frac{A^2}{4} \{ |X(f-f_c)|^2 + |X(f+f_c)|^2 \}$$

$$= \frac{A^2}{4} \{ G_X(f-f_c) + G_X(f+f_c) \}$$

$$G_Y(f) = \frac{A^2}{4} \{ G_X(f-f_c) + G_X(f+f_c) \}$$

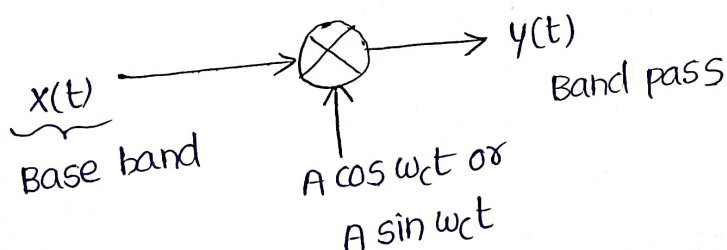
Important points :-



$$x(t) \xrightarrow{\text{ESD}} G_X(f)$$

$$y(t) \longrightarrow \frac{A^2}{4} [G_X(f-f_c) + G_X(f+f_c)]$$

$\Rightarrow$ Band pass power spectral density functions :-



$$x(t) \xrightarrow{\text{P.S.D}} S_X(f)$$

$$y(t) \xrightarrow{\text{P.S.D}} \frac{A^2}{4} \{ S_X(f-f_c) + S_X(f+f_c) \}$$

Mixing a WSSRP with a sinusoidal signal $A \cos(\omega_c t + \theta)$

consider a random process $x(t)$ is wide sense stationary random process and let assume $c(t) = A \cos(\omega_c t + \theta)$ where θ is a random variable which is uniformly distributed b/w 0 and 2π . $x(t)$ and θ are independent.

Random process Random variables

$\Rightarrow$ All random variables obtained from $x(t)$ are individually independent from θ .

$$x(t) \rightarrow \text{Multiplier} \rightarrow y(t) = A x(t) \cos(\omega_c t + \theta)$$

$$A \cos(\omega_c t + \theta)$$

$$y(t) = A x(t) \cos(\omega_c t + \theta)$$

$$E[x(t)x(t+T)] = R_{xx}(T) \xleftrightarrow{F.T} S_{xx}(f)$$

Now

$$E[y(t)y(t+T)] = E[A x(t) \cos(\omega_c t + \theta) A x(t+T) \cos(\omega_c (t+T) + \theta)]$$

$$= E[A^2 x(t) x(t+T) \cos(\omega_c t + \theta) \cos(\omega_c t + \omega_c T + \theta)]$$

$$= A^2 E[x(t) x(t+T) \cos(\omega_c t + \theta) \cos(\omega_c t + \omega_c T + \theta)]$$

$$= A^2 E[x(t) x(t+T)] E[\cos(\omega_c t + \theta) \cos(\omega_c t + \omega_c T + \theta)]$$

$$= A^2 R_{xx}(T)$$

$$E\left[\frac{1}{2} \cos(2\theta + 2\omega_c t + \omega_c T) + \frac{1}{2} \cos(\omega_c T)\right]$$

expectation = 0 deterministic value

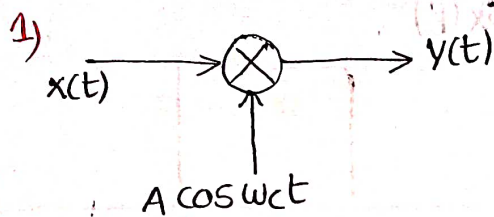
$$E[y(t)y(t+T)] = \frac{A^2}{2} R_{xx}(T) \cos(\omega_c T)$$

$$\Rightarrow R_{yy}(T) = \frac{A^2}{2} R_{xx}(T) \cos(\omega_c T)$$

$\Rightarrow$ By Fourier transform

$$S_{yy}(f) = \frac{A^2}{4} \{ S_{xx}(f - f_c) + S_{xx}(f + f_c) \}$$

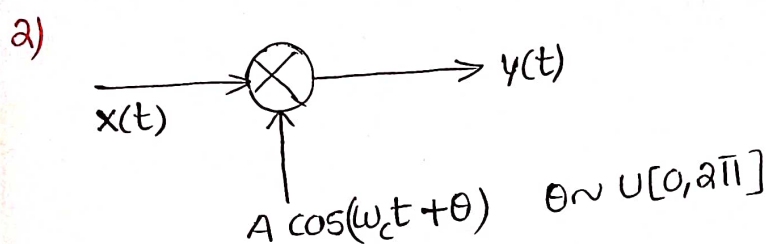
summary: $-0.1 + (0.0004 + 9) \cdot 0.01 - (0.0004 + 9) \cdot 0.01 = (9)_{10}$



(i) Autocorrelation function $R_y(T) = \frac{A^2}{2} R_x(T) \cos \omega_c T$

(ii) E.S.D: $G_y(f) = \frac{A^2}{4} \{ G_x(f-f_c) + G_x(f+f_c) \}$

(iii) P.S.D: $S_y(f) = \frac{A^2}{4} \{ S_x(f-f_c) + S_x(f+f_c) \}$



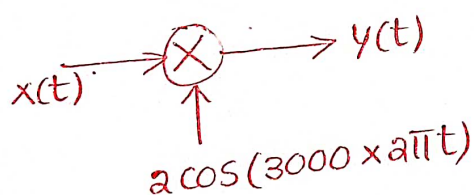
(i) Autocorrelation function: $R_y(T) = \frac{A^2}{2} R_{xx}(T) \cos \omega_c T$

