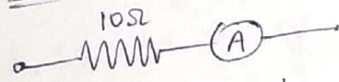


Unit-1. Random Process (or) Stochastic Process

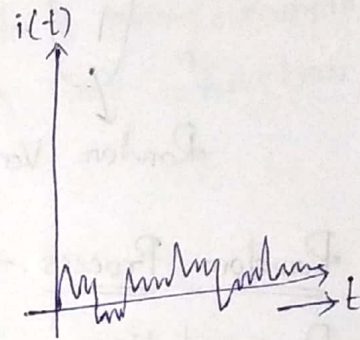
Let us take an example (T, H) - these random events - these are mapped to a real numbers (or) values (0,1) - this is called "Random Variable".

Random Process:- The above events when mapped to a function of time $x(t)$ (wave or signal). This is called "Random Process".

Ex:- Practical Example



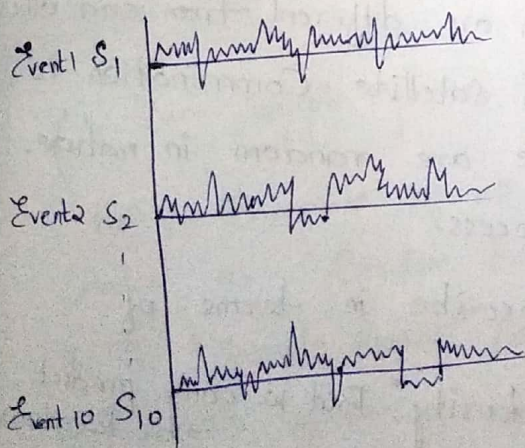
Ammeter is connected to a Resistor without voltage source. Then there exist a small current (less Amplitude) which is a "Random Current". This is similar to "Random Process".



X - Random Variable
 x - value / Element

Ensemble:- Means it is a collection (or) family of similar function.

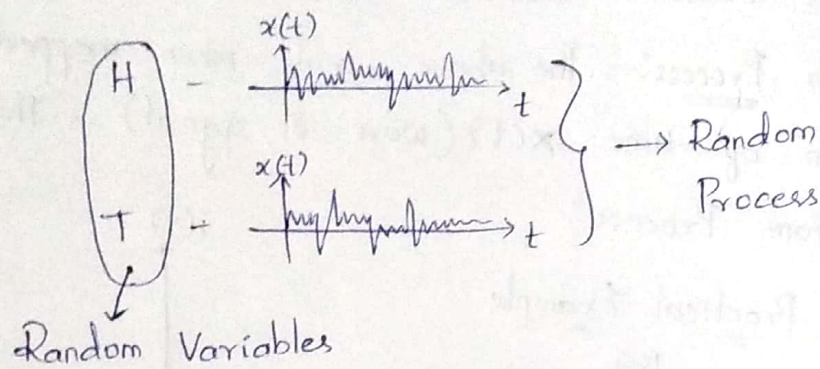
For example, 10 students are provided doing experiment (same) with this resistor & ammeter. The 'random current' will be different for 10 students. Everyach student gets different random current.



This Entire function is called as 'Random Process'.

* Random Variable:- The outcome of the random experiment is mapped into a number.

* Random Process:- The outcome of the random experiment is mapped to a waveform $[x(t)]$ which is a function of time.



Random Process:- (R.P)

Representation $\rightarrow X(t)$

* "Information Bearing Signal" means signal which contains information. Also called as "message signal".
(or) "Base band Signal".

Ex:- Voice Signal, Video Signal (or) data, all these are natural signals which are random.

* All communications are defined based on its frequency ranges.

Ex:- TV Communication, infrared Communication (or) microwave communication.

* Noise & interference both are different from each other.
For mobile communication, satellite communication is the interference. All these are random in nature.
So we go for Random Process.

These Random, we can describe in terms of

* "Power & Power Spectral density". But we can't predict this Random.

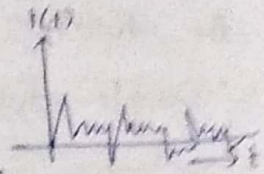
* If two persons are talking, it is interference.
But Noise is disturbance.

Mathematical Definition of a Random Process:

* Random process has 2 properties

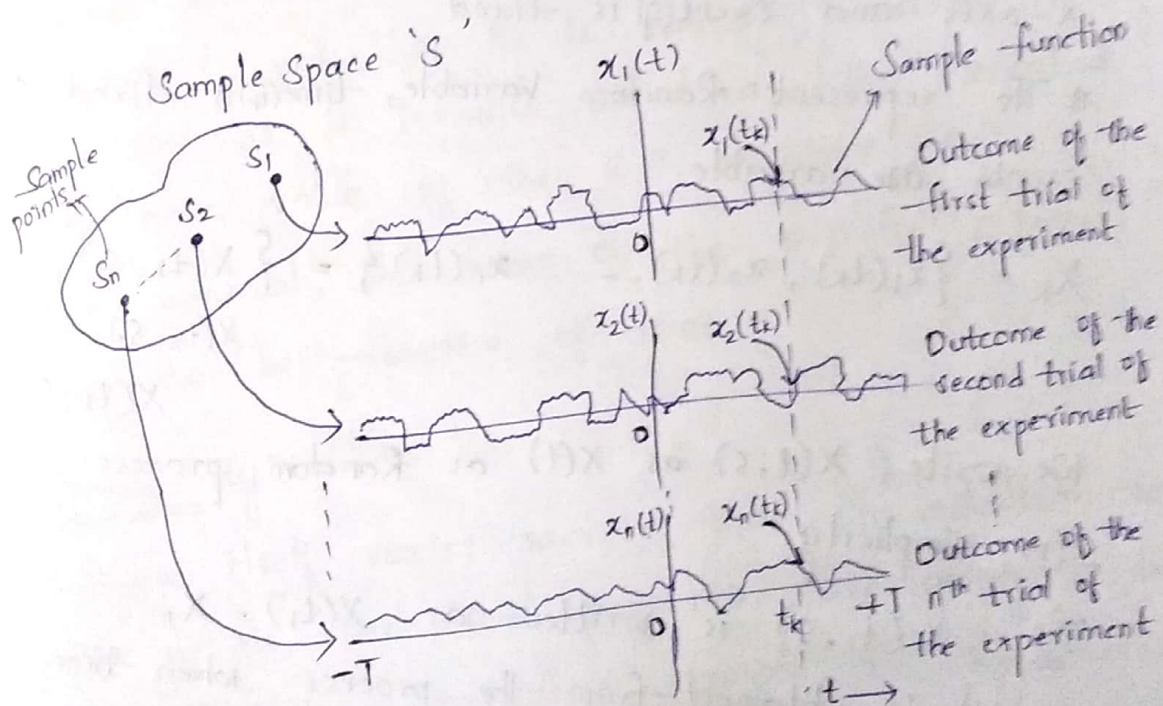
1. Random process is a function of time

2. They are random in nature in the sense that before conducting the experiment, it is not possible to exactly define the waveform that will be observed in future.



Now, Consider a sample space 'S' and its outcome is 's' (sample point) and for each & every sample point we are going to assign a time function i.e., $x(t, s)$ to every outcome.

time Event



An ensemble of sample fn's.

(∵ Each sample point is having a sample function)

Hence, Ensemble of all these sample functions is called as "Random Process".

* The "Sample Space" (or) "Ensemble" composed of functions of time is called "Random (or) Stochastic Process", defined as $X(t, s)$

* For fixed sample point s_j (Event), the graph of the function $X(t, s_j)$ vs time is called as a "Sample function of the Random process".

$$x_j(t) = X(t, s_j)$$

from fig, a set of sample functions

$$\{x_j(t) | j = 1, 2, \dots, n\}$$

* For a fixed time ' t_k ', Random Process represents a Random Variable.

i.e., Each sample fn contains some value (amplitude) at time ' t_k '.
(s) (Sample Value)

* To represent sample fn, time is a variable along x-axis and Event(s_j) is fixed.

* To represent Random Variable, time(t_k) is fixed & events are variable.

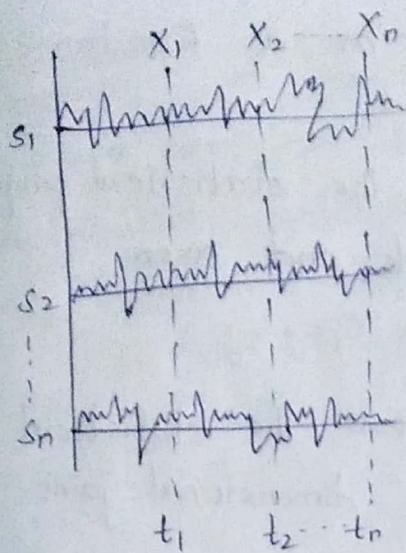
$$X_k = \{x_1(t_k), x_2(t_k), \dots, x_n(t_k)\} = \left\{ \begin{array}{l} X(t_k, s_1), \\ X(t_k, s_2), \dots \\ X(t_k, s_n) \end{array} \right\}$$

We write, $X(t, s)$ as $X(t)$ as Random process for simplicity.

Ex:- $X(t_1, s)$ is written as $X(t_1) = X_1$, which is obtained from the process when time is frozen (fixed) at $t = t_1$.

$$X_k = X(t_k, s) = X(t_k) \text{ where } k = 1, 2, \dots$$

Imp * A Random process can also represent a mere (only) number when t & s are both fixed.



At t_1 we have random variable ' X_1 ' & at t_2 we have ' X_2 '.
Similarly at t_n we have ' X_n '.
Even though these random variables are different at different time instants, the
"Probability density fn is same."

