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Unit-II

Power Spectrum (or)

Power Density Spectrum (or) Power Spectral density (PSD):

Parseval's Theorem:- if $x(t) \xleftrightarrow{FT} X(\omega)$ Then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Proof:

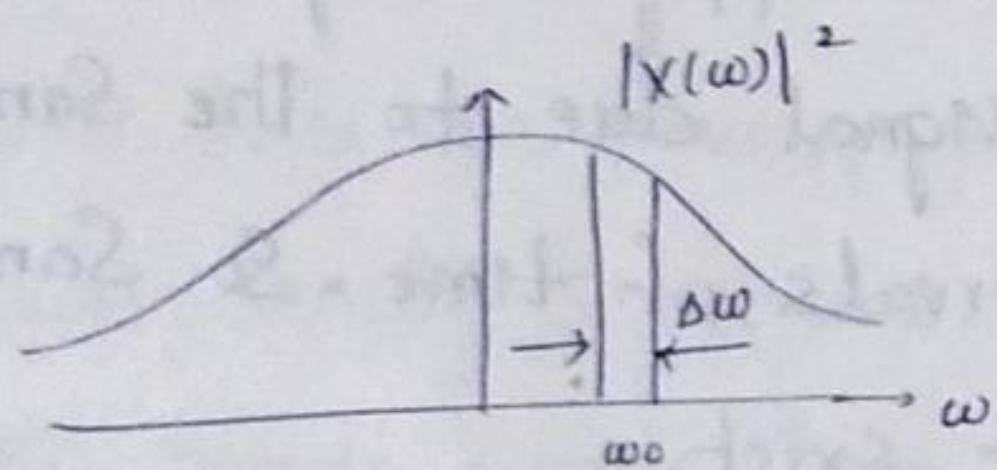
Signal Energy $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$E_x = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega$$

$$E_x = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega) d\omega$$

from this we can find the signal Energy.



$|X(\omega)|^2 \rightarrow$ Energy-density spectrum of the signal $x(t)$.

$\rightarrow$ Energy Spectral density of the signal $x(t)$ (per unit B.W in Hz) (E.S.D)

$$\left(\begin{array}{l} \therefore \text{Power} = \text{Avg of Energy} \\ \text{Power} = \frac{\text{Energy}}{\text{Time}} \end{array} \right)$$

$\rightarrow$ Therefore, We can determine the signal Energy from either time domain specification $x(t)$ or the Corresponding frequency-domain specification $X(\omega)$.

$\rightarrow$ The Energy of the signal $x(t)$ results from energies contributed by all Spectral Components of the signal $x(t)$.

All these is for deterministic signal. Now we move to

Fourier transform of power for Random signals.

→ Total energy may be determined either by computing the Energy per unit time ($|x(t)|^2$) or integrating over all time or by computing the Energy per unit frequency ($|x(\omega)|^2/2\pi$) & integrating over all frequencies.

Imp Why we need frequency domain description of the signal?

Because at the receiver side, for ex:- Our mobile will receive FM signal we must know the centered freq. i.e. 88 to 108 Hz are the carrier freq's and those are the ^{entered} carrier freq for Band-pass filter. This frequency domain description is important in designing the filters.

Without frequency description we can't design filter. We need Centered freq & Cutoff freq to design the filters. Simply time domain is not sufficient. We also require Spectrum domain. This freq domain we can apply directly for deterministic signal but not for Random signal due to the Sample functions usually takes large intervals of time. So Sample function Fourier transform does not exist.

$$\text{F.T.:- } X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow \text{Voltage Density Spectrum}$$

$\downarrow$
Volts-sec (or) Volts/Hz

This can be treated as Voltage Spectral density otherwise Voltage density Spectrum. i.e., How much Voltage is present over a freq or 1 Hz Bandwidth.

$$\text{I.F.T.:- } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

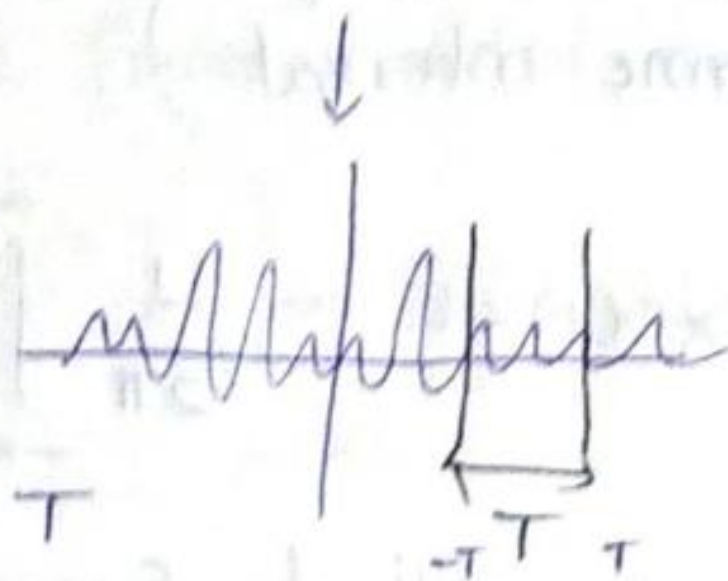
- * Since $X(\omega)$ doesn't exist for Random signal. Without freq description it is difficult to design at receiver.
- * Laplace Transform is somewhat applicable for Random signal. But I.L.T require Complex integration which is Very difficult.

Hence we are using power density Spectrum Concept to overcome not directly with some modifications.

Let us consider Random Process $X(t)$

Sample function $x(t)$ (or) $x_T(t)$

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & \text{else} \end{cases}$$



Here $x(t)$ is from $-T$ to T

finite value. So this $x_T(t)$ will have fourier transform.

* So, most of the Sample function will not have fourier transform.

So, that we are considering finite time interval - which exist fourier transform.

$$\therefore T \text{ is finite} \Rightarrow \int_{-T}^T |x_T(t)| dt < \infty \quad (\text{absolutely integrable})$$

* With this we develop power spectral density. Also called as Power density Spectrum.

* In Simple words, how the power is going to Vary with respect to freq. Otherwise with 1 Hz B.W how much of power is present.

Therefore,

Fourier Transform exist for $x_T(t)$

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt \quad \left(\because x_T(t) \text{ exist for } -T \text{ to } T \text{ only} \right)$$

$$X_T(\omega) = \int_{-T}^T x_T(t) e^{-j\omega t} dt$$

Now,

$$\text{Energy } E(T) = \int_{-T}^T |x_T(t)|^2 dt \quad \left(\because E = \int_{-\infty}^{\infty} |x(t)|^2 dt \right)$$

$$E(T) = \int_{-T}^T x^2(t) dt$$

$$E(T) = \int_{-T}^T x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \quad \left(\text{from parseval's Theorem} \right) \rightarrow \textcircled{1}$$

$$\text{power} = p(T) = \frac{1}{2T} \int_{-T}^T x^2(t) dt \quad \left(\because p(t) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt \right)$$

($\because$ Power is also Energy, that means we are calculating Energy for large time interval) ($\because$ T is finite here so, lim is not taken)

$$p(T) = \frac{1}{2T} \int_{-T}^T x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|x_T(\omega)|^2}{2T} d\omega \rightarrow (2)$$

Eq (1) $\Rightarrow$ is called Energy Spectral density

By integrating E.S.D we get Energy.

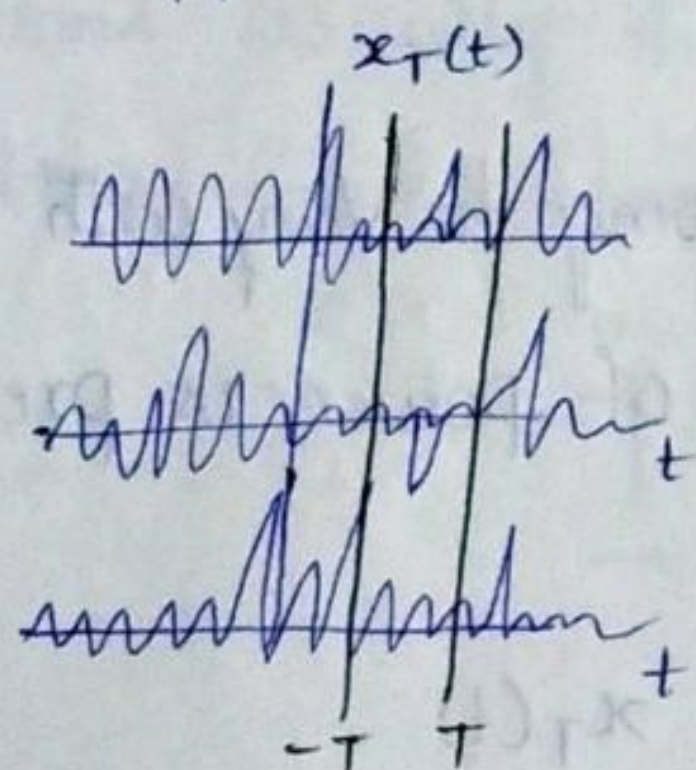
Eq (2) $\Rightarrow$ is called Power Spectral density } But it is X

By integrating P.S.D we get power.

$\therefore$ it is the power over some time interval not entire $x_T(t)$.

(So, to calculate power for entire $x_T(t)$ Sample-function, we need to take average sense.

$\Rightarrow$ if Suppose, if we take sample functions



The power will vary for each & every Sample fn for the given time interval $-T$ to T .

So, power will become as Random Variable

So, we need to take the average value,

to obtain power.

$$\text{Power of } X(t) \text{ over } [-T, T] = P_{XX} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \underbrace{E[x^2(t)]}_{\text{Mean}} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega \quad \text{Now we}$$

Here we obtained the power for entire random process.

($\because$ W.K.T, $\bar{x} = A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$ This is time avg of the mean square value.)

$$P_{xx} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x^2(t)] dt = A \underbrace{\left\{ E[x^2(t)] \right\}}_{\text{square value}}$$

* If $x(t)$ is WSS, then mean becomes constant & power becomes as $\overline{x^2}$.

$\therefore$ Power density Spectrum (or) Power Spectral density is defined as,

(PSD): $S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$ $\rightarrow$ for Random signal. $\rightarrow$ ①

Power is defined as,

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$\lim_{T \rightarrow \infty} \frac{|x(\omega)|^2}{2T} \text{ for Deterministic Signal.}$$

S_{xx} in terms of 'f',

$$S_{xx}(f) = \lim_{T \rightarrow \infty} \frac{E[|X_T(f)|^2]}{2T} \quad \left(\because \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E[|X_T(f)|^2]}{2T} \cdot 2\pi df \right) \rightarrow ②$$

$$\therefore ① = ②$$

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(f) \cdot 2\pi df = \int_{-\infty}^{\infty} S_{xx}(f) df$$

$$P_{xx} = \int_{-\infty}^{\infty} S_{xx}(f) df$$

Q. P.S.D $S_{xx}(\omega) = \frac{2a}{a^2 + \omega^2}$, Calculate the power.

Sol:

$$\therefore \text{In terms of '}\omega\text{' } \Rightarrow P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{2a}{a} \tan^{-1} \frac{\omega}{a} \right]_{-\infty}^{\infty} = \frac{1}{\pi} \left[\tan^{-1} \frac{\infty}{a} - \tan^{-1} \frac{-\infty}{a} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = \frac{1}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = 1 //$$

Q. P.S.D $S_{xx}(f) = \frac{2a}{a^2 + (2\pi f)^2}$, Calculate power.

Sol:

$$\therefore \text{In terms of 'f' } \Rightarrow P_{xx} = \int_{-\infty}^{\infty} S_{xx}(f) df$$

$$P_{xx} = \int_{-\infty}^{\infty} \frac{2a}{a^2 + (2\pi f)^2} df$$

$$= 2a \int_{-\infty}^{\infty} \frac{1}{a^2 + (2\pi f)^2} df$$

Q. Random process $X(t) = A_0 \cos(\omega_0 t + \theta)$, θ , uniformly distributed Random Variable $(0, \pi/2)$. Then P_{xx} .

Sol:

$$X(t) = A_0 \cos(\omega_0 t + \theta)$$

Uniformly distributed R.V $(0, \pi/2)$

Properties of the Power Density Spectrum (PSD):

$\therefore$ Power Spectral density $S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|x(\omega)|^2]}{2T}$

$\therefore \int_{-T}^T$ we take $\frac{1}{2T}$
 $\text{if } \int_{-\frac{T}{2}}^{\frac{T}{2}}$ we take $\frac{1}{T}$

1. $S_{xx}(\omega) \geq 0$

($\therefore E[\text{+ve Quantity}] = \text{+ve}$)

2. $S_{xx}(\omega)$ is Real.