(ii) P.S.D: $\frac{A^2}{4} \{ S_x(f-f_c) + S_x(f+f_c) \}$

#96) Power spectral density of WSSRP $x(t)$ is given as

$S_x(f) = 10 [u(f+1000) - u(f-1000)]$

Determine $S_y(f)$



$A=2 \quad f_c = 3 \text{ KHz}$

$S_y(f) = \frac{A^2}{4} \{ S_x(f-f_c) + S_x(f+f_c) \}$

$= \frac{2^2}{4} \{ S_x(f-3000) + S_x(f+3000) \}$

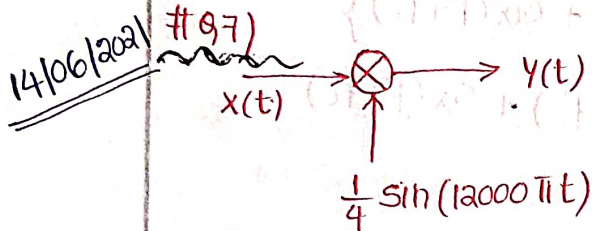
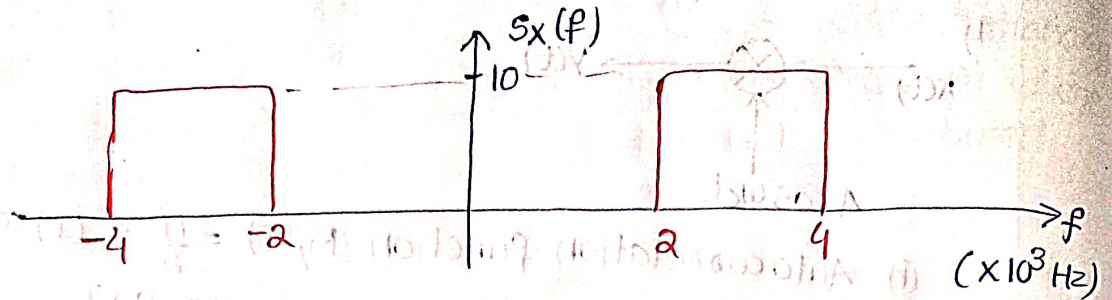
$= 10 [u(f+1000-3000) - u(f-1000-3000)] + 10$

$[u(f+1000+3000) - u(f-1000+3000)]$

$= 10 u(f-2000) - 10 u(f-4000) + 10 u(f+4000) - 10 u(f+2000)$

Soln

$$S_y(f) = 10 u(f+4000) - 10 u(f+2000) + 10 u(f-2000) - 10 u(f-4000)$$



$x(t)$ is WSSRP having autocorrelation function

$$R_{xx}(\tau) = \frac{1}{3} e^{-a|\tau|}$$

Soln

$$\frac{1}{4} \sin(2\pi \times 6000 t)$$

$$A = \frac{1}{4}$$

$$f_c = 6000 \text{ Hz}$$

$$\omega_c = 12000\pi \text{ rad/sec}$$

$$R_{xx}(\tau) = \frac{1}{3} e^{-a|\tau|}$$

$$R_{yy}(\tau) = \frac{A^2}{2} R_{xx}(\tau) \cos \omega_c \tau = \frac{1}{32} \left(\frac{1}{3} e^{-a|\tau|} \right) (\cos 12000\pi \tau)$$

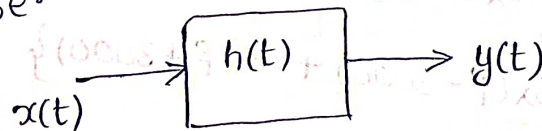
$$\text{Power of } y(t) = E[y^2(t)] = R_{yy}(0)$$

$$= \left(\frac{1}{32} \right) \left(\frac{1}{3} \right) (e^{-a|0|}) (\cos 12000\pi(0))$$

$$= \frac{1}{96}$$

Transmission of WSSRP through an LTI system :-

Any LTI system can be represented by unit impulse response.

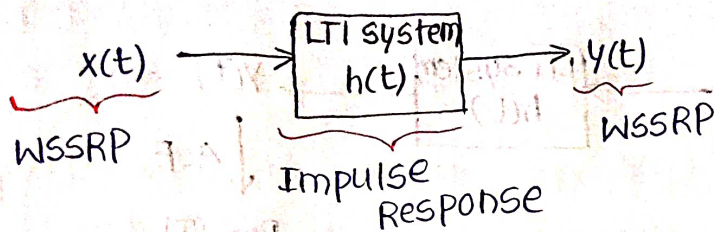


$$\Rightarrow y(t) = x(t) * h(t)$$

Transfer function of LTI system : $H(s)$

$$\frac{Y(s)}{X(s)} = H(s)$$

Here $h(t)$ is deterministic function of time.



$$1) y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

since $x(t)$ is WSSRP $\left\{ \begin{array}{l} \text{mean is constant} = \mu_x \\ \text{Autocorrelation function is function of } \tau = R_{xx}(\tau) \end{array} \right.$

let us find mean of $y(t)$

$$E[y(t)] = E\left[\int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau\right]$$

$$= \int_{-\infty}^{\infty} \left[E[x(t-\tau)] h(\tau) d\tau \right]$$

$$= \int_{-\infty}^{\infty} \mu_x h(\tau) d\tau$$

$$E[y(t)] = \mu_x \int_{-\infty}^{\infty} h(\tau) d\tau$$

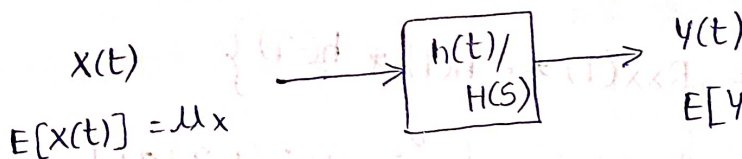
from laplace transform

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

when $s=0$

$$H(0) = \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$\Rightarrow E[y(t)] = \mu_x H(0)$$

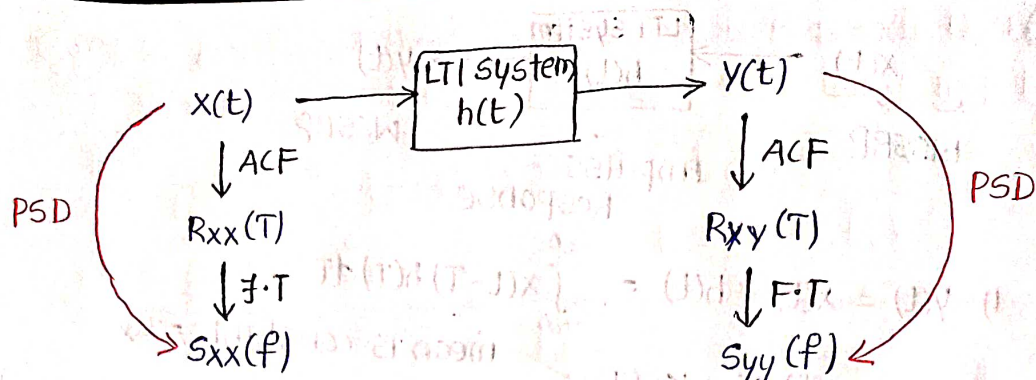


$$E[y(t)] = \mu_x \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$E[y(t)] = \mu_x H(s) \big|_{s=0}$$

$$E[y(t)] = \underbrace{\mu_x H(0)}_{\text{constant}}$$

When zero mean random process is provided to LTI system then it results a zero mean random process.