The Statistical Properties

- * The statistical properties of $X_1 = X(t_1)$ describes the statistical properties of the R.P at time ' t_1 '.
- * Once we obtained the R.V at time ' t_1 ', the expected value of X_1 is called "Ensemble Average" as well as it is also called as the expected (or) mean value of the R.P at time ' t_1 '.
- * In general, the mean of the R.P is " $E[X(t)]$ " and can be function of time.

Stationarity:-

- * Signal itself varies w.r.to time. In addition to that if its statistical properties are going to be vary, then there are less chances to characterize that signal (or) to analyze the signal.
- * If we are unable to analyze the signal, we don't know what we are receiving.
- * So, everytime if it is random in nature, we can't analyze the signal.
- * The condition for stationary is statistical properties should be constant w.r.to time."

* A Random Process becomes as a Random Variable when time is fixed.

* A Random Variable possess the statistical properties like mean value, moments, and so on. associated means and so on.

* 'n' random variable will possess the statistical properties related to the 'n'-dimensional joint density function.

(Theory in Peeble's Textbook)

Definition:-

* A R.P is said to be "Stationary" if all its statistical properties do not change with time.

* Statistical properties indicates that means, variances, moments^{etc} which are related to joint probability density function. (pdf).

* If its statistical properties depends on time, then R.P is said to be "Non-Stationary".

Ex for Stationary:-

* If the Probability density functions (Pdf) of all Random Variables ($X_1, X_2, \dots, X_n$) at time instants $t_1, t_2, \dots, t_n$ is same, then it is called as a 'Stationary'. i.e., Statistical properties of these random variables ($X_1, X_2, \dots, X_n$) are independent of time.

Distribution & Density functions:-

* At a particular time instant ' t_1 ', -the associated distribution function with random variable $X_1 = X(t_1)$.

* The distribution fn,
 $F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$ for any real number ' x_1 '.

This is the definition of Probability
Distribution fn (P.D) (R.V)

For two R.V, $X_1 = X(t_1)$ & $X_2 = X(t_2)$

The second order joint distribution fn,

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1; X(t_2) \leq x_2\}$$

Similarly,

For 'N' R.V, $X_1 = X(t_1)$, $X_2 = X(t_2)$, ..., $X_N = X(t_N)$

The N^{th} Order joint distribution fn,

$$F_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

* The First Order density fn is

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

The Second Order joint density fn,

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \cdot \partial x_2}$$

The N^{th} Order joint density fn,

$$f_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = \frac{\partial^N F_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N)}{\partial x_1 \cdot \partial x_2 \cdot \dots \cdot \partial x_N}$$

First Order Stationary Process:-

* At every instant of time, -the density function should be same, -then it is called as a first order Stationary Process.

$f_x(x_1; t_1) = f_x(x_1; t_1 + \Delta)$, $\rightarrow$ It is true for any value of 't', and any real number ' Δ '

* All the density-fn's should be independent of time.

* This density-function characterizes the mean value. So, Mean value should be independent of time.

The mean value, $E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_x(x_1; t_1) dx_1$

The mean value should be constant because it should not depend on time.

$$\boxed{\therefore \bar{X} = E[X_1] = \text{constant}}$$

Second Order & Wide Sense Stationary (WSS):

* A process is called a stationary of order two if its second order density-fn i.e., joint density function satisfies.

$$f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; t_1 + \Delta, t_2 + \Delta) \text{ for all } t_1, t_2 \text{ \& } \Delta$$

* This joint density-fn should be dependent on time differences but not on the absolute time i.e., it should not depend on t_1, t_2 . It should depend on the time difference of t_1, t_2 . ($t_2 - t_1$)

Correlation:

* It is the expected value of the product

$$E[X_1 X_2] = E[X(t_1) X(t_2)] = R_{xx}(t_1, t_2)$$

Correlation in general, it is

In general, Correlation is a fn of time ' t_1 ' & ' t_2 '

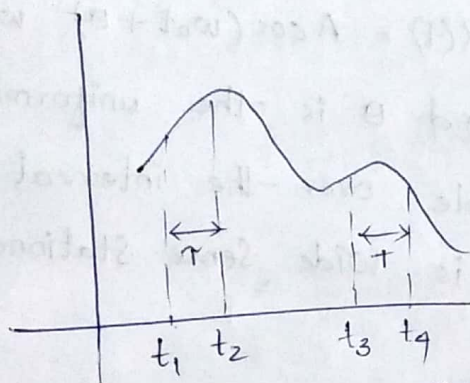
But, for Second Order Stationary, it should depend on time difference of t_1 & t_2 i.e., $(t_2 - t_1)$.

* Auto Correlation is calculated b/w the different time instants of same R.P. $X(t)$.

$$R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)] = R_{XX}(t_2 - t_1)$$

* 'Auto-Correlation function' of a 'second order stationary process' is a function only of a time differences not on the absolute time.

Let us take a low-pass signal,



Consider time instants t_1, t_2
let the time difference
b/w t_1 & t_2 be ' τ '.

This time difference should
be same if we consider
somewhere in the signal.

i.e., The variation within the same separation, the
should be same.

* Assume that the entire R.P. is divided into the
number of equal separation sections. Then everywhere
'Auto-Correlation should be same'. (Difference
should be same).

$$\tau = t_2 - t_1 \rightarrow \text{Time difference}$$

$$R_{XX}(t_1, t_1 + \tau) = E[X(t_1) X(t_1 + \tau)] = R_{XX}(\tau)$$

* If the R.P. satisfies the below condition,

$$\text{Mean} \leftarrow E[X(t)] = \bar{X} = \text{constant}$$

$$E[X(t) X(t + \tau)] = R_{XX}(\tau)$$

Then this R.P. is called Wide Sense Stationary (WSS)

* If mean is constant & Auto-Correlation is a function of time difference, then the R.P. 'X(t)' is called 'WSS'.

For
* Most of the practical Random Processes, in order to check the stationarity, upto second order it is sufficient.

In the second order, it indicates that mean is constant & Auto-Correlation is a fn of time difference.

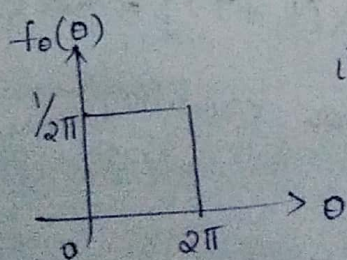
Problem:-

① A Random Process $X(t) = A \cos(\omega_0 t + \theta)$ where A, ω_0 are constants and θ is the uniformly distributed random variable over the interval $(0, 2\pi)$. Is this R.P. is Wide Sense Stationary (WSS)?

Sol:- Given that, $X(t) = A \cos(\omega_0 t + \theta)$
 θ is uniformly distributed random variable.

$$f_{\theta}(\theta) = \begin{cases} \frac{1}{2\pi} & ; 0 < \theta < 2\pi \\ 0 & ; \text{otherwise.} \end{cases}$$

We can't estimate the wave-form at the receiver side as ' θ ' is random variable.



(i) Mean Value, $\omega\pi$

$$E[X(t)] = \int_0^{2\pi} A \cos(\omega_0 t + \theta) \cdot \frac{1}{2\pi} \cdot d\theta$$

$$= \frac{A}{2\pi} \int_0^{2\pi} \cos(\omega_0 t + \theta) \cdot d\theta$$

$$= 0 \quad (\because \text{One cycle})$$

$$\therefore E[X(t)] = 0 = \text{Constant}$$

* At any time instant, the mean should be zero.

(ii) Auto-Correlation Function,

$$\begin{aligned} R_{xx}(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\left[\underbrace{A \cos(\omega_0 t + \theta)}_A \underbrace{A \cos(\omega_0 t + \omega_0 \tau + \theta)}_B\right] \\ &= \frac{A^2}{2} E\left[\cos(\omega_0 t + \theta + \omega_0 t + \omega_0 \tau + \theta) + \cos(\omega_0 t + \theta - \omega_0 t - \omega_0 \tau - \theta)\right] \end{aligned}$$

$$\left(\because \cos A \cdot \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]\right)$$

$$R_{xx}(t, t+\tau) = \frac{A^2}{2} E\left[\cos(2\omega_0 t + \omega_0 \tau + \theta) + \cos(-\omega_0 \tau)\right]$$

$$\left(\because \cos(-\theta) = \cos(\theta)\right)$$

$$= \frac{A^2}{2} E\left[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + \theta)\right]$$

$$= \frac{A^2}{2} \cos(\omega_0 \tau) + \frac{A^2}{2} E\left[\cos(2\omega_0 t + \omega_0 \tau + \theta)\right]$$

$$\left(\because E[\text{constant}] = \text{constant}\right. \\ \left.- \text{Avg value of constant is constant}\right)$$

$$= \frac{A^2}{2} \cos(\omega_0 \tau) + \frac{A^2}{2} \int_0^{2\pi} \cos(2\omega_0 t + \omega_0 \tau + \theta) \cdot \frac{1}{2\pi} d\theta$$

$$\boxed{\therefore R_{xx}(t, t+\tau) = \frac{A^2}{2} \cos(\omega_0 \tau)}$$

Here, Auto-Correlation is a fn of time difference ' τ ' but not fn of absolute times $(t, t+\tau)$

$\therefore X(t)$ is Wide Sense Stationary (WSS)

* If we consider two R.P.s $X(t)$, $Y(t)$, then we say it is a jointly Wide Sense Stationary when the below conditions are satisfied.