($\therefore$ || Modulus gives real Value)

3. $S_{xx}(-\omega) = S_{xx}(\omega)$ (Even fn)

Proof:

Let $x_T(t)$ be any Fourier Transformable signal.

$$x_T(t) \longleftrightarrow X_T(\omega)$$

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt$$

Put $\omega = -\omega$

$$X_T(-\omega) = \int_{-\infty}^{\infty} x_T(t) e^{j\omega t} dt$$

($\therefore$ for real $x_T(t) \rightarrow X_T^*(-\omega) = X_T(\omega)$)

$$\therefore X_T(-\omega) = \int_{-\infty}^{\infty} x_T(t) e^{j\omega t} dt = X_T^*(\omega)$$

$$S_{xx}(-\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T^*(-\omega) X_T(-\omega)]}{2T}$$

$$= \lim_{T \rightarrow \infty} \frac{E[X_T(\omega) X_T^*(\omega)]}{2T}$$

$$S_{xx}(-\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\therefore S_{xx}(-\omega) = S_{xx}(\omega)$$

4. $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = A [E[x^2(t)]] \Rightarrow$ direct from formula.

Imp 5. $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega \tau} d\omega = A [R_{xx}(t, t+\tau)] \rightarrow$ This is non-stationary.
 (-for non-stationary only it depends on absolute time)

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} A[R_{xx}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

* Power density spectrum is the time avg of Autocorrelation function
form a fourier transform pair. (A)

* if $x(t)$ is WSS, then,

$$A[R_{xx}(t, t+\tau)] = R_{xx}(\tau) \quad \therefore A[R_{xx}(t, t+\tau)] =$$

Here, $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xx}(\tau) d\tau$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

$$\therefore R_{xx}(\tau) \xleftrightarrow{FT} S_{xx}(\omega)$$

This famous theorem (or) relationship is called,
'Wiener - Khintchine Theorem'

Proof: Relationship between Power Spectrum and Auto Correlation fn.

$$A[R_{xx}(t, t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xx}(t, t+\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)x(t+\tau)] dt$$

$$F[A[R_{xx}(t, t+\tau)]] = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)x(t+\tau)] dt \cdot e^{-j\omega\tau} d\tau$$

$$= E \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \int_{-\infty}^{\infty} x(t+\tau) e^{-j\omega\tau} d\tau \cdot dt \right]$$

Time shifting
 $x(\omega) \cdot e^{j\omega t}$

$$= E \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \cdot e^{j\omega t} dt \cdot x(\omega) \right]$$

$$= E \left[\lim_{T \rightarrow \infty} \frac{1}{2T} x^*(\omega) x(\omega) \right] \quad \left(\because x(t) \text{ is real} \right)$$

$$x^*(t) = x(t)$$

$$FT[A[R_{xx}(t, t+\tau)]]_{xx(\omega)} = \lim_{T \rightarrow \infty} \frac{1}{2T} E[|x(\omega)|^2]$$

$$= S_{xx}(\omega)$$

for wss,

$$F.T[R_{xx}(\tau)] = S_{xx}(\omega)$$

This is Wiener-Khinchine Theorem.

* If we know the power spectrum we can recover the Autocorrelation fn for wss case.

* For ^{nm}Stationary case we can recover only the timeavg of Auto Correlation fn. This is the difference.

$$(6) S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)$$

(derivative of power spectrum = $\omega^2 \times S_{xx}(\omega)$.)

Proof:

$$x_T(t) \xleftrightarrow{F.T} X_T(\omega)$$

$$\frac{d}{dt} x_T(t) = \dot{x}_T(t) \xleftrightarrow{F.T} j\omega X_T(\omega) = \dot{X}_T(\omega)$$

$$S_{xx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$S_{\dot{x}\dot{x}}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|\dot{X}_T(\omega)|^2]}{2T}$$

$\therefore S_{\dot{x}\dot{x}} \rightarrow$ due to p.s.d dep - ends on $x(t)$

$$= \lim_{T \rightarrow \infty} \frac{E[|j\omega X_T(\omega)|^2]}{2T}$$

$$= \omega^2 \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\boxed{S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)}$$

Apply I.F.T for above eqn,

$$R_{\dot{x}\dot{x}}(\tau) = -\frac{\partial^2}{\partial \tau^2} R_{xx}(\tau)$$

$$\left(\because \frac{d^n x(t)}{dt^n} \xleftrightarrow{F.T} (j\omega)^n X(\omega) \right)$$

$$\frac{d^2 x(t)}{dt^2} \xleftrightarrow{F.T} -\omega^2 X(\omega)$$

Therefore, Power Spectral density &

Auto Correlation fns are fourier transformable pair.

($\therefore$ Mostly Random Signals are power signals. So it deals with Power spectral density).

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Cross - Power density Spectrum & its Properties:-For two real R.P's $x(t)$ & $y(t)$

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & \text{else} \end{cases} \quad y_T(t) = \begin{cases} y(t), & -T < t < T \\ 0, & \text{else} \end{cases}$$

 $x_T(t), y_T(t)$ is any sample-function.

$$\text{Cross Power } P_{xy}(T) = \frac{1}{2T} \int_{-T}^T x_T(t) \cdot y_T(t) \cdot dt \quad \left(\begin{array}{l} x_T(t) \xleftrightarrow{FT} X_T(\omega) \\ y_T(t) \xleftrightarrow{FT} Y_T(\omega) \end{array} \right)$$

$$= \frac{1}{2T} \int_{-T}^T x(t) \cdot y(t) \cdot dt$$

By using Parseval's Relation,

$$P_{xy}(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X_T^*(\omega) Y_T(\omega)}{2T} d\omega$$

$$\overline{P_{xy}}(T) = \frac{1}{2T} \int_{-T}^T E[x(t) y(t)] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega) X_T(\omega)]}{2T} d\omega$$

$$\text{Total Cross power } (P_{xy}) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xy}(t, t) dt = \frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega) X_T(\omega)]}{2T} d\omega$$

Cross Power Spectral density,

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T}$$

 $S_{xy}(\omega)$ is real,
but in general

$$\text{Total Cross power } P_{xy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) d\omega$$

 $S_{xy}(\omega)$ is Complex.

$$\text{Similarly, } S_{yx}(\omega) = \lim_{T \rightarrow \infty} \frac{E[Y_T^*(\omega) X_T(\omega)]}{2T}$$

$$P_{yx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yx}(\omega) d\omega$$

Properties:-

Cross power spectral density,

$$S_{xy}(\omega) = \lim_{T \rightarrow \infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T}$$

$$①) S_{xy}(\omega) = S_{yx}(\omega) = S_{yx}^*(\omega)$$

Proof:

$$S_{yx}^*(\omega) = \lim_{T \rightarrow \infty} \frac{E[y_T(\omega) x_T^*(\omega)]}{2T} = S_{xy}(\omega)$$

$$S_{yx}(-\omega) = \lim_{T \rightarrow \infty} \frac{E[y_T^*(-\omega) x_T(-\omega)]}{2T} = \lim_{T \rightarrow \infty} \frac{E[y_T(\omega) x_T^*(\omega)]}{2T} = S_{xy}(\omega)$$

$$ii) S_{yx}(\omega) = S_{xy}(-\omega) = S_{yx}^*(\omega)$$

$$② \text{ Re}[S_{xy}(\omega)] \text{ and } \text{Re}[S_{yx}(\omega)] \text{ are even fn's of } \omega.$$

Proof:-

$$\text{Re}[S_{xy}(\omega)] = \frac{1}{2} [S_{xy}(\omega) + S_{xy}^*(\omega)]$$

$$\begin{aligned} \text{Re}[S_{xy}(-\omega)] &= \frac{1}{2} [S_{xy}(-\omega) + S_{xy}^*(-\omega)] \\ &= \frac{1}{2} [S_{xy}^*(\omega) + S_{xy}(\omega)] \end{aligned}$$

$$\therefore \text{Re}[S_{xy}(-\omega)] = \text{Re}[S_{xy}(\omega)]$$

$$③ \text{ Im}[S_{xy}(\omega)] \text{ and } \text{Im}[S_{yx}(\omega)] \text{ are odd function of } \omega$$

Proof:-

$$\text{Im}[S_{xy}(\omega)] = \frac{1}{2} [S_{xy}(\omega) - S_{xy}^*(\omega)]$$

$$\begin{aligned} \text{Im}[S_{xy}(-\omega)] &= \frac{1}{2} [S_{xy}(-\omega) - S_{xy}^*(-\omega)] \\ &= \frac{1}{2} [S_{xy}^*(\omega) - S_{xy}(\omega)] \end{aligned}$$

$$\therefore \text{Im}[S_{xy}(-\omega)] = -\frac{1}{2} [S_{xy}(\omega) - S_{xy}^*(\omega)]$$

$$\therefore \text{Im}[S_{xy}(-\omega)] = -\text{Im}[S_{xy}(\omega)]$$

$$④ A[R_{xy}(t, t+\tau)] \xleftrightarrow{F.T} S_{xy}(\omega)$$

$$A[R_{yx}(t, t+\tau)] \xleftrightarrow{F.T} S_{yx}(\omega)$$

Cross power density spectrum and the time avg of the cross-correlation functions are a Fourier transform pair.

for jointly wss processes,

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau \quad (\because \text{for wss } t=t')$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau} d\tau$$

Therefore I.F.T also exist,

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega\tau} d\omega$$

$$R_{yx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yx}(\omega) e^{j\omega\tau} d\omega$$

Proof:-

$$A[R_{xy}(t_1, t_1 + \tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R_{xy}(t_1, t_1 + \tau) dt$$

Apply F.T on both sides,

$$F.T \{ A[R_{xy}(t_1, t_1 + \tau)] \} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} E[x(t) y(t + \tau)] e^{j\omega\tau} e^{-j\omega t} d\tau dt$$

$$= E \left[\lim_{T \rightarrow \infty} \int_{-T}^T x(t) \cdot \int_{-\infty}^{\infty} y(t + \tau) e^{-j\omega\tau} d\tau \cdot dt \right]$$

$$= E \left[\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) y_T(\omega) \cdot e^{j\omega t} dt \right] y_T(\omega) \cdot e^{j\omega t}$$

$$= E \left[\lim_{T \rightarrow \infty} \frac{1}{2T} X_T^*(\omega) y_T(\omega) \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} E [X_T^*(\omega) y_T(\omega)]$$

$$\therefore F.T [A[R_{xy}(t_1, t_1 + \tau)]] = S_{xy}(\omega)$$

for WSS,

$$F.T [R_{xy}(\tau)] = S_{xy}(\omega)$$

⑤ $S_{xy}(\omega) = 0$ & $S_{yx}(\omega) = 0$ if $x(t)$ & $y(t)$ are orthogonal.

Proof:

If $x(t)$ & $y(t)$ are orthogonal,

$$R_{xy}(\tau) = 0$$

$$F.T [R_{xy}(\tau)] = S_{xy}(\omega) = 0$$

$$R_{yx}(\tau) = 0$$

$$F.T [R_{yx}(\tau)] = S_{yx}(\omega) = 0$$

⑥ If $x(t)$ & $y(t)$ are uncorrelated & have constant means $\bar{x}$ & $\bar{y}$

then,

$$S_{xy}(\omega) = S_{yx}(\omega) = 2\pi \bar{x} \bar{y} \delta(\omega)$$

Proof:

$$C_{xy}(\tau) = R_{xy}(\tau) - E[x(t)] E[y(t + \tau)]$$

• Uncorrelation.

$$R_{xy}(\tau) = E[x(t)] E[y(t+\tau)] = \bar{x} \cdot \bar{y}$$

$\therefore$ If two processes are said to be independent, then they are uncorrelated. But if two processes are uncorrelated then they may or may not be independent.