1) If $x(t)$ is WSSRP then $y(t)$ will as be WSSRP

$$2) \mu_y = H(0) \mu_x = \mu_x \int_{-\infty}^{\infty} h(T) dT$$

3) Autocorrelation function :-

$$\star R_{yy}(T) = R_{xx}(T) \star h(T) \star h(-T) = S$$

$$\star \Rightarrow S_{yy}(T) = S_{xx}(f) H(f) H(-f)$$

4) cross correlation function :-

$$\star R_{xy}(T) = R_{xx}(T) \star h(T)$$

$$\star \Rightarrow S_{xy}(T) = S_{xx}(f) H(f)$$

$$\star R_{yx}(T) = R_{xx}(T) \star h(-T)$$

$$\star \Rightarrow S_{yx}(T) = S_{xx}(f) H(-f)$$

$$\text{Take } R_{yy}(T) = R_{xx}(T) \star h(T) \star h(-T)$$

$$R_{yy}(T) = R_{xx}(T) \star \{ h(T) \star h(-T) \}$$

$$R_{yy}(T) = R_{xx}(T) \star \left\{ \int_{-\infty}^{\infty} h(\alpha) h(\alpha - T) d\alpha \right\}$$

$$S_{yy}(f) = S_{xx}(f) H(f) H(-f)$$

Let $h(t)$ is real function

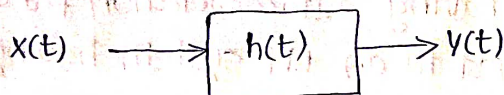
$$h(t) = h^*(t)$$

$$H(f) = H^*(-f)$$

$$H(-f) = H^*(f)$$

$$\text{Therefore } S_{yy}(f) = S_{xx}(f) H(f) H^*(f)$$

$$S_{yy}(f) = S_{xx}(f) |H(f)|^2$$



$$1) E[x(t)] = \mu_x$$

$$2) E[x(t)x(t+T)] = R_{xx}(T)$$

$$3) S_{xx}(f)$$

$$4) E[x^2(t)] = R_{xx}(0)$$

$$E[x^2(t)] = \int_{-\infty}^{\infty} S_{xx}(f) df$$

$$1) E[y(t)] = \mu_x H(0) = \mu_x \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$2) E[y(t)y(t+T)] = R_{yy}(T)$$

$$R_{yy}(T) = R_{xx}(T) * \int_{-\infty}^{\infty} h(\alpha) h(\alpha-T) d\alpha$$

$$3) S_{yy}(f) = S_{xx}(f) |H(f)|^2$$

$$4) E[y^2(t)] = R_{yy}(0)$$

$$= \left\{ R_{xx}(T) * \int_{-\infty}^{\infty} h(\alpha) h(\alpha-T) d\alpha \right\}_{T=0}$$

$$E[y^2(t)] = \int_{-\infty}^{\infty} S_{yy}(f) df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 S_{xx}(f) df$$

Assignment problems continuation:

- 3) A random process $x(t) = m$ where m is a random variable with mean value a and variance b , is applied to an LTI system with the following impulse response

$$h(t) = 5e^{-3t} u(t)$$

Find the mean value of output process.

soln



$$\mu_x = 2$$

$$\sigma_x^2 = 6$$

$$\mu_y = ?$$

$$E[y(t)] = \mu_y = \mu_x \int_{-\infty}^{\infty} h(\tau) d\tau$$

$$= 2 \int_{-\infty}^{\infty} 5e^{-3\tau} u(\tau) d\tau$$

$$= 2 \int_0^{\infty} 5e^{-3\tau} d\tau = \frac{10e}{-3} \Big|_0^{\infty}$$

$$E[y(t)] = \frac{-10}{3} \{0 - 1\} = \frac{10}{3}$$

Another method:- $h(t) = 5e^{-3t} u(t) \xleftrightarrow{L} H(s) = \frac{5}{s+3}$

$$H(0) = \frac{5}{3}$$

$$\mu_y = \mu_x H(0) = 2 \times \frac{5}{3} = \frac{10}{3}$$

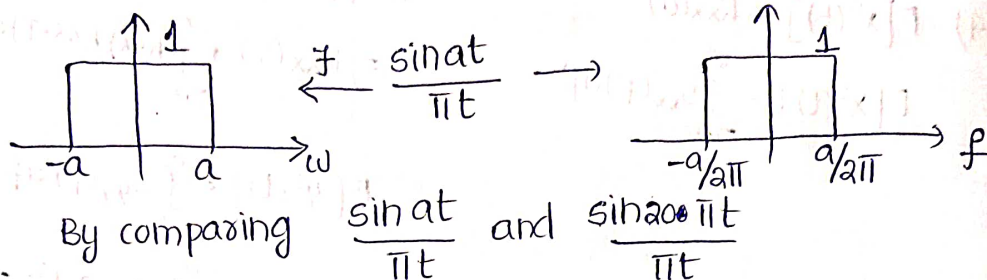
- 5) A random process having autocorrelation function $R_x(t) = \frac{\sin(20\pi t)}{\pi t}$ is applied to an LTI system having impulse response $h(t) = e^{-t}u(t)$. Find the mean square value of output response $y(t)$.

Soln

mean square value of $y(t)$ = power. $\approx E[y^2(t)] = ?$

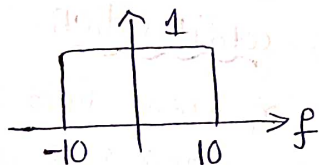
$$E[y^2(t)] = \int_{-\infty}^{\infty} S_{yy}(f) df$$

since $h(t)$ is real then $S_{yy}(f) = S_{xx}(f) |H(f)|^2$



we get $a = 20\pi$

$$\Rightarrow S_{xx}(f) =$$



$$S_{xx}(f) = \begin{cases} 1 & |f| \leq 10 \\ 0 & |f| > 10 \end{cases}$$

$$\frac{20\pi}{2\pi} = 10$$

$$h(t) = e^{-t}u(t)$$

$$H(\omega) = \frac{1}{1+j\omega} \quad H(f) = \frac{1}{1+j2\pi f}$$

$$|H(f)|^2 = \frac{1}{\sqrt{1+(2\pi f)^2}} = \frac{1}{1+4\pi^2 f^2}$$

Now

$$S_{yy}(f) = \frac{1}{1+4\pi^2 f^2} \quad \text{when } |f| \leq 10$$

Now

$$E[y^2(t)] = \int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2} df$$

$$= \int_{-10}^{10} \frac{1}{4\pi^2 \left(\frac{1}{4\pi^2} + f^2 \right)} df$$

$$= \frac{1}{4\pi^2} \int_{-10}^{10} \frac{1}{\left(\frac{1}{4\pi^2} \right)^2 + f^2} df$$

$$= \frac{1}{\pi^2} \tan^{-1} \left(\frac{f}{\frac{1}{2\pi}} \right) \Big|_{-10}^{10}$$

$$= \frac{2\pi}{4\pi^2} \tan^{-1}(2\pi f) \Big|_{-10}^{10}$$

$$= \frac{1}{2\pi} \left\{ \tan^{-1}(20\pi) - \tan^{-1}(-20\pi) \right\} \quad (\text{in radians})$$

$$E[y^2(t)] = 6.494 \text{ Watts}$$

- 6) A stationary random process $x(t)$ having autocorrelation function $R_x(T) = 5e^{-5|T|}$. Find the autocorrelation function of another random process $y(t)$, given as

$$y(t) = x(t) - 2x(t-1).$$

Also find the average power in $y(t)$.