* The $X(t)$ & $Y(t)$ ^{both} should be WSS individually.
In addition to that, Cross-Correlation should be a function of time difference not ~~on~~ the function of absolute times.

Cross Correlation function is,

$$R_{xy}(t_1, t_2) = E[X(t_1) Y(t_2)]$$

$$R_{xy}(t, t+\tau) = E[X(t) Y(t+\tau)] = R_{xy}(\tau)$$

Each R.P. should satisfy WSS property,
~~Mean Value,~~

$$E[X(t)] = \text{constant}$$

$$E[Y(t)] = \text{constant}$$

$$E[X(t) X(t+\tau)] = R_{xx}(\tau) \quad E[Y(t) Y(t+\tau)] = R_{yy}(\tau)$$

Nth Order & Strict-Sense Stationary (SSS)::-

* For 'N' R.V.s $X_i = X(t_i)$ where $i = 1, 2, \dots, N$

If N^{th} Order joint density function is invariant to a time origin shift, then the process is said to be Statistically Stationary in the Strict Sense

$$\text{i.e., } f_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = f_X(x_1, x_2, \dots, x_N; t_1+\Delta, \dots, t_N+\Delta)$$

for all $t_1, t_2, \dots, t_N$ & Δ

* Stationarity of order 'N' implies stationarity to all order $K \leq N$. A process stationary to all orders $N = 1, 2, \dots$ is called "Strict Sense Stationary" (SSS).

* All SSS processes are also WSS but NOT all WSS processes are SSS.

Time Averages and Ergodicity:-

* The time-Average of a quantity defined as

$$A[\cdot] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [\cdot] dt$$

where 'A' is used to denote the time average (E for statistical-Average).

$$\text{Time-Average} = A[x(t)] = \bar{x}$$

(A lower case letter indicates a sample-function)

$$\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \cdot dt$$

* Time-Auto-Correlation-function,

$$\begin{aligned} R_{xx}(\tau) &= A[x(t)x(t+\tau)] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt \end{aligned}$$

Ergodic Process:-

* Some Stationary random processes possess the property that almost every member of the ensemble exhibits the same statistical behaviour as the whole ensemble. Thus it is possible to determine this statistical behaviour by examining only one typical sample-function. Such processes are called "Ergodic".

Condition:-

Time-Averages = Statistical-Averages.

$$\begin{aligned} \text{i.e., } A[x(t)] &= E[x(t)] \\ \bar{x} &= \bar{X} \end{aligned}$$

$$R_{xx}(\tau) = R_{XX}(\tau)$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t+T) dt = E[x(t) x(t+T)]$$

* The R.P which satisfy the above conditions is called "Ergodic Processes".

Problem:-

② A R.P $X(t) = A \cos(\omega_0 t + \theta)$ where A, ω_0 are constants and θ is the uniformly distributed random variable over the interval $(0, 2\pi)$.
Is this R.P is Ergodic?

Sol:- Given,

$$X(t) = A \cos(\omega_0 t + \theta)$$

We already calculated the mean & Auto-Correlation in previous problem ①.

$$\text{Mean} \rightarrow E[X(t)] = 0 = \text{Constant}$$

$$\text{Auto-Correlation fn} \rightarrow R_{XX}(t, t+T) = \frac{A^2}{2} \cos(\omega_0 T)$$

For one sample function, θ is any value in the range $(0, 2\pi)$

let it be ϕ ,

$$x(t) = A \cos(\omega_0 t + \phi)$$

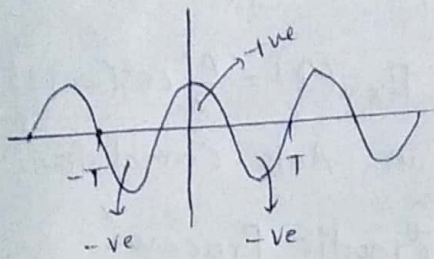
$$A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A \cos(\omega_0 t + \phi) dt$$

Here, T is not fundamental time period (It is very large value & it can be anything).

If T is F.T.P, then $\int_{-T}^T A \cos(\omega_0 t + \phi) dt$ would be zero ($\because$ One complete cycle).

But here, T is not F.T.P. So, this integration is

exactly not equal to zero.



Let us consider the wave-form with this time interval $(-T \text{ to } T)$.

After integrating, one +ve & one -ve get cancelled but another

-ve cycle remains. So, it is not equal to zero.

If we consider complete cycles, then it can be zero.

$\therefore$ Integration value may be zero or may not be zero (finite).

$$A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A \cos(\omega_0 t + \phi) dt$$

finite

As $T \rightarrow \infty$, $\frac{1}{2T}$ becomes zero

$$\boxed{\therefore A[x(t)] = 0 = E[x(t)]} \quad (\because E[x(t)] = 0)$$

$\therefore X(t)$ is Ergodic in Mean

The ^{Time} Auto-Correlation function,

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t+\tau) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T A \cos(\underbrace{\omega_0 t + \phi}_B) A \cos(\underbrace{\omega_0 t + \omega_0 \tau + \phi}_A) dt$$

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{A^2}{2T} \cdot \frac{1}{2} \left[\int_{-T}^T \underbrace{\cos(\omega_0 \tau)}_{\text{constant}} dt + \int_{-T}^T \cos(2\omega_0 t + \omega_0 \tau + 2\phi) dt \right]$$

$$\left(\because \cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \right)$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{4T} \left[\cos(\omega_0 \tau) \left[\int_{-T}^T dt \right] + \int_{-T}^T \cos(2\omega_0 t + \omega_0 \tau + 2\phi) dt \right]$$

$$= \lim_{T \rightarrow \infty} \frac{A^2}{4T} \cos(\omega_0 \tau) (2T) + \lim_{T \rightarrow \infty} \frac{A^2}{4T} \int_{-T}^T \underbrace{\cos(2\omega_0 t + \omega_0 \tau + 2\phi)}_{\text{finite}} dt$$

$$= 0 \quad \left(\because \text{As } T \rightarrow \infty, \frac{A^2}{4T} = 0 \right)$$

$$\therefore \boxed{R_{xx}(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau) = R_{xx}(\tau)}$$

$$(\because R_{xx}(\tau) = \frac{A^2}{2} \cos(\omega_0 \tau))$$

The R.P $X(t)$ is Ergodic in Auto-Correlation.

$\therefore$ "The R.P $X(t)$ is an Ergodic Process".

Jointly Ergodic:-

* Two random process $X(t), Y(t)$ are called "Jointly Ergodic" if they are individually ergodic and also have time cross correlation function, that equals to statistical Cross-Correlation function.

$$R_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) y(t+\tau) dt = R_{xy}(\tau) = E[x(t) y(t+\tau)]$$

Statistical Independence:-

* Two processes $X(t)$ & $Y(t)$ are "Statistically Independent", if the random variables $X(t_1), X(t_2), \dots, X(t_N)$ is independent of $Y(t'_1), Y(t'_2), \dots, Y(t'_M)$ for any choice of time $t_1, t_2, \dots, t_N, t'_1, t'_2, \dots, t'_M$.

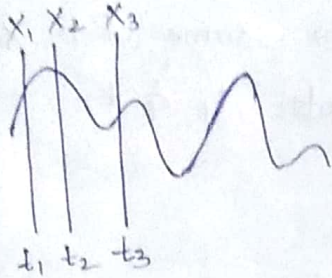
So, Independence requires that the joint pdf be factorable by groups of

$$f_{xy}(x_1, \dots, x_N, y_1, \dots, y_M; t_1, \dots, t_N, t'_1, \dots, t'_M) \\ = f_x(x_1, \dots, x_N; t_1, \dots, t_N) \cdot f_y(y_1, \dots, y_M; t'_1, \dots, t'_M)$$

Correlation:-

* The correlation b/w two random variables is defined as a expected value of their product.

Ex:- Let us consider a low-pass signal,



The correlation b/w two time instants, we find how it is varied. It is going to depend on the how rapidly this function is going to be varied.

The correlation depends on time separation of two time instants.

* If the time separation is less, correlation will be more i.e., they are highly correlated (Those points are similar). Ex:- Random Variables of t_1 & t_2 .

* If the time separation is more, correlation will be less i.e., they have more variation.

Ex:- Random Variables of t_1 & t_3 .

* In correlation function, the argument is nothing but time separation

* For R.V case, correlation was simply a number since

But here, every pair of R.Vs can be related by by the time separation between them & correlation will be a f_n (function) of this separation.

Auto Correlation:-

* If two R.Vs come from the same R.P, this function will be known as the "Auto Correlation function."

Cross Correlation:-

* If two R.V's come from different R.Ps, it will be called "Cross Correlation function".

Auto Correlation & its properties:-

* The R.Vs are considered from same R.P $X(t)$ at two different time instants t_1 & t_2 .

$$X_1 = X(t_1) \quad X_2 = X(t_2)$$

Auto Correlation R_{XX} is defined as,

$$R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$R_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 \cdot f_X(x_1, x_2; t_1, t_2) dx_1 dx_2$$

This definition is valid for any stationary (or) non-stationary R.Ps.

Let us consider time instants $t_1 = t$
 $t_2 = t + \tau$

$$R_{XX}(t, t + \tau) = E[X(t) X(t + \tau)]$$

For WSS, the Auto Correlation depends on time difference ' τ '.

$$\boxed{R_{XX}(\tau) = E[X(t) X(t + \tau)]} \quad \text{for all values of 't'}$$

Properties:-

$$1. \boxed{R_{XX}(0) = E[X^2(t)]}$$

$R_{XX}(0) = \bar{X}^2 = \text{Mean Square Value} = \text{Power of R.P}$

This mean square value of the process is obtained by putting ' $\tau = 0$ ' from $R_{XX}(\tau)$ by putting $\tau = 0$.