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \bar{x} \cdot \bar{y} \cdot e^{-j\omega\tau} d\tau$$

$$S_{xy}(\omega) = \bar{x} \bar{y} \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau} d\tau$$

$$\boxed{S_{xy}(\omega) = 2\pi \bar{x} \bar{y} \delta(\omega)}$$

$$\left(\because 1 \xrightarrow{F.T} 2\pi \delta(\omega) \right)$$

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⑦ Let R.P $x(t)$ & $\dot{x}(t) = \frac{d}{dt} x(t)$ then

$$R_{\dot{x}\dot{x}}(\tau) = \frac{\partial}{\partial \tau} R_{xx}(\tau)$$

$$S_{\dot{x}\dot{x}}(\omega) = j\omega S_{xx}(\omega)$$

Proof:

The Cross Power Spectrum density b/w $x(t)$ & $\dot{x}(t)$.

$$S_{\dot{x}\dot{x}}^*(\omega) = \lim_{T \rightarrow \infty} \frac{E[\dot{x}^*(\omega) \dot{x}(\omega)]}{2T}$$

$$= \lim_{T \rightarrow \infty} \frac{E[\dot{x}^*(\omega) j\omega x(\omega)]}{2T} \Rightarrow j\omega \lim_{T \rightarrow \infty} \frac{E[|x(\omega)|^2]}{2T}$$

$$\boxed{\begin{array}{l} S_{\dot{x}\dot{x}}(\omega) = j\omega S_{xx}(\omega) \\ \text{I.F.T} \\ R_{\dot{x}\dot{x}}(\tau) = \frac{\partial}{\partial \tau} R_{xx}(\tau) \end{array}}$$

$\Rightarrow$ Consider a real R.P $w(t)$,

$$w(t) = x(t) + y(t)$$

The Auto Correlation fn of $w(t)$ is,

$$R_{ww}(t, t+\tau) = E[w(t)w(t+\tau)]$$

$$= E[(x(t) + y(t))(x(t+\tau) + y(t+\tau))]$$

$$= R_{xx}(t, t+\tau) + R_{yy}(t, t+\tau) + R_{xy}(t, t+\tau) + R_{yx}(t, t+\tau)$$

Take Time Avg on b.s & Apply F.T on b.s.

$$S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega) + S_{xy}(\omega) + S_{yx}(\omega)$$

if $x(t)$ & $y(t)$ are Uncorrelated, then

$$R_{xy}(\tau) = E[x(t)] E[y(t+\tau)]$$

$$R_{xy}(\tau) = 0 = R_{yx}(\tau)$$

$\Rightarrow x(t)$ & $y(t)$ are Orthogonal to each other.

$$S_{xy}(\omega) = S_{yx}(\omega) = 0$$

$$S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$$

($x(t), y(t)$)

So, If two random processes are added together and those are Orthogonal to each other, Otherwise this Orthogonality is defined in another way first it should be Uncorrelated and then either of its mean is '0' then this cross correlation fn becomes '0', So that the Power Spectral density of the resultant o/p $S_{ww}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$.

$$\therefore \text{Power } \overline{W^2} = \overline{X^2} + \overline{Y^2}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ww}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$$

$$E[W^2] = P_{ww} = P_{xx} + P_{yy}$$

$$E[W^2(t)] =$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)y(t)] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) d\omega$$

For, R.P's, $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)y(t)] dt = 0$ (if $x(t)$ & $y(t)$ are Orthogonal)

For Deterministic signals,

$$\int_{-\infty}^{\infty} x(t)y(t) dt = 0$$

Then $x(t)$ & $y(t)$ are Orthogonal.

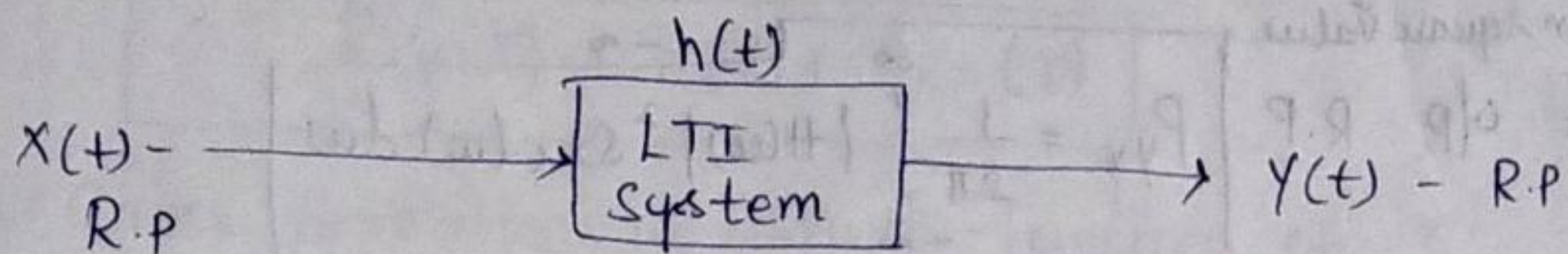
In text book
8th Chapter

LTI Systems with Random input:-

this topic

Consider an LTI system which is characterised by its impulse response $h(t)$ of the system.

* i/p to LTI Sys is R.P $x(t)$ & o/p also becomes R.P $y(t)$.



$$y(t) = h(t) * x(t) \quad (\text{Convolution})$$

$$Y(\omega) = H(\omega) \cdot X(\omega) \quad (\text{Multiplication})$$

The Output PSD is given by, $S_{yy}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|Y(\omega)|^2]}{2T}$

$$= \lim_{T \rightarrow \infty} \frac{E[|H(\omega) \cdot X(\omega)|^2]}{2T} \quad (\because H(\omega) \text{ is not a R.P.})$$

$$= |H(\omega)|^2 \lim_{T \rightarrow \infty} \frac{E[|X(\omega)|^2]}{2T}$$

$$\boxed{S_{yy}(\omega) = |H(\omega)|^2 \cdot S_{xx}(\omega)} \rightarrow \text{if } x(t) \text{ is WSS.}$$

o/p power spectral density = $|H(\omega)|^2$ times i/p power spectral density.

Apply I.F.T,

$$R_{yy}(\tau)$$

$$S_{yy}(\omega) = H(\omega) \cdot H^*(\omega) \cdot S_{xx}(\omega)$$

For, WSS i/p R.P.,

Apply I.F.T,

$$\boxed{R_{yy}(\tau) = h(\tau) * h(-\tau) * R_{xx}(\tau)} \quad \text{This is the}$$

Auto Correlation fn of $y(t)$.

* if i/p $x(t)$ is WSS the only we can apply I.F.T

* if i/p $x(t)$ is WSS then o/p $y(t)$ is also WSS.

$$R_{yy}(\tau) = h(\tau) * \left[\int_{-\infty}^{\infty} h(-\alpha) \cdot R_{xx}(\tau - \alpha) d\alpha \right]$$

$$\text{Let } \alpha = -\alpha \Rightarrow d\alpha = -d\alpha$$

$$= h(\tau) * \left[\int_{-\infty}^{\infty} h(\alpha) R_{xx}(\tau + \alpha) d\alpha \right]$$

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\beta) \cdot R_{xx}(\tau + \alpha - \beta) d\beta \cdot d\alpha \cdot h(\alpha) d\alpha$$

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) R_{xx}(\tau + \alpha - \beta) d\beta \cdot d\alpha$$

This is in terms of Integration.

This is about Auto Correlation fn.

The Avg power, $E[y^2(t)] = P_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$. This is the Mean square value

power of o/p. R.P

$$P_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{xx}(\omega) d\omega$$

Mean value:- $E[y(t)] =$

$$E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) d\alpha\right]$$

$$= \int_{-\infty}^{\infty} h(\alpha) E[x(t-\alpha)] d\alpha$$

if $x(t)$ is WSS,

$\downarrow$
Constant

$$E[x(t)] = E[x(t-\alpha)] = \bar{x} = \text{Constant}$$

$$\bar{y} = E[y(t)] = \bar{x} \int_{-\infty}^{\infty} h(\alpha) d\alpha$$

$\bar{y} = \text{Constant}$.

Area under impulse response

$\therefore y(t)$ is also WSS. proved.

Cross-Correlation functions of i/p $x(t)$ & o/p $y(t)$:-

Assume $x(t)$ & $y(t)$ are WSS.

$$R_{xy}(\tau) = E[x(t)y(t+\tau)]$$

$$R_{xy}(\tau) = E\left[x(t) \int_{-\infty}^{\infty} h(\alpha) x(t+\tau-\alpha) d\alpha\right]$$

$$= \int_{-\infty}^{\infty} h(\alpha) E[x(t)x(t+\tau-\alpha)] d\alpha$$

$$= \int_{-\infty}^{\infty} h(\alpha) R_{xx}(\tau-\alpha) d\alpha$$

$$R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

$$\therefore R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

||y,

$$R_{yx}(\tau) = E[y(t)x(t+\tau)]$$

$$R_{yx}(\tau) = E\left[\int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) x(t+\tau) d\alpha\right]$$

$$= \int_{-\infty}^{\infty} h(\alpha) E[x(t-\alpha)x(t+\tau)] d\alpha$$

$$= \int_{-\infty}^{\infty} h(\alpha) R_{xx}(\tau+\alpha) d\alpha$$

$$\therefore t_2 - t_1 = t + \tau - t + \alpha = \tau + \alpha$$

$$= h(\tau) * R_{xx}(\tau)$$

$$\therefore R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

And Cross Correlation is also a function of ' τ '

$$R_{yx}(\tau) = E[Y(t)X(t+\tau)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\alpha) X(t-\alpha) X(t+\tau) d\alpha\right]$$

$$= \int_{-\infty}^{\infty} h(\alpha) E[X(t-\alpha)X(t+\tau)] d\alpha$$

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} h(\alpha) R_{xx}(\tau+\alpha) d\alpha$$

$$= h(-\tau) * R_{xx}(\tau)$$

$$R_{yx}(\tau) = h(-\tau) * R_{xx}(\tau)$$

$$\Rightarrow \therefore R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$$

$$F.T = S_{xy}(\omega) = H(\omega) \cdot S_{xx}(\omega)$$

$$\Rightarrow \therefore R_{yx}(\tau) = h(-\tau) * R_{xx}(\tau)$$

$$FT = S_{yx}(\omega) = H^*(\omega) S_{xx}(\omega)$$

$$= S_{xy}^*(\omega) \text{ for real R. Process}$$

$\therefore$ The Cross Correlation fn depends only on ' τ '. So, $x(t)$ & $y(t)$ are "jointly WSS".

18/6/21

Problems:-

① A stationary R.P has a two sided power Spectral density given by $S_x(\omega) = \frac{24}{\omega^2 + 16}$. Find the mean square value of the process.

Sol:

$\therefore$ Mean Square Value is nothing but Power

$$\bar{X}^2 = P_x = P_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$\left(\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{24}{\omega^2 + 16} d\omega = \frac{12}{\pi} \int_{-\infty}^{\infty} \frac{1}{4^2 + \omega^2} d\omega$$

$$= \frac{12}{\pi} \cdot \frac{1}{4} \left[\tan^{-1} \frac{\omega}{4} \right]_{-\infty}^{\infty} = \frac{3}{\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$P_{xx}(\omega) = 3 \text{ Watts}$$

$$(a) \quad \therefore S_x(\omega) = \frac{24}{\omega^2 + 16}$$

$$\therefore e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2 + \omega^2} \quad \left(\because R_{xx}(\tau) \xleftrightarrow{FT} S_{xx}(\omega) \right)$$

$$R_{xx}(\tau) = 3 \cdot e^{-4|\tau|}$$

$$R_{xx}(0) = \overline{x^2} = P_x = 3 \text{ Watts}$$

2. A stationary random process has spectral density given below.

$$S_x(f) = 4\delta(f) + 18\delta(f-8) + 18\delta(f+8).$$

Find the autocorrelation function of the process.

Sol:

$$S_x(f) = 4\delta(f) + 18\delta(f-8) + 18\delta(f+8) \quad \left(\because F(\omega - \omega_0) = e^{j\omega_0 t} \right)$$

Apply I.F.T

$$\begin{aligned} \text{ACF} = R_{xx}(\tau) &= 4(1) + 18 \cdot e^{j2\pi 8\tau} + 18 \cdot e^{-j2\pi 8\tau} \\ &= 4 + 36 \cos(16\pi\tau) \end{aligned}$$

3. A R.P $x(t) = M$, where M is a random variable with mean value 2 and Variance 6, is applied to a LTI sys with the following impulse response. $h(t) = 5e^{-3t}u(t)$.

Find the mean value of the o/p process.