soln

Autocorrelation function of y - $R_{yy}(T) = ?$

$$E[y(t)y(t+T)] = E\left[\{x(t) - 2x(t-1)\} \{x(t+T) - 2x(t+T-1)\}\right]$$

$$= E\left[\{x(t)x(t+T) - 2x(t)x(t+T-1) - 2x(t-1)x(t+T) + 4x(t-1)x(t+T-1)\}\right]$$

$$= E[x(t)x(t+T)] - 2E[x(t)x(t+T-1)] - 2E[x(t-1)x(t+T)] + 4E[x(t-1)x(t+T-1)]$$

$$= R_{xx}(T) - 2R_{xx}(T-1) - 2R_{xx}(T+1) + 4R_{xx}(T)$$

$$R_{yy}(T) = 5R_{xx}(T) - 2R_{xx}(T-1) - 2R_{xx}(T+1)$$

$$\text{Power of } y(t) = E[y^2(t)] = R_{yy}(0)$$

when $T=0$

$$R_{yy}(0) = 5R_{xx}(0) - 2R_{xx}(-1) - 2R_{xx}(1)$$

$$R_{xx}(0) = 5e^{-5(0)} = 5$$

$$R_{xx}(-1) = 5e^{-5|-1|} = 5e^{-5}$$

$$R_{xx}(1) = 5e^{-5|1|} = 5e^{-5}$$

$$\text{Power of } y(t) = 5(5) - 2(5e^{-5}) - 2(5e^{-5})$$

$$= 25 - 10e^{-5} - 10e^{-5}$$

$$= 25 - 20e^{-5}$$

$$E[y^2(t)] = R_{yy}(0)$$

- 4) Find the autocorrelation function for following power spectral density $S_x(\omega) = \frac{\omega^2}{1+\omega^2}$

Soln

We knew that $R_x(\tau) \xleftrightarrow{F.T} S_x(\omega)$

$$S_x(\omega) = \frac{\omega^2}{1+\omega^2}$$

Long division method:

$$(1+\omega^2) \omega^2 \quad ($$

$$\Rightarrow \omega^2 + 1) \omega^2 \quad (1$$

$$\Rightarrow \frac{\omega^2}{1+\omega^2} = 1 - \frac{1}{1+\omega^2}$$

$$\delta(t) \xleftrightarrow{F.T} 1$$

$$e^{-a|t|} \xleftrightarrow{F.T} \frac{2a}{a^2 + \omega^2}$$

$$1 - \frac{1}{1+\omega^2} = 1 - \frac{2}{2(1+\omega^2)} \xleftrightarrow{F.T} \delta(\tau) - \frac{1}{2} e^{-|\tau|}$$

$$R_x(\tau) = \delta(\tau) - \frac{1}{2} e^{-|\tau|}$$

- 11) A white noise having autocorrelation function $R_x(\tau) = \delta(\tau)$ is applied to an LTI system with impulse response of $h(t) = e^{-at} u(t)$.

Find the average power of the random process at the system output.

Soln

$$E[y^2(t)] = \int_{-\infty}^{\infty} R_{yy}(0) = \text{output power} = \int_{-\infty}^{\infty} S_{yy}(f) df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$$

$$S_{yy}(\omega) = ?$$

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2 \quad (\text{since } h(t) \text{ is real})$$

$$h(t) = e^{-at} u(t) \xleftrightarrow{F.T} H(\omega) = \frac{1}{a + j\omega}$$

$$|H(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{4+\omega^2}$$

$$R_{xx}(T) = 10\delta(T)$$

$$S_{xx}(\omega) = 10$$

so now $S_x(f) = 10$

$$|H(\omega)|^2 = \frac{1}{\omega^2+2^2} = \frac{4}{4(\omega^2+2^2)} = \frac{2(2)}{4(\omega^2+2^2)}$$

Now

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

$$= 10 \left(\frac{4}{4(\omega^2+4)} \right)$$

$$S_{yy}(\omega) = \frac{5}{2} \left(\frac{4}{\omega^2+4} \right)$$

$\updownarrow$

$$R_{yy}(T) = \frac{5}{2} e^{-2|T|}$$

$$E[y^2(t)] = R_{yy}(T) \Big|_{T=0} = \frac{5}{2} e^{-2|T|} \Big|_{T=0} = \frac{5}{2} = 2.5 \text{ watts}$$

→ Assignment problems will continue after next topic:-

#98 consider the following LTI system



Given that

i) $x(t)$ is WSSRP

ii) $E[x(t)x(t+T)] = e^{-5|T|}$

iii) $h(t) = e^{-at} u(t)$

obtain a) nature of $y(t)$

b) P.S.D of $y(t)$

Soln

When an WSSRP pass through LTI system, it always results in a WSSRP

∴ $y(t)$ is WSSRP

$$S_{yy}(f) = S_{xx}(f) |H(f)|^2 \quad (\text{since } h(t) \text{ is real function of time})$$

$$h(t) = e^{-at} u(t) \quad H(\omega) = \frac{1}{a + j\omega} \quad |H(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$|H(\omega)| = \frac{1}{4 + \omega^2}$$

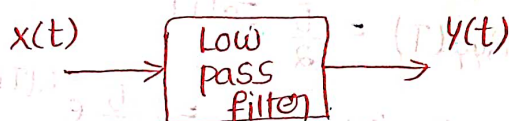
$$R_X(f) = e^{-5|f|} \xleftrightarrow{F.T} S_X(\omega) = \frac{10}{\omega^2 + 25}$$

$$S_Y(\omega) = S_X(\omega) |H(\omega)|^2$$

$$= \left(\frac{10}{\omega^2 + 25} \right) \left(\frac{1}{4 + \omega^2} \right)$$

$$S_Y(\omega) = \frac{10}{(\omega^2 + 25)(\omega^2 + 4)}$$

#99: consider an LTI system.