* If R.P. $X(t)$ has a zero mean, then this mean square value becomes as the 'variance'.

$$\sigma_x^2 = E[X^2(t)] - \underbrace{[E[X(t)]]^2}_{0 \rightarrow \text{Mean}} = E[X^2(t)] = R_{xx}(0)$$

Mean Square Value is the Power of the R.P.

2. $R_{xx}(\tau) = R_{xx}(-\tau)$

Even Symmetric about τ -axis

The Auto Correlation function $R_{xx}(\tau)$ is an even function of τ .

Proof:- We know, $R_{xx}(\tau) = E[X(t)X(t+\tau)]$

$$R_{xx}(-\tau) = E[X(t)X(t-\tau)]$$

Let $t-\tau = u, \Rightarrow t = u+\tau$ u is dummy variable

$$R_{xx}(-\tau) = E[X(u+\tau)X(u)] = R_{xx}(\tau)$$

$$\left(\because R_{xx}(\tau) = E[X(t)X(t+\tau)] \right)$$

$$\boxed{\therefore R_{xx}(-\tau) = R_{xx}(\tau)}$$

Advantage:-

* It is sufficient to derive only for +ve value of ' τ ' and the result for the -ve value of ' τ ' is going to be determined from its symmetry.

3. $|R_{xx}(\tau)| \leq R_{xx}(0)$

At $\tau=0$, we get the maximum magnitude value of $R_{xx}(\tau)$. Because, we know that if the time separation (τ) decreases, correlation increases.

Here $\tau=0$ i.e., time difference is zero.

* As τ increases, Auto Correlation decreases.

Proof:- Let us consider the non-negative quantity.

$$E[(X(t+\tau) \pm X(t))^2]$$

$$E[(X(t+\tau) \pm X(t))^2] \geq 0$$

$$E[X^2(t+\tau)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

According to 1st property,

$$E[X^2(t)] = R_{xx}(0)$$

for WSS case,

$$E[X^2(t+\tau)] = R_{xx}(0)$$

We know that,

$$E[X(t+\tau)X(t)] = R_{xx}(\tau)$$

$$\Rightarrow 2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$R_{xx}(0) \pm R_{xx}(\tau) \geq 0$$

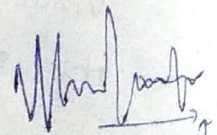
$$R_{xx}(0) \geq |R_{xx}(\tau)|$$

$$\text{Here, } (-R_{xx}(0) \leq R_{xx}(\tau) \leq R_{xx}(0))$$

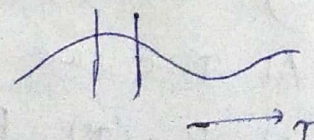
$$\boxed{\therefore |R_{xx}(\tau)| \leq R_{xx}(0)}$$

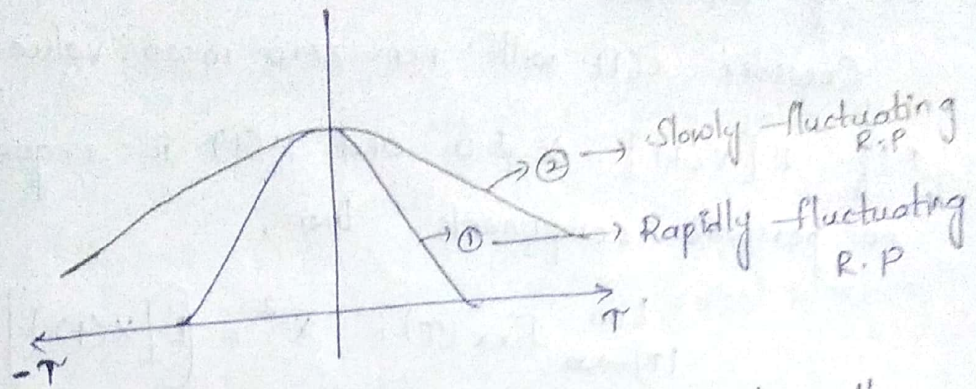
* If signal varies very rapidly, then the

Auto Correlation decreases very fast.



* If signal varies ^{very} slowly, then the auto-correlation increases.





- ① This is a auto-correlation function of a "rapidly fluctuating random process." i.e, signal is varying very fast. So, auto-correlation also decreases rapidly.
- ② If signal varies slowly, then the auto-correlation fn decreases very slowly as τ increases.
- * It is symmetric about ' τ ' axis.
 - * As τ increases, auto-correlation decreases.

4) If $x(t)$ has a periodic component, then $R_{xx}(\tau)$ will have a periodic component with same period.

Proof:- Consider a R.P ~~x(t)~~. $x(t)$, is a periodic with period T_0 .

$$x(t \pm T_0) = x(t) \rightarrow \textcircled{1}$$

$$x(t + \tau \pm T_0) = x(t + \tau) \rightarrow \textcircled{2}$$

$$R_{xx}(\tau) = E[x(t) \cdot x(t + \tau)]$$

$$R_{xx}(\tau \pm T_0) = E[x(t) \cdot x(t + \tau \pm T_0)]$$

$$= E[x(t) \cdot x(t + \tau)] \quad (\because \text{from } \textcircled{2})$$

$$\boxed{R_{xx}(\tau \pm T_0) = R_{xx}(\tau)}$$

$\therefore$ Auto Correlation fn is also periodic with period T_0 .

Ex:- (Problem 1)

5) ~~5) If $E[X(t)] \neq 0$~~

Consider $X(t)$ with non-zero mean value.

* If $E[X(t)] = \bar{X} \neq 0$ and $X(t)$ is ergodic with no periodic components then,

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = \bar{X}^2 = [E[X(t)]]^2$$

Proof:-

$$R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$$

* No periodic components and as $|\tau| \rightarrow \infty$
(Time Separation becomes very large)

then $X(t)$, $X(t+\tau)$ become independent as time difference is very large.

$$\begin{aligned} \lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) &= E[X(t)] \cdot E[X(t+\tau)] \\ &= \bar{X} \cdot \bar{X} \quad (\because E[XY] = E[X] \cdot E[Y]) \end{aligned}$$

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = (\bar{X})^2 = [E[X(t)]]^2$$

As there is square of mean value, we can't find the 'sign'.

6) If $X(t)$ is ergodic, with zero mean & has no periodic component then,

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = 0$$

Proof:- from 5th property, substitute mean $(\bar{X}) = 0$.

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = (\bar{X})^2 = (0)^2 = 0$$

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = 0$$

* If time separation (τ) is more, then the R.V becomes as independent

* As τ increases, the effect of past value doesn't depend on present value.

7) If the R.P $X(t)$ has a dc component 'A' (mean (or) Avg value) & $N(t)$ (~~Recd~~ Received signal is combination of unwanted & wanted signal) has a zero mean then $X(t) = A + N(t)$ then

$$R_{xx}(\tau) = A^2 + R_{NN}(\tau)$$

Proof:- $X(t) = A + N(t)$

$$\begin{aligned} R_{xx}(\tau) &= E[A(A + N(t)) (A + N(t + \tau))] \\ &= E[A^2 + AN(t + \tau) + AN(t) + N(t)N(t + \tau)] \\ &= A^2 + A \underbrace{E[N(t + \tau)]}_0 + A \underbrace{E[N(t)]}_0 \\ &\quad + E[N(t)N(t + \tau)] \end{aligned}$$

($\because N(t)$ has a zero mean)

$$= A^2 + E[N(t)N(t + \tau)]$$

$$\boxed{R_{xx}(\tau) = A^2 + R_{NN}(\tau)}$$

8) This property can be ~~detailed~~ explained detail later (i.e, after completion of power spectral density).

* $R_{xx}(\tau)$ cannot have an arbitrary shape.

This means that any arbitrary fn cannot be ACF.

Q) A R.P has the 'ACF' $R_{xx}(\tau) = \frac{4\tau^2 + 6}{\tau + 1}$. Find the mean-square value, the mean value & the variance of the process.

Sol:- Given, $R_{xx}(\tau) = \frac{4\tau^2 + 6}{\tau + 1}$

(i) Mean Square Value $R_{xx}(0) = \frac{4(0)^2 + 6}{0 + 1}$ (from property ①)
 $= 6$

Power of R.P $\boxed{\therefore E[x^2(t)] = R_{xx}(0) = 6}$

(ii) Mean Value,

$$\lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = \lim_{|\tau| \rightarrow \infty} \frac{4\tau^2 + 6}{\tau^2 + 1} = \lim_{|\tau| \rightarrow \infty} \frac{\tau^2 [4 + \frac{6}{\tau^2}]}{\tau^2 [1 + \frac{1}{\tau^2}]}$$

$$= \frac{4}{1} = 4 \quad \text{(from property ⑤)}$$

$$\therefore \lim_{|\tau| \rightarrow \infty} R_{xx}(\tau) = (\bar{x})^2 = 4$$

$$\Rightarrow \text{Mean Value } (\bar{x}) = \sqrt{4} = \pm 2$$

$$E[x(t)] = \boxed{\therefore (\bar{x}) = \pm 2}$$

(Using ⑤ property we can't find sign of mean value)

(iii) Variance $(\sigma_x^2) = E[x^2(t)] - [E(x(t))]^2$

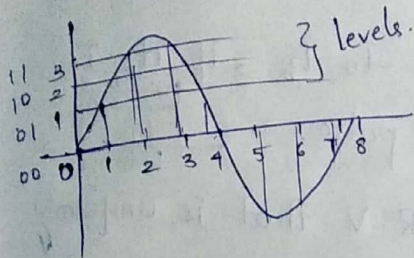
$$\sigma_x^2 = 6 - 4 = 2$$

$$\boxed{\therefore \sigma_x^2 = 2}$$

03/06/21

Auto Correlation Function of Random Binary Wave:-

A continuous time signal is converted to DTS by sampling & amplitude levels are converted using quantization.