Sol:

$$\begin{aligned} \text{o/p Mean Value } E[y(t)] &= \bar{x} \int_{-\infty}^{\infty} h(t) dt \\ &= 2 \int_{-\infty}^{\infty} 5e^{-3t} u(t) \cdot dt \end{aligned}$$

$$= 2 \int_0^{\infty} 5e^{-3t} dt = 10 \left[\frac{e^{-3t}}{-3} \right]_0^{\infty} = -\frac{10}{3} [e^{-\infty} - e^{-0}]$$

$$= -\frac{10}{3} [0 - 1] = \frac{10}{3}$$

4. Find the autocorrelation fn for the following power spectral density

$$S_x(\omega) = \frac{\omega^2}{(1+\omega^2)}$$

Sol:

$$R_x(\tau) \xleftrightarrow{FT} S_x(\omega)$$

Apply I.F.T

$$R_x(\tau) = \delta(\tau) - \frac{1}{2} e^{-|\tau|}$$

$$\therefore 1 - \frac{1}{2} \frac{2(1)}{1+\omega^2}$$

$$\left(\because e^{-a|t|} \xleftrightarrow{FT} \frac{2a}{a^2 + \omega^2} \right)$$

5. A R.P having autocorrelation $R_x(\tau) = \frac{\sin(20\pi\tau)}{\pi\tau}$ is applied to an LTI Sys having impulse response, $h(t) = e^{-t}u(t)$. Find the mean Square Value of the output.

Sol: Power is nothing but mean square value.

$$R_{yy}(\tau) = h(\tau) * h(-\tau) * R_{xx}(\tau) \rightarrow \text{in time domain (Complex)}$$

So, in freq domain,

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \quad \left(\because \text{sinc } x = \frac{\sin \pi x}{\pi x} \right)$$

$$\text{Given } R_{xx}(\tau) = \frac{\sin(20\pi\tau)}{\pi\tau} \quad \therefore \frac{W}{\pi} \text{sinc}(W\tau) \leftrightarrow \text{rect}\left(\frac{\omega}{2W}\right)$$

$$R_{xx}(\tau) = 20 \cdot \text{sinc}(20\pi\tau) \quad W = 20\pi$$

After F.T,

$$S_{xx}(\omega) = \pi \text{rect}\left(\frac{\omega}{40}\right)$$

$$W \text{sinc}(W\tau) \leftrightarrow \pi \text{rect}\left(\frac{\omega}{2W}\right)$$

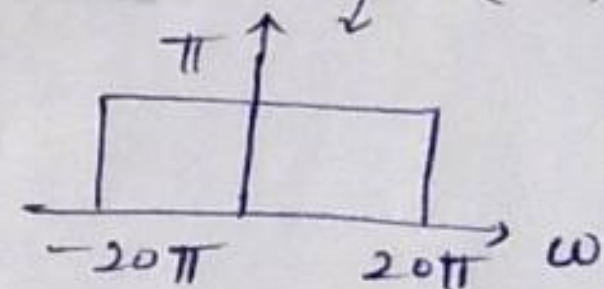
$$\leftrightarrow \pi \text{rect}\left(\frac{\omega}{40}\right)$$

$$h(t) = e^{-t}u(t)$$

$$H(\omega) = \frac{1}{1+j\omega}$$

$$|H(\omega)|^2 = \frac{1}{1+\omega^2}$$

$$S_{yy}(\omega) = \frac{1}{1+\omega^2} \cdot \pi \text{rect}\left(\frac{\omega}{40}\right)$$



$$\therefore \text{Power } (P_{yy}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-20\pi}^{20\pi} \pi \frac{1}{1+\omega^2} d\omega = \frac{1}{2\pi} \pi \left(\tan^{-1} \omega \right)_{-20\pi}^{20\pi}$$

$$= \frac{1}{2} \left(\tan^{-1} 20\pi - \tan^{-1}(-20\pi) \right) = 1.52$$

0.49 W

6. A Stationary R.P $x(t)$ having autocorrelation fn

$$R_x(\tau) = 5e^{-5|\tau|}$$

Find the auto correlation fn of another random process $y(t)$, given as $y(t) = x(t) - 2x(t-1)$. Also find the avg power in $y(t)$.

Sol:

$$R_x(\tau) = 5e^{-5|\tau|}$$

$$y(t) = x(t) - 2x(t-1)$$

$$R_y(\tau) = E[y(t)y(t+\tau)] = 5R_{xx}(\tau) - 2R_x(\tau+1) - 2R_x(\tau-1)$$

$$\text{Avg Power in } y(t) = R_y(0) = 5R_{xx}(0) - 2R_x(0+1) - 2R_x(0-1)$$

$$R_y(0) = 5(5) - 2(5)e^{-5} - 10e^{-5} = 25 - 10e^{-5}$$

$$\boxed{\text{Avg power} = 25 - 10e^{-5}}$$

7. A Stationary R.P $x(t)$ has an autocorrelation fn of the form

$$R_x(\tau) = 100e^{-\tau^2} \cos 2\pi\tau + 36$$

Find the mean value, mean square value, and the Variance of this process.

Sol:

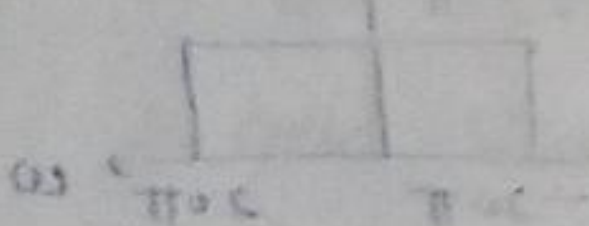
Mean Value:- $\lim_{T \rightarrow \infty} R_{xx}(\tau) = (\bar{x})^2$

$$(\bar{x})^2 = \lim_{T \rightarrow \infty} 100e^{-\tau^2} \cos 2\pi\tau + 36$$

$$= 100 \underbrace{e^{-\infty}}_0 \cos 2\pi(\infty) + 36$$

$$\bar{x} = \sqrt{36} = \pm 6 \quad \therefore \text{Mean Value} = \pm 6 //$$

Mean Square Value:-



$$\lim_{T \rightarrow \infty} R_{xx}(\tau) = \bar{x}^2$$

$$\bar{x}^2 = \lim_{T \rightarrow \infty} 100e^{-\tau^2} \cos 2\pi\tau + 36$$

$$\bar{x}^2 = 100 + 36 = 136 \quad \therefore \text{M.S.V} = 136 //$$

Variance:-

$$V(x) = -(\bar{x})^2 + \bar{x}^2$$

$$\therefore V(x) = \bar{x}^2 - (\bar{x})^2$$

$$= -(\pm 6)^2 + 136$$

$$= -36 + 136 = 100 //$$

8. A Stationary R.P has a spectral density of

$$S_x(\omega) = \frac{1}{\omega^4 + 5\omega^2 + 4}$$

mean square value

of the process. (Use Partial fractions to take I.F.T)

Sol:

$$R_x(\omega) \xrightarrow{\text{F.T}} S_x(\omega)$$

$$S_x(\omega) = \frac{(\omega^2 + 4)(\omega^2 + 1)}{\omega^4 + 5\omega^2 + 4}$$

$$S_x(\omega) = \frac{1}{(\omega^2 + 4)(\omega^2 + 1)} = \frac{A}{\omega^2 + 4} + \frac{B}{\omega^2 + 1}$$

$$A = (\omega^2 + 4) \frac{1}{(\omega^2 + 4)(\omega^2 + 1)} \Big|_{\omega = \sqrt{-4}} = \frac{1}{(\sqrt{-4})^2 + 1} = \frac{1}{-3} = -\frac{1}{3}$$

$$B = (\omega^2 + 1) \frac{1}{(\omega^2 + 4)(\omega^2 + 1)} \Big|_{\omega = \sqrt{-1}} = \frac{1}{(\sqrt{-1})^2 + 4} = \frac{1}{3} //$$

$$S_x(\omega) = \frac{-\frac{1}{3}}{(\omega^2 + 4)} + \frac{\frac{1}{3}}{(\omega^2 + 1)}$$

Apply I.P.T,

$$R_x(\tau) = -\frac{1}{3}$$

9. Find the power in freq band of 100 KHz centered at 0.5 MHz for the random process having two sided power Spectral density given below (Consider both positive and negative frequencies for power calculation).

Sol:

$$P_x(\omega) P_{xx}(f) = \int_{-\infty}^{\infty} S_x(f) \cdot df$$

$$P_x(f) = 2 \int_{450 \text{ KHz}}^{550 \text{ KHz}} S_x(f) df$$

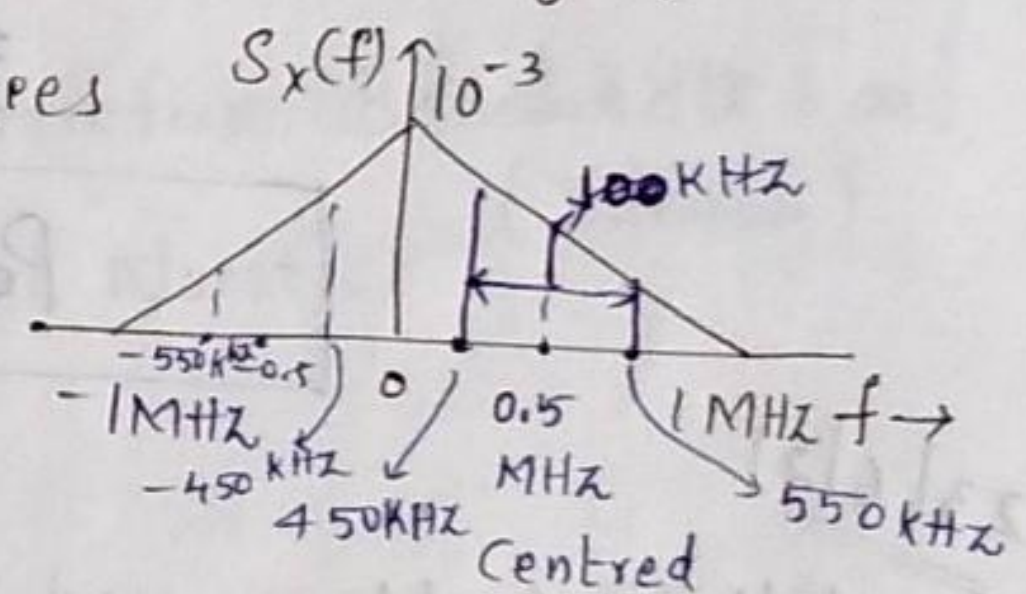
$$= 2 \cdot 10^{-9} \int_{450 \text{ KHz}}^{550 \text{ KHz}} (10^6 - f) df$$

$$= 2 \left[10^{-9} \left(10^6 f - \frac{f^2}{2} \right) \right]_{450 \text{ KHz}}^{550 \text{ KHz}}$$

$$= 2 \times 10^{-9} \left[10^6 (550 \times 10^3 - 450 \times 10^3) - \frac{1}{2} [(550 \times 10^3)^2 - (450 \times 10^3)^2] \right]$$

$$= 2 \times 10^{-9} \left[10^{10} (55 - 45) - \frac{1}{2} (10^8) [3025 - 2025] \right]$$

$$= 2 \times 10^{-9} \left(10^{11} - \frac{1}{2} 10^{11} \right) = 2 \times 10^{-9} \times \frac{1}{2} \times 10^{11} = 10^2 = 100 \text{ W}$$



follows even symmetry

(0, 10⁻³) (10⁶, 0)

$$\text{Slope} = \frac{0 - 10^{-3}}{10^6 - 0} = -10^{-3} \times 10^{-6} = -10^{-9}$$

$$y - 10^{-3} = -10^{-9}(f - 0)$$

$$S_x(f) = -10^{-9}f + 10^{-3}$$

$$= 10^{-9}(10^6 - f)$$

11. A white noise having autocorrelation fn $R_x(\tau) = 10\delta(\tau)$ is applied to an LTI sys with impulse response of $h(t) = e^{-2t}u(t)$. Find the avg power of the random process at the system output.