Given that i) $x(t)$ is WSSRP

ii) Low pass filter is ideal with cut off frequency f_c with pass band gain 2

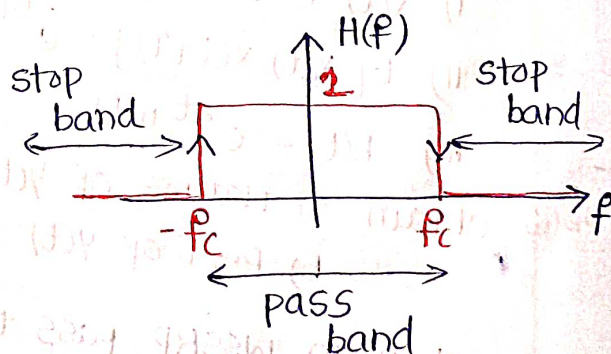
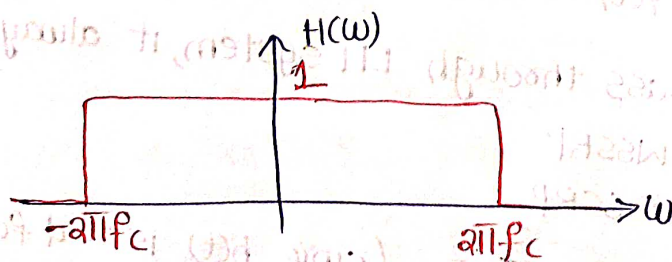
$$\text{iii) } E[x(t)x(t+T)] = A \delta(t)$$

calculate i) $E[y(t)y(t+T)]$ ii) $S_Y(f)$ iii) power in $y(t)$

soln

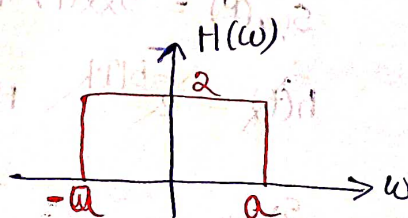
since the low pass filter is ideal, transition is very sharp.

Therefore

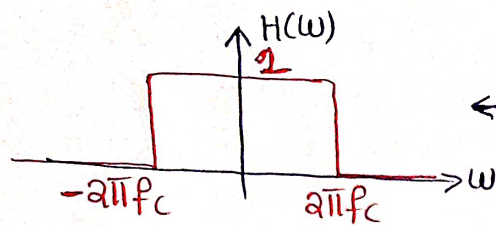


we knew that

$$2 \frac{\sin at}{\pi} \xleftrightarrow{F.T}$$



Therefore



$$\longleftrightarrow h(t) = 2 \frac{\sin(2\pi f_c t)}{\pi t}$$

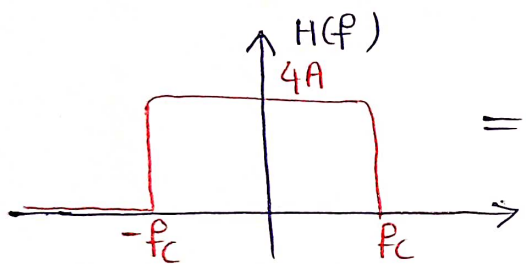
time domain expression of ideal low pass filter

$$R_{xx}(T) = A\delta(T) \longleftrightarrow S_{xx}(f) = A$$

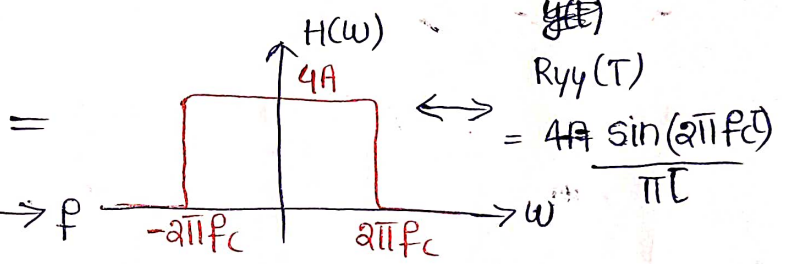
$$S_y(f) = S_x(f) |H(f)|^2$$

$$= A(2)^2$$

$$S_y(f) = \begin{cases} 4A & \text{when } |f| \leq f_c \\ 0 & \text{when } |f| > f_c \end{cases}$$



=

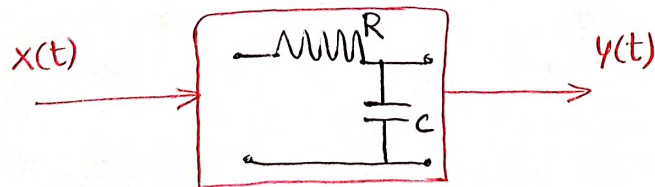


$$R_{yy}(T) = 4A \frac{\sin(2\pi f_c T)}{\pi T}$$

$$R_{yy}(T) = \frac{4A \sin(2\pi f_c T)}{\pi T}$$

$$P_y(t) = \int_{-\infty}^{\infty} S_y(f) df = \int_{-f_c}^{f_c} 4A df = 4A(f_c + f_c) = 8Af_c$$

#Q10) consider the following LTI system



Given that i) $x(t)$ is WSSRP

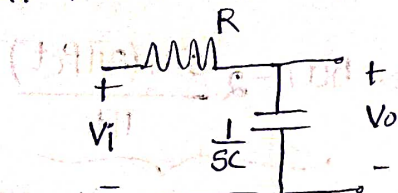
ii) Filter is practical RC LPF

iii) $R_{xx}(T) = A\delta(T)$

calculate i) $R_y(T)$ ii) $S_y(f)$ iii) $P_y(t)$

Soln

Practical RC low pass filter:-



$$H(s) = \frac{V_o}{V_i}$$

$$V_o = \frac{V_i \times \frac{1}{sC}}{R + \frac{1}{sC}}$$

$$V_o = \frac{V_i}{1 + sRC}$$

$$H(s) = \frac{V_o}{V_i} = \frac{1}{1 + sRC}$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$|H(f)|^2 = \frac{1}{1 + 4\pi^2 R^2 C^2 f^2}$$

$$S_y(f) = |H(f)|^2 S_x(f) = \frac{A}{1 + 4\pi^2 R^2 C^2 f^2}$$

$$e^{-a|T|} \longleftrightarrow \frac{2a}{a^2 + 4\pi^2 f^2}$$

$$S_y(f) = \frac{A}{1 + 4\pi^2 f^2 R^2 C^2} = \frac{A}{R^2 C^2 \left(\frac{1}{R^2 C^2} + 4\pi^2 f^2 \right)}$$