In CTS, b/w 0 to 1 time instants there will be infinite values.

In DTS, we eliminate all those values & take only discrete numbers. (time instants).

Amplitudes also discrete quantized to make as discrete numbers.

These discrete numbers are converted into binary form. depending upon the no. of levels.

Eg:- In order to represent 256 levels, 8 bits are required.

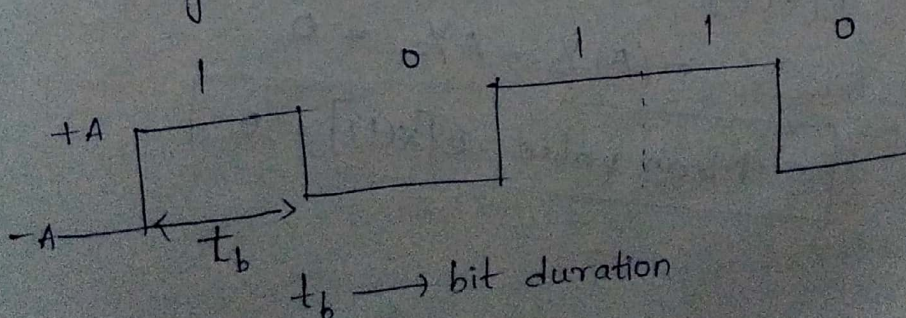
* The binary form will be completely in random nature. Eg:- 001110110

Random Signal

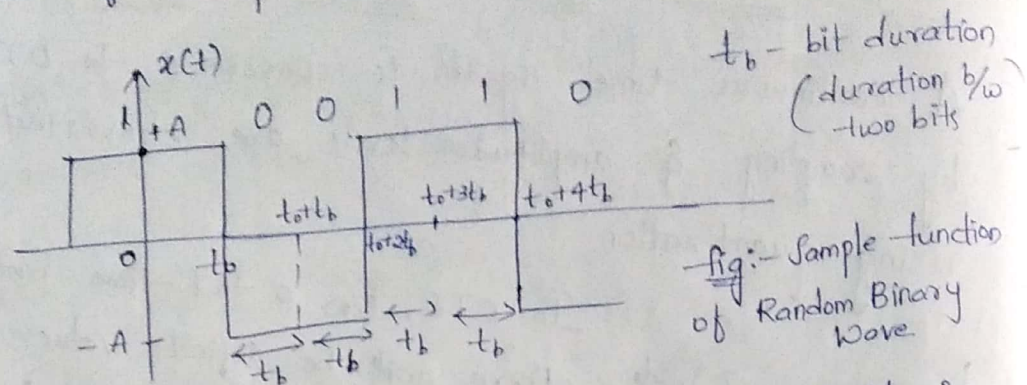
* For binary, only two symbols (0, 1) will be there. Representing these symbols as voltage levels

(∵ We convert these binary sequence into electrical before entering antenna).

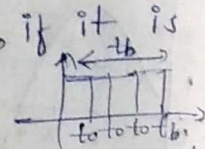
∴ Symbol 0 → -A volts
Symbol 1 → +A volts



Consider
One of the possible form of random process,



* t_0 can be anywhere b/w 0 to t_b , if it is greater than t_b , i.e., second pulse.



So, the starting position is R.V that is uniformly distributed in the interval 0 to t_b . because there is no synchronization. We are not assuming the pulse synchronization at the receiver side.

* We don't know starting time. So, we are assuming that t_0 is uniformly distributed R.V b/w 0 to t_b .

t_0 - origin of the pulse

* Bit intervals are independent to each other

* A typical sample function from a discrete, (Amplitudes are discrete, not time because before sending it to channel, the signal should be analog), stationary, zero mean R.P in which only two values $+A$, $-A$.

Probability $(+A) = 1/2$
Probability $(-A) = 1/2$
(Like head & tail)

Mean value $\rightarrow X: P(x)$

$$\therefore E[x(t)] = (+A)\left(\frac{1}{2}\right) + (-A)\left(\frac{1}{2}\right)$$

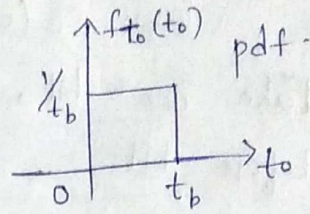
$$= (A)\frac{1}{2} - A\frac{1}{2} = 0$$

Mean value	$E[x(t)] = 0$
------------	---------------

* The sample $x(t)$ either can change from one value to other every t_b seconds or remains same.

* The time ' t_0 ' (starting time) is the R.V with respect to ensemble of possible time function, and is uniformly distributed over an interval of length ' t_b '.

$$f_{t_0}(t_0) = \begin{cases} 1/t_b & 0 \leq t_0 \leq t_b \\ 0 & \text{otherwise} \end{cases}$$



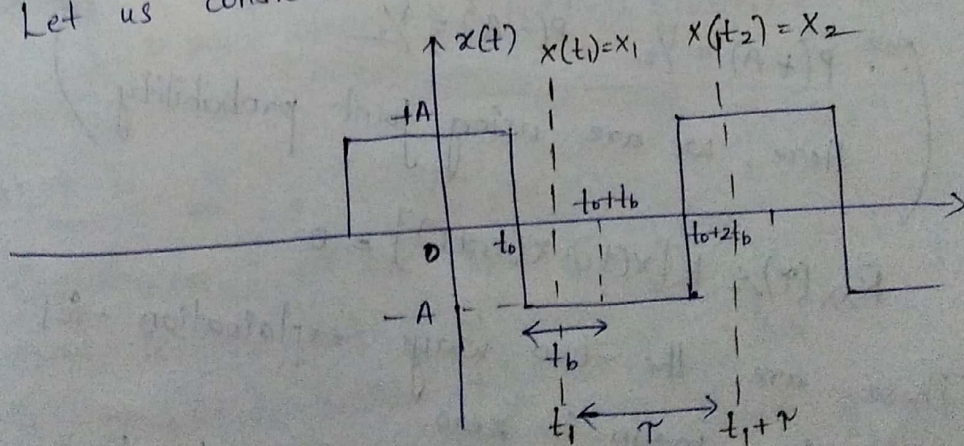
* The value of $x(t)$ in any one interval is statistically independent of its value in any other interval. $\left(\because \frac{1}{b-a} = \frac{1}{t_b-0} = \frac{1}{t_b} \right)$

Two cases:- (i) $|\tau| > t_b$ (ii) $|\tau| \leq t_b$

(i) $|\tau| > t_b$:-

If the time separation is greater than t_b , then ACF is,

Let us consider one R.V at time instant ' t_1 '



As pulses are independent to each other, the R.V's are also $(x_1 \& x_2)$ independent to each other.

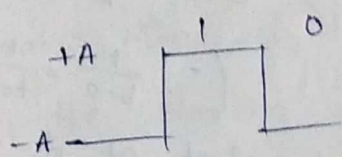
$\therefore$ If $|\tau| > t_b$, then t_1 & $t_1 + \tau$ cannot lie in the same interval. So, x_1, x_2 are statistically independent.

$$\therefore E[x(t)] = 0 = E[x(t_1)] = E[x(t_2)]$$

The ACF, $R_{xx}(\tau) = E[x(t_1)x(t_2)]$
 $= E[x(t_1)] \cdot E[x(t_2)] = 0$

(or)
 $R_{xx}(\tau) = E[x(t_1)x(t_1+\tau)] = E[x(t_1)] \cdot E[x(t_1+\tau)] = 0$

(or)
 Take one bit, the next bit doesn't depend on before bit.



There are four possibilities.

(In head & tail exp, if first trail is head, then second trail may be head or tail)

$$\begin{aligned} x(t_1)x(t_1+\tau) &= (+A) \times (+A) = A^2 \\ &= (+A) \times (-A) = -A^2 \\ &= (-A) \times (+A) = -A^2 \\ &= (-A) \times (-A) = A^2 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{4 possibilities}$$

$$E[x(t_1)x(t_1+\tau)] = A^2 \cdot \frac{1}{4} - A^2 \cdot \frac{1}{4} - A^2 \cdot \frac{1}{4} + A^2 \cdot \frac{1}{4} = 0$$

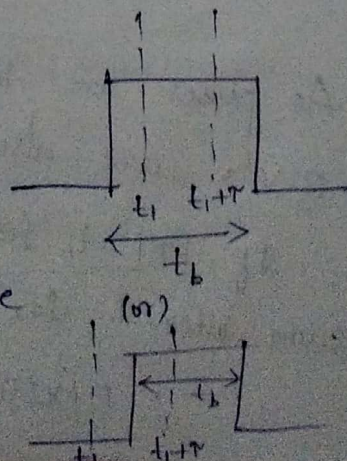
$\therefore P(+A) = \frac{1}{2} \quad P(-A) = \frac{1}{2}$
 Here, we are using joint probability

$$R_{xx}(\tau) = E[x(t_1)x(t_1+\tau)] = 0$$

These are the two ways explanation for mean i.e., mean is zero.