Sol:

$$\begin{cases} R_x(\tau) = 10\delta(\tau) & \text{timedomain} \\ h(t) = e^{-2t}u(t) \end{cases}$$

$$S_x(\omega) = 10(1) = 10$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$H(\omega) = \frac{1}{\omega + j2} \Rightarrow |H(\omega)|^2 = \frac{1}{(\sqrt{\omega^2 + 2^2})^2} = \frac{1}{\omega^2 + 4}$$

$$S_{yy}(\omega) = \frac{1}{\omega^2 + 4} \cdot 10$$

$$P_{yy}(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yy}(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + 4} \cdot 10 \cdot d\omega = \frac{10}{2\pi} \cdot \frac{1}{2} \left[\tan^{-1} \frac{\omega}{2} \right]_{-\infty}^{\infty}$$

$$= \frac{10}{2\pi} \left[\tan^{-1} \infty - \tan^{-1}(-\infty) \right] = \frac{10}{2\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$= \frac{10}{2\pi} \cdot \pi = 2.5 \text{ W}$$

$$\therefore \text{Power} = 2.5 \text{ W}$$

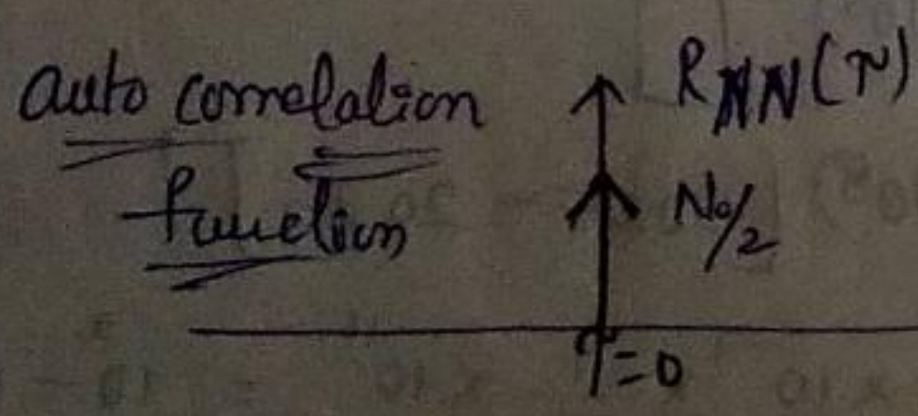
23/6/21

White light:- White light Contains equal amount of all freqs in the Visible spectrum.

White Noise:- The spectral density is constant for all frequencies.

White Noise
+t +\tau

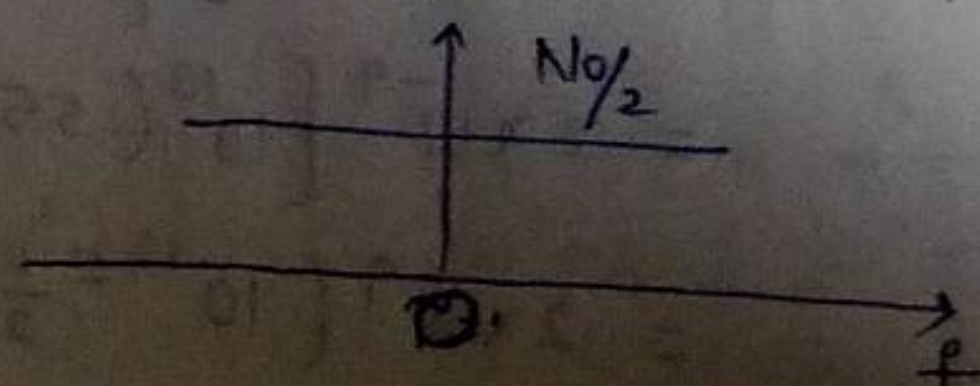
-freq is different at different particular time instants



White Noise
+t +\tau

-freq remain same

Power Spectral density
 $S_{NN}(f)$

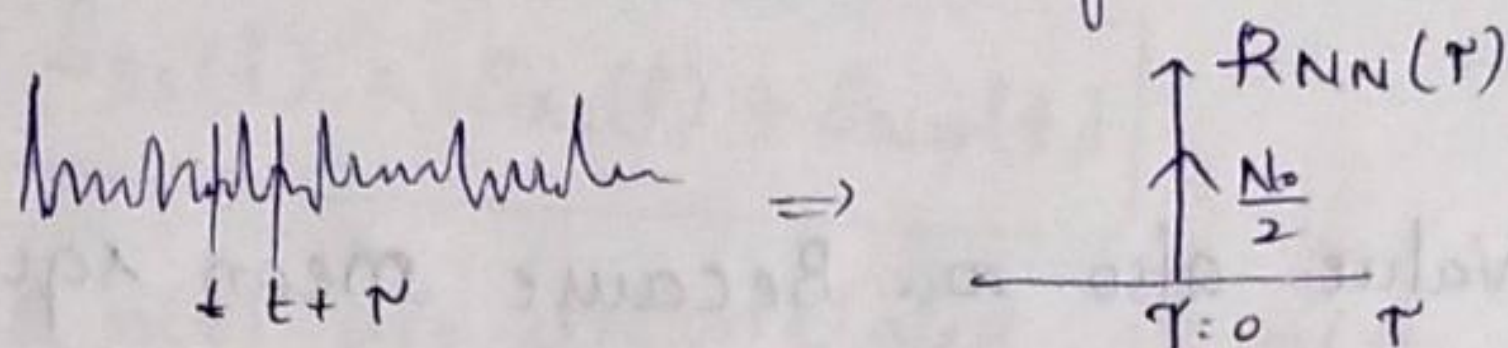


∴ Power Spectral density for white noise $(S_{NN}(f)) = \frac{N_0}{2}$
 that means it has equal power for each & every freq.
 where, N_0 :- Units :- Watts/Hz.

Auto correlation function $R_{NN}(\tau) = \frac{N_0}{2} \delta(\tau)$

* Auto correlation fn of delta white noise is just a $\delta(\tau)$ delta function whose area is $\frac{N_0}{2}$ (spectral density).

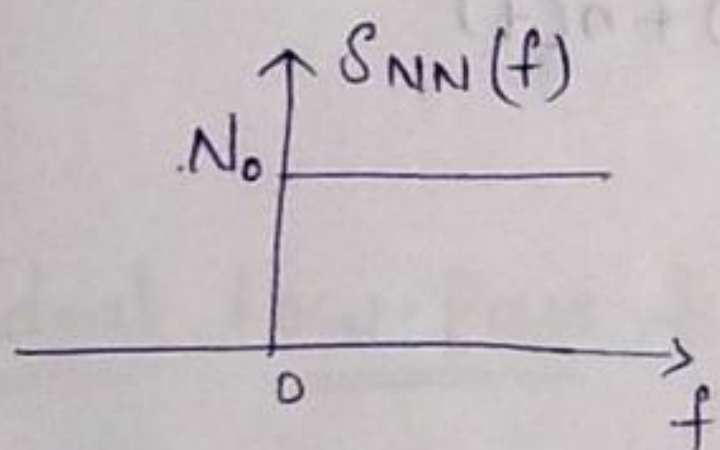
* Auto correlation fn exists only at $\tau=0$ because it is uncorrelated.



Since, we know,

power spectral density is even fn of ω . So we calculate for only one side & multiplied by 2.

* $\frac{N_0}{2}$ is the two sided Spectral density.



N_0 - one sided power spectral density.

* White noise is a wide band signal. Therefore Bandwidth $= \infty$.

* Practical ex. for white noise is "Thermal Noise". (Ideally)

Note:-

$$\text{Power} = \frac{V^2}{R}$$

But Here we consider power $= E[V^2]$

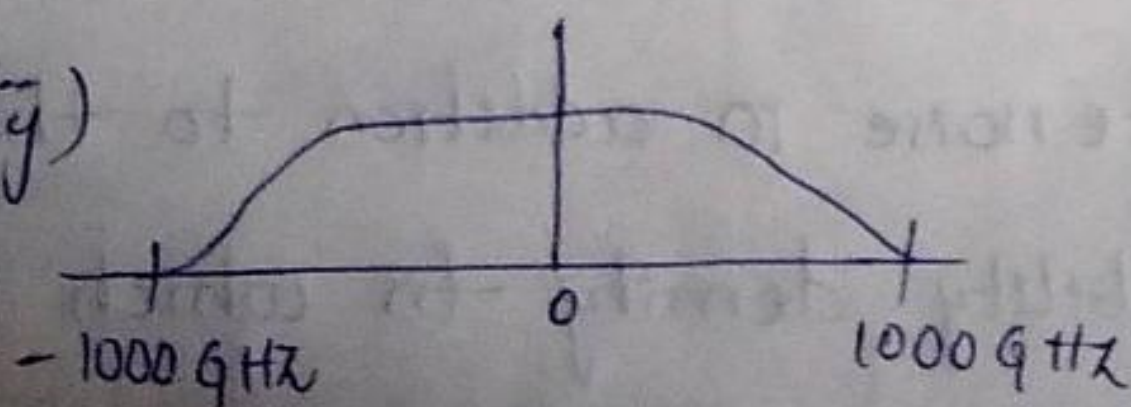
where 'V' is any random Variable. By taking $R=1\Omega$.

But in problems we consider 'R' value. If for current $E[I^2]$

By taking $R=1$ in $I^2 R$.

* Ex:- Thermal Noise. Has Bandwidth $\cong 10^{12}$ Hz = 1000 GHz

(practically)



* White noise is also Unrealizable physically.

Power :-

Power of white Noise $P_{NN}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{NN}(\omega) d\omega$

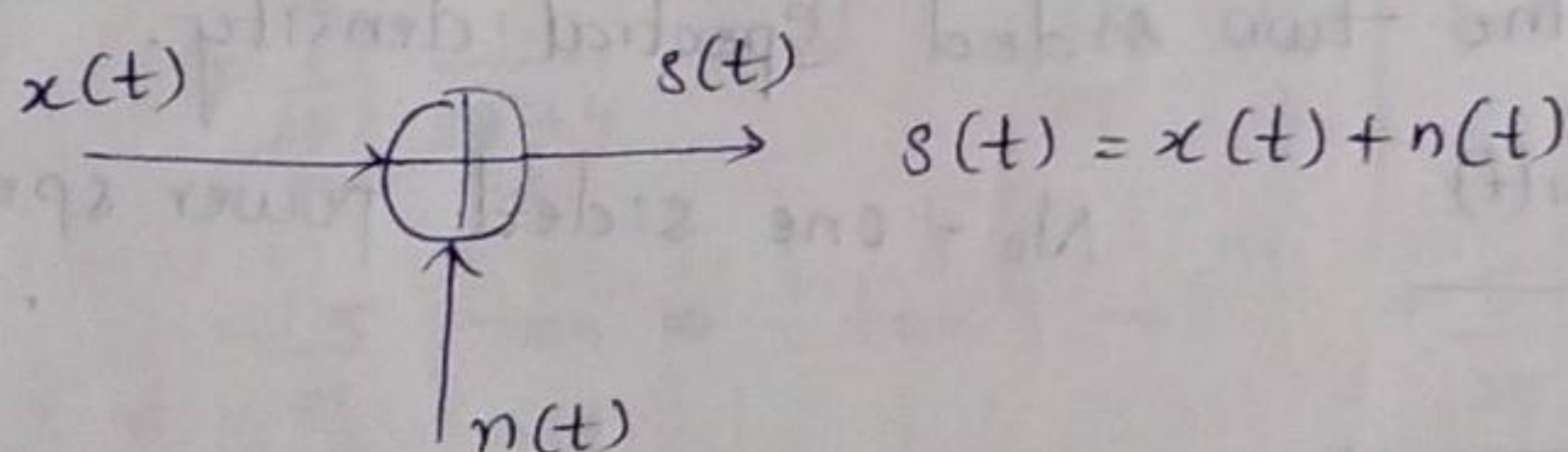
$$P_{NN}(f) = \int_{-\infty}^{\infty} S_{NN}(f) df$$

$$\left(P_{NN} = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty \right) \rightarrow \text{Avg power of white Noise is also infinity.}$$

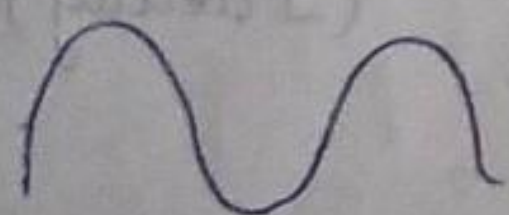
- * Since Random signal is uncorrelated so, that only we are getting Auto Correlation fn as impulse (exist at $\tau=0$ only) for white Noise.
- * Mean square Value also ∞ . Because mean square Value & Power are same, Because $R=1\Omega$.

Additive white Gaussian Noise :- (AWGN) :-

- * This is the Ultimate Noise in Communication system.



In CRO if we observe the Sinusoidal Signal it looks like



But it exist as



if we zoom.

This is due to temperature & the Voltage is nothing but thermal Voltage & this Noise is nothing but thermal Noise. It is applicable to both Current & Voltage. Both are random in nature & this Value is Very small because thermal Noise is Very small. And this randomness is superimposing on Original sinusoidal. Therefore Thermal noise is additive in nature. Hence, we defined, AWGN as, Additive, white noise in addition to that if the random process the probability density fn which is nothing but gaussian fn. so it is called as AWGN.

- * we already know that white Noise is Uncorrelated.
- * So, if Samples are Uncorrelated then they are orthogonal or not? Independent directly indicates $\rightarrow$ Uncorrelation but Uncorrelation not always means it is Independent.