$$S_y(f) = \frac{A}{RC} \left[\frac{2 \left(\frac{1}{R^2 C^2} \right)}{\frac{1}{R^2 C^2} + 4\pi^2 f^2} \right] \longleftrightarrow R_y(T) = \frac{A}{2} e^{-\frac{1}{RC}|T|}$$

$$S_y(f) = \frac{A}{2RC} \left[\frac{2/RC}{\left(\frac{1}{RC} \right)^2 + (2\pi f)^2} \right] \longleftrightarrow R_y(T) = \frac{A}{2RC} e^{-\frac{|T|}{RC}}$$

$$\text{Power } E[y^2(t)] = R_y(0) = \frac{A}{2RC} e^0 = \frac{A}{2RC}$$

Assignment problems continuation:-

7.1-6) Let A_0 and B_0 be independent random variables. We form the random process $x(t) = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)$ where ω_0 is real constant.

- show that if A_0 and B_0 are uncorrelated with zero means and equal variance then $x(t)$ is wide sense stationary
- Find the autocorrelation function of $x(t)$
- Find the power density spectrum.

soln

- a) For $x(t)$ to be WSSRP $\begin{cases} \mu_x \text{ is constant} \\ E[x(t)x(t+T)] = R_{xx}(T) \end{cases}$

And given that $E[A_0] = E[B_0]$

$$\text{and } \sigma_A^2 = \sigma_B^2 = \sigma^2$$

mean of $x(t)$:

$$\begin{aligned} E[x(t)] &= E[A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t)] \\ &= E[A_0] \cos \omega_0 t + E[B_0] \sin \omega_0 t \\ &= 0 + 0 \end{aligned}$$

$$E[x(t)] = 0 \quad (\text{Independent of } t)$$

Autocorrelation :

$$\begin{aligned} E[x(t)x(t+T)] &= E[(A_0 \cos(\omega_0 t) + B_0 \sin \omega_0 t)(A_0 \cos(\omega_0 t + \omega_0 T) \\ &\quad + B_0 \sin(\omega_0 t + \omega_0 T))] \\ &= E[A_0^2 \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 T) + A_0 B_0 \cos(\omega_0 t + \omega_0 T) \\ &\quad \sin(\omega_0 t) + A_0 B_0 \cos(\omega_0 t) \sin(\omega_0 t + \omega_0 T) + B_0^2 \sin(\omega_0 t) \\ &\quad \sin(\omega_0 t + \omega_0 T)] \end{aligned}$$

$$\begin{aligned} &= E[A_0^2 \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 T)] + E[A_0 B_0 \cos(\omega_0 t + \omega_0 T) \\ &\quad \sin(\omega_0 t)] + E[A_0 B_0 \cos(\omega_0 t) \sin(\omega_0 t + \omega_0 T)] + \\ &\quad + E[B_0^2 \sin(\omega_0 t) \sin(\omega_0 t + \omega_0 T)] \end{aligned}$$

$$= E[A_0^2] \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 T) + \xrightarrow{=0}$$

$$E[A_0 B_0] \cos(\omega_0 t + \omega_0 T) \sin(\omega_0 t) + \xrightarrow{=0}$$

$$E[A_0 B_0] \cos(\omega_0 t) \sin(\omega_0 t + \omega_0 T) +$$

$$E[B_0^2] \sin(\omega_0 t) \sin(\omega_0 t + \omega_0 T)$$

$$= E[A^2] \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 T) + E[B^2] \sin(\omega_0 t) \sin(\omega_0 t + \omega_0 T)$$

$$= \sigma^2 \{ \cos(\omega_0 t) \cos(\omega_0 t + \omega_0 T) + \sin(\omega_0 t) \sin(\omega_0 t + \omega_0 T) \}$$

$$R_{xx}(T) = \sigma^2 \{ \cos(\omega_0 T) \}$$

$$R_{xx}(T) = \sigma^2 \{ \cos(\omega_0 T) \} \text{ function of } T \text{ only}$$

$\Rightarrow$ mean is constant, $R_{xx}(T)$ is function of T only

$\Rightarrow x(t)$ is WSSRP:

b) Autocorrelation function of $x(t)$ is $R_{xx}(T) = \sigma^2 \cos \omega_0 T$

c) Power spectral density of $x(t) = ?$

$$R_{xx}(T) \xleftrightarrow{F.T} S_{xx}(f)$$

$$1 \longleftrightarrow 2\pi \delta(\omega)$$

$$1 \cdot e^{j\omega_0 T} \longleftrightarrow 2\pi \delta(\omega - \omega_0)$$

$$1 \cdot e^{-j\omega_0 T} \longleftrightarrow 2\pi \delta(\omega + \omega_0)$$

Therefore :-

$$\sigma^2 \cos(\omega_0 T) \longleftrightarrow \frac{\sigma^2}{2} \{ 2\pi \delta(\omega - \omega_0) + 2\pi \delta(\omega + \omega_0) \}$$

$$\sigma^2 \cos(\omega_0 T) \longleftrightarrow \frac{\sigma^2}{2} \{ \delta(f - f_0) + \delta(f + f_0) \}$$

(By $\delta(f)$ function scaling property)

$$\Rightarrow S_{xx}(f) = \frac{\sigma^2}{2} \{ \delta(f - f_0) + \delta(f + f_0) \} \text{ (OR)}$$

$$\Rightarrow S_{xx}(\omega) = \frac{\sigma^2 \pi}{2} \{ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \}$$

7.1-12) Determine whether of the following functions can and cannot be power density spectrums. For those that are not, explain why?

a) $\frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$ b) $\exp[-(\omega - 1)^2]$ c) $\frac{\omega^2}{\omega^4 + 1} \delta(\omega)$ d) $\frac{\omega^4}{1 + \omega^2 + j\omega_0}$