(ii) $|\tau| < t_b$

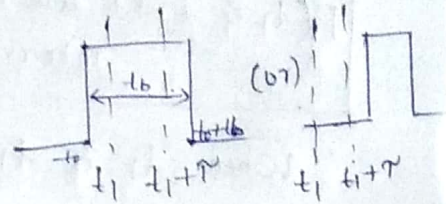
If $|\tau| < t_b$ then, $t_1, t_1+\tau$ may or may not be in same interval depending upon the value of t_b .



(a) τ is +ve:-

$$(t_1 + \tau - t_1 = \tau)$$

Considering $t_1, t_1 + \tau$ in the same interval, but τ is +ve
 Same interval means $t_1, t_1 + \tau$
 should lie b/w t_0 to $t_0 + t_b$
 i.e., $t_0 \leq t_1$ & $t_1 + \tau \leq t_0 + t_b$



$$t_1 + \tau - t_b \leq t_0$$

$$X(t_1) \cdot X(t_1 + \tau) = A \cdot A = A^2$$

$$(b) (-A) \cdot (-A) = A^2$$

$$\Rightarrow t_1 + \tau - t_b \leq t_0 \leq t_1$$

$\therefore$ Probability (t_1 & $t_1 + \tau$ are in same interval)

$$= P(t_1 + \tau - t_b \leq t_0 \leq t_1)$$

$$= \frac{t_1 - (t_1 + \tau - t_b)}{t_b} = \frac{t_b - \tau}{t_b}$$

($\therefore t_0$ is a uniformly distributed R.V over the interval 0 to t_b)

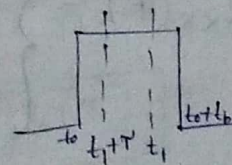
$$= \frac{t_b - \tau}{t_b}$$

$$\left(\begin{aligned} P(3 \leq x \leq 4) \\ &= \frac{4-3}{5} \\ &= \frac{1}{5} \end{aligned} \right) \quad \text{Diagram showing a rectangle from 0 to 5 with a shaded region from 3 to 4.$$

$$\therefore P(t_1 \text{ \& } t_1 + \tau \text{ are in same interval}) = P(t_1 + \tau - t_b \leq t_0 \leq t_1) = \frac{t_b - \tau}{t_b}$$

(b) τ is -ve:-

Consider $t_1, t_1 + \tau$ are in same interval but τ is -ve (i.e., $t_1 - (t_1 + \tau) = -\tau$)



$$t_0 \leq t_1 + \tau \text{ \& } t_1 \leq t_0 + t_b$$

$$t_1 - t_b \leq t_0$$

$P(t_1 \text{ \& } t_1 + \tau \text{ are in same interval})$

$$\Rightarrow t_1 - t_b \leq t_0 \leq t_1 + \tau$$

$$= P(t_1 - t_b \leq t_0 \leq t_1 + \tau)$$

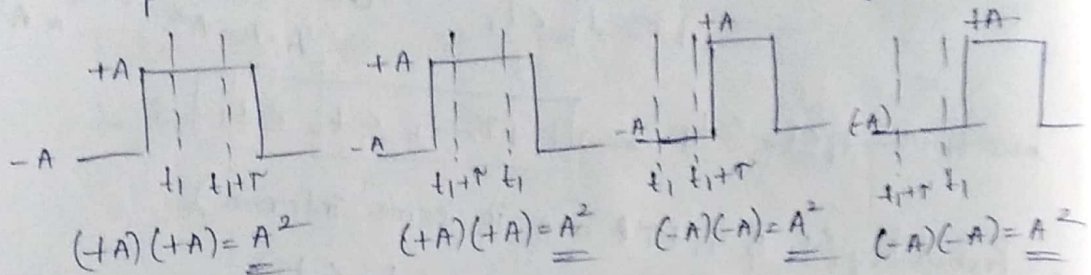
$$= \frac{t_1 + \tau - (t_1 - t_b)}{t_b} = \frac{t_b + \tau}{t_b}$$

$$\therefore P(t_1 \text{ \& } t_1 + \tau \text{ are in same interval}) = P(t_1 - t_b \leq t_0 \leq t_1 + \tau) = \frac{t_b + \tau}{t_b}$$

In general, (By (a) & (b))

$$P(t_1 \text{ \& } t_1 + \tau \text{ are in same interval}) = \frac{t_b - |\tau|}{t_b}$$

∴ When t_1 & $t_1 + \tau$ are in same interval, the product of $x(t_1) \times x(t_1 + \tau)$ is always A^2 .



$$R_{xx}(\tau) = A^2 \cdot P(t_1 \text{ \& } t_1 + \tau \text{ are in same interval})$$

$$= A^2 \left(\frac{t_b - |\tau|}{t_b} \right)$$

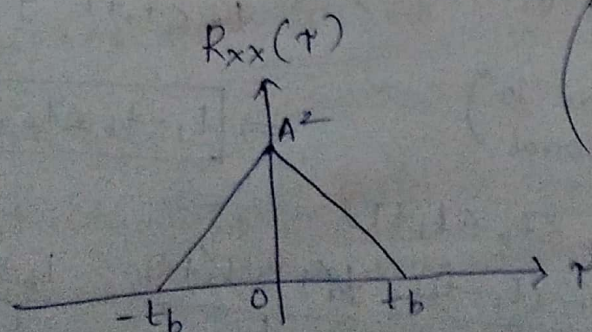
$$R_{xx}(\tau) = A^2 \left(1 - \frac{|\tau|}{t_b} \right) \quad 0 \leq |\tau| \leq t_b$$

t_b is constant value (bit duration)

Here, $R_{xx}(\tau)$ is depending upon only two time instants t_1 & $t_1 + \tau$. (Difference b/w two time instants i.e., τ)

∴ It is Wide Sense Stationary Time Separation.

$$\therefore R_{xx}(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{t_b} \right) & 0 \leq |\tau| \leq t_b \\ 0 & |\tau| > t_b \end{cases}$$



(∴ Triangular function)
 $= \begin{cases} 1 - \text{slope} \cdot |\tau| & |\tau| \leq t_b \\ 0 & |\tau| > t_b \end{cases}$
 where s is slope

04/06/21 Cross-Correlation function & its properties:-

Let two R.P.s $X(t)$ & $Y(t)$, the cross correlation fn is $R_{xy}(t, t+\tau) = E[X(t) \cdot Y(t+\tau)]$

* If $X(t)$ & $Y(t)$ are atleast jointly wide sense stationary.

$$R_{xy}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

* If two R.P.s are statistically independent,

$$R_{xy}(t, t+\tau) = E[X(t)] \cdot E[Y(t+\tau)]$$

* for atleast WSS,

$$R_{xy}(\tau) = \bar{X} \cdot \bar{Y} = \text{Constant}$$

* If $R_{xy}(t, t+\tau) = 0$, then $X(t)$ & $Y(t)$ are called orthogonal processes.

Properties:- $R_{xy}(\tau) = E[X(t) Y(t+\tau)] \rightarrow \text{Atleast WSS}$

$$\textcircled{1} R_{xy}(-\tau) = R_{yx}(\tau) \quad R_{yx}(\tau) = E[Y(t) \cdot X(t+\tau)]$$

It is symmetric but not even or odd.

Proof:- $R_{yx}(-\tau) = E[Y(t) X(t-\tau)] \quad \begin{matrix} t-\tau = u \\ t = u+\tau \end{matrix}$

$$R_{yx}(-\tau) = E[Y(u+\tau) X(u)]$$

$$R_{yx}(-\tau) = E[X(u) \cdot Y(u+\tau)]$$

$$\boxed{\therefore R_{yx}(-\tau) = R_{xy}(\tau)}$$

$$\textcircled{2} |R_{xy}(\tau)| < \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

In Auto correlation, we consider only one R.P. So, we can say that max value will be at $\tau = 0$.

But here, we have two R.P.s. Each R.P. will have max value at different τ values. So, we cannot say that max value is at $\tau = 0$.

* The cross correlation function does not necessarily have its max value at $\tau=0$.

Proof:- $R_{xx}(\tau)$ & $R_{yy}(\tau)$ are ACFs of $x(t)$ & $y(t)$ respectively.

$$\text{Let } E \left[\left[\frac{x(t)}{\sqrt{R_{xx}(0)}} \pm \frac{y(t+\tau)}{\sqrt{R_{yy}(0)}} \right]^2 \right] \geq 0$$

$$E \left[\frac{x^2(t)}{R_{xx}(0)} + \frac{y^2(t+\tau)}{R_{yy}(0)} \pm \frac{2x(t)y(t+\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \right] \geq 0$$

$$\frac{E[x^2(t)]}{R_{xx}(0)} + \frac{E[y^2(t+\tau)]}{R_{yy}(0)} \pm \frac{2E[x(t)y(t+\tau)]}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

By Auto Correlation property ①,

$$E[x^2(t)] = R_{xx}(0) \quad E[y^2(t+\tau)] = R_{yy}(0)$$

$$\therefore \frac{R_{xx}(0)}{R_{xx}(0)} + \frac{R_{yy}(0)}{R_{yy}(0)} \pm \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq 0$$

$$2 \pm \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

$$2 \left(1 \pm \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \right) \geq 0$$

$$1 \pm \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

$$1 \geq \mp \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}}$$

$$\sqrt{R_{xx}(0) \cdot R_{yy}(0)} \geq |R_{xy}(\tau)|$$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

Note:-

Geometric mean of $a, b = \sqrt{ab}$

Arithmetic mean of $a, b = \frac{a+b}{2}$

$$(3) |R_{xy}(\tau)| \leq \frac{1}{2} [R_{xx}(0) + R_{yy}(0)]$$

Proof:- Geometric mean $\leq$ Arithmetic mean.