* If it is gaussian then these are

$$s(t) = x(t) + n(t) \quad \text{Orthogonal to each other.}$$

* If those are Orthogonal then Power Spectral density, is

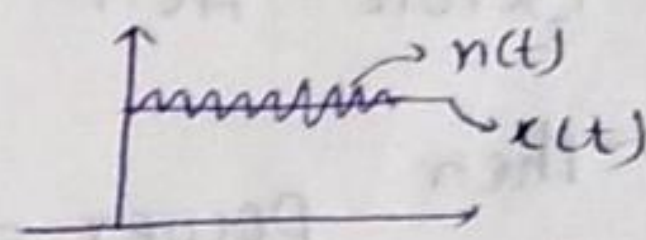
$$\underline{\text{Imp}} \quad \boxed{S_{ss}(f) = S_{xx}(f) + S_{nn}(f)}$$

$\therefore x(t)$ & $n(t)$ are uncorrelated

i.e, $x(t)$ & $n(t)$ are uncorrelated, if $n(t)$

is Gaussian then $x(t)$ & $n(t)$ are

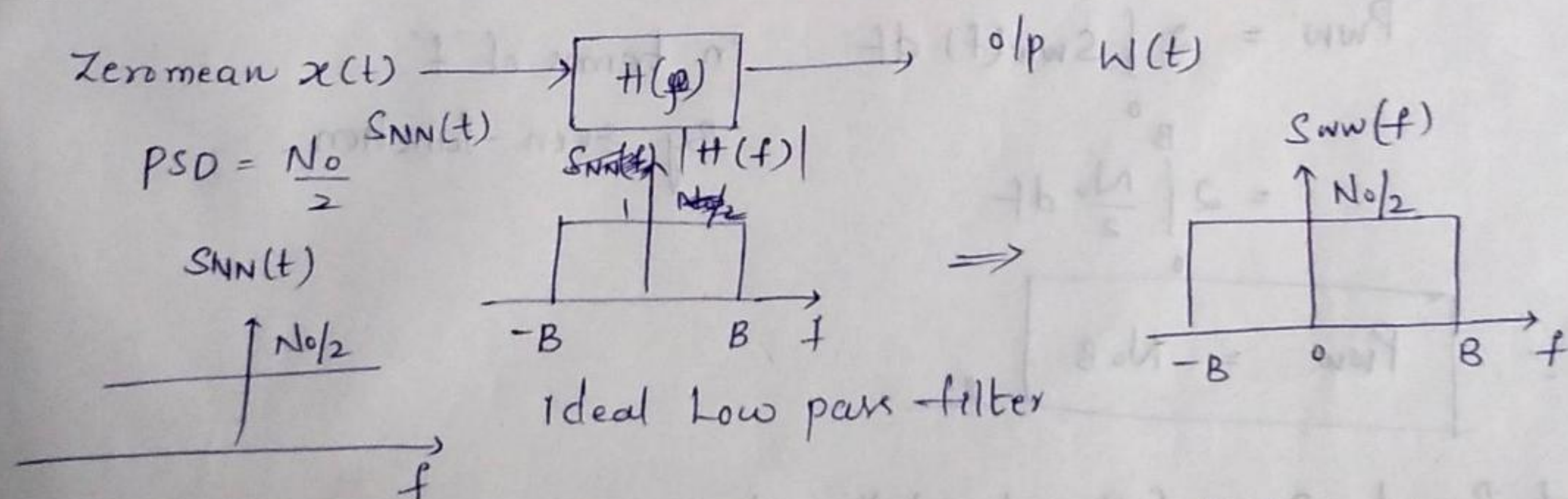
Orthogonal.



— then power = mean square value $\} = \overline{s^2} = \text{Addition of individual powers}$

$$\text{i.e, } \boxed{\overline{s^2} = \overline{x^2} + \overline{n^2}}$$

Ideal Low-Pass filter White Noise:-



* This is about when white noise is passed through a system, resultant power spectral density $S_{ww}(f)$.

$$\therefore S_{ww}(f) = |H(f)|^2 S_{nn}(f)$$

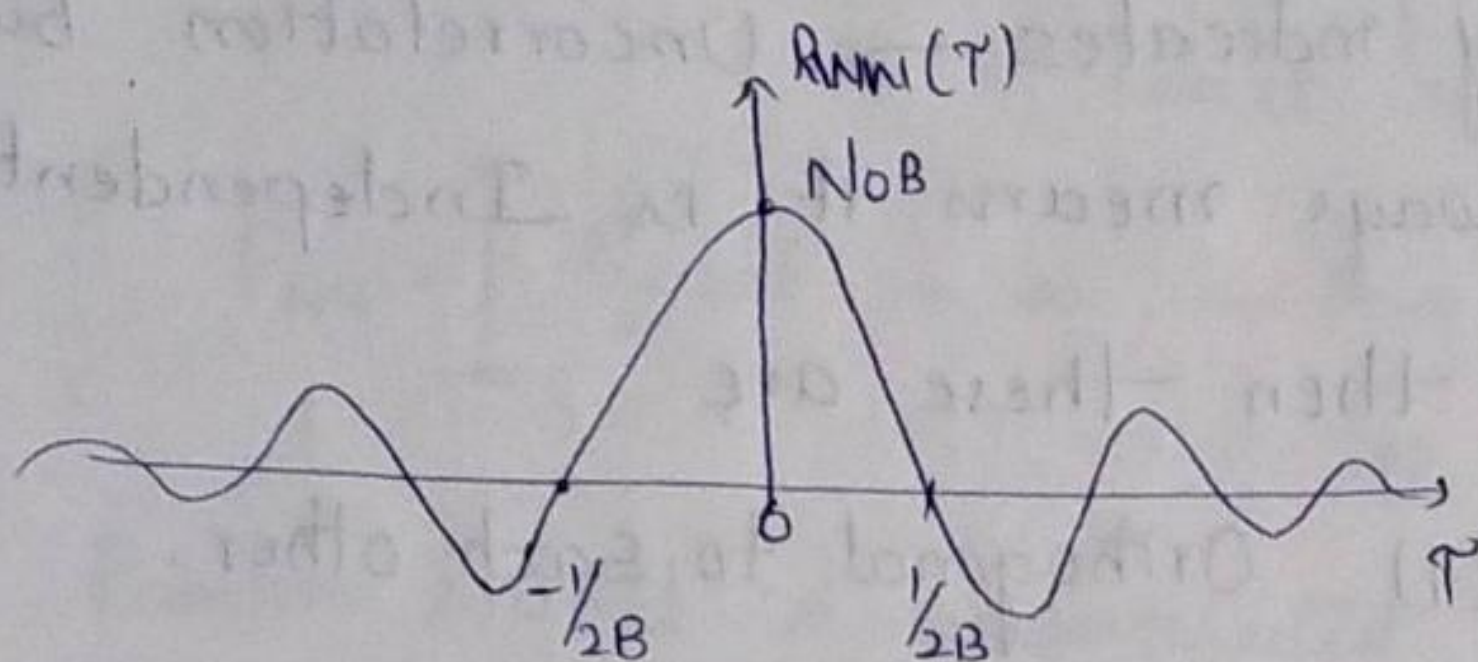
$$\text{Power Spectral density} = S_{ww}(f) = \begin{cases} \frac{N_0}{2} & , -B < f < B \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore \text{FT}^{-1} \{ S_{xx}(w) \} = R_{xx}(\tau)$$

then, Autocorrelation function $\} = R_{ww}(\tau) = \int_{-B}^B \frac{N_0}{2} \cdot e^{j2\pi f\tau} df$

By solving we get,

$$= N_0 B \text{ Sinc}(2B\tau)$$



This is the Auto Correlation fn of "Ideal low pass filtered white Noise".

* Before A.C.F $R_{NN}(\tau)$ is impulse $\uparrow (\delta)$. But Now it is sinc fn.
then power = ?

Power can be solved in 2 ways.

i) $P_{ww} = R_{ww}(0) =$ we get power (or)

ii) $P_{ww} = \int_{-\infty}^{\infty} S_{ww}(f) df$ (or) $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ww}(\omega) d\omega$

$$P_{ww} = R_{ww}(0) = N_0 B$$

$$P_{ww} = 2 \int_0^{\infty} S_{ww}(f) df \quad \text{in terms of 'f'}$$

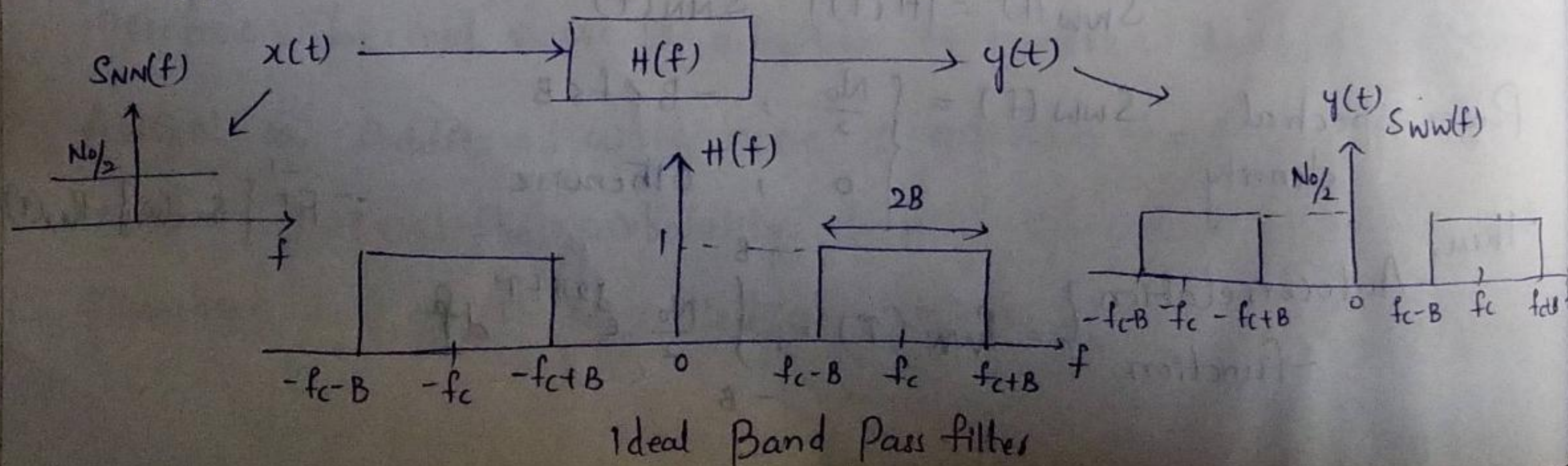
By even function.

$$= 2 \int_0^B \frac{N_0}{2} df$$

$$P_{ww} = N_0 B$$

Ideal Band-pass filtered white Noise:-

It is nothing but what happens when it pass Band-pass filter.



$$S_{ww}(f) = |H(f)|^2 S_{nn}(f)$$

$$i) \text{ PSD} = S_{ww}(f) = \begin{cases} \frac{N_0}{2} & ; \quad -f_c - B \text{ to } -f_c + B \text{ \& } f_c - B \text{ to } f_c + B \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

$$ii) \text{ ACF} = R_{ww}(\tau) = 2 \int_{f_c - B}^{f_c + B} \frac{N_0}{2} \cdot e^{j2\pi f\tau} \cdot df$$

The RMS Bandwidth:-

$\therefore$ Probability density fn

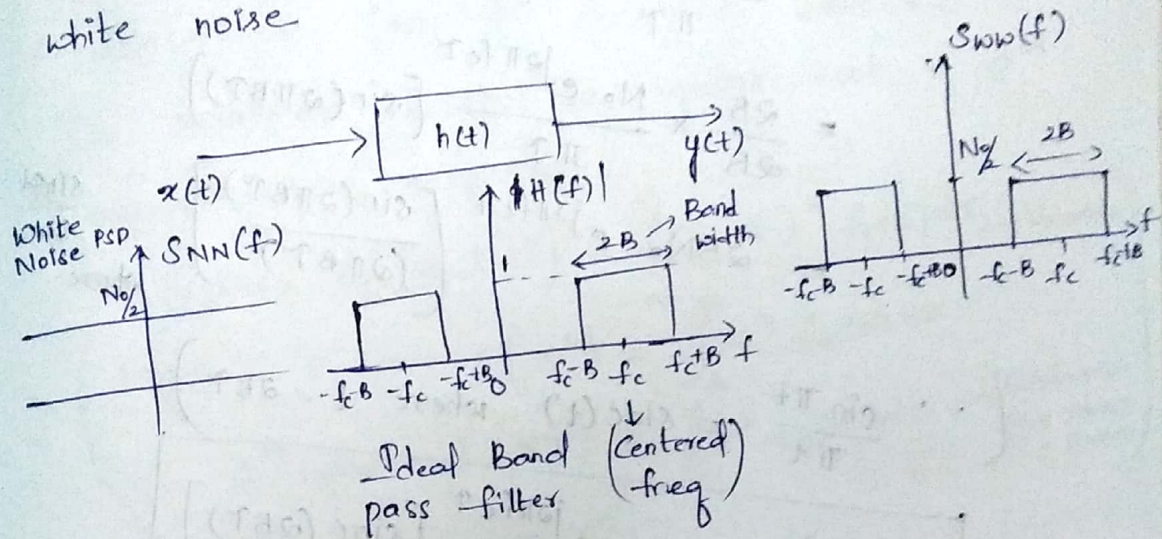
23/06/21

Unit-II (Continuation)