Soln

a) $\frac{\omega^2}{\omega^6 + 3\omega^2 + 3}$ $\begin{cases} \text{even function} \\ S_{xx}(\omega) \geq 0 \forall \omega \end{cases} \Rightarrow \text{valid P.S.D function}$

b) $\exp(-(w-1)^2)$ $\begin{cases} \text{Not an even function} \\ S_{xx}(\omega) \geq 0 \forall \omega \end{cases} \Rightarrow \text{Not valid P.S.D function}$

explanation:

$$S_{xx}(\omega) = e^{-(\omega-1)^2} = e^{-\omega^2 - 1 + 2\omega}$$

$$S_{xx}(\omega) = e^{-(\omega-1)^2} = e^{-\omega^2 - 1 - 2\omega}$$

$$S_{xx}(\omega) \neq S_{xx}(-\omega)$$

c) $\frac{\omega^2}{\omega^4 + 1} - \delta(\omega)$ $\begin{cases} \text{even function of } \omega \\ S_{xx}(\omega) \not\geq 0 \forall \omega \end{cases} \Rightarrow \text{Not valid PSD function}$

explanation: when $\omega = 0$

$$S_{xx}(\omega) = \frac{\omega^2}{\omega^4 + 1} - \delta(\omega) \Big|_{\omega=0} = 0 - \delta(0) = -\delta(0)$$

Negative Amplitude

d) $\frac{\omega^4}{1 + \omega^2 + j\omega^6}$ $\begin{cases} \text{even function of } \omega \\ S_{xx}(\omega) \geq 0 \forall \omega \text{ but not real function} \end{cases} \Rightarrow \text{Not a valid P.S.D function}$

7-1-15) A random process is given by

$$x(t) = A_0 \cos(\omega t + \theta)$$

where A_0 is real constant, ω is random variable with density function $f_{\omega}(\cdot)$ and θ is a random variable uniformly distributed on interval $(0, 2\pi)$ independent of ω .

show that the power spectrum of $x(t)$ is

$$S_{xx}(\omega) = \frac{\pi A_0^2}{2} \{ f_{\omega}(\omega) + f_{\omega}(-\omega) \}$$

Let us find correlation functions individually.

i) Take θ is random variable & treat ω as constant

$$R_{xx}(t, t+T) = E[A_0 \cos(\omega t + \theta) A_0 \cos(\omega t + \omega T + \theta)]$$

$$= A_0^2 E[\cos(\omega t + \theta) \cos(\omega t + \omega T + \theta)]$$

$$= \frac{A_0^2}{2} E\{ \cos(2\omega t + 2\theta + \omega T) + \cos(\omega T) \}$$

$$\begin{aligned}
&= \frac{A^2}{2} \left\{ \cos \Omega T + \int_{-\infty}^{\infty} \cos(2\Omega t + 2\theta + \Omega T) f_{\theta}(\theta) d\theta \right\} \\
&= \frac{A^2}{2} \left\{ \cos \Omega T + \int_0^{2\pi} \cos(2\theta + 2\Omega t + \Omega T) \frac{1}{2\pi} d\theta \right\} \\
&= \frac{A^2}{2} \left\{ \cos \Omega T + \frac{1}{4\pi} \sin(2\theta + 2\Omega t + \Omega T) \Big|_0^{2\pi} \right\} \\
&= \frac{A^2}{2} \left\{ \cos \Omega T + \frac{1}{4\pi} (\sin(2(2\pi) + 2\Omega t + \Omega T) - \sin(2\Omega t + \Omega T)) \right\} \\
&= \frac{A^2}{2} \left\{ \cos \Omega T + \frac{1}{4\pi} [\sin(2\Omega t + \Omega T) - \sin(2\Omega t + \Omega T)] \right\} \\
R_{xx}(t, t+T) &= \frac{A^2}{2} \cos \Omega T \rightarrow \text{free from } \theta
\end{aligned}$$

ii) Now take Ω as random variable

$$R_{xx}(T) = \int_{-\infty}^{\infty} \frac{A^2}{2} \cos \Omega T f_{\Omega}(\Omega) d\Omega$$

We know that $R_{xx}(T)$ and $S_{xx}(\omega)$ form Fourier transform pairs.

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(T) e^{-j\omega T} dT$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{A^2}{2} (\cos \Omega T) f_{\Omega}(\Omega) e^{-j\omega T} d\Omega dT$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{A^2}{2} \left[\frac{e^{j\Omega T} + e^{-j\Omega T}}{2} \right] f_{\Omega}(\Omega) e^{-j\omega T} dT d\Omega$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{4} \int_{-\infty}^{\infty} (e^{j\Omega T} + e^{-j\Omega T}) e^{-j\omega T} dT f_{\Omega}(\Omega) d\Omega$$

frequency shifting property

$$= 2\pi \delta(\omega - \Omega) + 2\pi \delta(\omega + \Omega)$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{4} \{ 2\pi \delta(\omega - \Omega) + 2\pi \delta(\omega + \Omega) \} f_{\Omega}(\Omega) d\Omega$$

$$S_{xx}(\omega) = \frac{A^2 \pi}{2} \{ f_{\Omega}(\omega) + f_{\Omega}(-\omega) \}$$

7-1.16) If $x(t)$ is stationary process, find the power spectrum of $y(t) = A_0 + B_0 x(t)$ in terms of power spectrum of $x(t)$ if A_0, B_0 are real constants.

Soln

$$R_{xx}(T) = E[x(t)x(t+T)]$$

$$y(t) = A_0 + B_0 x(t)$$

$$R_{yy}(T) = E[y(t)y(t+T)]$$

$$= E[(A_0 + B_0 x(t))(A_0 + B_0 x(t+T))]$$

$$= E[A_0^2 + A_0 B_0 x(t+T) + A_0 B_0 x(t) + B_0^2 x(t)x(t+T)]$$

$$= E[A_0^2] + A_0 B_0 E[x(t+T)] + A_0 B_0 E[x(t)] + B_0^2 E[x(t)x(t+T)]$$

$$R_{yy}(T) = A_0^2 + A_0 B_0 \mu_x + A_0 B_0 \mu_x + B_0^2 R_{xx}(T)$$

$$R_{yy}(T) = A_0^2 + 2A_0 B_0 \mu_x + B_0^2 R_{xx}(T)$$

Now applying Fourier transform on both sides:

$$S_{yy}(\omega) = A_0^2 2\pi \delta(\omega) + 2A_0 B_0 \mu_x (2\pi \delta(\omega)) + B_0^2 S_{xx}(\omega)$$

$$\underbrace{S_{yy}(\omega)}_{\text{PSD of } y(t)} = A_0^2 2\pi \delta(\omega) + 4A_0 B_0 \mu_x \pi \delta(\omega) + B_0^2 \underbrace{S_{xx}(\omega)}_{\text{PSD of } x(t)}$$

7-1.32) Assume a random process has a power spectrum

$$S_{xx}(\omega) = \begin{cases} 4 - \frac{\omega^2}{9} & |\omega| \leq 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find a) the average power b) RMS bandwidth and c) the autocorrelation function of the process.