for $R_{xx}(0), R_{yy}(0)$

Geometric mean $= \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$

Arithmetic mean $= \frac{1}{2} [R_{xx}(0) + R_{yy}(0)]$

$$\Rightarrow \sqrt{R_{xx}(0) \cdot R_{yy}(0)} \leq \frac{1}{2} [R_{xx}(0) + R_{yy}(0)]$$

By property (2), $[|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}]$

$$\therefore |R_{xy}(\tau)| \leq \frac{1}{2} [R_{xx}(0) + R_{yy}(0)]$$

* The "Correlation Matrix" of R.P.s $x(t)$ & $y(t)$ is

$$R(t, t+\tau) = \begin{matrix} x & \begin{bmatrix} R_{xx}(t, t+\tau) & R_{xy}(t, t+\tau) \\ R_{yx}(t, t+\tau) & R_{yy}(t, t+\tau) \end{bmatrix} \\ y \end{matrix}$$

Atleast joint wss case,

$$R(\tau) = \begin{bmatrix} R_{xx}(\tau) & R_{xy}(\tau) \\ R_{yx}(\tau) & R_{yy}(\tau) \end{bmatrix}$$

Covariance Functions:-

For the same signal, variance is second central moment

$$\sigma_x^2 = E[(X - \bar{X})^2]$$

* The Auto Covariance function is defined by

$$C_{xx}(t, t+T) = E\left[\{X(t) - E[X(t)]\}\{X(t+T) - E[X(t+T)]\}\right]$$

$$\left[\because \text{Cov}(X, Y) = E[(X - \bar{X})(Y - \bar{Y})] \right]$$

* Variance of R.P can be obtained by making $T=0$.

$$\sigma_x^2 = E\left[\{X(t) - E[X(t)]\}^2\right]$$

$$C_{xx}(t, t+T) = E\left[X(t) \cdot X(t+T) - X(t) \cdot E[X(t+T)] - E[X(t)] \cdot X(t+T) + E[X(t)] \cdot E[X(t+T)]\right]$$

$$C_{xx}(t, t+T) = E[X(t)X(t+T)] - E[X(t)]E[X(t+T)] - E[X(t)]E[X(t+T)] + E[X(t)]E[X(t+T)]$$

$$C_{xx}(t, t+T) = R_{xx}(t, t+T) - E[X(t)] \cdot E[X(t+T)] \quad \text{--- (1)}$$

For processes atleast wss,

$$C_{xx}(\tau) = R_{xx}(\tau) - \bar{X} \cdot \bar{X} = R_{xx}(\tau) - (\bar{X})^2$$

* The Cross-Covariance function for 2 R.Ps $X(t)$ & $Y(t)$ is defined by.

$$C_{xy}(t, t+T) = E\left[\{X(t) - E[X(t)]\}\{Y(t+T) - E[Y(t+T)]\}\right]$$

$$\text{for atleast wss, } C_{xy}(t, t+T) = E[X(t) \cdot Y(t+T) - X(t) \cdot E[Y(t+T)] - E[X(t)] \cdot Y(t+T) + E[X(t)] \cdot E[Y(t+T)]]$$

$$= E[X(t) \cdot Y(t+T)] - E[X(t)] \cdot E[Y(t+T)] - E[X(t)] \cdot E[Y(t+T)] + E[X(t)] \cdot E[Y(t+T)]$$

$$C_{xy}(t, t+T) = R_{xy}(t, t+T) - E[X(t)] \cdot E[Y(t+T)] \quad \text{--- (2)}$$

$$\text{for atleast wss, } C_{xy}(\tau) = R_{xy}(\tau) - \bar{X} \cdot \bar{Y}$$

* For two R.P's if $C_{xy}(t, t+\tau) = 0$

Then they are called uncorrelated.

from (2),

$$C_{xy}(t, t+\tau) = R_{xy}(\tau) - E[x(t)] \cdot E[y(t+\tau)]$$

$$0 = R_{xy}(\tau) - E[x(t)] \cdot E[y(t+\tau)]$$

$$R_{xy}(\tau) = E[x(t)] \cdot E[y(t+\tau)]$$

∴ If 2 R.P's are uncorrelated, then they are independent also.

* The independent processes are uncorrelated, but the converse case is not necessarily true.

Covariance Matrix,

08/06/21

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \dots & C_{NN} \end{bmatrix}$$

→ This is for N-random Variable case.

The diagonal elements — Auto Covariance
Remaining elements — Cross Covariance

$$C_{ij} = E[(x_i - \bar{x}_i)(x_j - \bar{x}_j)] = \begin{cases} \sigma_{x_i}^2, & i=j \\ C_{x_i x_j}, & i \neq j \end{cases}$$

(Explained Assignment problems)

09/06/21

Mean Ergodic processes:-

* For WSS, $x(t)$ is the R.P with a constant mean value $\bar{x}$. It is called as a Mean Ergodic.

* If the statistical average $\bar{x}$ for any sample function $x(t)$ is time average, then it is ergodic

$$E[x(t)] = \bar{x} = A[x(t)] = \bar{x} \text{ with probability '1'}$$

Let us assume 4 conditions for $x(t)$.

1, $x(t)$ has a finite constant mean value $\bar{x}$ for all values of 't'.

2, $x(t)$ is bounded i.e., $|x(t)| < \infty$ for all 't' & all $x(t)$ comprising $x(t)$.

3, $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)| dt < \infty$

4, $x(t)$ is a regular process.

$$\text{i.e., } E[|x(t)|^2] = R_{xx}(t, t) < \infty$$

for a real WSS process,

$$E[(x(t))^2] = R_{xx}(0) < \infty$$

Since $\bar{x}$ is finite, $C_{xx}(0) = R_{xx}(0) - (\bar{x})^2$ is also less than infinity

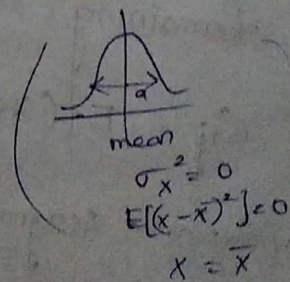
$$\text{i.e., } C_{xx}(0) = R_{xx}(0) - (\bar{x})^2 < \infty$$

Time-Avg Value: of sample fn :-

$$\bar{x} = A[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$$

Time Avg of a R.P.:-

$$\text{R.V } A_x = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) \cdot dt$$



The mean value,

$$\bar{A}_x = E[A_x] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)] \cdot dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \bar{x} [t]_{-T}^T$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \bar{x} [0]$$

$$\boxed{\bar{A}_x = \bar{x}}$$

Chebyshev's inequality:-

$$P\{|A_x - \bar{A}_x| < \epsilon\} \geq 1 - \left(\frac{\sigma_{A_x}^2}{\epsilon^2}\right) \quad \text{for any } \epsilon > 0$$

If $\sigma_{A_x}^2$ is zero, then irrespective of ϵ value

$$A_x = \bar{A}_x$$

Now, for $\epsilon > 0$, no matter how small, if $\sigma_{A_x}^2 = 0$

then $A_x = \bar{A}_x$ with probability '1'

(Proof is in peebles textbook)

$\Rightarrow$ The R.V A_x is a constant

$$\begin{aligned} \sigma_{A_x}^2 &= E[(A_x - \bar{A}_x)^2] \\ &= E\left[\left\{\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt - \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)] dt\right\}^2\right] \\ &= E\left[\left\{\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (x(t) - \bar{x}) dt\right\}^2\right] \\ &= E\left[\lim_{T \rightarrow \infty} \left(\frac{1}{2T}\right)^2 \int_{-T}^T \int_{-T}^T (x(t_1) - \bar{x})(x(t_2) - \bar{x}) dt_1 dt_2\right]\right] \\ &= E\left[\lim_{T \rightarrow \infty} \left(\frac{1}{2T}\right)^2 \int_{-T}^T \int_{-T}^T C_{xx}(t_1, t_2) dt_1 dt_2\right] \end{aligned}$$

for WSS, $C_{xx}(t_1, t_2) = C_{xx}(\tau)$

$$\tau = t_2 - t_1 \quad d\tau = dt_2$$

limits:- $\tau = -T - t_1$ (O.L. limit)

$\tau = T - t_1$ (U. limit)

$$\sigma_{A_x}^2 = E\left[\lim_{T \rightarrow \infty} \left(\frac{1}{2T}\right)^2 \int_{-T-t_1}^{T-t_1} \int_{-T}^T C_{xx}(\tau) dt_1 d\tau\right]$$

Changing the order of integration,

$$\sigma_{A_x}^2 = E\left[\lim_{T \rightarrow \infty} \left(\frac{1}{2T}\right)^2 \int_{t_1=-T}^T \int_{-T-t_1}^{T-t_1} C_{xx}(\tau) d\tau dt_1\right]$$

∴ It is satisfying both conditions.

∴ $R_{xx}(\tau) = C_{xx}(\tau) = e^{-2\alpha|\tau|}$ is mean-ergodic

11/06/21 Gaussian Random Variable:-

The probability density function of Gaussian R.V. 'x'.

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \cdot \exp\left[-\frac{(x-\bar{x})^2}{2\sigma_x^2}\right] \quad -\infty < x < \infty$$

$N(\mu, \sigma^2) \rightarrow$ Representation of Gaussian R.V

Why it is important?

* Most of the practical physical phenomenon finally leads to Gaussian distribution.

For the theoretical analysis also the Gaussian R.V is a good model.

* In order to deal with large no. of R.V's,

Gaussian R.V is very comfortable. As $n \rightarrow \infty$ this R.V is very comfortable.