(from 12/06/21 to 23/06/21 → Saranya Notes)

Remaining topics:Ideal Band-pass filtered white Noise:-

The % of Band-pass filter when the i/p is white noise



$$S_{WW}(f) = |H(f)|^2 \cdot S_{NN}(f)$$

$$= (1)^2 \cdot \frac{N_0}{2} = \frac{N_0}{2}$$

$$\therefore S_{WW}(f) = \begin{cases} \frac{N_0}{2} & |f_c - B| < f < |f_c + B| \\ 0 & \text{otherwise} \end{cases}$$

The Auto Correlation $R_{WW}(\tau) = \int_{-f_c+B}^{f_c+B} \frac{N_0}{2} \cdot e^{j2\pi f\tau} df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \cdot e^{j2\pi f\tau} df$

$$= 2 \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \cdot e^{j2\pi f\tau} df \quad (\because \text{Even Symmetry})$$

$$= \frac{N_0}{2} \int_{f_c-B}^{f_c+B} e^{j2\pi f\tau} df = N_0 \left[\frac{e^{j2\pi f\tau}}{j2\pi\tau} \right]_{f_c-B}^{f_c+B}$$

$$= \frac{N_0}{j(2\pi\tau)} \left[e^{j2\pi(f_c+B)\tau} - e^{j2\pi(f_c-B)\tau} \right]$$

$$= \frac{N_0}{j(2\pi\tau)} \left[e^{j2\pi f_c \tau} \cdot e^{j2\pi B \tau} - e^{j2\pi f_c \tau} \cdot e^{-j2\pi B \tau} \right]$$

$$= \frac{N_0 \cdot e^{j2\pi f_c \tau}}{\pi \tau} \left[\frac{e^{j2\pi B \tau} - e^{-j2\pi B \tau}}{2j} \right]$$

$$= \frac{N_0 \cdot e^{j2\pi f_c \tau}}{\pi \tau} \left[\sin(2\pi B \tau) \right]$$

$$= \frac{2B}{2B} \times \frac{N_0 \cdot e^{j2\pi f_c \tau}}{\pi \tau} \left[\sin(2\pi B \tau) \right]$$

$$= 2BN_0 \cdot e^{j2\pi f_c \tau} \left[\frac{\sin(2\pi B \tau)}{(2\pi B \tau)} \right] \quad \text{where } \frac{\sin t}{t} = \text{sinc}(t)$$

$$\left(\because \frac{\sin \pi t}{\pi t} = \text{sinc}(t) \text{ where } t = 2B\tau \right)$$

$$R_{ww}(\tau) = 2BN_0 \cdot e^{j2\pi f_c \tau} \left[\text{sinc}(2B\tau) \right]$$

$$\text{Power } (P_{ww}(\tau)) = R_{ww}(0) = 2BN_0 \cdot e^0 \left[\text{sinc}(0) \right]$$

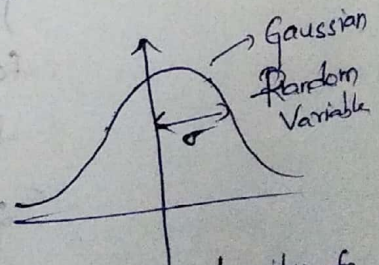
$$P_{ww}(\tau) = 2BN_0$$

$$\left(\because \text{sinc}(0) = 1 \text{ \& } e^0 = 1 \right)$$

Root Mean Square Band Width:-

* Variance is how much it is varied from mean ~~square~~ value.

$$\text{Standard deviation } (\sigma) = \sqrt{V(x)}$$



* Power density resembles the probability density fn but the difference is area.

i.e., Area under pdf is '1' $\left(\int_{-\infty}^{\infty} f_x(x) dx = 1 \right)$

but $\int_{-\infty}^{\infty} S_{xx}(f) \cdot df$ is not necessarily '1' ($\because$ power can be any value)

The normalization is doing in order to get unity.

* For the normalized power spectrum, measure of spread (standard deviation) is "RMS B.W" denoted by W_{rms} . (Units: - rad/s)

* $S_{xx}(\omega)$ is Even fn & Mean Value = 0

Its standard deviation is square root of 2nd moment about mean (μ_2)

$$\left(\begin{array}{l} \therefore \mu_n = E[(x - \bar{x})^n] = \int_{-\infty}^{\infty} (x - \bar{x})^n \cdot f_x(x) dx \rightarrow \text{Variance} \\ \text{Mean is '0'} \Rightarrow E[x^n] = \int_{-\infty}^{\infty} x^n \cdot f_x(x) dx \\ 2^{nd} \text{ moment} \rightarrow E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot f_x(x) \cdot dx \rightarrow \text{Variance when mean is zero} \end{array} \right)$$

$$\therefore \sigma = \sqrt{E[x^2]}$$

$$\Rightarrow W_{rms}^2 (\text{RMS BW}) = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{xx}(\omega) \cdot d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega) \cdot d\omega}$$

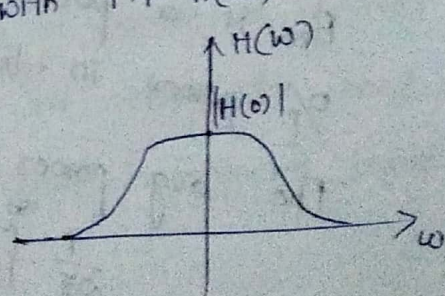
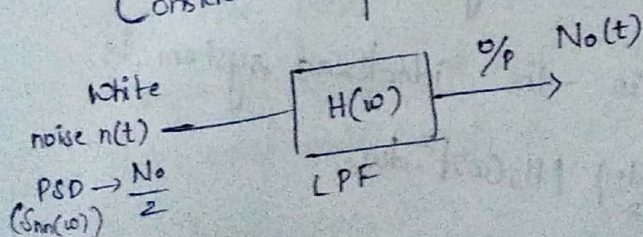
$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{xx}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega}$$

($\therefore$ Area of power spectral density is not always '1'. So, we are normalizing with respect to area)

25/06/21

Noise Band width:-

Consider a practical L.P.F with T.F $H(\omega)$



$$S_{N_0 N_0}(\omega) = |H(\omega)|^2 S_{nn}(\omega) = \frac{N_0}{2} |H(\omega)|^2$$

The total average power,

$$P_{N_0 N_0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_0}{2} |H(\omega)|^2 d\omega \rightarrow \textcircled{+}$$

if the sys impulse response is real, $|H(\omega)|^2$ will be an even function of ω .

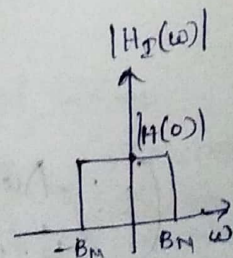
$$P_{N_0 N_0} = \frac{1}{2\pi} \cdot \frac{N_0}{\cancel{N_0}} \int_0^\infty |H(\omega)|^2 d\omega$$

$$P_{N_0 N_0} = \frac{N_0}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \rightarrow \textcircled{1}$$

* The % power from the ideal LPF is equal to the % power of these practical filters.
The required band width is ? $\therefore$

* Now consider an ideal LPF, i.e. equivalent to the actual system (practical LPF) in the sense that both produce the same o/p avg power when they both are excited by the same white noise source & $|H(\omega)|^2$ is also same in both the systems.

$$|H_I(\omega)|^2 = \begin{cases} |H(0)|^2 & |\omega| < B_N \\ 0 & |\omega| > B_N \end{cases}$$



✓
Ideal LPF
T.F

B_N is a positive constant selected to make
% power in the two system are equal.

The avg power in the idealized system is

$$\frac{r}{2\pi} \int_{-\infty}^{\infty} \left(\frac{N_0}{2}\right) \cdot |H_I(\omega)|^2 \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-B_N}^{B_N} \frac{N_0}{2} |H_I(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{N_0}{2} \int_0^{B_N} |H(\omega)|^2 d\omega \quad (\because \text{Even Symmetry})$$

$$= \frac{N_0}{2\pi} \int_0^{B_N} |H(\omega)|^2 d\omega = \frac{N_0}{2\pi} |H(\omega)|^2 [\omega]_0^{B_N}$$

$$= \frac{N_0}{2\pi} |H(0)|^2 \cdot B_N \rightarrow (2)$$

To find B_N (BW) we equate both avg powers (Ideal & practical LPF).

$$(1) = (2)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{N_0}{2} \right) |H(\omega)|^2 d\omega = \frac{N_0}{2\pi} |H(0)|^2 \cdot B_N$$

($\because$ from $-B_N$ to B_N
 $|H_I(\omega)|$ value is $|H(0)|$)

$$\frac{N_0}{2\pi} \int_0^{\infty} |H(\omega)|^2 d\omega = \frac{N_0}{2\pi} |H(0)|^2 \cdot B_N$$

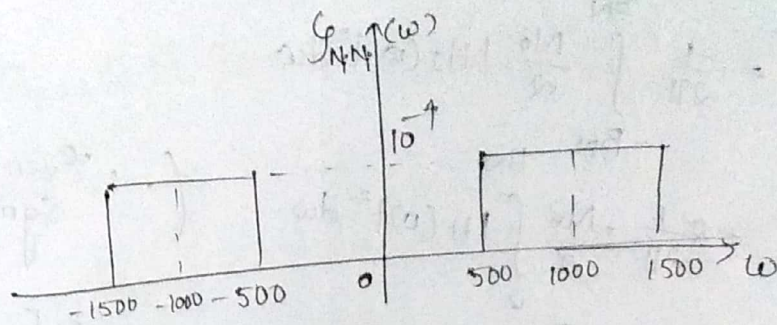
$$B_N (\text{Noise BW}) = \frac{\int_0^{\infty} |H(\omega)|^2 d\omega}{|H(0)|^2}$$

' B_N ' is called 'Noise Bandwidth' of the system.

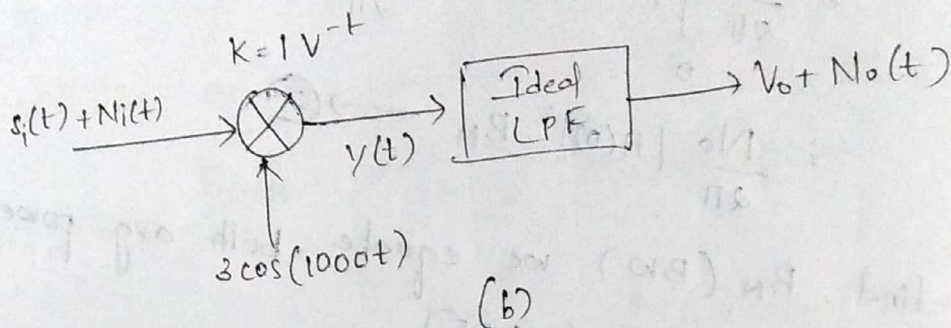
(Problems are in peebles textbook pg. No. :- 287 (Chapter 8))

28/06/21

7.6-10) A signal $s_i(t) = 2.3 \cos(1000t)$ plus an input noise process $N_i(t)$ having the power spectrum shown in Figure are applied to the product device shown in (b). The ideal LPF acts only to remove all spectral components (signal & noise) that are outside the band $|\omega| < 500 \text{ rad/s}$ and doesn't effect components inside the band.



(a)



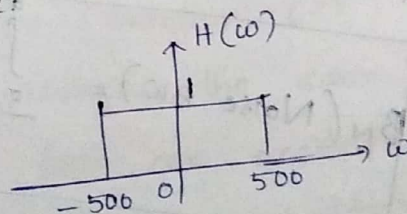
(b)

(a) Find the % dc level V_o .