Soln

a) Average power $P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$

$$= \frac{2}{2\pi} \int_0^6 \left(4 - \frac{\omega^2}{9}\right) d\omega$$

$$= \frac{1}{\pi} \left\{ 4\omega - \frac{\omega^3}{27} \right\}_0^6$$

$$P = \frac{1}{\pi} \{ 24 - 8 \} = \frac{16}{\pi}$$

b) $\text{RMS Bandwidth} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega) d\omega$

$$= \frac{1}{2\pi} \int_{-6}^6 \omega^2 \left(4 - \frac{\omega^2}{9}\right) d\omega = \frac{1}{2\pi} \int_{-6}^6 \left(4\omega^2 - \frac{\omega^4}{9}\right) d\omega$$

$$= \frac{1}{2\pi} \left\{ \frac{4}{3} \omega^3 - \frac{\omega^5}{45} \right\}_{-6}^6$$

$$= \frac{1}{2\pi} \left\{ \left(\frac{4}{3} (216) - \frac{7776}{45} \right) - \left(\frac{4}{3} (-216) + \frac{7776}{45} \right) \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{1728}{3} - \frac{15552}{45} \right\}$$

$$= \frac{1}{2\pi} \left\{ 576 - \frac{1728}{5} \right\}$$

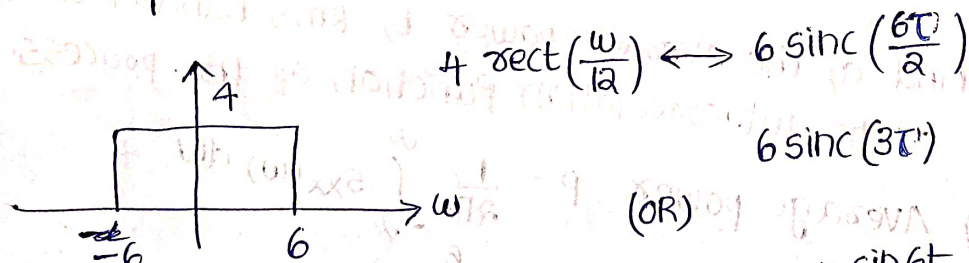
$$= \frac{1}{\pi} \left\{ 288 - \frac{864}{5} \right\}$$

$$= \frac{576}{5\pi}$$

$$= \frac{4^2 \times 16^2}{5\pi}$$

c) $S_{xx}(\omega) = 4 - \frac{\omega^2}{9} \quad |\omega| \leq 6$

$S_{xx_1}(\omega) = 4 \quad |\omega| \leq 6$



$$4 \text{rect}\left(\frac{\omega}{12}\right) \leftrightarrow \frac{4 \sin 6t}{\pi t}$$

$$S_{xx_2}(\omega) = \frac{\omega^2}{9} \quad |\omega| \leq 6 \leftrightarrow \frac{1}{2\pi} \int_{-6}^6 \frac{\omega^2}{9} e^{j\omega T} d\omega \quad \text{LATE}$$

$$= \frac{1}{2\pi} \left[\frac{\omega^2}{9} \int e^{j\omega T} d\omega - \int \frac{2}{9} \omega \int e^{j\omega T} d\tau \right]$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[\frac{\omega^2}{9} \frac{e^{j\omega\tau}}{j\tau} - \frac{2}{9} \left\{ \int \omega \frac{e^{j\omega\tau}}{j\tau} d\omega \right\} \right] \\
&= \frac{1}{2\pi} \left[\frac{\omega^2}{9} \frac{e^{j\omega\tau}}{j\tau} - \frac{2}{9} \left\{ \omega \frac{e^{j\omega\tau}}{(j\tau)^2} - \int \frac{e^{j\omega\tau}}{(j\tau)^2} d\omega \right\} \right] \\
&= \frac{1}{2\pi} \left[\frac{1}{j\tau} \frac{\omega^2}{9} e^{j\omega\tau} - \frac{2}{9} \frac{1}{(j\tau)^2} \omega e^{j\omega\tau} + \frac{2}{9} \frac{e^{j\omega\tau}}{(j\tau)^3} \right]_{-6}^6 \\
&= \frac{1}{2\pi} \left[\frac{1}{j\tau} (36 - 36) (e^{j\tau 6} - e^{-j\tau 6}) \right. \\
&\quad \left. - \frac{2}{9(j\tau)^2} \left\{ 6 e^{j\tau 6} - 6 e^{-j\tau 6} \right\} + \frac{2}{9(j\tau)^3} \left\{ e^{j\tau 6} - e^{-j\tau 6} \right\} \right] \\
&= \frac{1}{2\pi} \left[\frac{36}{9j\tau} (e^{j\tau 6} - e^{-j\tau 6}) - \frac{2}{9(j\tau)^2} (6 e^{j\tau 6} - 6 e^{-j\tau 6}) + \right. \\
&\quad \left. \left(\frac{2}{9} \right) \frac{1}{(j\tau)^3} (e^{j\tau 6} - e^{-j\tau 6}) \right] \\
&= \frac{1}{2\pi} \left[\frac{72}{9\tau} \left(\frac{e^{j\tau 6} - e^{-j\tau 6}}{2} \right) + \frac{12 \times 2}{9\tau^2} \left(\frac{e^{j\tau 6} + e^{-j\tau 6}}{2} \right) + \right. \\
&\quad \left. \left(\frac{2}{9} \right) \left(\frac{1}{\tau^3} \right) \left(\frac{e^{j\tau 6} - e^{-j\tau 6}}{2} \right) \right]
\end{aligned}$$

$$= \frac{1}{2\pi} \left[\frac{72}{9\tau} \sin 6\tau - \frac{2 \times 2}{9\tau^3} \sin 6\tau + \frac{24}{9\tau^2} \cos 6\tau \right]$$

$$= \frac{1}{18\pi} \left[72 \frac{\sin 6\tau}{\tau} - 4 \frac{\sin 6\tau}{\tau^3} + \frac{24}{\tau^2} \cos 6\tau \right]$$

$$= \frac{-4}{18\pi} \left[\frac{\sin 6\tau}{\tau} \left(\frac{1}{\tau^2} - 18 \right) - \frac{6}{\tau^2} \cos 6\tau \right]$$

finally $R_{xx}(\tau) \rightarrow \mathcal{F}^{-1} \{ S_{xx_1}(\omega) - S_{xx_2}(\omega) \}$

$$R_{xx}(\tau) = 6 \sin(3\tau) + \frac{4}{18\pi} \left\{ \frac{\sin 6\tau}{\tau} \left(\frac{1}{\tau^2} - 18 \right) - \frac{6}{\tau^2} \cos 6\tau \right\}$$