* The linear combination of ^{Jointly} Gaussian R.V will give rise to another R.V.

* With Gaussian R.V, if we define the R.P

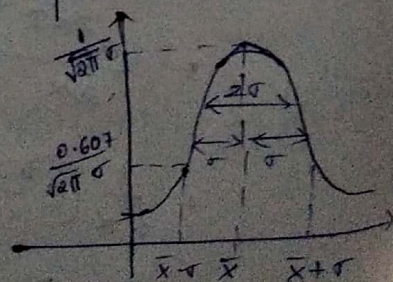
From the R.P, if we define R.V, then the complete description can get through first & second order moments i.e, mean, variance & covariance & autocorrelation. (no need of nth order)

* We can do statistical analysis for the linear & non-linear systems through this Gaussian R.V.

Graphically represented as,

* It is symmetric about mean.

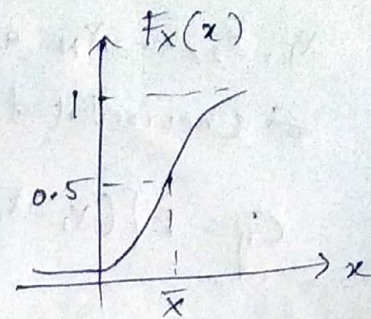
* Max value occurs only at mean value.



The distribution function $F_X(x)$,

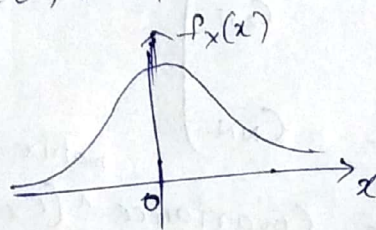
* Max value is 1

* The value at $\bar{x}$ (mean) is 0.5



$\Rightarrow N(0, 1) \rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Rightarrow$ Normalized Gaussian R.V

$\mu=0, \sigma^2=1$



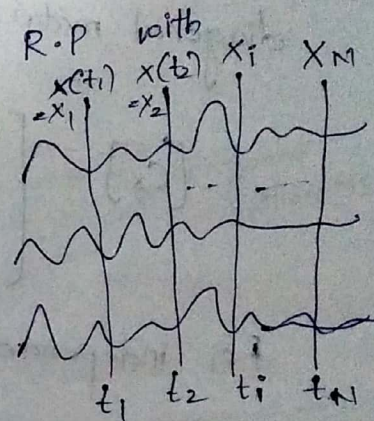
(or) Standard Gaussian R.V

Gaussian Process:-

Let us consider a continuous sample functions shown in fig.

Total 'N' R.V's.

* If these R.V's are jointly gaussian then this process is called Gaussian Process.



* If the samples of a R.P's are joint Gaussian R.V's, then the process is a Gaussian Random process.

* The joint density is given by.

$$f_X(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = \frac{\exp\left\{-\frac{1}{2} [x - \bar{x}]^T [C_X]^{-1} [x - \bar{x}]\right\}}{\sqrt{(2\pi)^N |C_X|}}$$

Proof:- For simplicity, assuming these R.V's are statistically independent.

Imp For Gaussian R.V's, if the R.V's are statistically independent, they are uncorrelated. And the vice-versa is also true only for Gaussian R.V's

$X_1, X_2, \dots, X_N$ are statistically independent

$\Rightarrow$ Uncorrelated.

$$C_{ij} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] = \begin{cases} \sigma_{x_i}^2 & i=j \\ 0 & i \neq j \end{cases}$$

$= 0$ for independent R.V.s

The covariance matrix of 'N' R.V.s.

$$[C_x] = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \dots & C_{NN} \end{bmatrix} \quad \text{matrix}$$

for uncorrelatedness, the covariance $[C_x]$ becomes diagonal matrix. (Uncorrelated means independent also)

$$[C_x] = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N^2 \end{bmatrix}$$

For independent, $f_{xy}(x, y) = f_x(x) \cdot f_y(y)$

(Product of individual density fn's)

The joint density fn can be written as.

$$f_x(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = \prod_{i=1}^N f_{x_i}(x_i) \quad \xrightarrow{\text{Product}}$$

where, $f_{x_i}(x_i) = \frac{1}{\sqrt{2\pi} \sigma_i} \cdot \exp\left[-\frac{(x_i - \bar{x}_i)^2}{2\sigma_i^2}\right]$ ($N \times$ independent R.V. Gaussian)

$$f_x(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{1}{(\sqrt{2\pi})^{N/2} \cdot \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_N} \exp\left\{-\sum_{i=1}^N \frac{(x_i - \bar{x}_i)^2}{2\sigma_i^2}\right\}$$

($\therefore$ Multiplied N times. $\Rightarrow (\sqrt{2\pi})^N = (2\pi)^{N/2}$)

$e^a \cdot e^b = e^{a+b}$

We define $[x - \bar{x}]$ as a matrix

$$[x - \bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}$$

$$[x - \bar{x}]^T [C_x]^{-1} [x - \bar{x}] = [x - \bar{x}]^T \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_N^2} \end{bmatrix} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}$$

$$= \sum_{i=1}^N \frac{(x_i - \bar{x})^2}{\sigma_i^2} \rightarrow (2)$$

$$|[C_x]| = \begin{vmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N^2 \end{vmatrix} = \sigma_1^2 \cdot \sigma_2^2 \cdot \dots \cdot \sigma_N^2$$

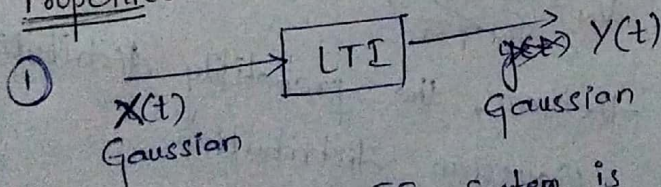
($\because$ det of diagonal matrix = Product of diagonal elements)

$$|[C_x]|^{1/2} = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_N \rightarrow (3)$$

Sub (2) & (3) in (1),

$$f_x(x_1, x_2, \dots, x_N; t_1, t_2, \dots, t_N) = \frac{1}{\sqrt{(2\pi)^N |C_x|}} \exp \left\{ -\frac{1}{2} [x - \bar{x}]^T [C_x]^{-1} [x - \bar{x}] \right\}$$

Properties:-



If the I/p of LTI System is Gaussian & then o/p is also gaussian.

② Wide Sense Stationary implies that strict Sense Stationary. (Valid only for Gaussian)

Because Joint pdf depends on first & second order

Statistical properties.

For WSS, $\bar{X}_i = E[X(t_i)] = \bar{X}$ Constant

$$R_{xx}(t_i, t_k) = R_{xx}(t_k - t_i)$$

$$C_{xx}(t_i, t_k) = C_{xx}(t_k - t_i)$$

Imp Central Limit Theorem:-

* When independent R.Vs are added, then their properly normalized sum tends towards a normal or Gaussian distribution

We have to assume those R.Vs are

1) X_i are statistically independent

2) X_i have same distribution with mean μ_x & variance σ_x^2

Based on these two statements we can say that a R.V is i.i.d (Independent, Identically & Independently distributed).

$\therefore X_i \rightarrow$ i.i.d R.Vs.

$$\text{If } Y_i = \frac{1}{\sigma_x} (X_i - \mu_x), i = 1, 2, \dots, N$$

$$\text{then } E[Y_i] = \frac{E[X_i - \mu_x]}{\sigma_x} = \frac{\mu_x - \mu_x}{\sigma_x} = 0$$

$$\text{Var}(Y_i) = E\left[\left(\frac{X_i - \mu_x}{\sigma_x}\right)^2\right] = E\left[\frac{(X_i - \mu_x)^2}{\sigma_x^2}\right] = \frac{\sigma_x^2}{\sigma_x^2} = 1$$

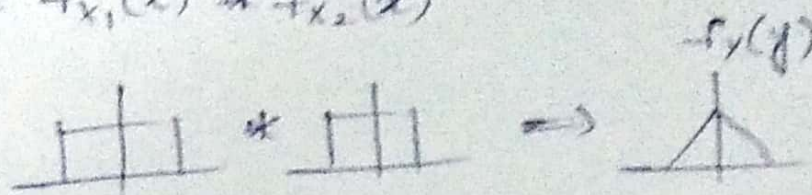
* Define the R.V, $V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i$

From Central limit theorem, the probability distribution of V_N become Gaussian distribution.

$N(0, 1)$ in the limits as $N \rightarrow \infty$

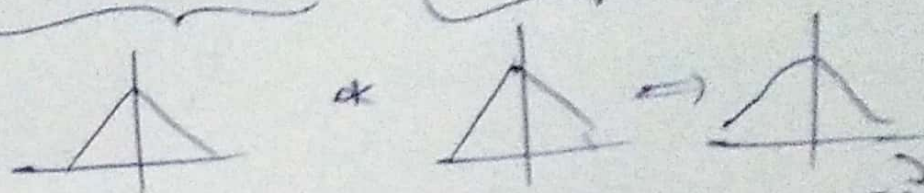
eg:- Let us consider $Y = X_1 + X_2$ where X_1 & X_2 are statistically independent & it is uniform pdf

-then $f_Y(y) = f_{X_1}(x) * f_{X_2}(x)$



If $Y = X_1 + X_2 + X_3 + X_4$ (All are uniform)

-then $f_Y(y) = \underbrace{f_{X_1}(x) * f_{X_2}(x)} * \underbrace{f_{X_3}(x) * f_{X_4}(x)}$



We took uniform R.V. but the sum lead to the Gaussian R.V.