(b) Sketch the Power density spectrum of $N_o(t)$

(c) What signal power to average noise power ratio, $V_o^2 / E[N_o^2(t)]$, occurs at %? Note that this ckt acts as a detector of signal's amplitude in presence of noise.

Sol:- The LPF is,



$$s_i(t) = 2 \cdot 3 \cos(1000t)$$

$N_i(t) \rightarrow \text{fig(a)}$

$[s_i(t) + N_i(t)] \times [3 \cos(1000t)] \rightarrow$ When this is passed through ideal LPF, the frequencies $|\omega| > 500$ are eliminated ($\because$ LPF range is $|\omega| < 500$)

$N_i(t)$ range is $1500 < \omega < 1500$

$N_i(t)$ value becomes zero after passing through LPF.

$$\therefore [s_i(t) + N_i(t)] \times [3 \cos(1000t)] \rightarrow (a) V_o = s_i(t) \times 3 \cos(1000t)$$

($\because$ They asked only %p dc level V_o)

$$V_o = 2.3 \cos(1000t) \times 3 \cos(1000t)$$

$$= 2.3 \times 3 \cdot (\cos^2(1000t)) = 2.3 \times 3 \left[\frac{1 + \cos 2(1000t)}{2} \right]$$

$$= 2.3 \times 3 \left[\frac{1}{2} \right] \quad (\because \text{Allows only range of } |\omega| < 500)$$

$$V_o = 3.45$$

(b) Power density spectrum of $N_o(t) = ?$

$$\text{Let } Y(t) = X(t) + N(t)$$

$$N(t) = N_i(t) \times 3 \cos(1000t)$$

The Auto Correlation fn,

$$R_{NN}(\tau) = E[(N_i(t) \times 3 \cos(1000t)) \cdot (N_i(t+\tau) \times 3 \cos(1000(t+\tau)))]$$

$$= \frac{9}{2} E[N_i(t) \cdot N_i(t+\tau)] [\cos(1000\tau) + \cos(2000t + 1000\tau)]$$

$$(\because \cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)])$$

Expectation term of second term is zero, because we are assuming Auto Correlation fn (ACF) as a Wide Sense Stationary (WSS), which depends only on ' τ '.

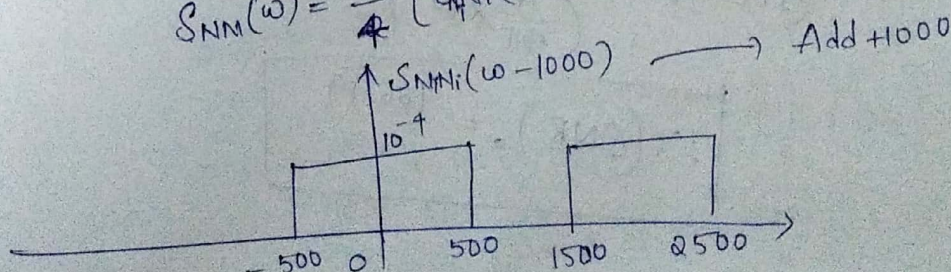
$$R_{NN}(\tau) = \frac{9}{2} R_{N_i N_i}(\tau) \cdot \cos(1000\tau)$$

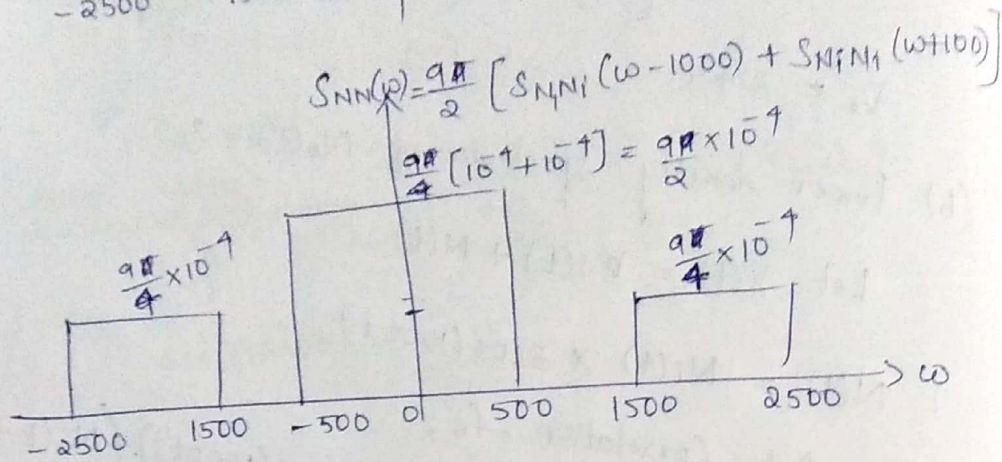
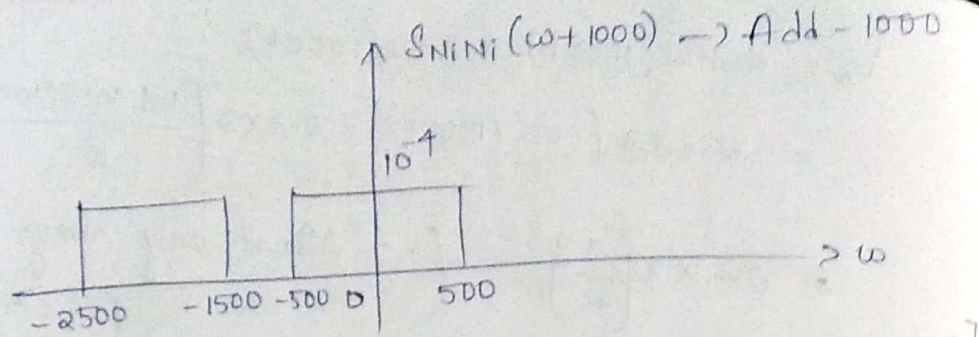
$$(\because E[N_i(t) \cdot N_i(t+\tau)] = R_{N_i N_i}(\tau))$$

Taking Fourier transform,

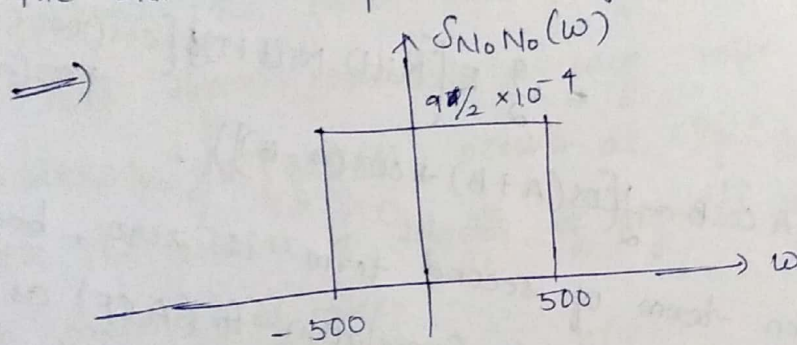
$$S_{NN}(\omega) = \frac{9}{2} [S_{N_i N_i}(\omega) * (\frac{1}{2} \delta(\omega - 1000) + \delta(\omega + 1000))]$$

$$S_{NN}(\omega) = \frac{9}{4} [S_{N_i N_i}(\omega - 1000) + S_{N_i N_i}(\omega + 1000)]$$





The $S_{NN}(\omega)$ is passed through Ideal LPF.



This is power density spectrum of $N_0(t)$.

(c) Signal to Noise Ratio at the o/p is,

$$(SNR)_o = \frac{\text{Signal power}}{\text{Noise Power}} = \frac{(V_o)^2}{E[N_o^2(t)]}$$

PSD $\rightarrow$ Noise Power $= \frac{1}{2\pi} \int_{-500}^{500} \frac{9}{2} \times \pi \times 10^{-4} d\omega = \frac{9}{4\pi} \times 10^{-4} \times 10^3$

$E[N_o^2(t)] = \frac{9}{4\pi} \times 10^{-1}$

$$(SNR)_o = \frac{(3.45)^2}{\left(\frac{9}{4\pi} \times 10^{-1}\right)} = 166.19$$

$$\therefore (SNR)_o = 166.19$$

8.4-11) White noise with power density $N_0/2 = 6(10^{-6})$ W/Hz is applied to an ideal filter (gain = 1) with bandwidth W (rad/s). Find W so that the %'s avg noise power is 15 watts.

Sol:- Given, $\frac{N_0}{2} = 6(10^{-6})$ W/Hz.

%'s Avg Noise Power ($P_{N_0 N_0}$) = 15 Watts

$$P_{N_0 N_0} = \frac{1}{2\pi} \int_0^W (6 \times 10^{-6}) d\omega = \frac{(6 \times 10^{-6})}{\pi} [W]$$

$$15 = \frac{(6 \times 10^{-6})}{\pi} (W)$$

$$W = \frac{\frac{15 \pi}{6 \times 10^{-6}}}{2} = \frac{5\pi}{2 \times 10^{-6}}$$

$$\boxed{\therefore W = \frac{5\pi}{2} \times 10^6}$$

8.5-6) White noise with power density $N_0/2$ is applied to a lowpass network for which $|H(\omega)| = 2$; it has a noise bandwidth of 2 MHz. If the average % noise power is 0.1 W in a 1 Ω resistor. What is N_0 ?

Sol:- $|H(\omega)| = 2$ $B_N = 2 \text{ MHz}$
 % Noise power for idealised LPE = $\frac{N_0}{2\pi} \int_0^{B_N} |H(\omega)|^2 d\omega$

$$\left(\therefore \% \text{ Noise power} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N_0}{2} |H_S(\omega)|^2 d\omega \right)$$

$$0.1 = \frac{N_0}{2\pi} \int_0^{B_N} (2)^2 d\omega$$

$$0.1 = \frac{N_0}{2\pi} \left[\omega \right]_0^{B_N} \Rightarrow 0.1 = \frac{2N_0}{\pi} [B_N]$$

$$\Rightarrow 0.1 = \frac{2N_0}{\pi} [2 \times 10^6]$$

$$0.1 = \frac{2N_0}{\pi} [B_N] \quad (\because B_N = 2M\text{Hz})$$

$$0.1 = \frac{2N_0}{\pi} [2 \times 10^6]$$

$$N_0 = \frac{0.1 \pi}{4 \times 10^6} = 0.07 \times 10^{-6}$$

3) Consider $X(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(2\pi k f_c t) + \beta_k \sin(2\pi k f_c t)$
 where all α_k, β_k are independent zero-mean Gaussian random variables with variance of α_k, β_k equal to $\frac{1}{2^k}$. What is the autocorrelation $R_{XX}(\tau)$ of this random process?

Sol:- Given, α_k, β_k are independent zero-mean gaussian random variables

A/c to the property of Gaussian random variable, independent means uncorrelated random variables.
 $\&$ uncorrelated means independent

$$X(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(2\pi k f_c t) + \beta_k \sin(2\pi k f_c t)$$

$$R_{XX}(\tau) = E \left[\left\{ \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(2\pi k f_c t) + \beta_k \sin(2\pi k f_c t) \right\} \times \left\{ \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(2\pi k f_c (t+\tau)) + \beta_k \sin(2\pi k f_c (t+\tau)) \right\} \right]$$

$$\text{Given mean zero} \Rightarrow E[\alpha_k] = E[\beta_k] = 0$$

$$\therefore E[\alpha_k \beta_k] = 0 \quad (\because \text{Uncorrelated})$$

$$E[\alpha_k^2] = E[\beta_k^2] = \frac{1}{2^k} \rightarrow \text{Variance}$$

$$E[\alpha_0^2] = \frac{1}{2^0} = 1$$

$$R_{xx}(\tau) = E \left[\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_k^2 \cos(2\pi k f_c t) \cos(2\pi k f_c (t+\tau)) + \sum_{k=1}^{\infty} \beta_k^2 \sin(2\pi k f_c t) \sin(2\pi k f_c (t+\tau)) \right]$$

$$R_{xx}(\tau) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} \cos(2\pi k f_c \tau)$$

$$\left(\begin{array}{l} \therefore E[\alpha^2] = E[\beta^2] = \frac{1}{2^k} \\ \cos A \cos B + \sin A \sin B = \cos(A-B) \\ \cos(2\pi k f_c t) \cos(2\pi k f_c t + 2\pi k f_c \tau) \\ + \sin(2\pi k f_c t) \sin(2\pi k f_c t + 2\pi k f_c \tau) \\ = \cos(2\pi k f_c \tau) \end{array} \right)$$